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NOTES ON THE THEORY OF LARGE-SCALE DISTURBANCES IN ATMOSPHERIC FLOW WITH APPLICATIONS TO NUMERICAL WEATHER PREDICTION

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ABSTRACT

The problem of predicting the behavior of large-scale disturbances in the mean horizontal flow of the earth's atmosphere, which is directly connected with the problem of predicting the day-to-day changes of surface weather conditions, has been studied from the standpoint of formulating and solving the hydrodynamical equations which govern the flow. Owing to the difficulty of solving the complete system of equations (whose very generality implies the existence of several irrelevant, but possible, types of solutions), it is convenient to develop a "scale theory" whereby the various possible types of atmospheric motion, each corresponding to a distinct type of solution, can be distinguished and classified. As it turns out, each type of motion is characterized by its phase speed and frequency. The large-scale disturbances, for example, are distinguished from all other types of motion by the fact that their characteristic phase speed is much less than that of sound waves and of high-speed internal gravity waves.

By explicitly introducing this information into a mean vorticity equation for adiabatic flow, it is then possible to reduce the system to a single equation from which the extraneous solutions have been excluded and which is otherwise free of major difficulties. The resulting "prognostic equation," which governs the large-scale motions of a fictitious two-dimensional fluid whose velocity is a vertically integrated mean value of the horizontal component of velocity in the real three-dimensional atmosphere, forms the basis for a method of numerical prediction.

An iterative scheme, based on the solutions of a succession of linear equations, has been proposed for solving the nonlinear prognostic equation. In the course of developing this method, the complete solutions for forced oscillations induced by irregular terrain and for linear transient disturbances have been presented in readily computable form, in terms of known initial values and the appropriate Green's functions. Finally, the prediction formulas for large-scale transient disturbances have been applied to observed initial data, with generally favorable results.

FOREWORD

This is the first of two reports on recent researches in the problem of numerical weather prediction currently being carried out at the Atmospheric Analysis Laboratory of the Geophysical Research Division, Air Force Cambridge Research Center. This report deals primarily with the theoretical aspects of the problem and represents the author's own efforts to shed a little light on this difficult subject. The second report, to be published in the near future, after completion of the present phase of the program, will summarize the results of several rather laborious attempts to test the theory. The latter work, because of its magnitude and many ramifications, is necessarily a group effort and will be reported accordingly. It will include a descriptive study of the conservation and generation of mean horizontal circulation, as well as a full account of our attempts to apply the theory to the problem of predicting the mean horizontal flow.

The further one explores the difficulties of the special problem of weather prediction, the more evident is the necessity of discussing general questions of method, predictability and ultimate aims. A somewhat heterodox approach to the problem cannot, in fact, be justified without reviewing the relative merits and disadvantages of several possible methods. The first and, to some extent, the second sections of this report have, therefore, degenerated into a sort of essay on meteorological manners and morals. It is not expected that every reader will be interested in those portions. Those who do read them, however, should do so with the realization that they are tentative, exploratory and essentially speculative. Readers concerned only with practical applications might do well to skip to the fourth, fifth, seventh and eighth sections, turning back to intervening sections for definitions.

Throughout this report there appear frequent references to the recent papers of J. G. Charney of the Institute for Advanced Study whose work, perhaps more than any other, has clarified the fundamental problems of numerical weather prediction. His contributions to this field are so numerous that it would be difficult even to acknowledge them all, let alone elaborate on them. It is, therefore, appropriate to recognize a general debt of gratitude to Dr. Charney who, through many lively discussions, has influenced the author's viewpoint and attitude toward the problem. Special thanks are due to Mr. Louis Berkofsky and Miss Agnes Galligan for carrying out the laborious and unrewarding task of tabulating the Green's function for the two-dimensional form of the linearized vorticity equation.

CONTENTS

ILLUSTRATIONS

LIST OF SYMBOLS*

Symbol	*Meaning or Definition*
a	arbitrary amplitude factor
a_n	Fourier sine coefficient
A_ϕ	characteristic amplitude of ϕ-disturbance
A_v	$\mathbf{V}' = A_v\overline{\mathbf{V}}$
A_ζ	$\zeta' = A_\zeta\bar{\zeta}$
b_n	Fourier cosine coefficient
c	phase speed, characteristic phase speed
c_g	speed of surface gravity waves or of internal gravity waves
c_m	speed of modified gravity waves
c_n	Newtonian speed of sound at sea level
c_s	Laplacian speed of sound
C	a circle in the (x, y) plane
E	$E = \int_0^{p_0} RT\,dp$
f	characteristic double frequency
g	gravitational acceleration
G	a Green's function
h	height of terrain above mean sea level, elevation of sea surface
H	depth of an ocean, height of a density discontinuity
$H(s, y)$	$H(s, y) = L\{h(x, y)\}$
$H(x, y)$	$H(x, y) = \nabla^2 z_0 - \mu^2 z_0$
i	integer index
I	a Green's function
$I_n(\ )$	Bessel function of order n, first kind, imaginary argument
j	integer index
J	Jacobian determinant or a kernel function
$J_n(\ )$	Bessel function of order n, first kind, real argument
k	$k^2 = m^2 + s^2$
$\mathbf{K}$	vertically-directed unit vector
K	a kernel function
$K_n(\ )$	Bessel function of order n, second kind, imaginary argument
l	coordinate along a path of integration
L	characteristic half-wavelength, interval of Fourier expansion
m	an integer or $m^2 = \beta\tau^{-1}U^{-1}$
$M(x, y)$	$M(x, y) = -\mu^2 U\,\partial h/\partial x$
n	an integer, or coordinate normal to a path of integration
$N_i(x, y)$	a quantity at the ith stage of iteration

* Some symbols carry several different meanings. In general, such ambiguities have been minimized by defining each meaning of the symbols in context.

