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THE INFLUENCE OF EROSIVE BURNING ON ACOUSTIC INSTABILITY IN SOLID PROPELLANT ROCKET MOTORS

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ABSTRACT

The influence of erosivity on the acoustic stability of solid fuel rockets is examined. Allowance is made for a frequency-dependent erosion constant. The theory verifies the stabilizing influence of the steady flow erosivity for propellants having positive erosion constants, with the contrary prediction for those having negative erosion constants. The influence of acoustic erosion is examined and the order in which it contributes is shown to depend on the symmetry of the flow field in the rocket, being quite different for rockets with nozzles at the end as compared to rockets with nozzles in the center. Degenerate and quasi-degenerate cases where erosivity might become a predominant effect are discussed.

TABLE OF CONTENTS

I. INTRODUCTION

A burning propellant [illegible] possesses the ability of amplifying acoustic disturbances at its surface over a relatively wide frequency band. If the net acoustic losses in the rocket motor are sufficiently small, as they frequently are near the resonant frequencies of the rocket cavity, the motor acts as an acoustic oscillator. The high amplitude pressure oscillations which then result produce a wide variety of secondary phenomena.[1]

Empirical studies have led to a few general observations, among which is the very interesting one that it is the fluctuation in gas pressure (in contrast to gas velocity) which appears to be of prime consequence in determining the stability. Corresponding to this observation, many theoretical approaches have ignored the possible effects of erosive velocity on burning rate and have actually achieved a considerable progress toward understanding the phenomenon. These approaches are concerned chiefly with the acoustic stability of the burning propellant. The acoustic stability is determined by whether or not a pressure wave (of some arbitrary frequency) incident on the burning surface alters the burning rate in such a way as to inhibit or reinforce the incident wave. The boundary conditions at the burning propellant surface then are expressible in the usual admittance form and the motion of the gas is simply described in terms of the usual acoustic modes of the system. Ultimately, of course, the acoustic

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[1] See, for example, "Analysis of Results of Combustion Research on Solid Propellants", E. W. Price Solid Propellant Rocket Research Conference, Princeton University, Princeton, New Jersey, Jan. 28-29, 1960, ARS preprint No. 1058-60.

stability rests upon the sign of the real part of the acoustic admittance, which is the indicator of amplification or attenuation of a sound wave at a surface. A variety of phenomena now seem to be accounted for in terms of such analyses.[2,3,4,5]

In spite of the empirical observation of the prime importance of the acoustic pressure, and the success of theories based thereon, one cannot help

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[2] "Combustion Instability: Acoustic Interaction with a Burning Surface", R. W. Hart and F. T. McClure, J. Chem. Phys. 30, 1501 (1959). Also APL report TG-309. "Effect of Solid Compressibility on Combustion Instability". J. F. Bird, L. Haar, R. W. Hart, and F. T. McClure, scheduled for April issue, J. Chem. Phys. Also APL report TG-335-1.

[3] "On Acoustic Resonance in Solid Propellant Rockets", F. T. McClure, R. W. Hart, and J. F. Bird, scheduled for May issue, J. Appl. Phys. Also APL report TG-335-2, "Solid Propellant Rocket Motors as Acoustic Oscillators". F. T. McClure, R. W. Hart, and J. F. Bird. Solid Propellant Rocket Research Conference, Princeton University, Princeton, New Jersey, Jan. 28-29, 1960, ARS preprint No. 1049-60. Also APL Report TG-335-3.

[4] "High Frequency Combustion Instability in Solid Propellant Rockets. Part I". Sin-I Cheng. Jet Propulsion 24 27-32 (1954).

[5] "Unstable Burning Phenomenon in Double-base Propellants", Theos A. Angelus, Solid Propellant Rocket Research Conference, Princeton University, Princeton, New Jersey, Jan. 28-29, 1960. Also Allegany Ballistics Laboratory Report ABL Z-9.

but wonder if acoustic velocity might not be important to the stability, at least under some circumstances. Since the steady state burning rate depends on both the pressure and the erosive velocity (the gas stream velocity component parallel to the burning surface) the theoretical basis of the empirical observation and its probable generality deserves elucidation.

Although the role of acoustic erosive velocity has not been entirely ignored, for the most part the analyses[6] in which it plays a part have been directed toward the question of <u>inherent</u>, rather than acoustic instability. Inherent stability is concerned with whether or not an arbitrary perturbation on the burning rate decays, rather than grows, exponentially with time. A "propellant" which is inherently unstable would not be expected to possess a "steady state" of combustion under any ordinary circumstances. On the other hand, we recognize that a propellant for which transients are damped, i.e., which possesses a steady state, may or may not act as an acoustic amplifier for incident sound waves.

The bearing of erosive velocity on acoustic stability was discussed briefly by Cheng[7] who concluded that "the dependence of the burning rate on the drifting velocity alone is not likely to excite instability..." Since the scope of his analysis was rather limited, this conclusion cannot be accepted as definitive.

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[6]See, for example, L. Green and W. Nachbar, "Analysis of a Simplified Model of Solid Propellant Resonant Burning", J. Aero. Space Sciences <u>26</u>, 518 (1959).

[7]Sin-I Cheng, "High-Frequency Combustion Instability in Solid Propellant Rockets," Part 1, (particularly Appendix II), Jet Propulsion <u>24</u>, 102-109 (1954).

The subject which we shall consider, then, is that of acoustic stability in the presence of a burning rate dependent on both pressure and erosive velocity. We intend to ask for the conditions under which such a burning surface amplifies or attenuates an incident sound wave. The question of stability against finite disturbances is quite another problem, which will not be treated here.

II. ANALYTICAL FORMULATION

In order to suppress unnecessarily complicating distractions, we shall simplify the geometry to that of a circular cylindrical cavity containing a cylindrical grain (outer radius b, inner radius a). We shall neglect the small doppler effect produced by the rather slow mean flow of burned gas and regard the burned gases as a homogeneous medium insofar as acoustic motion is concerned. For our present study, which is concerned merely with the relative importance of pressure and erosive velocity, we will select a method by which we may also neglect acoustic losses in the gas and in the nozzle.

2.1 The End Face Boundary Conditions

We shall treat only the case of rigid, lossless ends so that we have

$$(u_z)_{z=0} = 0 \quad \text{and} \quad (u_z)_{z=L} = 0 \, , \tag{1a}$$

where $z = 0$ and $z = L$ are the coordinates of the end faces in a cylindrical system whose axis coincides with that of our cylindrical cavity, and $\underset{\sim}{u}$ is the acoustic velocity. These boundary conditions correspond to those encountered in some research type rocket motors with side nozzles, and may be thought to approximate end nozzle rockets without end cavities and with high port to throat ratios.

2.2. The Boundary Conditions at the Burning Surface

The formulation of the erosive boundary condition at the propellant surface is, of course, the crux of the problem, and at the very beginning we are somewhat nonplussed by the absence of a time-dependent theory of the erosion of the burning boundary layer. We may elect to proceed in either of two directions. On the one hand, we might elect to stop and attack the development of such a theory, or on the other hand, we might attempt to formulate the acoustic analysis in sufficiently general terms that the results of a future theory could be inserted into the analytical structure. These two paths are rather distinct. The former course would apparently involve a conglomeration of chemistry and aerodynamics whereas the latter is primarily an application of acoustical principles. Rather than concern ourselves with the predictions of a particular burning theory, we elect to pursue the latter course and focus our attention on the development of a rather general acoustic structure. By following this path, we incidentally may hope to be able to form an estimate as to the importance of a time-dependent erosive burning theory.

