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Research Report

Partially Coherent Diffraction by a Circular Aperture

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ELECTROMAGNETIC RADIATION LABORATORY PROJECT 5635

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Abstract

Diffraction patterns of a circular aperture illuminated by partially coherent light can be calculated for a variety of correlation functions and radially varying intensities. The calculation is facilitated by using a theorem, based on the Wolf formulation of coherent theory, that relates the Fraunhofer intensity of a quasimonochromatic spatially stationary source to the Fourier transform of the product of the source autocorrelation function and the normalized source mutual intensity. Curves are presented for the case of a circular aperture with uniform intensity and exponential correlation.

Partially Coherent Diffraction by a Circular Aperture

1. INTRODUCTION

Although the theory of partially coherent light has been the subject of considerable study,[1, 2] relatively little work has been devoted to partially coherent diffraction even though an adequate mathematical formalism exists. Diffraction studies have to date been confined to the one-dimensional slit. Parrent and Skinner[3] have examined the diffraction pattern of a primary slit source with uniform intensity and exponential correlation, and Bakos and Kantor[4] have theoretically and experimentally studied the diffraction pattern of a slit illuminated by a parallel incoherent slit of finite width.

One reason that the study of partially coherent diffraction has been limited is the difficulty of evaluating the multiple surface integrals in the Wolf-Parrent expressions for the intensity in the Fraunhofer region of a plane partially coherent quasimonochromatic source. Schell's[5] recent derivation of a Fourier transform theorem valid for quasimonochromatic spatially stationary sources considerably facilitates the calculation of partially coherent diffraction patterns.

The first part of this paper presents a derivation of the transform theorem that closely parallels Schell's original. A familiarity with the Wolf formulation of partial coherence is assumed, and only essential background material is reviewed.

The transform theorem is then applied to the case of the circular aperture. It is shown that diffraction patterns for a variety of correlation functions and of radially varying intensities can be obtained from line integrals easily evaluated by numerical methods. Curves for a circular aperture with uniform intensity and exponential correlation are presented.

Scalar theory is used throughout and so the results are applicable either to sources with no preferred direction of polarization[6] or to linearly polarized sources.

2. A FOURIER TRANSFORM THEOREM FOR PLANE QUASIMONOCHROMATIC PARTIALLY COHERENT SOURCES

Given an analytic signal representation V(t) of the light disturbance at time t at a point in a time-stationary field, a measure of the correlation of the field is the mutual coherence function

$$\Gamma(P_1, P_2, \tau) \equiv \Gamma_{12}(\tau) = \langle V_1(t+\tau) V_2^*(t) \rangle , \tag{1}$$

where $V_1(t)$ and $V_2(t)$ represent the disturbance at points P_1 and P_2, and the sharp brackets indicate that an infinite time average is taken. A normalized form of the mutual coherence function, called the complex degree of coherence, is defined by

$$\gamma(P_1, P_2, \tau) \equiv \gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{I_1^{\frac{1}{2}} I_2^{\frac{1}{2}}} , \tag{2}$$

where $I_1 = \Gamma_{11}(0)$ and $I_2 = \Gamma_{22}(0)$ are the time-averaged light intensities at P_1 and P_2. As a result of the normalization, $|\gamma_{12}(\tau)| \leq 1$.

The freespace propagation of the mutual coherence function is described by a pair of wave equations

$$\nabla_m^2 \Gamma_{12}(\tau) = \frac{1}{c^2} \frac{\partial^2 \Gamma_{12}(\tau)}{\partial \tau^2} , \qquad m = 1, 2 . \tag{3}$$