LIST OF SYMBOLS (*Continued*)

Symbol	*Meaning or Definition*
p	atmospheric pressure
P	a path of integration
q	an integer
r	$r^2 = (x - \xi)^2 + (y - \eta)^2$
R	gas constant, or radius of a circle
s	variable of the Laplace transform
S	an annular region of integration
t	elapsed time, or a dummy variable of integration
T	absolute temperature, characteristic half-period, or the Green's function for linear transient disturbances
u	eastward horizontal component of velocity, or $u = m(y - \eta)$
u'	$u = U + u'$
U	mean value of u integrated over a horizontal area
v	northward horizontal component of velocity
V	$V = \lvert \mathbf{V} \rvert$
$\mathbf{V}$	horizontal component of vector velocity
$\mathbf{V}'$	$\mathbf{V} = \overline{\mathbf{V}} + \mathbf{V}'$
w	vertical component of velocity
x	locally Cartesian coordinate directed toward east
x'	same as above relative to a moving origin
y	locally Cartesian coordinate directed toward north
$Y_n(\quad)$	Bessel function of order n, second kind, real argument
z	height above mean sea level, height of an isobaric surface, height of an isentropic surface, a dummy variable of integration, or $z^2 = t^2 + u^2$
z_F	height disturbance due to linear transient disturbances
z_M	height disturbance due to irregular terrain
z_N	height disturbance due to nonlinear effects
$\bar{z}$	area average of 500-millibar contour height
Z	$Z = \zeta + \lambda$ or $Z = e^{\beta^* x/2\alpha} L\{z\}$
Z'	$Z = \overline{Z} + Z'$
α	wave number in x-direction, or $\alpha = sm^{-1}$
β	$\beta = \partial\lambda/\partial y$
β^*	$\beta^* = \beta + \tau l/\mu^2$
γ	$\gamma = C_p C_V^{-1}$
Γ	a circle in the (x, y) plane
Δ	finite difference
ϵ	a small constant
ζ	vertical component of relative vorticity
η	a dummy variable of integration corresponding to y
θ	potential temperature, $p^{*}\rho^{-1}$, or an angular coordinate

LIST OF SYMBOLS (*Continued*)

Symbol	*Meaning or Definition*
κ	$\kappa = \gamma^{-1}$ or $\kappa = \sin \frac{1}{2}\theta$
λ	Coriolis parameter
μ	$\mu^2 = \lambda^2 c_n^{-2} V_0 V^{-1}$
ν	$\nu^2 = \mu^2 + (\beta^*/2s)^2$
ξ	arbitrary phase angle, or a dummy variable of integration corresponding to x
ρ	density of air, or radius of a circle
σ	$\sigma^2 = \beta^* rt$
τ	$\tau = 1 + \overline{A_\nu A_\xi}$, or a dummy variable of integration
χ	angle between **V** and some fixed horizontal line
ϕ	representative dependent variable, dummy variable of integration
Φ	the Green's function for forced oscillations
ψ	stream function for mean flow
	Used systematically, subscripts will generally denote conditions at some particular level or along some surface, or will indicate the manner in which an operation is to be carried out, as follows:
d	conditions along an isentropic surface at height d
h	conditions at the ground surface
p	differentiation with p held fixed
z	differentiation with z held fixed
θ	differentiation with θ held fixed
0	conditions at mean sea level, or at $t = 0$
1,2	conditions below and above a density discontinuity
	Special Operators
∇	horizontal component of vector derivative
$L\{\ \}$	Laplace transformation
$L^{-1}\{\ \}$	Inverse Laplace transformation $LL^{-1}\{\phi\} = \phi$
$(\ )*(\ )$	convolution operator (Faltung integral)
$\overline{(\ )}$	$\overline{(\ )} = p_h^{-1}\int_0^{p_h} (\ )\, dp$

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1.00 INTRODUCTION AND GENERAL REMARKS

THE PREDICTION PROBLEM

1.01. This paper is an attempt to deal with certain limited aspects of the problem of numerical weather prediction—a problem which, in its most general formulation, is roughly equivalent to the "forecasting problem" proposed by V. Bjerknes (1919) in his celebrated work on physical hydrodynamics. As far as ultimate aims are concerned, this problem is not essentially different from the classical conception of the general prediction problem. The present approach to it, however, departs far enough from the classical theory to warrant a fairly complete discussion of the nature of the problem itself.