The erosive boundary condition may be obtained from the general relationship expressing the flux density of hot gas generation, m. Restricting our attention to the periodic case (i.e., the transients have already decayed), we may express m as a function of pressure, the erosive velocity, and their time derivatives.* Thus,

$$m = m(\bar{P}; p_a, (p_a)_t, \ldots ; |v_e|, |v_e|_t, \ldots) \qquad (1b)$$

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*Integrals with respect to time would also be required in the transient case.

where $\bar{P}$ is the mean pressure, p_a is the acoustic pressure at the burning surface ($r=a$), v_e is the erosive velocity (i.e., gas velocity parallel to burning surface) and where the subscript t's indicate partial differentiation with respect to time. The erosive burning depends on the absolute magnitude of v_e, rather than on v_e itself, because the effect of erosion is presumed to be independent of the direction of the erosive velocity.

2.3 Perturbation Form of the Erosive Boundary Condition

Assuming the burning rate to be an analytic function of $|v_e|$, its time derivatives, and p and its time derivatives, we may expand about the point $(\bar{P}; 0, 0, 0, \ldots; 0, 0, 0, 0 \ldots)$, i.e., about the non-oscillatory state with no erosion. Thus, we have

$$m = \Bigg[m(\bar{P}; 0, 0, \ldots) + \left\{ \frac{\partial m}{\partial p_a}\bigg|_0 p_a + \frac{\partial m}{\partial (p_a)_t}\bigg|_0 (p_a)_t + \cdots \right\} + \left\{ \frac{\partial m}{\partial |v_e|}\bigg|_0 |v_e| + \frac{\partial m}{\partial |v_e|_t}\bigg|_0 |v_e|_t + \cdots \right\} + \left\{ \text{higher order non-linear terms} \right\} \Bigg] \tag{1c}$$

where, to avoid unwieldy notation, we have displayed only one of the multitude of subscripts "zero" attached to each of the partial derivatives. Insofar as the onset of instability is concerned, we may neglect higher order terms provided the steady state erosive burning law is adequately approximated by a linear function of $|v_e|$.

The first term on the rhs of Eq. (1c) is merely the dc burning rate in the absence of erosion, which we designate by $\bar{m}_0$. The first of the bracketed sums expresses the contribution to gas flux of a fluctuating pressure with no erosion. In the notation of ref. 2, this bracket will be denoted by $(\bar{m}_0\,\mu_1)$. The second bracket on the rhs of Eq. (1c) describes the effect of an erosive velocity, with no pressure fluctuation. (We may recognize the term $\frac{\partial m}{\partial |v_e|_0}$ as $\bar{m}_0 k$, where k is the usual (dc) erosion constant.) The important point here is that the pressure and erosive contributions to burning rate are now separated, with the erosive part expressed as a linear function of $|v_e|$ and its time derivatives. The pressure part has, of course, been studied separately, elsewhere.[2]

Finally, we complete the perturbation formulation by recalling that m may be expressed as

$$-m = \left[\rho(v_r - v_s)\right]_{r=a} , \tag{1f}$$

where ρ is the hot gas density, v_r is the hot gas radial velocity, and v_s is the regression speed of the solid surface, with radial velocities measured positive in the outward radial direction. We have

$$v_s = \bar{v}_s + Y_s\,p_a ,$$

where the bar denotes the time average, Y_s = specific acoustic admittance of the solid surface, which is zero if the solid is rigid, and p_a = the acoustic pressure at the propellant surface, where r = a. We use

$$\rho = \bar{\rho}\left[1 + \frac{p}{\gamma\bar{P}}\right] , \tag{1g}$$

where γ is the hot gas specific heat ratio and $\bar{P} = \bar{\rho}\,\frac{c^2}{\gamma}$ is the mean chamber pressure, with c the sound velocity, $\bar{\rho}$ the mean density of product gases. Equations (1f) and (1g), with the definitions of $\bar{m}_0$ and $\bar{m}_0\mu_1$ given above, when substituted into Eq. (1c), yield

$$\left(\frac{-u_r}{c}\right)_{r=a} - M_a\left(\mu_1 - \frac{\epsilon}{\gamma}\right) + \frac{\bar{\rho}cY_s}{\gamma}\epsilon = \frac{-M_a k\left|\bar{v}_e\right|\epsilon}{\gamma}$$

$$+\left(\frac{1}{\bar{\rho}c}\right)\left[\left[\left|v_e\right| - \left|\bar{v}_e\right|\right]\cdot\frac{\partial m}{\partial\left|v_e\right|}\bigg|_0 + \left|v_e\right|_t\frac{\partial m}{\partial\left|v_e\right|_t}\bigg|_0 + \cdots\cdots\cdots\right]_{r=a} \tag{1h}$$

where we have also used $\epsilon \equiv \left(\frac{p}{\bar{P}}\right)_{r=a}$ and $M_a \equiv \left(\frac{\bar{m}_0}{\bar{\rho}c}\right)$.

For the present case, one further simplification of the boundary equation also applies. Noting that the magnitude of the erosive velocity may be expressed in terms of its (cylindrical coordinate) components as

$$\left|v_e\right| = \sqrt{(\bar{v}_e + u_z)^2 + u_\theta^2} \quad ,$$

we expand the radical to see that, for the arbitrarily small acoustic amplitudes relevant to stability theory,

$$\left|v_e\right| \approx \left|\bar{v}_e + u_z\right| = \left|\bar{v}_e\right| + u_z \text{ Signum } (\bar{v}_e), \tag{1i}$$

where Signum $(\bar{v}_e) = +1$ or -1 according to whether $\bar{v}_e$ is positive or negative. In the following, then, we will use the linear boundary condition obtained by substituting Eq. (1i) into Eq. (1h), i.e.,

$$-\left(\frac{u_r}{c}\right)_{r=a} - M_a\left(\mu_1 - \frac{\epsilon}{\gamma}\right) + \frac{\bar{\rho}cY_s\epsilon}{\gamma} = -M_a k\left|\bar{v}_e\right|\frac{\epsilon}{\gamma} + \frac{1}{\bar{\rho}c}\text{ Signum } (\bar{v}_e)$$

$$\left\{u_z\frac{\partial m}{\partial\left|v_e\right|}\bigg|_0 + (u_z)_t\frac{\partial m}{\partial\left|v_e\right|_t}\bigg|_0 + \cdots\cdots\right\} \tag{2}$$

in which $\bar{v}_e$ is still to be regarded as an arbitrary funct' n of z .

2.4 Fourier Transformation

It is most expedient to take advantage of the simplifications which result when we consider only harmonic solutions. Accordingly, we may define the time-independent tilda'd functions by the relationships

$$p = (\text{Real of})\ \tilde{p}\, e^{i\omega t}$$

$$\underset{\sim}{u} = (\text{Real of})\ \underset{\sim}{\tilde{u}}\, e^{i\omega t}\ , \text{ and}$$

$$\mu_1 = (\text{Real of})\ \tilde{\mu}\, e^{i\omega t}\ . \qquad (2a)$$

The boundary equation (Eq. (2)) then takes the form

$$-\left(\frac{\tilde{u}_r}{c}\right)_a - \left[M_a\left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma}\right) - \frac{\bar{\rho} c}{\gamma}\, Y_s\right]\tilde{\epsilon} = -\frac{M_a k\, |\bar{v}_e|\, \tilde{\epsilon}}{\gamma} + M_a K\, \tilde{u}_z \text{ Signum } (\bar{v}_e) \qquad (2b)$$

where K, because of this definition, may readily be obtained from a time-dependent theory of erosion if it should become available. We may note, in particular, that $K(\omega \rightarrow 0) \rightarrow k$, and refer to K as the "ac erosion constant". The end face boundary conditions (Eq. (1a)) are easily transformed to read

$$\tilde{u}_z\Big|_{z=0} = \tilde{u}_z\Big|_{z=L} = 0 \qquad (2c)$$

and, of course, the wave equation to be solved for these boundary conditions transforms to the Helmholz equation, $\nabla^2 \tilde{p} + \left(\frac{\omega}{c}\right)^2 \tilde{p} = 0$.

III. ANALYTICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM

The analytical problem has now been defined. If the solution were obtainable by several of the more obvious and routine approaches, we would merely display the result, and assert that it easily could be verified by substitution into the Helmholz and the boundary equations. A rather special approach seems necessary, however, so that it seems desirable briefly to sketch the method of solution. The method begins in the standard way. We note that the solution will be expressible as a sum of the elementary (separable) solutions of the Helmholz equation expressed in cylindrical coordinates:

$$\frac{\tilde{p}}{\bar{P}} = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} p_{\ell,j} \, J_j(\alpha_\ell r) \cos\left(\frac{\ell \pi z}{L}\right) \cos(j\theta) \tag{3a}$$

$$\tilde{u}_r = \frac{-c^2}{i\gamma\omega} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} p_{\ell,j} \alpha_\ell J_j'(\alpha_\ell r) \cos\left(\frac{\ell \pi z}{L}\right) \cos(j\theta) \tag{3b}$$

$$\tilde{u}_z = \frac{\pi c^2}{i\gamma\omega L} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} p_{\ell,j} \, \ell \, J_j(\alpha_\ell r) \sin\left(\frac{\ell \pi z}{L}\right) \cos(j\theta) \tag{3c}$$

with

$$\alpha_\ell^2 = \frac{\omega^2}{c^2} - \left(\frac{\pi \ell}{L}\right)^2 ,$$

The appearance of the individual sine and cosine terms indicates that we have forseen the axial boundary conditions expressed by Eq. (2c).

The usual approach would be to substitute the field expression (Eqs. (3a), (3b), (3c))into the boundary equation (Eq. (2b)). In the absence of erosion ($k = K = 0$), this substitution quickly yields the usual result that the characteristic frequencies would be determined by the values of ω for which

$$\left\{ -(\alpha_n a) J_j'(\alpha_n a) + \frac{i \gamma a \omega}{c} \left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma} - \frac{\bar{\rho} c Y_s}{\gamma M_a} \right) M_a J_j(\alpha_n a) \right\} = 0 \quad . \tag{4a}$$

We note that the system would be neutrally stable (real ω) only if

$$\text{Real of} \left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma} \right) = \frac{\bar{\rho} c}{\gamma M_a} \text{ Real of } Y_s \, , \tag{4b}$$

and that the system would be damped only if

$$\text{Real} \left(\frac{\tilde{\mu}}{\tilde{\epsilon}} - \frac{1}{\gamma} \right) < \frac{\bar{\rho} c}{\gamma M_a} \text{ Real of } Y_s \, .$$

In the presence of erosion, however, the analytical difficulties begin immediately. Commencing as before, we note that in order to determine the characteristic frequencies (for which a solution for the p_n's is at least possible), we would have to discover the roots of an exceedingly intractable infinite determinant.

At this point, a bit of reflection is enough to convince us that we did not really want specifically to know about the effect of erosion on the characteristic frequencies, anyway. The information which we really seek is the relative importance of erosion compared to pressure in determining stability.

Let us, then, attempt a less standard, but more direct approach. Suppose we recognize that we would have a rather satisfactory answer to our question if we knew for a given erosion constant how much the admittance (as measured, say, by $\tilde{\mu}/\tilde{\epsilon}$) of the burning surface would have to be altered in order to restore the characteristic frequency to its value for zero erosion. The important advantage in formulating this particular question is that the characteristic frequencies of the system then correspond exactly to those for zero erosion, which are very easy to determine!

In order to discuss the contribution of erosive velocity to stability, we must first digress for a moment in order to display the nomenclature for labeling the acoustic modes. To specify a particular mode, we must give angular, radial and axial mode numbers. Thus, a mode is represented in the usual way by a three-element array, the first number representing the number of axial half waves, the second the number of radial interior (velocity) nodes, and the third the number of angular nodes. In order to avoid unnecessary repetition of subscripts, we shall designate the modes by $\underset{\sim}{N} = N, R, \Theta$.

In order to carry through this approach, then, we shall write $k = \beta K$, and develop the solution as a power series in K.* We write, for the $\underset{\sim}{N}$th mode,

$$\begin{pmatrix} \tilde{\mu} \\ \tilde{\epsilon} \end{pmatrix} = \begin{pmatrix} \tilde{\mu} \\ \tilde{\epsilon} \end{pmatrix}_{K=0} + \sum_{j=1}^{\infty} \left(\frac{K c a M_a}{L} \right)^j D_{j,\underset{\sim}{N}} \tag{4c}$$

(where the expansion parameter $\left(\frac{K c a M_a}{L} \right)$, rather than merely K itself, is used for later convenience). Thus, if we can determine the

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*The value of β would have to be determined from a time-dependent erosion theory and in general would be a function of propellant composition, operating conditions and frequency. If there were frequency intervals for which k was much greater than K, the appropriate expansion for our problem would of course be in powers of k, with $K = \beta' k$.

coefficients $D_{j,\underset{\sim}{N}}$ we shall have found the comparison between the effects of erosive velocity and pressure. Restricting our attention to the conditions for neutral stability, for which ω's are real (and given by Eq. (4a)), one can now readily verify that the solution to our boundary value problem is expressed by Eq. (3a) with the p_ℓ given for the $\underset{\sim}{N}$th mode of the system by

$$p_{\ell,\underset{\sim}{N}} = \delta_{j,\Theta} \sum_{n=0}^{\infty} \left(\frac{aKcM_a}{L} \right)^n p_\ell^{(n)} \underset{\sim}{N} , \qquad (5a)$$

where the $p^{(n)}_{\ell,N}$ are found from the recursion formula

M_p = port mach number.

for $\ell \neq N$

$$p^{(n)}_{\ell,\underset{\sim}{N}} = \left\{ \frac{\dfrac{i\,\omega_{\underset{\sim}{N}}\gamma a M_a}{c}\, J_\Theta(\alpha_{\ell,\underset{\sim}{N}}a) \displaystyle\sum_{m=0}^{n-1} p^{(m)}_{\ell,\underset{\sim}{N}}\, D_{n-m,\underset{\sim}{N}} + \sum_{\nu=0}^{\infty} p^{(n-1)}_{\nu,\underset{\sim}{N}} J_\Theta(\alpha_{\nu,\underset{\sim}{N}}a)\left[\pi\nu c_{\nu,\ell} - \frac{i\rho\,\omega_{\underset{\sim}{N}} L M_p b_{\nu,\ell}}{c}\right]}{\alpha_{\ell,\underset{\sim}{N}}a\, J'_\Theta(\alpha_{\ell,\underset{\sim}{N}}a) - \dfrac{i\gamma a\omega_{\underset{\sim}{N}}}{c} M_a J_\Theta(\alpha_{\ell,\underset{\sim}{N}}a)\left[\left(\dfrac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 - \dfrac{1}{\gamma} - \dfrac{\rho c}{\gamma M_a} Y_s\right]} \right\} \tag{5b}$$

where

$$D_{n,\underset{\sim}{N}} = \frac{i c}{\omega_{\underset{\sim}{N}}\gamma a M_a J_\Theta(\alpha_{N,\underset{\sim}{N}}a)} \sum_{\nu=0}^{\infty} p^{(n-1)}_{\nu,\underset{\sim}{N}} J_\Theta(\alpha_{\nu,\underset{\sim}{N}}a)\left[\pi\nu c_{\nu,N} - \frac{i\rho\omega_{\underset{\sim}{N}} L M_p b_{\nu,N}}{c}\right] \tag{5c}$$

with $p^{(n)}_{N,\underset{\sim}{N}} = \delta_{n,0}$ and $p^{(0)}_{\ell,\underset{\sim}{N}} = \delta_{\ell,N}$, (5d)

where δ is the Kroenecker delta, $\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$.

