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ABSTRACT

Acoustic instability in solid propellant rocket motors is a question of the balance of the acoustic gains and losses in the system. With special reference to transverse modes, this question is examined with respect to those loss and gain mechanisms for which some limited information is available. It is interesting that, depending on which mechanisms are predominant, the critical conditions for instability may be quite different functions of the parameters characterizing the rocket. A possible mechanism consistent with the main results of Brownlee and Marble is presented.

I. INTRODUCTION

Acoustic instability in solid propellant motors is a well recognized phenomenon.[1,2,3,4] Typically a motor spontaneously breaks into oscillation in one or another of its characteristic modes. The consequences are seldom desirable and often catastrophic to the mission of the engine.

A large part of our experimental knowledge of the phenomenon of acoustic instability in solid propellant motors is derived from empirical correlations between the appearance of instability and the values of design parameters. For example, the rather generally accepted statement that the tendency toward instability is less at higher operating pressure is such a correlation. The operating pressure is commonly controlled by selecting the K_n (ratio of powder burning surface area to nozzle throat area) of the system. Thus, the correlation might well have been with K_n. In perhaps the most systematic examination of geometrical factors published, Brownlee and Marble[2] examined a particular system in which they varied both K_n and the port diameter, D_p, and then were able to divide the K_n - D_p plane into stable and unstable regions with respect to the mode of oscillation (first tangential) predominant in their system.

Such correlations with geometrical factors might be important in two ways. On the one hand, they might provide the designer with some guidance, and on the other, they might provide the researcher with some clues. In either case, it is vital to know the degree to which general validity may be attached to the correlations.

As has been previously stressed,[1] acoustic instability is a property of the whole system, rather than of a component. In this sense, a solid propellant rocket motor may be thought of as a complex acoustic cavity. Within this cavity there will be two acoustically rather homogeneous regions (the solid and the gas regions) separated by a thin burning zone, which may amplify sound. The question of acoustic stability or instability is resolved at least in principle by determining whether the influx of acoustic power introduced by amplification at the burning surface is or is not exceeded by the loss of acoustic power via the several damping mechanisms. Thus, the correlations with design parameters must somehow reflect the combined properties of the acoustic sources and sinks in the system.

It is the purpose of this paper to examine the general validity of empirical correlations of instability with variations in geometrical parameters. Naturally, we shall limit ourselves to those factors about which we have some a priori knowledge and to systems whose analysis is reasonably tractable. In particular, the last qualification causes us to focus our attention on the transverse modes of motors which are relatively long compared to their diameter.

Several acoustic loss mechanisms are considered theoretically. Since attention is directed primarily toward transverse modes, the major losses will usually arise from: molecular relaxation phenomena in the propellant gases; small, medium and large particles in the gas, each depending in a characteristic way on frequency; viscoelastic losses in the solid propellant itself.

With respect to the acoustic source we shall assume this to be the burning surface and in particular will stress the response to pressure variations of the form suggested by theory.[1,5,8] We shall only touch briefly on the possible contributions from erosivity. Amplification in the volume of the gas is not specifically considered; however, it presumably could be accommodated as "negative" damping.

In spite of the paucity of information, it turns out to be possible to display the effects of these various losses in terms of K_n - D_p stability diagrams portraying regions of stability and instability in the parameter space of quantities describing the rocket motor conditions. These stability diagrams differ rather widely in character, depending on which of the several loss mechanisms predominate. In fact, it is quite conceivable that a correlation might take diametrically opposed forms in different systems which emphasize different mechanisms of damping.

II. THE BALANCE OF GAINS AND LOSSES

1. The Rocket Motor as an Acoustic Cavity

In the ordinary acoustic cavity, one considers acoustic damping arising from thermal losses to conducting walls, viscous losses to the wall, viscous, thermal and relaxation losses in the body of the gas, and mechanical losses to the non-rigid cavity wall. In addition, if the cavity is not entirely enclosed, loss through the opening would be considered. For the case of a cavity containing two media, (e.g., the solid propellant and the gas), one would also determine the viscoelastic loss in the solid.*

Prerequisite to the determination of acoustic losses is the determination of the acoustic field. Referring to Fig. 1, and comparing that rocket

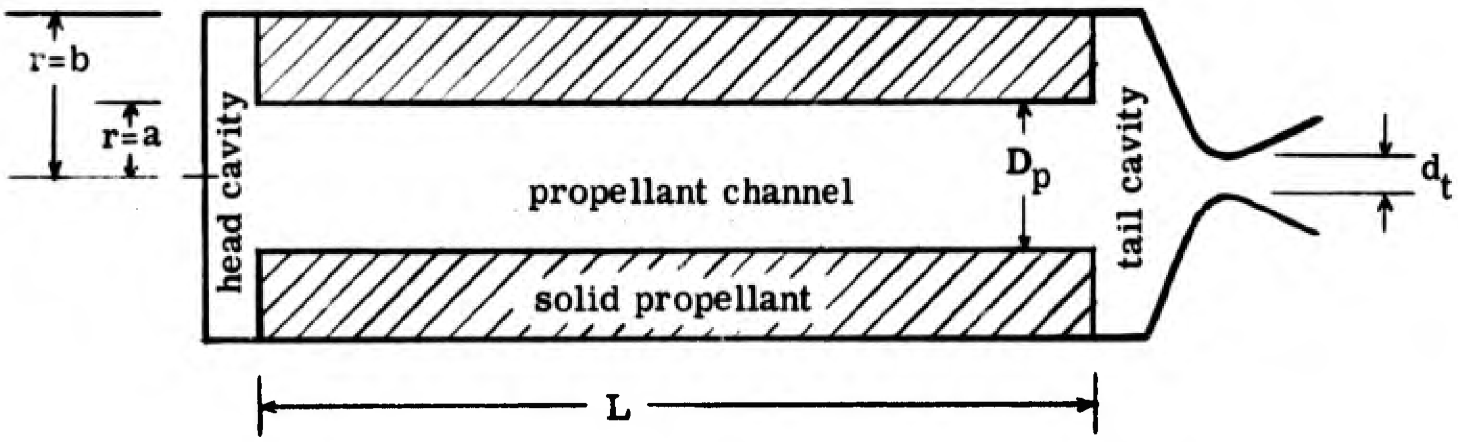

Fig. 1

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*Damping through radiation might also be mentioned (the quasi-adiabatic compression of the gas by the sound wave alters the gas temperature). This loss is difficult to estimate with accuracy, but it appears to us to be very small even at the elevated temperatures of current rocket motors.