1.02. In order to motivate the choice of special problems to be studied here and to clarify its nature and extent, the scope of this work will be narrowed as the difficulties of the more general problem become apparent. First, by analyzing those difficulties and reviewing the limited means at our disposal to overcome them, we shall attempt to state a problem which is neither so special that it is trivial nor so general that it cannot be solved. Second, although it is not one of the purposes of this paper to present a comprehensive critique of method, it is at least necessary to consider the relative merits and disadvantages of several possible lines of attack on the problem. Finally, it is not only essential to state the problem as a real and sensible question, but to specify what shall be taken as a satisfactory solution.

1.03. The general statement of the prediction problem, taken as it stands, is so inclusive that its complete solution must describe all the aspects of behavior which any fluid can possibly exhibit. To mention only a few, it would include convection, aerodynamic and other boundary effects, the propagation of sound and gravity waves, as well as those phenomena usually considered to be more typically meteorological. The overwhelming difficulties of the general problem are immediately clear if it is only realized that it embraces several classes of problems which are still unsolved, although they are very special and perfectly well defined. In its general form, therefore, the prediction problem cannot be completely solved.

1.04. On the other hand, it is not obviously necessary to solve the problem in its most general form. Since the aim of the meteorologist is confined to predicting those aspects of the atmosphere which are peculiarly meteorological in character, some of the difficulties of the general problem are only apparent. It is probably safe to say that the existence of sound waves, for example, has little or nothing to do with the course of meteorological events, and that other difficult aspects of the general behavior of the atmosphere are likewise not essential to the weather producing mechanism. The first concern, therefore, is to rephrase the prediction problem, deliberately introducing those specializations that make it explicitly meteorological. We shall, in fact, adopt the point of view that the fundamental problem of weather prediction has not been stated as a meaningful question unless the terms of the problem distinguish it from problems of acoustics, the aerodynamics of supersonic flow and other irrelevant questions implicit in the general problem.

1.05. The most obvious and straightforward way in which the prediction problem might be specialized is to examine the inner structure of the problem in complete detail, in the hope of finding some inherently natural basis for breaking it down into less inclusive (and correspondingly less formidable) component problems. Such a program as this, however, would require an exhaustive catalogue of all possible modes of behavior, some of which might be excluded from the very outset. Moreover, the selection of any special problem must be based, at least partially, on economic considerations which are external to the problems themselves. It appears logical and natural, therefore, to begin with a discussion of external constraints on the prediction problem, emphasizing those which have acted to specialize the problem in the past.

ECONOMIC BACKGROUND OF THE PROBLEM

1.06. The general problem of weather prediction is far from new. In one form or another and for various reasons—possibly the grandness of the scale of events, the obvious economic value of successful predictions, the layman's natural and sometimes rather alarming preoccupation with the weather, or perhaps the sheer appeal of a difficult problem—it has held the attention of meteorologists, mathematicians, physicists, professional forecasters and amateur weather prophets alike for some centuries. The desirability of introducing meteorological factors into agricultural, commercial and industrial planning is evident and has long been recognized. As an indication of the growing public demand for meteorological information, it is sufficient to mention that almost every national government maintains some sort of weather forecasting service as an integral part of its executive body.

1.07. Despite the fairly obvious advantages of efficiency to be gained by simply knowing what to plan for, the operational phases of weather prediction have not received material support in proportion to the widespread interest in accurate estimates of meteorological factors. There were probably very sound economic and psychological reasons for this lack of support in the past. First, aside from the admitted unreliability of weather predictions and the enormous expense of maintaining an adequate network of observing posts, the economic effects of meteorological factors were not very well understood and accordingly could not be weighted quantitatively. Second, in those few areas of economic activity where it was possible to assign a calculable weight to the meteorological factor, it was only sufficient to affect over-all efficiency and was not in itself decisive in determining total success or total failure.

1.08. During recent years the economic value of accurate weather prediction and its importance to human safety have been heightened by the rapidly increasing scale of commercial and military aircraft operations and by the recognition and introduction of meteorological factors in military planning. These, of course, have been selected deliberately as examples of human activity whose success or failure—not merely their efficiency—is affected decisively by the weather. There are urgent social, economic and geopolitical reasons for wishing to know the future state of the atmosphere and a corresponding increase in support, both material and moral, has been given to improving our knowledge of it. The civil and military weather services of the government have together built up and maintained a dense network of observing stations, which produces, as a by-product of its routine activities, an invaluable mass of measurements for study and research. At the same time, the military services have sponsored an extensive program of research in the fundamental problem of weather prediction. It is quite fair to say that more than half of all meteorological research in this country bears directly on the prediction problem and has as its ultimate objective the successful prediction of weather.

1.09. It is not very surprising that the course of meteorology as a science and weather forecasting as a profession have been influenced strongly by so extensive a background of economics. As in all other areas of economic endeavor, the production of the forecaster—the content of the information he provides and the form in which he presents it—is determined directly by consumer demand. Less directly, perhaps, but to an equally great extent, commercial interests and military requirements also influence the forecaster's choice of variables to be measured, his technique of analysis, and basic methods of prediction. The over-all effect of these constraints has been to confine the outlook of the practicing meteorologist to those few fields of problems which are important from the standpoint of operations, and to methods which are optimum from the standpoints of reliability, initial expense of development and continuing expense of application.