The Fourier coefficients $b_{\nu,\ell}$ and $c_{\nu,\ell}$ are defined by

$$c_{\nu,\ell} = \frac{2}{(1+\delta_{\ell,0})L}\int_0^L dz\ \mathrm{Signum}\,(\bar{v}_e)\cos\left(\frac{\ell\pi z}{L}\right)\sin\left(\frac{\nu\pi z}{L}\right), \text{ and} \tag{5e}$$

$$b_{\nu,\ell} = \frac{2}{(1+\delta_{\ell,0})M_p c L}\int_0^L dz\,|\bar{v}_e|\cos\left(\frac{\ell\pi z}{L}\right)\cos\left(\frac{\nu\pi z}{L}\right), \text{ with} \tag{5f}$$

M_p = port mach number.

This completes the solution for the non-degenerate case. Calculation is not difficult in spite of the rather unwieldy expressions because of the typically small value of $\left(\frac{aKcM_a}{L}\right)$. It will usually be entirely adequate to terminate with the $p^{(2)}$'s. In a degenerate case, where more than one mode of motion corresponds to a single frequency, the denominator of Eq. (5b) vanishes for some $\ell \neq N$, and a slightly more general solution must then be obtained.

IV. RESULTS OBTAINED FROM THE ANALYTICAL SOLUTION OF BOUNDARY VALUE PROBLEM (NON-DEGENERATE CASE)

The question to be considered in this section is the effect of an erosive velocity on stability. As has been discussed, we have decided to measure this effect by determining the increment in the response function $(\tilde{\mu}/\tilde{\epsilon})$ which would be necessary to re-establish the neutral stability condition, when erosion is suddenly "turned on". The reference scale is provided by the observation that the real part of $\tilde{\mu}/\tilde{\epsilon}$ can ordinarily be expected to lie between the dc pressure exponent (<1) and a value perhaps several tenths greater than unity.[2] Of course, we must keep in mind that the question of instability or stability is determined by whether the real part of $\tilde{\mu}/\tilde{\epsilon}$ does or does not exceed a critical value, and therefore the erosive term, though small, would nevertheless become a critical factor in determining stability for cases such that $(\tilde{\mu}/\tilde{\epsilon})_0$ were close to the critical value.

Examining the magnitude of the expansion parameter $\left(\frac{aM_aKc}{L}\right)$, we suspect that, largely because of the small mach number ($M_a \sim 10^{-2}$) of the hot gases leaving the burning zone, it usually will be quite satisfactory to consider only the lowest order non-vanishing terms even for highly erosive propellants.

Let us consider the linear term first. Substituting Eq. (5d) into Eq. (5c), and the result into Eq. (4c), we find rather quickly that (we also use the identity $\beta K = k$),

$$\frac{\tilde{\mu}}{\tilde{\epsilon}} - \left(\frac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 = \frac{kc}{\gamma} \; M_p \, b_{NN} + \frac{iKc^2 \pi N}{\gamma \omega L} \; c_{N,N} + \text{higher order terms.} \qquad (6)$$

In discussing Eq. (6) two cases need to be distinguished, namely, the ordinary rocket with the nozzle at the end and special rockets with a side nozzle at the center. In the first case $\bar{v}_e$ is a positive quantity whereas in the second case it is positive in one end of the rocket and negative in the other.

For the usual case of positive $\bar{v}_e$ the coefficient $c_{N,N} = 0$. Thus, to first order, the contribution of the acoustic erosion is zero. For the typical cylindrical rocket $\bar{v}_e \simeq M_p c z/L$ we obtain $b_{NN} = \frac{1}{2}$. Thus, we have already determined in this first order term the predominant effect of mean erosive velocity on the onset of instability. It is clear that, with kc typically less than about 4 even for very erosive propellants, and M_p typically not greater than perhaps 1/10, the effect of dc erosion is to be "compensated for" by altering $\tilde{\mu}/\tilde{\epsilon}$ by ~ 1.5. Thus, we should expect to find enhanced stability for propellants having a positive dc erosion constant. For unusually high erosive velocity, and highly erosive propellants, this stabilizing effect could be very substantial. Such an effect has indeed been observed. On the other hand, we would expect the opposite effect for negative erosion constants, namely, a greater tendency toward instability. In order to discuss the effect of the acoustic velocity for the end nozzle case we must go to second order. This we will do after discussion of the first order treatment for the center nozzle case.

For the center nozzle rocket the situation is quite different. Here

$$\bar{v}_e \simeq \frac{2M_p cz}{L}, \qquad z < \frac{L}{2},$$

$$\bar{v}_e \simeq -\frac{2M_p c(L-z)}{L}, \qquad z > \frac{L}{2}.$$

This steady state velocity distribution leads to

$$c_{N,N} = \frac{2}{N\pi} \qquad \text{for N odd}$$

$$c_{N,N} = 0 \qquad \text{for N even;}$$

while

$$b_{NN} = \frac{1}{2} - \frac{2}{\pi^2 N^2} \qquad \text{for N odd}$$

$$= \frac{1}{2} \qquad \text{for N even}.$$

Thus we see that for N even (e.g., the even axial modes) the center nozzle rocket gives the same results as the end nozzle rocket. However, for N odd (e.g., the odd axial modes) the situation is strikingly different. Here the contribution of the acoustic erosion could be extremely important. In particular, for the first axial mode we find

$$\frac{\tilde{\alpha}}{\tilde{\epsilon}} - \left(\frac{\tilde{\alpha}}{\tilde{\epsilon}}\right)_0 = \frac{kcM_p}{\text{[illegible]}}\left(\frac{1}{2} - \frac{2}{\pi^2}\right) + \frac{2\,Kc}{\text{[illegible]}}$$

and we note that the ordinarily stabilizing (for k > 0) effect of the steady erosion is reduced by 40%. More important, the imaginary part of K need not be large to make an extremely significant contribution. For example, if imag $(Kc) \simeq -\pi/2$, the acoustic term would be $1/\gamma$, which is the usual criterion on $(\tilde{\mu}/\tilde{\epsilon})$ for an amplifying surface

Provided k is positive, the damping due to the $b_{N,N}$ term is greater for even modes than for odd modes. The first order acoustic term contributes to the instability if $(iK) < 0$, and to the damping if $(iK) > 0$. Thus, for low frequency (and $k > 0$) one would expect this term to contribute to instability for the odd modes.* One cannot help but wonder whether this effect does not contribute to the experimental observation that the predominantly unstable axial modes in the center nozzle of Price et al are the odd modes. However, at much higher frequencies the odd modes might be, on the contrary, stabilized.

Returning to the discussion of the end nozzle case, we shall investigate the contribution of acoustic erosion by displaying the second order terms. We find

$$c_{\nu,\ell} = \frac{2\nu}{\pi} \frac{[1-(-1)^{\nu+\ell}]}{[1+\delta_{\ell,0}][\nu^2-\ell^2]} \qquad \text{for } \nu \neq \ell \tag{6a}$$

and

$$c_{\ell,\ell} = 0 \, .$$

For $\bar{v}_e = M_p \frac{cz}{L}$.

$$b_{\nu,\ell} = -\frac{2[1-(-1)^{\nu+\ell}][\nu^2+\ell^2]}{\pi^2[1+\delta_{\ell,0}][\nu^2-\ell^2]^2} \quad \text{if } \nu \neq \ell \tag{6b}$$

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*These remarks do not pertain to the established oscillation at high amplitude because the non-linearities, particularly those due to erosion itself, would then generate the even modes.