The subscript m indicates that the Laplacian operates on the coordinates of the point P_m. The boundary-value problem for a plane aperture with a known distribution of the mutual coherence function has been solved.[2] In the field of the aperture,

$$\Gamma(P_1, P_2, \tau) = \frac{1}{(2\pi)^2} \int_\sigma \int_\sigma \frac{\cos\theta_1 \cos\theta_2}{r_1^2 \, r_2^2} \, \Omega(r_1, r_2, \tau) \, \Gamma(S_1, S_2, \tau - \frac{r_1 - r_2}{c}) \, d\sigma_1 d\sigma_2 , \tag{4}$$

where S_m is an aperture point, P_m is a field point, r_m is the distance from

S_m to P_m, θ_m is the angle between the normal to the aperture and the line from S_m to P_m, and Ω is the differential operator

$$\Omega = 1+\frac{r_1-r_2}{c}\frac{\partial}{\partial\tau} - \frac{r_1 r_2}{c^2}\frac{\partial^2}{\partial\tau^2} .$$

The field is assumed to vanish over the aperture plane exterior to the aperture.

The form of Eq. (4) appropriate to the important case of a quasimonochromatic field with small path differences is[2]

$$\Gamma(P_1, P_2, \tau) = \frac{e^{-2\pi i\bar{\nu}\tau}}{(2\pi)^2}\iint_{\sigma\sigma}(1-i\bar{k}r_1)(1+i\bar{k}r_2)\frac{\cos\theta_1\cos\theta_2}{r_1^2 r_2^2}\Gamma(S_1, S_2, 0)\,e^{i\bar{k}(r_1-r_2)}d\sigma_1 d\sigma_2 ,$$

$$\frac{\Delta\nu}{\bar{\nu}} \ll 1,\ \mathrm{Max}\left|\tau-\frac{r_1-r_2}{c}\right| \ll \frac{1}{\Delta\nu} , \tag{5}$$

where $\bar{\nu}$, $\bar{k}$, and $\Delta\nu$, respectively denote the mean frequency, mean wavenumber, and spectral width. The quantity $\Gamma(S_1, S_2, 0) = I(S_1)^{\frac{1}{2}} I(S_2)^{\frac{1}{2}} \gamma(S_1, S_2, 0)$ is the mutual intensity function of the aperture.

From Eq. (5), the intensity in the Fraunhofer region of the source is obtained by letting the field points P_1 and P_2 coincide, setting $\tau=0$, and making the standard farfield approximations.* Thus,

$$I(P) \equiv I(\hat{p}, \theta) = \frac{\cos^2\theta}{\bar{\lambda}^2 R^2}\iint_{\sigma\sigma}\Gamma(S_1, S_2, 0)\,e^{i\bar{k}\sin\theta\hat{p}\cdot(\underset{\sim}{S_2}-\underset{\sim}{S_1})}d\sigma_1 d\sigma_2 . \tag{6}$$

The quantities in Eq. (6) are defined with reference to Fig. 1. The aperture source σ is located in the x_1y_1 plane and the diffraction pattern is described with respect to the identically oriented x_2y_2 coordinates; $\theta = \cos^{-1} z/R$, where R is the distance from the origin O_1 in the plane of the source to the field point P, and z is the distance from O_1 to the field point O_2 perpendicular to the plane of the aperture; $\underset{\sim}{S_1}$ and $\underset{\sim}{S_2}$ are the vectors to the source points from O_1, and $\hat{p}$ is the unit vector in the direction from O_2 to P.

A simple geometric calculation applied to the assumption of small path differences gives the following condition for the maximum angle θ_{max} for which Eq. (6) is valid:

$$\frac{\Delta\nu}{\bar{\nu}}\frac{D\sin\theta_{max}}{\bar{\lambda}} \ll 1 . \tag{7}$$

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* It is assumed that $R \gg D$ and $R \gg \pi D^2/4\bar{\lambda}$, where D is the maximum dimension of the source.