1.10. To cite an example, the viewpoint of the forecaster underwent a pronounced change immediately following the development of "all weather flying" equipment. As the need for meteorological information gradually shifted from factors that affect airport control to those that affect air navigation, the forecaster came to concentrate more and more on the configuration of flow in the upper troposphere and less on the tangible, moist aspects of weather at lower levels. Confronted with the problem of predicting winds at high operating altitudes, he has been forced to make maximum use of data from radio-balloon ascents by devising new techniques of analysis and representation. Similarly, he has found it convenient to introduce entirely new concepts to deal with the special problem of high altitude wind prediction.

1.11. Those same economic factors have also exerted a powerful influence on the development of basic systems of meteorological measurement. It is doubtful that measurements on the vast scale of the atmosphere would ever have been undertaken out of pure scientific curiosity, without some strong external motive for doing so. In the instance mentioned above, the increased demand for accurate wind predictions alone lent considerable impetus to the expansion of the network of meteorological observations and to its vertical extension by a system of radio-balloon soundings. It is evident that the economic value of meteorological information will inevitably control the density and geographical extent of an observing network whose expense, because of its very size, is a major consideration. To a somewhat lesser extent, that constraint has also acted to focus attention on certain aspects of the corresponding scientific problem and to fix the problem of weather prediction within definite limits of feasibility.

1.12. In discussing the manner in which external constraints serve to specialize the prediction problem, it might be profitable to examine the viewpoint of the practicing weather forecaster, who is continually subjected to those constraints and who presumably maintains his position by exercising his knowledge of the problem. Since the skill of the forecaster is essentially positive, it is reasonably safe to accept his estimate of what is important to the problem, if not his methods and results, as somewhere near correct. The forecaster might even be regarded as the arbiter of meteorological opinion in matters where common experience and opinion are most appropriate.

CONSTRAINT OF OBSERVABILITY

1.13. The forecaster's viewpoint is strongly colored by his realization that the complete state of the atmosphere is neither observed nor observable, for there is only a finite amount of time and effort to be expended in observing it, even if it were otherwise feasible to do so. The fact is that purely economic constraints set a low upper limit on the density and geographical extent of the observation network. This alone has a marked effect on the forecaster's choice of variables to be predicted. Judging from his well-known and often deplored tendency to state his predictions in very general terms, the forecaster is trying to predict variables which are "representative" of an interval of time or a region of space, rather than the values that will actually occur at each instant and at every point. The forecaster is simply recognizing that it is futile and illusory to try to predict the state of the atmosphere in greater detail than the resolving power with which it can be observed. Moreover, although the forecaster himself rarely goes about it in such an objective fashion, his attitude may be interpreted as an indication that one should predict some sort of mean values of the state variables, in the sense that they are representative of conditions over a finite interval of time or a finite region of space.

1.14. Looking at the problem from yet another point of view—from the standpoint of mathematical physics—the original statement of the prediction problem is little more than a testament of faith that a solution exists. Apart from the fact that it has no special context and contains no hint as to what aspects of the atmosphere are relevant, the general problem is framed in terms which have no counterpart in observable reality. The question becomes meaningful (and the differences of several possible viewpoints are partially reconciled) if the problem is restricted to that of predicting the mean local state of the atmosphere, integrated over intersecting volumes whose linear dimensions are several times greater than the distance between adjacent observation points. No matter which view of the general problem one takes, and however hidden this assumption may be, *it must be assumed that such mean values display some sort of statistical stability.* That is to say, the mean values of any two finite random selections of variables from the infinite aggregate must not differ by more than some small fraction of the total variability. Under these conditions, the true mean values can be approximated by finite sums. If this condition is not met, our present observations are inadequate to describe the state of the atmosphere at all, and the prediction problem is hopeless from any standpoint.

1.15. At least twice before in the history of the physical sciences, we have been confronted with similar difficulties, i.e., our inability to observe the complete state of a system and to predict its state to the last detail. The cases in point are the kinetic theory of gases and the Reynolds theory of turbulence. In each of these instances, the physicist has resorted to the purely mathematical device of deriving principles that apply to certain *statistics* of a state, from the physical laws that presumably describe it in complete detail. In the former case, owing to the impossibility of observing the position and velocity of every molecule of a fluid, Maxwell extracted from the Newtonian equations of motion for each *individual* molecule a set of partial differential equations which describe the behavior of certain statistical properties of an *aggregate* of molecules, for example, pressure, temperature, density and mean velocity. If the statistics of the aggregate display sufficient stability, then it is permissible to think of the hydrodynamical equations as governing the state of a fictitious continuous medium. For similar reasons, Reynolds found it convenient to integrate the Navier-Stokes equations in such a way that they refer to integrated or mean values of the original dependent variables. Two points should be made clear. First, these techniques are applicable if and only if the statistics of the state are stable in the sense outlined earlier. Second, it is important to realize that the general method of reframing a problem in terms of statistical functions of the state variables is simply an expedient to make up for our inability to observe the state in complete detail.