and

$$b_{\ell,\ell} = \frac{1}{2} \quad .$$

Substitution of these expressions into the general solution finally yields, for this case,

$$\frac{\tilde{\mu}}{\tilde{\epsilon}} - \left(\frac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 = \frac{kcM_p}{2\gamma} \cdot \frac{4M_a a}{\gamma L}\left(\frac{Nc}{i\omega_{\underset{\sim}{N}}L}\right) \sum_{\substack{\ell=0 \\ (\ell \neq N)}}^{\infty} \frac{J_\Theta(\alpha_{\ell,\underset{\sim}{N}}a)\left[1-(-1)^{\ell+N}\right]^2}{\left[1+\delta_{N,0}\right]\left[N^2-\ell^2\right]^2\left[1+\delta_{\ell,0}\right]} \quad \text{X}$$

$$\left\{ \frac{\left[KcN + M_p\frac{kc}{\pi^2}\left(\frac{i\omega_{\underset{\sim}{N}}L}{Nc}\right)\frac{(N^2+\ell^2)}{(N^2-\ell^2)}\right]\left[\frac{-kc\ell^2}{N} + M_p\frac{kc}{\pi^2}\left(\frac{i\omega_{\underset{\sim}{N}}L}{Nc}\right)\frac{(N^2+\ell^2)}{(N^2-\ell^2)}\right]}{\left(\frac{\alpha_{\ell,\underset{\sim}{N}}a}{N}\right)J'_\Theta(\alpha_{\ell,\underset{\sim}{N}}a) - \frac{a\gamma M_a i\omega_{\underset{\sim}{N}}}{cN} J_\Theta(\alpha_{\ell,\underset{\sim}{N}}a)\left[\left(\frac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 - \frac{1}{\gamma} - \frac{\bar{\rho}c}{\gamma M_a}Y_s\right]} \right\} \tag{7}$$

While Eq. (7) may appear simpler than the exact solution, this appearance is somewhat deceptive and one must be careful about generalizing from it as to importance of the erosive effects. These obviously depend upon the nature of the propellant (particularly with respect to the values of its erosion constants K and k) and upon the geometry and the mode under consideration. If, for example, the denominator in the sum should fortuitously become very small for some particular ℓ and $\underset{\sim}{N}$, the result would be a large effect on the stability, either to increase or decrease it depending on the sign of the real part of the term involved. The extreme case of one of these denominators actually being zero corresponds to the degenerate case discussed in the section V.

In many typical rocket geometries, however, the denominators will be more or less uniform in value over the frequency range of interest and in that event one can expect less drastic results. In the case of $\underset{\sim}{N} = N, 0, 0$, or essentially pure axial modes, we notice that $\frac{\omega_{\underset{\sim}{N}} L}{Nc} = \pi$ and as a consequence, for the usual motors and propellants, the second term in each square bracket (k terms) will make a negligible contribution if the axial number N is not very large. The contribution of the first terms (K terms), however, needs examination since they may have larger coefficients. To gain some idea of the size of the effect let us consider the special case with $\left[(\tilde{\mu}/\tilde{\epsilon})_0 - (1/\gamma) - (\bar{\rho}\, c\, Y_s/\gamma M_a)\right] = 0$, which corresponds to the unperturbed solution (see Eq. (4a)) having a pressure antinode at the surface. Assuming the rocket to be a cylinder with $\bar{v}_e = M_p cz/L$ we obtain

$$\frac{\tilde{\mu}}{\tilde{\epsilon}} - \left(\frac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 \simeq \frac{kcM_p}{2\gamma} + (1 + \delta_{N,0})^{-1} \frac{2iM_a(Kc)^2}{\gamma\pi^2} \times$$

$$\sum_{\substack{\nu=0 \\ \neq N}}^{\infty}{}' \frac{2}{1+\delta_{\nu,0}} \frac{\nu^2\left[1 - (-1)^{\nu+N}\right]^2 J_0\left(\frac{N\pi a}{L}\sqrt{1 - \nu^2/N^2}\right)}{(\nu^2 - N^2)^2\sqrt{1 - \nu^2/N^2}\; J_1\left(\frac{N\pi a}{L}\sqrt{1 - \nu^2/N^2}\right)} \quad . \tag{8a}$$

When $\frac{N\pi a}{L}$ is small, the sum converges rather rapidly because of the high power in the denominator, and clearly, the largest contributions will ordinarily come from the terms $\nu = N \pm 1$, except for the case $N = 1$ when only the $\nu = N + 1$ terms need be retained. In particular, for the lowest axial mode, we find (with $N = 1$, Θ 0, and $\nu = 2$, and $\frac{LM_a}{a} = \frac{M_p}{2}$),

$$\frac{\tilde{\mu}}{\tilde{\epsilon}} - \left(\frac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 \simeq \frac{kcM_p}{2\gamma}\left[1 - \frac{iKc}{\beta}\left(\frac{128}{27\pi^3}\right)\right] . \tag{8b}$$

For this particular case the acoustic contribution is proportional to the imaginary part of K/β (i.e., (K^2/k)) so that there is no contribution to stability unless K is complex. The importance of the result depends, of course, on the size of this component of K/β. If it should be of the same order as k itself, i.e., kc say about 4, the acoustic contribution would be approximately one-half the dc contribution. Thus for positive erosion the net effect would be to stabilize the rocket.

Investigation of Eq. (7) for radial and tangential modes shows that the acoustic term is quite small for typical geometries. Thus, we see, in contrast to the center nozzle rocket, there is a wide range of conditions over which we expect erosivity to contribute little to the question of stability or instability. For rockets in which axial modes of very high N are not heavily damped, this generalization may not be valid. Moreover, it may fail dramatically for conditions approaching the degenerate case discussed in the next Section.

V. THE DEGENERATE CASE

5.1 Degeneracy and Quasi-Degeneracy

If we were to consider maximizing the acoustic erosive effect, we would refer to Eq. (7), and note that if the denominator were to become very small for some term, say for $\ell = M$, then the influence of erosion might indeed become overwhelming. Of course, the real part of the denominator will vanish whenever the frequencies of two characteristic modes of the system coincide. Since the internal radius of the usual solid propellant rocket chamber increases with time as the propellant burns, the frequencies of the radial and tangential modes decrease with time while the pure axial mode frequencies are independent of chamber radius. This state of affairs is indicated in Figs. 1 and 2, for a hypothetical motor with $L = 5\pi$ cm, $c = 10^5$ cm/sec.* In Fig. 1 we have plotted the radial and axial frequencies, and in Fig. 2, the first tangential frequencies, corresponding to the radial velocity node boundary condition on the burning surface.**

In general, at each intersection point in Figs. 1 and 2, the real part of the denominator vanishes in one of the p_ℓ's occurring in the sum representation of the solution. But the system is not truly degenerate at these intersections unless the imaginary part of the denominator also vanishes there, i.e.,

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*The figure is easily scaled to represent chambers of other lengths and radii. If both a and L are multiplied by a common factor, the frequency scale is to be divided by that factor.

** Although these "quasi-modes" are not the true modes of the cavity because the solid propellant itself also indulges in the acoustic motion, it has been shown elsewhere that the frequencies of the modes of the composite system which have large gas pressure amplitude - and therefore large erosive acoustic velocity, cf. (Eq. (3c)) at the propellant surface lie close to these gas quasi-mode frequencies.[3]