The restriction to spatially stationary sources, which is central to this paper, is now made. It is assumed that the normalized mutual intensity function of the source, $\gamma(S_1, S_2, 0)$, depends only on the (oriented) distance $\underline{S} = \underline{S}_2 - \underline{S}_1$ between the source points S_1 and S_2. Since no restriction is made regarding the intensity distribution of the source, the aperture mutual intensity function becomes

$$\Gamma(S_1, S_2, 0) = I(S_1)^{\frac{1}{2}}\, I(S_2)^{\frac{1}{2}}\, \gamma(\underline{S}_2 - \underline{S}_1, 0) \ . \tag{8}$$

(This would be the form of the mutual intensity if, for example, the aperture were in the paraxial region of, and at a large distance from, a parallel-plane incoherent source.[7]) Equation (8) is substituted in Eq. (6) and the transformation

$$\underline{S}_1 = \underline{S}_1; \quad \underline{S}_2 = \underline{S}_1 + \underline{S} \tag{9}$$

is made. The Jacobian of this transformation is 1,* and the expression for the Fraunhofer intensity becomes

$$I(\hat{p}, \theta) = \frac{A\cos^2\theta}{\bar{\lambda}^2 R^2} \int_{\sigma_{\underline{S}}} \gamma(\underline{S}) e^{i\bar{k}\sin\theta\hat{p}\cdot\underline{S}} \left[\frac{1}{A}\int_{\sigma'(\underline{S})} I(\underline{S}_1)^{\frac{1}{2}}\, I(\underline{S}_1 + \underline{S})^{\frac{1}{2}} d\sigma\right] d\underline{S} \ . \tag{10}$$

In Eq. (10), A is the area of the aperture and $\gamma(\underline{S}, 0)$ has been written $\gamma(\underline{S})$ for conciseness; $\sigma_{\underline{S}}$ denotes the range of $\underline{S}$ (for a circular aperture of radius a, for example, $\sigma_{\underline{S}}$ is a circle of radius 2a), and $\sigma'(\underline{S})$ is that region of the aperture to which S_1 is restricted in order that $\underline{S}_1 + \underline{S}$ may lie on the aperture. The bracketed quantity is the autocorrelation function of the source amplitude $C(\underline{S})$. Since for a finite aperture $C(\underline{S})$ is equal to zero for $\underline{S}$ greater than some finite bound determined by the aperture dimensions, Eq. (10) is valid with no restriction on the range of $\underline{S}$. Hence,

$$I(\hat{p}, \theta) = \frac{A\cos^2\theta}{\bar{\lambda}^2 R^2} \int_{|\underline{S}| \leq \infty} \gamma(\underline{S})\, C(\underline{S})\, e^{i\bar{k}\sin\theta\hat{p}\cdot\underline{S}} d\underline{S} \ . \tag{11}$$

This completes the proof of the theorem that (apart from an obliquity factor) the Fraunhofer intensity of a plane quasimonochromatic spatially stationary source is proportional to the Fourier transform of the product of the source autocorrelation function and the normalized source mutual intensity. The dependence on the distance R in Eq. (11) can be removed by considering the intensity per unit solid angle $P(\hat{p}, \theta)$:

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*This is most easily seen by expanding Eq. (9) in cartesian components.

$$P(\hat{p}, \theta) = \frac{A\cos^2\theta}{\bar{\lambda}^2} \int_{|S|\leqslant\infty} \gamma(\underline{S})\, C(\underline{S})\, e^{i\bar{k}\sin\theta\hat{p}\cdot\underline{S}} d\underline{S} \quad . \tag{12}$$

In the case of the coherent limit $\gamma(\underline{S}) = 1$, Eq. (12) reduces to the standard expression for the Fraunhofer intensity of a plane aperture illuminated by a monochromatic point source. The diffraction pattern for the limiting case of an incoherent source, as Parrent[8] has shown, cannot be obtained simply by letting the complex degree of coherence approach zero for any pair of arbitrarily close source points, but is instead found by taking the aperture dimensions very large compared with the coherence interval.