1.16. Since the meteorologist is now confronted with precisely the same sort of difficulty, it appears reasonable to adopt similar methods for expressing the fundamental laws of hydrodynamics in terms of variables which are averaged over a large space aggregate of nonobservables, and which are therefore representative of the observed statistics of the aggregate. In fact, one might hazard the guess that one of the next important advances of meteorological science will be brought about by introducing statistical concepts into the hydrodynamical theory of large-scale atmospheric motions. The desirability of such a procedure will be discussed further in Section 8.00.

REPRESENTATIVE VARIABLES

1.17. Returning to the viewpoint of the weather forecaster, it is also significant that he does not find it necessary to know all the variables which characterize the initial state of the atmosphere, in order to predict its mean state. If his main concern is to predict the general configuration of the pressure distribution, for example, he usually considers only the initial values of pressure, independent of all other state variables. Such considerations are usually sufficient to give a rough idea of the mean state of the atmosphere, because the mean wind and temperature are approximately related to the pressure distribution through semi-empirical rules, such as the so-called geostrophic and hydrostatic relations.

1.18. There are sound reasons which underlie—or at least justify—the forecaster's selection of the pressure distribution as the best single indicator of the mean state of the atmosphere. In the first place, of all the variables that are normally conceded to characterize the physical state of the atmosphere, pressure can be measured most accurately. It should be noted also that the composition of the atmosphere, aside from determining the gross temperature distribution and mean circulation, is important only if there is continual change of phase, with a resulting capture or release of energy. Although changes of phase are undoubtedly operative in modifying the state of the atmosphere, it is equally certain that they are not an essential part of the mechanism by which they themselves are originally generated. First of all, it is necessary to inquire how the initial disturbances, which must precede changes of phase, are developed and maintained. It is probably safe to say that the kinematic and thermodynamic history of the true atmosphere, with the previously stated qualifications, will not differ radically from that of a fictitious atmosphere which is initially identical in all other respects, but absolutely devoid of moisture. And granting that the atmosphere does contain moisture, it is reasonable to assume that the circulation of the atmosphere is much more effective in producing changes of phase than vice versa.

1.19. It remains to decide which of the kinematic and thermodynamic variables is most representative of the meteorological state of the atmosphere. Temperature and density can be eliminated from consideration, because it is inherent in the present system of measurement that they are related directly to the pressure distribution through the equation of state and the condition for hydrostatic equilibrium. The question is thus reduced to choosing between pressure and the kinematic variables. The fact that further limits the choice is this: The variations of pressure associated with disturbances of various scales generally decrease in magnitude with decreasing scale, whereas the corresponding variations of wind speed are of the same general order of magnitude, independent of scale. This implies that pressure measurements are the least sensitive to disturbances whose scale is less than the mesh size of the observation network, and most representative of conditions over a region whose linear dimension is greater than the mesh size. It is not very surprising, therefore, that the forecaster habitually thinks of the atmosphere in terms of the pressure distribution. Of all the variables that describe the physical state of the atmosphere, pressure is the most representative of conditions which extend over scales equal to or greater than the distance between adjacent observation stations.

1.20. Up to this point, the several possible directions of specialization have been considered without regard to the methods by which the prediction problem might be solved. The next concern is to discuss the advantages and applicability of several methods that have been tried in the past. Before going on to a discussion of method, however, it is appropriate to summarize the previous discussion by restating the problem in less general terms. The remainder of this paper will deal almost exclusively with the problem of predicting atmospheric pressure. Moreover, because limitations on observability also limit the detail in which one can predict the state of the atmosphere, further discussion will be restricted to pressure disturbances whose characteristic scale is several times greater than the distance between adjacent observing stations.

THE BASIC METHODS OF PREDICTION

1.21. Considered for the number of methods that have been applied to it, the problem of weather prediction is one of the most remarkable of the fundamental problems of meteorology. The spectrum of methods ranges from the most powerful techniques of mathematical physics to the crudest and most subjective kind of empiricism. That they display such great variety is not very surprising, since it is natural that methods of prediction should evolve with the science. But it is certainly curious that almost all those methods are still in use. It is relevant, therefore, to review some of the methods in current use, synthesizing from them a few basic methods which are common to all.

1.22. The methods that have previously been applied to the prediction problem are, in essence, variations and combinations of two basically different techniques. These are the methods of mathematical physics and the statistical method. It should be realized that there is no real and clear-cut distinction between these methods, considered as equally legitimate variants of the scientific method, and that, in that sense, they both tend to the same ultimate end. For the limited purpose of this discussion, however, it is still possible to draw valid distinctions between two essentially different routes of approach, whether or not they eventually lead to a common end.

1.23. The method of mathematical physics, as discussed here, consists primarily in the suitable mathematical formulation of certain fundamental physical principles, which govern the behavior of any fluid—the laws of conservation of momentum, energy, mass and composition, along with an equation of state. It goes without saying, unless the question is entirely trivial, that those laws are actually known and that the simultaneous system of differential equations embodying those principles is capable of solution. Subject to appropriate boundary and initial conditions, the solution of this mathematical problem may be regarded as a prediction of the future state of the atmosphere.