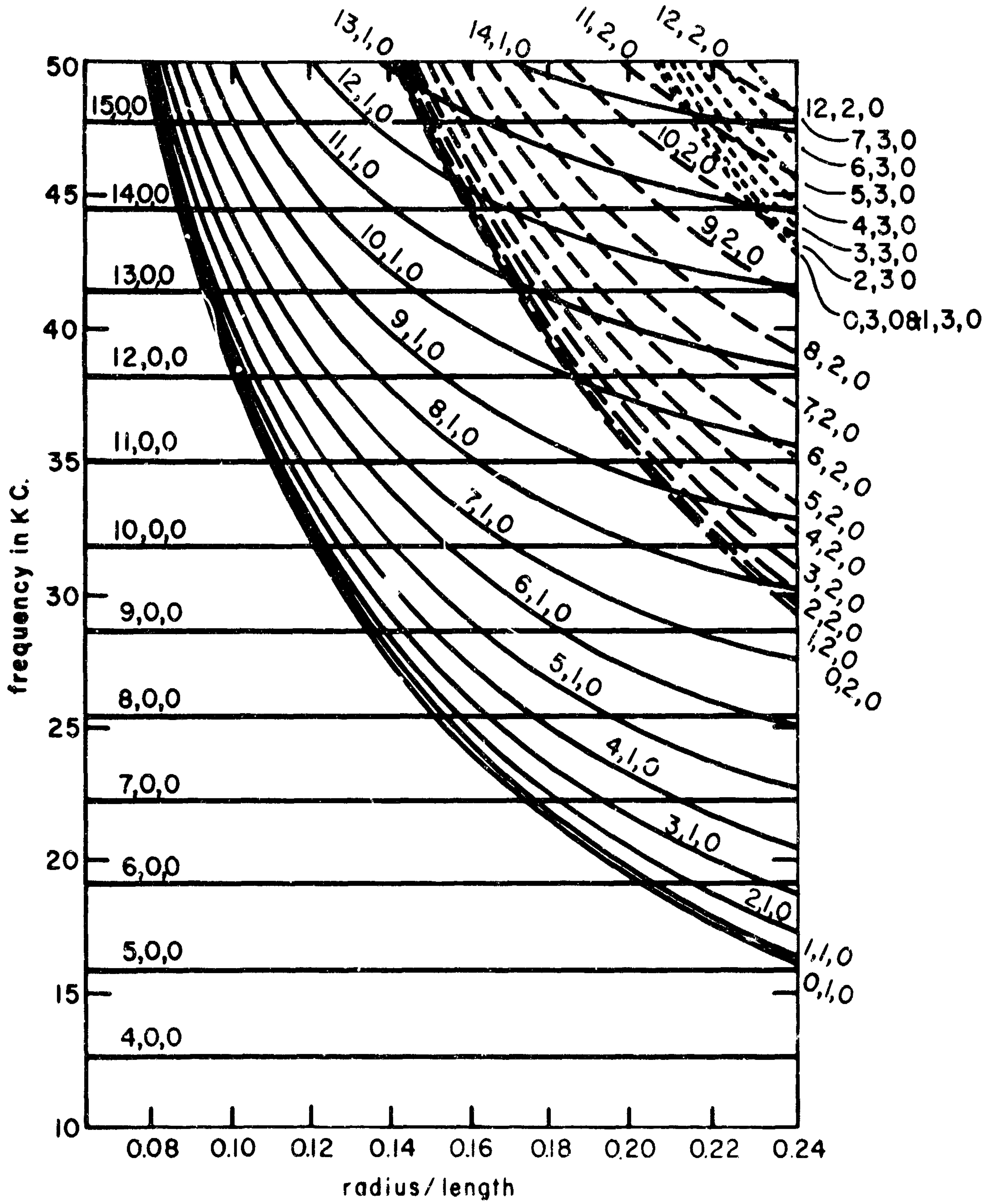

Fig. 1 FREQUENCY VERSUS GEOMETRY FOR THE AXIALLY SYMMETRIC MODES OF A GAS CONTAINED IN A RIGID-WALLED CYLINDER (GAS QUASI-MODES)

The axial, radial, and angular mode numbers are shown for each curve, in that order.

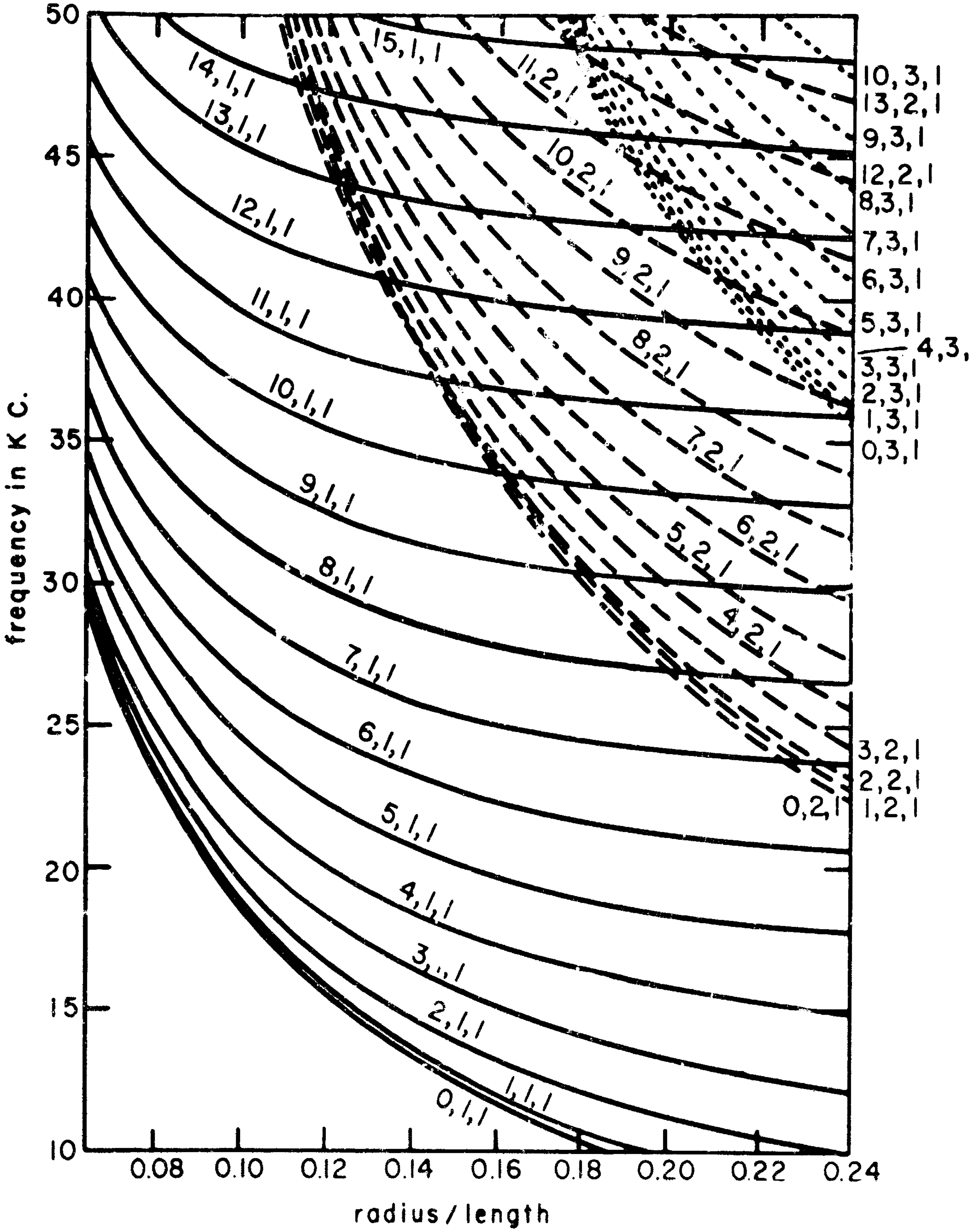

Fig. 2 FREQUENCY VERSUS GEOMETRY FOR THE FIRST AZIMUTHAL MODES OF A GAS CONTAINED IN A RIGID-WALLED CYLINDER (GAS QUASI-MODES)
The axial, radial, and angular mode numbers are shown for each curve, in that order.

unless the acoustic losses "accidentally" balance the gains for that particular geometry. However, we see that during the course of burning, many points of "quasi-degeneracy" may appear, where the denominator of a term in the sum is especially small because its real part vanishes. But we shall expect a true degeneracy, characterized by the vanishing of both real and imaginary parts, to be a rare occurrence.

Unfortunately, at this time it seems impossible to calculate the expected effect on stability at the points of quasi-degeneracy. The impasse arises because we need to know the imaginary part of the denominator which will, in general, be a function of the losses in the normal mode being considered. The detailed computation of these losses poses formidable difficulties and no experimental information bearing directly on this point is available.