motor with the usual acoustic cavity, we note that the motor presents at least two new difficulties of a rather fundamental nature, namely, a flow field and a sonic nozzle. In a general way the nozzle represents three simultaneous transitions of importance to acoustic theory. First, there is a transition from a large channel to a small one. From the standpoint of the propagation of transverse waves in a pipe, such a transition has relatively poor transmission for the fundamental modes of the larger pipe, though not necessarily for the overtones. Second, there is a transition in density and temperature which results in a drop in the characteristic impedance of the medium of almost a factor of two. This should also contribute to the reflection at the nozzle plane. Third, there is a jump in flow velocity from a moderate subsonic to sonic value. The consequences of this change are hard to predict and it is really not known whether the first two effects are thereby emphasized or diminished. While an approximate theory has been published for axial modes,[6] only incomplete and tentative studies have been carried out for transverse modes.[7] These suggest that the attenuation of transverse modes by the nozzle may be small. For the purpose of this paper, which is limited to the question of the fundamental transverse modes, the above considerations give us some comfort in our decision to proceed without estimating the nozzle losses.

Further complexity of a more practical than fundamental nature is introduced into the rocket motor problem through the common presence of head and tail cavities. These cavities, of course, can considerably complicate exact determination of the acoustic fields, but the head cavity (usually containing the igniter) is, at least, frequently omitted in rockets used for research purposes.

Before taking up the gains and losses in detail, it will be helpful to recall the basic elements of their calculation. When acoustic losses in cavities are relatively small (i.e., for high Q cavities), it is usually convenient to take advantage of perturbation techniques. Thus, one evaluates the acoustic field which would exist in the absence of losses (or gains), and then determines the losses (or gains) which would result in the presence of these "zero order" fields. For the purposes of determining the unperturbed field, we shall ignore the head and tail cavities, regarding the head plane as a velocity node and retaining generality at the port plane only to the extent that it may be either a velocity node or a pressure node. The latter boundary condition is of little importance to transverse modes for the usual small values of the ratio port radius to grain length. Then the general acoustic mode in the main channel has pressure (p) and velocity variations (u) given by the real parts of

$$p = \bar{P} p_o e^{i\omega t} \cos(m\phi) \cos\left(\frac{h\pi z}{2L}\right) J_m(\alpha r)$$

$$\underset{\sim}{u} = \frac{i}{\rho\omega} \text{ grad } p \qquad (1)$$

where $\alpha^2 = (\omega/c)^2 - (h\pi/2L)^2$, and $\bar{P} = \rho c^2/\gamma$ is mean chamber pressure, ρ is the product gas density, γ is the specific heat ratio, c is the sound velocity in the gas, $\bar{P} p_o$ is the amplitude of the acoustic pressure, and ω is the angular frequency ($\omega = 2\pi f$). Here, h even corresponds to closed end, h odd to open end. From (1), the time average sound energy in the chamber is

$$E = \frac{\pi}{8} \frac{La^2}{\gamma} \bar{P} \left[2^{\delta_{h,o} + \delta_{m,o}} \right] |p_o|^2 J_m^2(\alpha a) \left[F_m(\alpha a) + \frac{\alpha c^2}{a\omega^2} \frac{J'_m(\alpha a)}{J_m(\alpha a)} \right], \quad (2)$$

with the abbreviation $F_m(x) \equiv 1 - \frac{m^2}{x^2} + \left(\frac{J'_m(x)}{J_m(x)} \right)^2$.

The power transfer across a surface will, of course, be evaluated by determining the time average of the integral over the surface of the pressure times normal component of velocity. (The velocity will be expressed in terms of the pressure by means of the acoustic admittance of the surface.)

2. The Pressure Response of the Burning Surface

It appears that the sensitivity of the burning surface to acoustic pressure is the primary cause of acoustic instability in solid propellant rockets, at least in the high frequency transverse modes. In general, the acoustic response of the burning surface is described by an admittance which may be expressed by[8]

$$Y(a) = - \frac{\bar{v}}{\bar{P}} \left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma} \right) \quad (3)$$

where $\bar{v}$ is the mean product gas velocity normal to the burning surface, $\tilde{\mu}$ is the ratio of the complex amplitude of the fluctuating burning rate (per unit area) to the mean burning rate, and $\tilde{\epsilon}$ is the corresponding ratio of acoustic pressure amplitude at the burning surface to the mean pressure. For the acoustic field expressed by Eq. (1), this admittance leads to a gain (or loss) of acoustic power at the burning surface (r = a) given by:

($\langle\ \rangle$ denotes time average)

$$\dot{E}_p = a \int_0^{2\pi} d\varphi \int_0^L dz \left\langle \mathrm{Re}\left(p(a)\right) \mathrm{Re}\left(u_r(a)\right) \right\rangle \qquad (4a)$$

$$= -\frac{\pi a L}{4} \left(2^{\delta_{m,o} + \delta_{h,o}}\right) \left|p_o\right|^2 \bar{P}^2 J_m^2 (\alpha a) \; \mathrm{Re}\,(Y(a)) \qquad (4b)$$

$$\dot{E}_p = \frac{\pi}{4} aL \left(2^{\delta_{m,o} + \delta_{h,o}}\right) \left|p_o\right|^2 J_m^2 (\alpha a) \frac{\bar{m} c^2}{\gamma} \mathrm{Re}\left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma}\right) \qquad (4c)$$

where $\bar{v} = \bar{m}/\rho$ is the normal flow velocity of hot gases from the flame front. We note that if the propellant response function, real part of $\left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma}\right)$, is positive, then the burning propellant amplifies. Since propellants are found typically to produce instability over a wide frequency band, the response function is supposed to be a positive but rather slowly varying function of frequency in the relevant frequency domain. Although response functions have not yet been measured experimentally, various estimates as to order of magnitude do exist,[4] as well as theoretical calculations based on rather simplified hypothetical propellants.[8] In general, for amplifying surfaces, it appears reasonable to expect the value to lie in the range between zero and unity.

3. Erosive Response of the Burning Surface

In a previous paper[9] we gave some consideration to the effect of erosion on the acoustic response of a burning surface. The approach was chosen with the hope of providing a general framework for inspecting erosive effects in the limit of small (acoustic) perturbations. It escaped our attention at that time, however, that the treatment lost some of its intended generality when certain higher-order terms (such as $\frac{\partial^2 m}{\partial p \partial |v_e|}$) were dropped from Eq. (1c) of ref. 9. This would be valid only for a burning law in which the effects of pressure and erosive velocity are additive, whereas in the common empirical representation (for steady-state burning) these effects are multiplicative. Hence we are impelled to so modify our previous analysis that it will encompass more realistic burning laws than the purely additive one.

Fortunately, only a slight alteration in approach and a redefinition of symbols allows our previous derivation of the burning boundary condition to be carried over to the more general case. Thus, we again assume that the flux of hot gas from the burning surface is a quite general function of pressure at r = a, erosive velocity, and all their time derivatives (denoted by subscript "t"):

$$m = m\,(P(a),\ P_t(a),\ \cdots\cdots\ ;\ |v_e|\ ,\ |v_e|_{t},\ \cdots\cdots). \qquad (5a)$$

This expression we now expand about the steady-state* conditions $P(a) = \bar{P}$, $|v_e| = |\bar{v}_e|$, to obtain

$$m = m(\bar{P}, |\bar{v}_e|) + \left[\overline{\left(\frac{\partial m}{\partial P(a)} \right)} + \overline{\left(\frac{\partial m}{\partial P_t(a)} \right)} \frac{\partial}{\partial t} + \cdots \right] (P(a) - \bar{P})$$
$$+ \left[\overline{\left(\frac{\partial m}{\partial |v_e|} \right)} + \overline{\left(\frac{\partial m}{\partial |v_e|_t} \right)} \frac{\partial}{\partial t} + \cdots \right] (|v_e| - |\bar{v}_e|) , \tag{5b}$$

where we have retained all terms up to first-order in the acoustic variables $P(a) - \bar{P} \equiv \epsilon \bar{P}$, $|v_e| - |\bar{v}_e| = u_z$ signum $(\bar{v}_e)$. By its definition, m can also be written in the acoustic approximation as

$$m = \bar{m} + \bar{m} \frac{\epsilon}{\gamma} + \bar{\rho} (u_s - u_r(a)) \tag{5c}$$

where $\bar{m} \equiv m(\bar{P}, |v_e|)$, and u_s is the fluctuating normal velocity of the transpiring solid surface. Combining Eqs. (5b) and (5c), and introducing harmonic time variation ($\epsilon = \tilde{\epsilon} e^{i\omega t}$, $\underline{u} = \tilde{\underline{u}} e^{i\omega t}$, etc.) we obtain the desired condition at r = a:

$$-\left(\frac{\tilde{u}_r(a)}{c} \right) - \frac{\bar{m}}{\bar{\rho} c} \left(\tilde{\mu}_p - \frac{\tilde{\epsilon}}{\gamma} \right) + \frac{\bar{\rho} c}{\gamma} Y_s \tilde{\epsilon} = \frac{\bar{m}}{\bar{\rho} c} K(\omega) \tilde{u}_z \text{ signum } (\bar{v}_e) , \tag{5d}$$

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*This is the essential departure from the derivation of ref. 9 where the expansion was about the non-oscillatory state with no erosion, ($P(a) = \bar{P}$, $|v_e| = 0$) so that some higher terms in the expansion might remain of first order.

where Y_s is the admittance of the solid surface, and

$$\frac{\tilde{\mu}_p}{\tilde{\epsilon}} \equiv \frac{\bar{P}}{\bar{m}}\left[\left(\overline{\frac{\partial m}{\partial P(a)}}\right) + i\omega\left(\overline{\frac{\partial m}{\partial P_t(a)}}\right) + \cdots\right], \tag{5e}$$

$$K(\omega) \equiv \frac{1}{\bar{m}}\left[\left(\overline{\frac{\partial m}{\partial |v_e|}}\right) + i\omega\left(\overline{\frac{\partial m}{\partial |v_e|_t}}\right) + \cdots\right], \tag{5f}$$

are the pressure response function and acoustic erosivity, respectively, for the burning propellant.