1.24. The statistical method, on the other hand, seeks to establish a direct correspondence between the state of the system at some arbitrarily chosen initial moment and its state at any time in the future, simply by analyzing the past history of the atmosphere to find out what has happened before in similar circumstances. To put it a little more precisely, this method offers a means of estimating the probability that any of a number of mutually exclusive events will occur in the future. The postulate which makes the method operative is that those probabilities may be identified with the observed frequencies of those same events in the past, following combinations of variates identical (or similar) to that which characterizes the given initial state.

1.25. To illustrate the way in which these two basic methods, under various guises and with varying degrees of objectivity, have been applied to the prediction problem, it is simplest to examine an accepted and fairly typical pattern of meteorological research. This, for lack of a better name, will be called the synoptic method. As its name might indicate, one of its principal aims is to present a concise description or synopsis of the state of the atmosphere at a given instant—so concise that certain selected aspects can be apprehended immediately and as a whole. In this respect, the synoptic method is essentially descriptive, and necessary from the standpoint of discovering which aspects of the atmosphere are relevant to the problem of weather prediction. In the same sense, it is not a prediction technique at all, but a method of representing the state of the atmosphere, usually graphically, according to certain preconceptions of what is especially important.

By its enforced association with the prediction problem, nevertheless, the "synoptic" method has come to refer to methods of scientific research and weather prediction as well.

1.26. In the view of the synoptic meteorologist, it is another essential feature of his method that he has continually sought to classify his experience by selecting a limited number of "lump" variables or indices to characterize the meteorological state and behavior of the atmosphere. As Charney (1949) points out, this is done evidently in the hope of reducing the number of degrees of freedom, while providing an inherently natural and adequate system of classification. The meteorologist has, for example, invented such gestalt concepts as "high," "low," "front" and "jet stream" to give a rough description of the mean state of the atmosphere and has introduced the notions of "cyclogenesis," "frontolysis," "blocking action" and the like to describe its behavior. Some of these fictions—for example, the "low"—have become so deeply ingrained in the thinking of meteorologists that they are frequently spoken of as real physical entities, capable of continued independent existence, but subject to their own peculiar laws of interaction.

1.27. The remaining aspect of the synoptic method consists in seeking to discover the laws governing the behavior of these meteorological constructs, or to discover and establish prognostic relationships between the "lump" variables, which characterize the meteorological state of the atmosphere prior to one given instant, and its state at some time in the future. Although such relationships are often suggested by the qualitative application of well-known physical principles, they most frequently emerge from the accumulated experience of the practicing forecaster as empirical rules-of-thumb.

1.28. From this point of view, the synoptic method contains nothing, aside from special techniques for representing the state of the atmosphere, that is not already contained in essence in the methods of statistics and mathematical physics. If there is any real difference, it lies in the subjectivity with which either or both of the two basic methods are applied. In fact, through common usage, "synoptic" has become more or less synonymous with "empirical." That is not to say, however, that the synoptic method is not perfectly scientific, and useful in isolating significant relationships from a mass of extraneous detail. The real point is that it consists mainly in descriptive analysis and classification of the recorded history of the atmosphere in the past, and partly in the qualitative application of quantitative physical principles. We shall confine our attention, therefore, to the two methods outlined earlier.

1.29. Resuming discussion of the two basic methods of attack, it is relevant to note that the statistical and mathematical-physical statements of the prediction problem are, at least in a certain limited sense, quite similar. It is implicit in the statements of both that the problem of weather prediction is essentially an initial value problem. In other words, whether the future state of the atmosphere is completely determined by its state at any one instant, or whether the distribution of probabilities of several alternative events is fixed by a single combination of variates, the burden of significance is placed on the moment of latest information. Moreover, although the forecaster habitually takes recourse to data at a succession of moments to extrapolate past behavior, he still has it in mind that the data at one time are actually sufficient.

1.30. The choice of methods is not to be founded on similarities, however, but on basic differences. In this case the real distinction between them is that the statistical method is a probabilistic approach to the problem, whereas the mathematical-physical method is essentially deterministic. To discuss the relative merits and disadvantages of the two methods, therefore, one is forced to consider the nature and extent of our positive knowledge of the atmosphere. This question is made difficult by the coexistence of probabilistic and deterministic elements in comparable degree.

1.31. Although our observations of the state of the atmosphere are far from complete, it is safe to say that we do possess some positive knowledge of the physical principles which govern the behavior of fluids in general and the atmosphere in particular. It would certainly be unreasonable to suppose that the meteorological behavior of the atmosphere is any more mysterious and unaccountable, because of its large scale, than the acoustic and aerodynamic properties of the very same medium. The latter have been deliberately chosen as examples of atmospheric behavior to which mathematical-physical methods have already been applied with great success. The atmosphere, in short, is a fluid which differs in no essential respect from any other fluid and is subject to the same general physical laws.