We may, however, expect to gain insight as to whether or not erosion can have a large influence at configurations corresponding to such quasi-degeneracies by investigating the limiting case of true degeneracy.

5.2 The Degenerate Case

Let us now suppose that two normal modes of the system are neutrally stable. In other words, let the inner radius $\underline{a}$ approach a value such that the denominator of Eq. (7) vanishes for some ℓ, say $\ell = M$. Then, we must re-solve the boundary value problem to develop a solution for this degenerate case.

The analytical method remains relatively unchanged; we again expand in powers of the expansion parameter, and equate the coefficients of equal powers in the boundary equation. This time, of course, the unperturbed motion must include two, rather than one, zero order terms, in order that the new sum to be obtained will not contain any terms with zero denominators. We write

$$\frac{\tilde{p}}{\bar{P}} = p_{N,\underset{\sim}{N}} \left\{ J_{\Theta}(\alpha_{N,\underset{\sim}{N}} r) \cos\left(\frac{\pi N z}{L}\right) + J_{\Theta}(\alpha_{M,\underset{\sim}{N}} r) \left(\frac{A J_{\Theta}(\alpha_{N,\underline{N}} a)}{J_{\Theta}(\alpha_{M,\underline{N}} a)} \right) \cos\left(\frac{\pi M z}{L}\right) \right\} \cos$$

$$+ \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} p_{\ell} J_j(\alpha_{\ell,\underset{\sim}{N}} r) \cos\left(\frac{\pi \ell z}{L}\right) \cos(j\theta) \qquad (9a)$$

$$(\ell \neq M, N \text{ when } j = \Theta)$$

where A is to be determined in such a way that the solution approaches the bracketed term as the erosion vanishes ($K \rightarrow 0$). The velocities $\hat{v}_r$ and $\hat{u}_z'$ are obtained in the usual way (Eqs. (3b), (3c)), and substitution into the boundary condition equation at $r = a$ finally yields

$$A = \frac{1}{2}\left[\frac{\beta M_p}{f_{M,N}} \frac{i\omega L}{c} (b_{N,N} - b_{M,M})\right] \pm i \sqrt{\frac{-f_{N,M}}{f_{M,N}} - \left(\frac{\beta M_p i\omega L}{2c f_{M,N}}\right)^2 (b_{NN} \cdot b_{MM})^2} \qquad (9b)$$

where

$$f_{N,M} = \pi N c_{N,M} - \beta M_p \frac{i\omega L}{c} b_{N,M} \qquad (9c)$$

The upper and lower signs correspond to the two possible choices for A in Eq. (9a), and define the two orthogonal combinations of the degenerate modes, which may be obtained by substituting these A's into Eq. (9a). The increment in the response function necessary to re-establish the neutral stability condition when erosion is "turned on" is finally found to be

$$\frac{\tilde{\mu}}{\tilde{\epsilon}} - \left(\frac{\tilde{\mu}}{\epsilon}\right)_0 = \frac{kcM_p}{2\gamma}(b_{N,N} + b_{M,M}) \mp \sqrt{(Kc)^2\left(\frac{c}{\omega L\gamma}\right)^2(-f_{N,M}f_{M,N}) + \left[\frac{kcM_p}{2\gamma}(b_{N,N} - b_{M,M})\right]^2} \tag{9d}$$

Noting that, at least for typically small M_p, $(-f_{N,M}\, f_{M,N})$ is positive, we see that the mode corresponding to the choice of the lower sign is made more stable by erosion, while the mode corresponding to the choice of the upper sign is made less stable. Each mode will look more or less like an N mode or an M mode according to whether or not $\frac{A\, J_\Theta(\alpha_{N,\tilde{N}}a)}{J_\Theta(\alpha_{M,\tilde{N}}a)}$ is substantially less than, or greater than, unity.

An appreciation of the order of magnitude involved may be obtained by evaluating Eq. (9d) for a few cases. If we consider the conventional end nozzle rocket, the $b_{N,M}$ may be obtained from Eq. (6b), and the $c_{N,M}$ from Eq. (6a). One quickly finds that the increment in $\tilde{\mu}/\tilde{\epsilon}$ arising from acoustic erosion will be sometimes large and sometimes small (relative to $1/\nu$), with the larger contributions occurring for N close to M. If we continue to take $Kc = 4$, $L = 5\pi$ cm, $c = 10^5$ cm/sec, and $\gamma = 1.2$, we find, for example, that for the 5, 2, 1 and 6, 1, 1 intersections, the radical in Eq. (9d) yields the value 1.8. Thus, such true degeneracies, if they were to occur, could enable erosion to exert a profound effect on stability. The importance of having M and N nearly equal can be shown by reducing the value of M from 5 to 1 in the above example, with the consequent diminishing of the contribution from 1.8 to 0.1.

In the neighborhood of degeneracy we see that we might expect acoustic erosion sometimes to have a large effect on the question of stability or instability. The question naturally arises why this effect has not made its presence obvious in experimental studies. Of course, one should note that unless the results were relatively catastrophic, it would be quite easy to miss the connection between degenerate modes unless one were specifically looking for it. However, there are probably more important reasons why the effect has not been noticed. In the first place, we should recall that the losses will generally be rather different in two modes even when they have the same frequency. As a result, true degeneracies will be relatively rare compared with the milder quasi-degeneracies. In the second place, we see that even the quasi-degeneracies are not particularly common in the small thin motors typically used in research studies except at very high frequencies where one would expect the general damping to be great enough to keep things under control.

In looking for this phenomenon, the principal clue would be the observation that two modes, one or both of which is individually stable, together become unstable when their frequencies cross.

Price has observed catastrophic interaction for modes having the same frequency in some of his experiments. It appears, however, that this is an interaction between certain axial and tangential modes. In linear theory, axial and tangential modes are orthogonal because of their different angular distribution, and thus should not be coupled. At finite amplitude, however, in the presence of mean flow and erosion, this exact orthogonality will no longer exist. It is possible that Price's observations represent the same phenomenon where, however, the coupling arises from the non-linearities implicit in the large amplitude case.

The mode maps obtained experimentally by Angelus provide an appropriate display of data for which a number of crossing points of mode frequencies exist. To the best of our knowledge, none of his experiments have shown any particular oscillations attributable to these intersections, nor have they shown the catastrophic behavior observed by Price. It would appear, therefore, either that the acoustic erosion constant was quite small in these frequency regions, or that the losses in the intersecting modes were such that nothing approaching a true degeneracy existed. At this time there is no information available to decide which of these reasons was dominant. Whether or not phenomena associated with mode degeneracies have actually occurred in researches of other investigators is essentially not determinable unless the data are processed in a manner similar to that used by Angelus.

5.3 Selection Rules

Up to this point, we have not considered the possibility that some degeneracies (or quasi-degeneracies) will fail to contribute because of zeros in the numerators of the critical terms in Eq. (7). Because of the potentially large degenerate effects, however, the selection rules introduced by zeros in the numerator of Eq. (7) should be mentioned.

Referring to Eq. (5e) and (5f), we see that the $b_{\ell,m}$ and $c_{\ell,m}$ will vanish for $\ell + m$ either odd or even, depending on symmetry of the function for which they are the Fourier coefficients. Thus, for example, analysis shows that if

$\left|\bar{v}_e\right|$ can be made into an antisymmetric function about the midpoint of the chamber by adding a constant, then the $b_{\ell, m}$ vanish for $\ell + m$ even. Analogously, if $\left|\bar{v}_e\right|$ can be made symmetric about L/2 by adding a constant, then the $b_{\ell, m}$ vanish for $\ell + m$ odd. With respect to the $c_{\ell, m}$, we find that if Signum $(\bar{v}_e)$ is symmetric about $z = L/2$, then they vanish for $\ell + m$ even (or for $\ell + m$ odd if antisymmetric), so that we should find, at least to first order, no erosive contributions corresponding to such "forbidden" intersections.