For simplicity consider the one-dimensional slit problem (the discussion carries over immediately to the two-dimensional aperture). The intensity per unit solid angle is given by the one-dimensional form of Eq. (12),

$$P(\theta) = \frac{2a\cos^2\theta}{\bar{\lambda}^2} \int_{-\infty}^{\infty} \gamma\left(\frac{x}{L}\right) C(x)\, e^{i\bar{k}\sin\theta\, x} dx \ , \tag{13}$$

where 2a is the width of the slit, x is the distance between two points on the slit, and L is the correlation interval. Let $x' = x/2a$ so that lengths are normalized to the slit width. Then

$$P(\theta) = \frac{2aL\cos^2\theta}{\bar{\lambda}^2} \int_{-\infty}^{\infty} \frac{2a}{L}\, \gamma\left(\frac{2ax'}{L}\right) C(2ax')\, e^{i2\bar{k}a\sin\theta\, x'} dx' \quad .$$

Now assume that

$$\lim_{a\to\infty} \frac{2a}{L}\, \gamma\left(\frac{2ax'}{L}\right) = c\delta(x') \ ,$$

where c is a finite constant depending on the form of γ. This assumption is valid for many of the commonly encountered correlation functions such as

$$\gamma\left(\frac{x}{L}\right) = e^{\frac{x}{L}},\ e^{-\left(\frac{x}{L}\right)^2},\ \left(\frac{\sin\frac{x}{L}}{\frac{x}{L}}\right).$$

Then as a increases, $P(\theta)$ approaches the form

$$P(\theta) = \frac{2aL\cos^2\theta}{\bar{\lambda}^2}\, c\,C(0) \ ,$$

which gives the linear dependence of intensity on aperture area conventionally associated with an incoherent source.

3. CIRCULAR APERTURE DIFFRACTION PATTERNS

The preceding results can now be applied to the case of the circular aperture. It is assumed that the aperture intensity varies radially only, and thus the autocorrelation function $C(\underline{S})$ depends only on $|\underline{S}|$, the distance between two points on the aperture. If in addition it is assumed that the source correlation $\gamma(\underline{S})$ is likewise a function of only the magnitude of $\underline{S}$, then the diffraction pattern is circularly symmetric, the vector $\hat{p}$ can be taken without loss of generality to be the unit vector in the positive x direction, and the expression for the intensity per unit solid angle given in Eq. (12) becomes

$$P(\theta) = \frac{\pi a^2}{\bar{\lambda}^2} \int_0^{2a} d\rho \int_0^{2\pi} d\phi \, \rho\gamma(\rho) \, C(\rho) \, e^{i\bar{k}\theta\rho\cos\phi} \tag{14}$$

In Eq. (14), $\rho = |S|$, and a is the aperture radius assumed to be much larger than $\bar{\lambda}$ so that the diffraction pattern will occur within a very small angle for which $\cos^2\theta \approx 1$ and $\sin\theta \approx \theta$. The angular integration yields

$$P(\theta) = \frac{2\pi^2 a^2}{\bar{\lambda}^2} \int_0^{2a} d\rho \, \rho\gamma(\rho) \, C(\rho) \, J_0(\bar{k}\theta\rho) \, . \tag{15}$$

Since the autocorrelation function is independent of the direction of the displacement it is convenient to choose the displacement in the negative x direction. By the definition in Eq. (10),

$$C(\rho) = \frac{4}{\pi a^2} \int_{\rho/2}^{a} dx \int_0^{\sqrt{a^2-x^2}} dy \left[I\left(\frac{x^2+y^2}{a^2}\right)\right]^{\frac{1}{2}} \left[I\left(\frac{(x-\rho)^2+y^2}{a^2}\right)\right]^{\frac{1}{2}} \, . \tag{16}$$

Evaluation of the integrals in Eqs. (15) and (16) is simplified if the integration variables are first normalized with respect to the aperture radius. Then

$$P(\theta) = \frac{2\pi^2 a^4}{\bar{\lambda}^2} \int_0^{2} d\rho' \, \rho'\gamma(a\rho') \, C\,(\rho') \, J_0(\bar{k}a\theta\rho') \, , \tag{15a}$$

with

$$C(\rho') = \frac{4}{\pi}\int_{\frac{1}{2}\rho}^{1} dx' \int_{0}^{\sqrt{1-x'^2}} dy' \left[I(x'^2+y'^2)\right]^{\frac{1}{2}} \left[I\left((x'-\rho')^2+y'^2\right)\right]^{\frac{1}{2}} \qquad (16\underline{a})$$

and

$$\rho' = \rho/a, \; x' = x/a, \; y' = y/a \quad .$$

The integration in Eq. (16a) can be performed for almost any radial intensity distribution since the intensity (or its square root) can if necessary be approximated by an even polynomial in the radial variable to obtain a closed form expression for $C(\rho')$ in terms of elementary functions. The determination of the diffraction pattern is thus reduced to evaluating the single line integral in Eq. (15a). Although for an arbitrary form of $\gamma(\rho)$ an analytic solution cannot in general be obtained, the integral can easily be evaluated by numerical methods.