Comparison of Eq. (5d) with the relation it supersedes (Eq. (2b) of ref. 9) reveals that the previous formulation of erosive effects can be generalized by making three modifications in it:

(a) Set the "dc erosion constant", k, in ref. 9 equal to zero;

(b) Allow M_a in ref. 9 to be a weak function of z;

(c) Interpret the response function $\tilde{\mu}/\tilde{\epsilon}$ as evaluated with, rather than without, steady erosion, and the K of ref. 9 as now defined in Eq. (5f) above. Both of these functions now may be dependent on z.*

Because of the change (a) the conclusions of ref. 9 about the contribution of dc erosion to stability are not necessarily applicable to any but an additive burning law. To the extent that (b) and (c) introduce z dependence,

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*Note that for a multiplicative burning law, $m = f_1(P, P_t, \ldots) \times f_2(|v_e|, |v_e|_t, \ldots)$, the z dependence of $\tilde{\mu}_p/\tilde{\epsilon}$ is only that arising from the small steady state pressure drop along the channel.

the conclusions based on symmetry considerations are somewhat weakened in the general case, but will still retain gross validity. It is important to note, however, that the conclusions regarding the contribution of ac erosion to stability remain essentially unchanged.

Turning now to the problem at hand, we will recall that reference 9 showed that if the magnitude of the ac erosion constant was not much greater than its zero frequency value, its effect on the stability of transverse modes was quite small. Then, for a general burning law, the main effect of erosion on the transverse modes will be contained in the modification it produces in the mean burning rate $\overline{m}$.

4. Relaxation Losses in the Gas

Several sources of acoustic attenuation will be recognized in the body of gas filling the propellant channel. There is, of course, absorption through ordinary gas viscosity and heat conduction. However, it appears that a far more important loss for rocket motors may be that arising from the transfer of acoustic energy into internal energy of the molecular constituents.[5]

The relaxation loss of sound energy in the burnt gases can be expressed in terms of the corresponding attenuation constant, α_g, by

$$\dot{E}_g = -2\alpha_g c E = -\frac{\pi}{4} La^2(2\delta_{m,o} + \delta_{h,o}) |p_o|^2 J_m^2(\alpha a) \left[F_m(\alpha a) + \frac{\alpha c^2}{a\omega^2} \frac{J_m'(\alpha a)}{J_m(\alpha a)} \right] \frac{\overline{P}c}{\gamma} \alpha_g \quad (6)$$

where we have used Eq. (2) for the mean sound energy.

It is unfortunate that no experimental information which bears directly on the relaxation loss in hot propellant gases is available. An order of magnitude estimate has been made, however,[5] by considering only the nitrogen component of the product gases, which turns out to become a much more effective absorber at rocket motor temperatures than it is at ordinary temperature. Assuming a 10% abundance of N_2, the attenuation constant is then determined readily from data presented in ref. 12. We find (for $2500^{\circ}K$)

$$\alpha_g \approx 7.3 \times 10^{-5} \frac{f^2}{\overline{P}} \tag{7}$$

where f is the frequency. (This estimate is essentially the same as a previous one made by the authors, which was based on less complete data.)[5] It should perhaps be mentioned that relaxation loss can be extremely sensitive to chamber temperature, and this point should be kept in mind when we consider hotter propellants. Further, one should be cautioned that this type of relaxation can be very sensitive to the presence of small amounts of gas such as H_2, H_2O, etc. which are effective in energy transfer. The propellant gas is, of course, abundantly supplied with a variety of such species.

At this point we should note that relaxations in chemical reactions could be included here by specifying an appropriate α_g for them. Relaxations in the shift of equilibrium would be expected to contribute a positive α_g (damping) while those related to incomplete reactions might result in either positive or negative values of α_g .

5. Attenuation by Solid Particles in the Gas*

The attenuation constant for small spherical particles (radius R, number density N) suspended in a gas has been calculated in ref. 10. Loss arises both from heat transfer and momentum transfer. The major contribution arises from viscous damping and is expressed in terms of an attenuation constant given by

$$\alpha_p = \frac{3\pi R}{c} \frac{N}{\rho} \eta(1+z) \left[\frac{16z^4}{16z^4 + 72z^3\delta + 81(2z^2+2z+1)\delta^2} \right] \tag{8}$$

where $\delta = \rho/\rho' << 1$, $z = R(\omega\rho/2\eta)^{1/2}$, η is gas viscosity, and primes refer to the solid substance. Three regions of approximation can be distinguished:**

(i) Small particles ($z^2 << \frac{9}{4}\delta$): $\alpha_p \approx \frac{12\pi}{81} \frac{\omega^2}{\eta c} \frac{N}{\rho} R^5 \rho'^2$

(ii) Medium particles ($\frac{9}{4}\delta << z^2 << 1$): $\alpha_p \approx \frac{3\pi\eta}{c} \frac{N}{\rho} R$ (9)

(iii) Large particles ($z^2 >> 1$): $\alpha_p \approx \frac{3\pi}{c} \sqrt{\frac{\omega\rho\eta}{2}} \frac{N}{\rho} R^2$

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*The possible importance of particle damping has previously been discussed in various reports from Aeronutronic.

**Note that the attenuation is a maximum with respect to particle radius (for given mass abundance, etc.) if $\omega R^2 \approx \frac{9}{2} \frac{\eta}{\rho'}$. For $\omega/2\pi = 5{,}000$ cps and typical physical constants this corresponds to $R \approx 2$ microns.