As a result, the selection rules are quite different in the end nozzle and center nozzle cases. For center nozzle rockets, we find the the b's and c's vanish if the sum of their indices is odd, while for the end nozzle case, these coefficients vanish if the sum of their indices is even. Thus, in center nozzle rockets, large erosive contributions from degenerate configurations become possible only at those configurations which were forbidden in the end nozzle rockets and vice versa.

VI. DISCUSSION

The initial objective of this study was merely to see whether acoustic theory would indeed confirm the rather widely held interpretation of experiments to the effect that the influence of erosion on stability is usually a minor one except for the general tendency of high steady erosion to stabilize. We have met with success insofar as this objective is concerned. But we have also found that under certain circumstances which are presently rather exceptional, rather than common, erosion should be very important in determining the answer to the stability question.

For the most part, the effects of erosive velocity have been separated into two components which arise on the one hand from the erosive component of the mean flow velocity, and on the other hand from the erosive component of the acoustic velocity. With respect to the mean flow velocity effects, the analysis indicates that if the dc erosion constant, k, and the mean flow velocity, $\bar{v}_e$, are sufficiently great that $|k||\bar{v}_e| > \sim 1/3$, then in end nozzle rockets, stability should be significantly enhanced for erosive propellants having $k > 0$, and significantly diminished for propellants having $k < 0$. Experimental evidence does substantiate the first of these two results, but we are unaware of experimental data relating to negative erosion constants. In any case, data necessary for a quantitative comparison with experiment do not yet exist. The odd axial modes of side nozzle rockets should be somewhat less sensitive to $k\bar{v}_e$ in linear analysis. We know of no quantitative data to confirm or deny this result, except again to note the fact that non-mesa propellants in side nozzle motors seem to prefer to initiate oscillation in odd axial modes.

The situation with respect to the effect of the erosive component of the acoustic velocity is numerically somewhat less satisfactory. The analysis indicates that the acoustic erosion constant should indeed usually have little bearing on the stability question - except in two kinds of exceptional circumstances. Of these two, the first pertains to the side nozzle rocket, and the second to geometrical configurations for which the system becomes acoustically degenerate.

With respect to the side nozzle case, we have found that acoustic erosion contributes in first order, whereas in usual end nozzle rockets, this effect makes its first appearance in the second order terms. Whether acoustic erosion increases or decreases stability (insofar as the first order term is concerned) depends on the imaginary part of the ac erosion constant, K. Thus, its effect must vanish at sufficiently low frequencies where $K \rightarrow k$, and at sufficiently high frequencies where the system becomes unable to respond to a sufficiently rapidly varying erosive velocity. Unfortunately, it appears that nothing is yet known regarding the relevant frequency interval.

With respect to the degenerate configurations, a realistic quantitative calculation in the immediate neighborhood of these "accidental" degeneracies not only would require a time-dependent erosive burning theory, but also a difficult and detailed treatment of the visco-elastic losses for the degenerate modes. We have attempted neither of these tasks. However, an order of magnitude calculation has shown that, at least in principle, the effect of acoustic erosion could be overwhelming in the case of certain degeneracies. We should perhaps mention a third atypical configuration, namely rockets having unusually high port Mach numbers. Here, too, the acoustic erosive velocity may substantially affect the balance of gains and losses, with the sign of the effect depending on the phase shift associated with the acoustic erosion.

Finally, the above generalizations must be tempered by the thought that for some future generation of rockets with high burning rates, highly erosive propellants and short fat motors many features not previously observed may well become apparent.

With respect to the elementary concepts, and to accounting for the existence of the phenomena through a physical picture, one may now easily visualize their occurrence with the aid of a little hindsight. The effect of the dc erosion constant is simplest to account for, so we shall discuss it first. In order to do so, then, we may suppress the acoustic erosion by setting $K(\omega) = 0$ for $\omega \neq 0$. Then, referring to the boundary condition at the burning surface, we see that the turning on of a steady uniform erosive velocity merely increases the usual steady mass flow by $(1 + k\,|\bar{v}_e|)$. How much will we then have to alter $\tilde{\mu}/\tilde{\epsilon}$ in order to restore neutral stability? Since μ was defined as the fractional increment in mass flow without erosion, we would expect to find the increment in $\tilde{\mu}/\tilde{\epsilon}$ to be $k\,|\bar{v}_e|\left(\frac{\tilde{\mu}}{\tilde{\epsilon}}\right)_0 = k\,|\bar{v}_e|\left(\frac{1}{\gamma}\right)$, just as is given by Eq. (6). The coefficient $b_{N,N}$ which is equal to unity only for the constant flow case just considered, may be regarded as describing an appropriate average of the erosive velocity over the mode.

The possibility of a large acoustic interaction for the degenerate cases might also have been anticipated. For simplicity, let us now suppress the dc erosion by setting $k = 0$. For an end nozzle rocket, any single mode with even ℓ has the acoustic pressure and radial velocity symmetric about the center of the chamber, whereas the acoustic axial (erosive) velocity is then antisymmetric about the midpoint. The reverse situation applies if ℓ is odd. Thus, the mass flow perturbation introduced by the axial velocity tends to destroy the pressure distribution which gives rise to it. But if we have two modes, (at the same frequency), one with even and one with odd ℓ, then the erosive perturbation of one mode can tend to reinforce (or cancel) the pressure and radial velocity oscillations of the other, and we might expect a relatively large effect. For the center nozzle rocket, however, the antisymmetry of the steady state flow field plays a dominant role. For odd modes, the pressure distribution is antisymmetric and the steady flow field will keep the velocity distribution substantially antisymmetric also. Thus, large interaction between the erosion and pressure contributions to stability would be expected even in the absence of degeneracy. On the other hand, if the mode is even, the situation is not essentially different from the end nozzle case. Of course, the dc and ac effects are not always so neatly separable, so that this backward look has qualitative significance, only. But it should probably serve to caution one against any ill-conceived generalizations of these results to other configurations with new boundary conditions.

In conclusion, it should probably be noted that the present analysis is hardly a "theory", but rather, a mathematical machine designed to translate an input consisting of the effect of acoustic erosion on burning rate into an output consisting of the resulting effect on acoustic stability. We have investigated the output of this machine for some ad hoc inputs. Further progress requires knowledge of the inputs which may be encountered in reality. This implies either a time-dependent theory of erosion, or direct experimental measurement of the dependence of burning rate on fluctuating gas velocity.

A number of simplifying assumptions have been made which would need examination in particular cases. The cross sectionally uniform flow field assumption is certainly incorrect, but, apart from its axial symmetry, the results depend rather insensitively on (space) averages of $\bar{v}_e$, so that perhaps the model is not seriously in error in this regard. The neglect of the doppler shifts arising from the mean flow field may become significant for high port Mach numbers (comparable to unity), however, and the present analysis should not be applied in such instances. Applicability of the analysis is further limited to the axial boundary conditions which we have treated and it is not too clear to what extent the results would be modified by other end boundaries. Undoubtedly the selection rules would be quite different, in general. The neglect of acoustic losses in the gas and in the nozzle is probably most significant at high frequency and, in the case of end nozzle rockets, for modes with high axial wave numbers. These losses are significant in determining which frequencies will appear. However, we have been interested in the relative importance of the pressure and velocity effects, so that this neglect does not seem to be a significant handicap.

The acoustic boundary condition at the burning surface has been represented by an admittance depending on frequency but not otherwise on the detailed nature of the mode being considered. Since the solid itself may be a participant in the acoustic motion, this representation is an oversimplification. The solid has both dilatation and shear motion, and these reflect into a component (Y_s) of the surface admittance presented to the gas which is not solely frequency dependent. We have not included this additional realism in the calculations presented herein, and must remind the reader that there may be occasion upon which this feature may be quite important.

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