As a representative example, the diffraction pattern of a uniformly illuminated circular aperture is calculated for an assumed exponential correlation

$$\gamma(\rho) = e^{-\rho/L} = e^{-\frac{a}{L}\rho'} = e^{-\alpha\rho'} \quad . \qquad (17)$$

Here L is the correlation interval and $\alpha = a/L$ is the number of correlation intervals contained in the aperture radius. The integration in Eq. (16a) with $I = 1$ gives as the autocorrelation function

$$C(\rho') = \frac{1}{\pi}\left[\pi - \rho'\left(1-\frac{\rho'^2}{4}\right)^{\frac{1}{2}} - 2\sin^{-1}\frac{\rho'}{2}\right]. \qquad (18)$$

Equations (17) and (18) are then substituted in Eq. (15a) and the diffraction pattern calculated for several values of the parameter α and a fixed aperture radius. When $\alpha = 0$, the aperture is fully coherent and the expression for $P(\theta)$ reduces to the standard formula:

$$P(\theta) = \frac{\pi^2 a^4}{\bar{\lambda}^2}\left[\frac{2J_1(Y)}{Y}\right]^2, \qquad Y = \bar{k}a\theta \quad .$$

The behavior of $P(\theta)$ for increasing α (decreasing L) is shown in Fig. 2, in which the curves have been normalized with respect to the maximum of the coherent pattern at $\theta = 0$. It is clear that as L decreases, the central fringe of the coherent pattern decreases in intensity and widens, the troughs between the bright fringes fill in, and the diffraction pattern tends toward the form of the incoherent pattern.

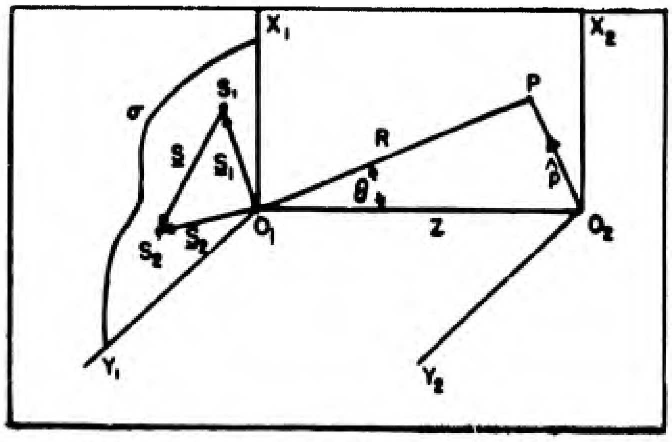

Figure 1. Reference geometry for calculating partially coherent diffraction patterns.