Since for typical cases in a rocket motor the thermal damping constant lies between $\frac{2}{3}\,\frac{\gamma-1}{\gamma}$ and $\frac{3}{2}\,\gamma(\gamma-1)\left(\frac{c_p'}{c_p}\right)^2$ times Eq. (8), where the c_p's are the specific heats at constant pressure, we shall neglect it. Hence the power loss due to suspended particles will be expressed by (cf. Eq. (6))

$$-\dot{E} = \frac{\pi}{4}\, La^2 (2^{\delta_{m,o}+\delta_{h,o}})\, |p_o|^2\, J_m^2(\alpha a)\left[F_m(\alpha a) + \frac{\alpha c^2}{a\omega^2}\,\frac{J_m'(\alpha a)}{J_m(\alpha a)}\right]\frac{\bar{P}c}{\gamma}\,\alpha_p\;. \tag{10}$$

6. Thermal and Viscous Losses at the Wall

Assuming the head wall to be rigid, the acoustic power dissipated through viscosity is[11]

$$\sqrt{\frac{\omega\rho\eta}{2}}\int_0^{2\pi} d\phi \int_0^a r dr \left\langle (\mathrm{Re}\, u_r)^2_{z=0} + (\mathrm{Re}\, u_\varphi)^2_{z=0}\right\rangle \;.$$

In addition, assuming the wall is a perfect conductor, there is a thermal loss rate[11]

$$\frac{1}{\bar{P}}\sqrt{\frac{\omega}{2\gamma}\,\frac{\kappa}{\rho c_v}}\left(\frac{\gamma-1}{\gamma}\right)\int_0^{2\pi} d\,\phi \int_0^a r dr \left\langle (\mathrm{Re}\, p)^2_{z=0}\right\rangle$$

where κ is thermal conductivity of the gas and c_v its specific heat. Adopting the ideal relation $\kappa = \eta c_v$, the total rate of change of sound energy due to head wall losses is

$$\dot{E} = -\frac{\pi}{4}\, a^2 \sqrt{\frac{\eta}{2\rho\omega}}\, |p_o|^2 J_m^2(\alpha a)\, \frac{\bar{P}\omega}{\gamma}\left(\frac{\alpha c}{\omega}\right)^2 (1+\delta_{m,o})$$
$$\left[\left(1+\frac{\gamma-1}{\sqrt{\gamma}}\,\frac{\omega^2}{\alpha^2 c^2}\right)F_m(\alpha a) + \frac{2}{\alpha a}\,\frac{J_m'(\alpha a)}{J_m(\alpha a)}\right]\;. \tag{11}$$

The further loss due to transmission through the head plate can be neglected since it is estimated to be only a few per cent relative to the above (also cf. ref. (11)). Losses of magnitude similar to Eq. (11) also would be expected due to the tail cavity wall. Evaluation of the wall losses for representative values of the parameters indicates that such losses would ordinarily be ignorable, except for cigarette type burners (cf. Sec. III).

7. Solid Phase Absorption

The loss of acoustic energy in the body of the solid propellant should be expected to have a significant effect on the stability of the system. A completely general treatment of an absorptive medium is, of course, an unreasonable goal. As pointed out earlier, we will depend on perturbation methods. It is interesting that for damping by the solid, we can handle two extreme cases, namely, the cases where the damping length in the solid is either long or short compared to the web thickness.

For the case of small viscous loss (long damping length) the solid must be treated as an acoustic medium bounded by the gas medium. The motions of this two medium system are calculated on a loss-free basis, and this result is then used to determine the losses which would ensue. In previous papers,[1,5] we reported studies based on this standard perturbation treatment. Not unexpectedly, the effect of the solid turns out to be dominant for such geometries that the solid participates heavily in the motion. Such geometries occur from time to time during the course of burning and lead to a source of intermittancy in the instability. Experimental observations at a number of laboratories of effects due to acoustic motion in the solid show that, at least for a number of propellants, the assumption of a relatively long damping length is valid.

If, on the other hand, the viscous damping in the solid is quite large (short damping length), the solid will not sustain resonant motion. The impedance mismatch at the solid-gas interface will be large. Consequently, nearly all of the acoustic energy incident on the solid surface will be reflected back into the gas, while that small amount which is transmitted into the solid will be absorbed. The loss in the solid will be limited to the amount of energy which can be transmitted across the boundary between two materials of different characteristic admittances. The solid then presents to the gas an admittance whose real part is

$$\frac{1}{\rho_s c_s} = \left[\frac{\rho_g c_g}{\rho_s c_s}\right] \frac{1}{\rho_g c_g} , \qquad (\rho_s c_s >> \rho_g c_g) \qquad (12)$$

where the subscripts s and g refer to the solid and gas respectively. (Of course, we assume that c_s is the sound velocity in the solid appropriate to the mode under consideration.) Now the burning surface also presents an admittance

$$- \frac{\bar{v}}{\bar{P}} \left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma} \right) = - \frac{1}{\rho_g c_g} \left[\frac{\bar{m} c_g}{\bar{P}} \left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma} \right) \right] \qquad (13)$$

Thus the significance of the solid loss is determined by the relative magnitudes of the coefficient in the square brackets in Eqs. (12) and (13).

Let us assume for rough typical values

$\overline{m} = 1.2 \text{ gm/cm}^2 \text{ sec}, \qquad c_g = c_s = 10^5 \text{ cm/sec}, \qquad \overline{P} = 400 \text{ psi},$

$\rho_s = 1.6 \text{ gm/cm}^3, \qquad \rho_g = 3.8 \times 10^{-3} \text{ gm/cm}^3 ,$

then we have

$$\left[\frac{\rho_g c_g}{\rho_s c_s}\right] = 2.4 \times 10^{-3} , \qquad \left(\frac{\overline{m} c_g}{\overline{P}}\right) = 4.4 \times 10^{-3} .$$

Thus we see that the gains and loss are comparable and the surface amplification for such a propellant would be effectively lowered by the loss in the solid. However, the sharply peaked intermittances characteristic of the long damping length case would not be expected.

In what follows we shall not give detailed consideration to the effect of the solid. The first case (long damping length) has already been discussed in some detail.[5] We do not have adequate information to know how common the second case might be, nor what the dependences of the appropriate propellant properties on frequency and pressure are, to handle the problem quantitatively at this time.