1.32 Accepting this point of view, it seems only reasonable to accord the forecasting problem the same consideration that one would give to more commonplace problems—for example, that of betting on a game of chance. Given the positive knowledge that the dice are loaded, one would have no hesitation in casting probabilistic considerations to the winds and betting on the favored sides, even though their appearance is not certain in every play. Similarly, if we have positive knowledge of the general principles that govern the behavior of the atmosphere, it is logical and consistent with normal judgment to exploit that knowledge by regarding the atmosphere as completely controlled by that strong element of determinism. In a manner of speaking, the behavior of the atmosphere is heavily "loaded" in favor of Newtonian physics.

1.33. It can still be argued that purely statistical methods might lead to results approaching positive information. However, if one has any faith at all in the general validity of the laws of mechanics, he is tempted to suspect that the most concise result of an exhaustive statistical study would simply show that the hydrodynamical laws are almost certainly valid.

1.34. It is also arguable that the hydrodynamical equations, however applicable or well known they are, may yield no directly verifiable information, because of the extreme mathematical difficulty of solving and deriving observable consequences from them. Until recently, this has been a valid (and frequent) objection to applying the methods of mathematical physics to the prediction problem. Because of the lack of sufficiently powerful methods of mathematical analysis, the theoretical meteorologist has been forced to make a number of concessions, primarily for the sake of convenience, and, more often than not, the special assumptions introduced to facilitate solution have completely obscured the question of the validity of the general equations. In any case, this objection refers to a fault of the mathematician and meteorologist, not to a fault of the equations.

1.35. During the past few years, high-speed automatic computing machines have been developed which are capable of performing a single multiplication, complete with the necessary transfer and storage of information, within a matter of micro- or milli-seconds. Thus, for the first time, it appears economically feasible to carry out the numerical integration of the complete hydrodynamical equations within a small fraction of the human lifetime. Granting that it would probably provide greater insight into the innermost nature of things to solve the equations by analytic methods and granting that one would really prefer, for aesthetic reasons, to solve them in that way, the mathematical methods at present at our disposal are not adequate to deal with the problem. Meanwhile it appears feasible to apply brute machine force to at least some aspects of the problem of weather prediction, by integrating the hydrodynamical equations numerically. It is probably safe to say that this fact alone has been a major factor in the recent rebirth of interest in the problem of numerical weather prediction, and possibly in theoretical meteorology in general. A few meteorologists and mathematicians have gone so far as to envision a completely automatic weather-forecasting machine, analogous to the Tide Machine, into which data will be fed directly and which inexorably and with great exactitude will calculate out the entire future course of the atmosphere.

1.36. In view of the foregoing considerations of method, it appears most reasonable to approach the prediction problem from the standpoint of mathematical physics, rather than from the standpoint of statistics. Before finally restating the problem, however, it is necessary to consider what shall be taken to constitute a satisfactory solution to the prediction problem.

1.37. It is almost characteristic of statistical hypotheses that they consist of a large number of apparently unrelated results. For this reason alone, statistical theories contribute little to our understanding of the external physical world, if only in the objective sense that it is difficult to apprehend a great many relationships simultaneously. A physical theory, on the other hand, usually consists of a relatively small number of statements and is framed in mathematical terms so concise that the formal aspects of the theory can be grasped simultaneously and as a whole. Partly for this reason, and possibly because statistical theories do not satisfy our instincts for an imposed order, it might be anticipated that no statistical theory will ever be accepted as the final solution of the prediction problem. This is not to say that statistical theories are not valid. They are simply not so satisfying.

1.38. Another question connected with the form of the final solution concerns ultimate accuracy or the irreducible minimum of error. This point has some bearing on the extent to which the solution is made determinate by the conditions of the prediction problem and was briefly touched on during the previous discussion of observability of the initial state. It has also been discussed at some length by Schumann (1950) in a recent pair of articles in *Weather*, in which he suggests that the difference between the apparent upper limit on forecasting accuracy and the Laplacian ideal of complete determinism is due to something like the Heisenberg uncertainty principle. Although it is certainly true that the ultimate accuracy of predictions hinges on the observability of the "true" state of a system, reference to the uncertainty principle is misleading and has simply obscured the issue. In the first place, it applies strictly only to a system whose state is significantly altered by the mere act of measuring it, so that it is not applicable to the macroscopic behavior of the atmosphere. Second, the difference between Heisenberg's uncertainty and Laplace's certainty is very small in any case. The reason for elaborating on this seemingly irrelevant detail is that, as the d nsity of initial data is indefinitely increased, the corresponding *ultimate* accuracy will probably approach a limit which, for all intents and purposes, amounts to determined certainty. From this standpoint, it is not inconsistent to apply an essentially deterministic method to the prediction problem.

THE PROBLEM RESTATED

1.39. Having discussed the difficulties and constraints on the general problem, and having touched briefly on general questions of method and predictability, we are now in a position to justify the choice of problems to be studied in the remainder of this paper. To state it briefly, the problem is to predict the pressure or mean horizontal circulation of the atmosphere, by integrating the equations of classical hydrodynamics (suitably modified if necessary) subject to given boundary and initial conditions. As specified earlier, we shall confine our attention to the large-scale, slowly moving disturbances which are apparently associated with the more tangible aspects of weather.