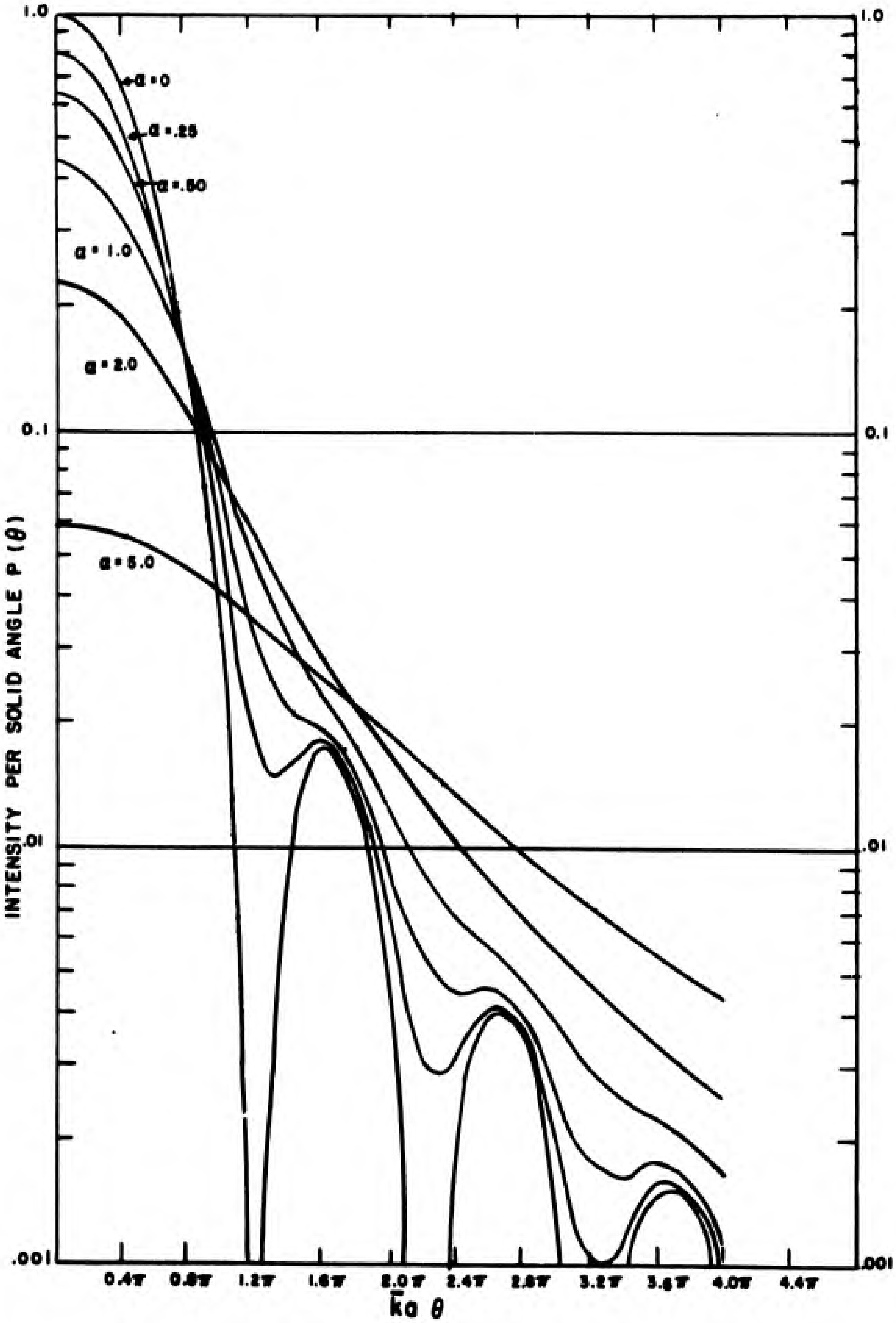

Figure 2. Circular aperture diffraction patterns.

References

1. M. BORN and E. WOLF, Principles of Optics, Pergamon Press, London, 1959, Chap. 10.
2. G.B. PARRENT, J. Opt. Soc. Am. **49**:787, 1959.
3. G.B. PARRENT and T.J. SKINNER, Optica Acta **8**:93 1961.
4. J. BAKOS and K. KANTOR, Il Nuovo Cimento **22**:519, 1961.
5. A.C. SCHELL, Doctoral Dissertation, Massachusetts Institute of Technology, September 1961.
6. H. OSTERBERG, Appendix in Phase Microscopy, Wiley, NY., 1951, p. 238, quoted in R. BARAKAT, Progress in Optics, E. WOLF, Ed., North-Holland Publishing Co., Amsterdam, 1961, p. 74.
7. H.H. HOPKINS, Proc. Roy. Soc., London (A)**217**:408, 1953.
8. G.B. PARRENT, Proceedings of the Symposium on Electromagnetic Theory and Antennas, Technical University of Denmark, 1962. To be published.