III. STABILITY DIAGRAMS IN THE K_n - D_p PLANE

1. Criterion for Acoustic Stability

Apparently the only empirical stability diagrams published for a solid propellant rocket motor are those of Brownlee and Marble (Ref. 2). For their particular set of experiments they were able to divide the K_n - D_p plane (K_n is the ratio of burning area to throat area, and D_p is the internal diameter of the cylindrical charge) into stable and unstable regions by a stability limit curve. We shall adopt their type of diagramming and consider the kinds of stability diagrams which might be expected from theoretical considerations of certain acoustic losses.

In order for a rocket motor to be acoustically stable, the losses must equal or exceed the gains. Treating the losses as outlined in Sec. II and assuming that the solid propellant surface is sufficiently immobile that the mode is adequately approximated as a "gas quasi-mode" (i.e., $J_m'(\alpha a) = 0$; cf. refs. 1 and 5 for elucidation of this point), we have as the stability criterion

$$K_n^{1/2}\,\mathrm{Re}\left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma}\right) \leq C_G\left(\frac{K_n^{\frac{1-3n}{2-2n}}}{a}\right) + C_W\left(\frac{a^{1/2}K_n^{\frac{2-3n}{2-2n}}}{L}\right)$$
$$+ C_i\left(\frac{K_n^{3/2}}{a}\right) + C_{ii}\left(aK_n^{3/2}\right) + C_{iii}\left(a^{1/2}K_n^{\frac{4-3n}{2-2n}}\right) \qquad (14)$$

where the coefficients are given in Table I. We have used the area ratio $K_n \equiv 8aL/D_t^2 = \bar{P}/\bar{m}c^*$ (where D_t is the nozzle throat diameter, c^* the propellant characteristic exhaust velocity), and assumed for steady state burning a rate law of the usual form $\overline{m} = \rho_s a_b \bar{P}^n$ where a_b will depend on erosive velocity (for a multiplicative law) and conditioning temperature, as well as on specific propellant properties. In the special case of a tangential mode, and negligible erosion, the C's are independent of K_n and $a (= D_p/2)$.

2. Examples Based on Gas Phase Losses

Here, we wish to illustrate the effect in the $K_n - D_p$ plane of the losses discussed in items 4, 5 and 6 of Sec. II. To best illumine the striking difference of these various mechanisms we will treat each in turn as the single predominant factor (except for the wall losses, which will be shown to be negligible for the particular example).

For simplicity we shall take advantage of the relatively broad frequency response of propellants and regard the propellant response function as a constant, independent of frequency over the relatively narrow frequency band traversed by a given mode during the course of burning in a fairly typical motor. We have arbitrarily assigned this constant the plausible value of 1/5. This choice does not affect the shape of the stability boundaries in our examples. For the sake of definiteness, we shall confine our attention to the stability of the first tangential mode. Insofar as they are known, we have endeavored to assign to the various parameters the values corresponding to the experiment of Brownlee and Marble, in order that a comparison with their results might be attempted. The coefficients C appearing in the criterion are exhibited in Table II with the numerical values used for the required parameters.

Table I

Coefficients in Stability Criterion

Coefficient and Damping Mechanism	Analytic Expression
C_G Chamber Gas	$1.85 \times 10^{-6}\, \omega^2 a^2 \left(1 - \frac{m^2}{\alpha^2 a^2}\right)\left(\frac{c^*}{c}\right)(\rho_s a_b c^*)^{-\frac{1}{1-n}}$
C_W Head Wall	$\sqrt{\frac{\eta\omega a}{2\gamma}}\left[\left(\frac{\alpha' c}{\omega}\right)^2 + \frac{\gamma-1}{\sqrt{\gamma}}\right]\left(1 - \frac{m^2}{\alpha^2 a^2}\right)\left(\frac{c^*}{c}\right)(1 + \delta_{h,o})^{-1}(\rho_s a_b c^*)^{-\frac{1}{2-2n}}$
C_i Small Particles (Smoke)	$\frac{12\pi}{81}\, \frac{(\omega a)^2}{\eta c}\, \frac{N}{\rho}\, R^5 \rho'^2 \left(1 - \frac{m^2}{\alpha^2 a^2}\right)\left(\frac{c^*}{c}\right)$
C_{ii} Medium Particles	$\frac{3\pi\eta}{c}\, \frac{N}{\rho}\, R \left(1 - \frac{m^2}{\alpha^2 a^2}\right)\left(\frac{c^*}{c}\right)$
C_{iii} Large Particles	$\frac{3\pi}{c^2}\sqrt{\frac{\omega a \eta \gamma}{2}}\, \frac{N}{\rho}\, R^2 (\rho_s a_b c^*)^{\frac{1}{2-2n}} \left(1 - \frac{m^2}{\alpha^2 a^2}\right)\left(\frac{c^*}{c}\right)$

Table II
(all units are for cgs system)

$a_b = 2.74 \times 10^{-3}$	$\gamma = 1.24$	$C_G = 3.98$
$n = 1/3$	$b = 6.35$	$C_i = 3.9 \times 10^{-3}$
$\rho_s = 1.6$	$L = 78.7$	$C_{ii} = 2.0 \times 10^{-3}$
$c^* = 1.29 \times 10^5$	$\rho' = 3$	$C_{iii} = 1.6 \times 10^{-6}$
$c = 0.94 \times 10^5$	$\eta = 6.64 \times 10^{-4}$	$C_w = 3.4 \times 10^{-2}$

First tangential mode, m = 1, h = 0.

Smoke particle radius, R = 1/2 micron, mass fraction in gas = 1%.

Medium size particles, R = 5 microns, mass fraction in gas = 10%.

Large size particles, R = 70 microns, mass fraction in gas = 10% .

It should be noted that for this particular geometry the wall losses are very small, so that we shall not discuss them further here. For different geometries, such as might occur in end burners, for example, the exposed wall area might well be large enough to make such losses quite significant.

The various loss mechanisms in the gas phase lead to quite different stability laws as illustrated in Fig. 2. The unstable region in each case is indicated by the arrow. The general nature of these results is also summarized in Table III. It should be re-emphasized that these results are for the first tangential and a propellant with a pressure exponent of the burning law of one third. It also should be noted that we have assumed a_b as a constant and thus have implicitly neglected the effect of steady erosion. A multiplicative erosion law would result in a_b being a slow function of D_p and/or K_n, and would accordingly change the shape of the stability lines somewhat.

Reference to Fig. 2 or Table III shows that the different loss mechanisms can limit the unstable regime to different parts of the K_n - D_p plane. Of particular interest here is the common generalization that instability tends to be less at higher pressures. Now, except for the modifying effect of erosion, there is a one to one correspondence between K_n and pressure. Thus, the damping arising from various particles in the gas phase would be consistent with this generalization. Damping due to relaxation processes in the gas itself, however, could lead to the opposite result, if the

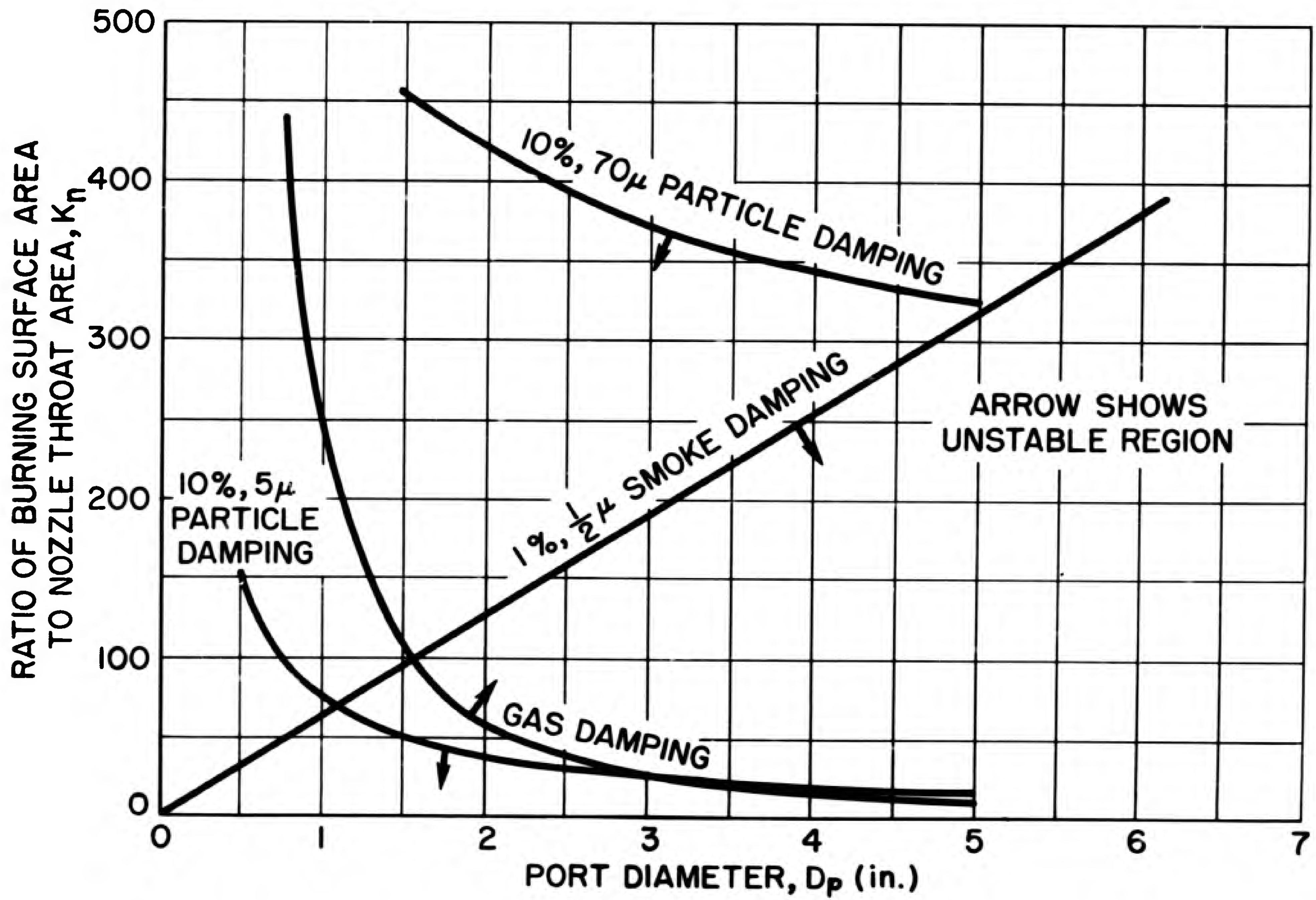
RATIO OF BURNING SURFACE AREA TO NOZZLE THROAT AREA, K_n
500
400
300
200
100
0
0
1
2
3
4
5
6
7
PORT DIAMETER, D_p (in.)
10%, 70μ PARTICLE DAMPING
10%, 5μ PARTICLE DAMPING
1%, ½μ SMOKE DAMPING
GAS DAMPING
ARROW SHOWS UNSTABLE REGION

Fig. 2

Table III

Loss mechanism	Form of the stability line	Portion of the $K_n - D_p$ plane that is unstable
Smoke-damping	$K_n \propto D_p$	Lower-right
Particles of size a few microns	$K_n \propto D_p^{-1}$	Lower-left
Particles of size a few tens of microns	$K_n \propto D_p^{-2/7}$	
Wall-damping (exceptional case)	$K_n \propto D_p^{-2}$	
Gas-damping	$K_n \propto D_p^{-2}$	Upper right

relationship between the frequency and relaxation loss falls in the region assumed in Sec. II-4. There are two points which should be made here. First, the experimentally determined stability map may provide some indication as to which types of losses may be significant. Second, it should not be considered too surprising if some experiments (with different propellants and configurations) should be found to provide counter-examples of some of the empirical generalizations, including the one just mentioned. This merely serves to stress again the point that the phenomenon is a result of the balance of at least two processes, and its properties are determined jointly by these processes, not by one of them alone. When we change from one experimental set-up to another, we may be changing the distribution of emphasis among the various processes involved.

Considerations of scaling are rather interesting. For a given tangential mode, we see that the motor length tends to be unimportant so long as it does not become so short that wall losses (or perhaps nozzle losses) become important. The diameter D_p of the port, on the other hand, is a vital parameter. We see from Eq. (14) that, relative to amplification due to pressure response, gas relaxation losses would be proportional to D_p^{-1}, small particle damping to D_p^{-1}, medium size particle damping to D_p, and large size particle damping to $\sqrt{D_p}$. Thus, scaling up a motor would be expected to make it either less or more stable in the first tangential mode according to whether the first two or the last two of these damping mechanisms predominated. Of course, additional complications would arise from other factors such as the frequency dependence of the propellant response function.

With respect to longitudinal modes, where the frequency of a particular longitudinal mode is approximately proportional to $1/L$, we refer to Eq. (14) and Table I to see a quite different effect. For this case, the effectiveness of gas damping and small particle damping are proportional to D_p/L^2, medium-size particle damping to D_p and large particle damping to $D_p/\sqrt{L}$. Thus, if a motor were scaled up uniformly in all dimensions the effect of the first two losses would be to allow greater instability while the effect of the last two would be to decrease the region of instability. Which would actually happen would of course depend on which loss or losses were dominant. On the other hand, if the motor were scaled by increasing its length alone, it would be expected to become less stable if gas relaxation, small particle damping or large particle damping were dominant, but unchanged if medium particles were controlling. Again we have neglected any change due to frequency sensitivity of the response function. Perhaps even more important here is the neglect of acoustic erosivity, which is expected to play a more critical role in the longitudinal modes than in the transverse. However, it is clear that care must be exercised in making empirical generalizations and that axial and transverse modes should be considered separately.

3. The Results of Brownlee and Marble

It is of interest to inquire as to the possible interpretation of the experimental results of Brownlee and Marble in terms of the preceding discussion. A brief review of their main result is thus in order. At the instant of firing the geometry of the rocket is described by a point in the K_n - D_p plane. For a cylindrical internal burning charge, inhibited on the ends, $K_n = 4 D_p L/D_t^2$. Thus as the charge burns, the representative point moves along a straight line through the origin in the K_n - D_p plane. For different throat areas a fan of such lines results. Brownlee and Marble found that their motor would oscillate in the first tangential mode sometime during the course of burning provided the initial representative point for the motor lay beneath a certain one of these lines. Thus, their stability line was similar to that arising from small particle damping.

In fact, the particular "smoke" line illustrated in the preceding section was chosen to coincide with the stability line of Brownlee and Marble. We used parameters appropriate to their system insofar as they were known, and then adjusted the "smoke" content so that, in conjunction with our assumed response function, the slope of the stability line would be correct.

Thus we see that it is possible to postulate an apparently reasonable mechanism which encompasses the main results of Brownlee and Marble. A secondary feature of their work is a suggestion of a slight upward curvature of their stability line near the origin. Including this in the theory would require further embellishment. A number of factors could contribute. For example, the response function might be increasing with frequency (small D_p means high frequency in a transverse mode), or acoustic erosivity may play a part, or a weak amplification in the gas might be responsible. The situation is simply not such that it is possible to distinguish any of a number of possible mechanisms for this effect.

It is not our intent to insist that our explanation of the general nature of the Brownlee and Marble experiments is by any means unique or even necessarily correct. We merely wish to indicate that a few plausible simple assumptions suffice to account for their results. Further, a number of other equally simple assumptions, namely, the other loss mechanism discussed in Sec. II, 4, 5, 6 would be in considerable conflict with the results.

One question might well be raised. Is the assumption of 1% by mass of 1/2 μ particles unreasonable, for a propellant which we have been told "isn't very smoky"? We can estimate the optical attenuation length, due to scattering, for such a gas. In the chamber we find an attenuation length (distance to reduce light intensity by a factor of 1/e) of approximately 5 cm. However, in the exhaust after expansion this would be increased to a value nearer 1 meter. This should not give a very "smoky" exhaust, particularly since the refraction effects of the temperature and density gradients would be large enough to provide considerable obscuration.

IV. CONCLUDING REMARKS

It should be recognized that our knowledge of the properties of all the acoustic loss and gain mechanisms within a rocket motor is still far too fragmentary to allow formulation of a complete and thorough criterion of instability of direct value to the design engineer. Nevertheless, we have shown that some of the factors can be handled well enough to provide a bare outline of how these criteria may evolve. In the process we have seen further causes for disagreement between experimentalists, if they are not aware of the various factors involved, and incidentally have provided a plausible explanation of the results of one systematic experimental study.

Of course, in most real cases, the situation will be considerably more complicated. The response function will be a function of frequency, acoustic erosivity will play its role, and very probably several loss mechanisms will be comparable, with emphasis shifted from one to another with change in the system or the mode. As can be imagined, the stability diagrams in the K_n - D_p plane could become very complex indeed. A diagram as simple as that of Brownlee and Marble might well be the exception rather than the rule. In fact, the suggestion of such additional complexity is in their work, both with respect to the curvature of the stability line at the lower end and in the hint of a possible new region of stability at low K_n .

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