2.00 HISTORICAL BACKGROUND AND FUNDAMENTAL DIFFICULTIES OF THE PROBLEM

RICHARDSON'S EXPERIMENT

2.01. The problem of integrating the hydrodynamical equations to predict the meteorological state of the atmosphere is far from new. As early as 1917 Richardson attempted to predict local pressure changes by stepwise numerical integration. His method consisted in estimating all space derivatives as finite differences between the initial values at various loc[illegible]ons, and in computing the instantaneous local time derivatives from the primitive hydrodynamical equations. The equations of motion, for example, express the local variations of the velocity components in terms of space derivatives only, and the continuity equation gives the local variation of density. Next, regarding the instantaneous local time derivatives as finite differences, Richardson simply extrapolated the variables a short time into the future to generate a new set of initial values, whence the process could be repeated ad infinitum.

2.02. The results, as presented in Richardson's "Numerical Weather Prediction" (1922), indicated pressure changes one or two orders of magnitude greater than those actually observed. This discrepancy was discouraging enough to cause widespread pessimism about the possibility of predicting the state of the atmosphere by integrating the hydrodynamical equations numerically or by any other mathematical means. Richardson's experiment was quite successful, however, in demonstrating certain difficulties which are inherent in his method in particular and are, to some extent, present in any method. It would have been quite surprising, in fact, if the results of his experiment had turned out positive.

DIFFICULTIES INHERENT IN THE PHYSICAL SYSTEM

2.03. Aside from approximative errors in the equations, the possible sources of error may be lumped under three main headings. First, a large source of error lies in the incompleteness and inaccuracy of the initial data. Second, there may be some peculiarity of a physical system which makes the problem of predicting its behavior an inherently difficult one. If, for example, the system is very near the state of complete mechanical equilibrium at all times, then our estimate of its departure from equilibrium, based on incomplete or inaccurate observations of its state, may contain errors as large as the true departures. Since the whole problem of predicting the state of a system revolves around our ability to estimate its departures from equilibrium, such innate characteristics of the physical system may conceal a large source of error. Finally, even with the most accurate and complete observations of the state of the atmosphere and with the most well-behaved physical system, small errors in the initial data may be magnified by the particular mathematical method one chooses for solving the hydrodynamical equations.

2.04. The first of these sources of error, which is common to all methods, has already been discussed at some length in Section 1.00. Although such errors cannot be completely removed, they can be minimized by predicting the mean state of the atmosphere, integrated over an aggregate of points in the network of meteorological observations, or at least by confining attention to disturbances whose characteristic dimension is several times the distance between adjacent points in the network. The remaining sources of difficulty, on the other hand, stem from circumstances over which there is more control, and there is some point to discussing them in detail. Although both of the latter sources of error have been previously discussed by Charney (1949), some of the facts concerning their existence and true nature are sufficiently inobvious to bear repetition and further elaboration.

2.05. It is a frequent complaint of the meteorologist that it is next to impossible to compute representative values of the local time derivatives, as given by the primitive hydrodynamical equations, in terms of actually observed initial values. He observes, for example, that the nongeostrophic accelerations result from a small imbalance between two large forces, the Coriolis and pressure forces, and that the error in estimating either of those forces is therefore about as large as the true nongeostrophic momentum change. Similarly, he observes that the local changes in density are also given as small differences between individually large components of mass accumulation and, as a consequence, that the computed local change in pressure is likewise extremely sensitive to small errors in the initial data. In every case, the local time derivatives are given by the raw hydrodynamical equations as small differences between individually large terms, whence the errors in estimating any one of the large terms—and the resulting errors in the computed time derivatives—are generally of the same order of magnitude as the true values of the local time derivatives. All of these difficulties, however, are simply different manifestations of the same essential fact. Considered as the medium for propagating large-scale slowly moving disturbances, the atmosphere is always and everywhere close to the state of complete mechanical equilibrium.

2.06. To illustrate this, let us consider some of the consequences of postulating that the pressure, Coriolis and gravitational forces are almost in equilibrium, i.e., that the atmosphere is very nearly in geostrophic and hydrostatic balance. In other words, we are supposing that the acceleration terms in the equations of motion

$$\frac{d\mathbf{V}}{dt} + \mathbf{K} \times \lambda\mathbf{V} + \rho^{-1}\nabla p = 0$$

and

$$\frac{dw}{dt} + g + \rho^{-1}\frac{\partial p}{\partial z} = 0$$

are small in comparison with either of the remaining terms. This is stated in mathematical form in the following approximate equations:

$$\rho\mathbf{V} \simeq \mathbf{K} \times \lambda^{-1}\nabla p \qquad (1)$$

$$g\rho \simeq -\frac{\partial p}{\partial z}.$$

The first equation carries the direct implication that the total horizontal divergence of momentum is actually quite small, for it implies that $\lambda^{-1}p$ is almost a momentum stream function. However, the separate components of momentum divergence in the two horizontal directions, which are reflected in the terms

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = \nabla \cdot \rho\mathbf{V}$$

are, in general, quite large. Thus the difficulty of estimating the horizontal momentum divergence stems directly from the balance between the horizontal components of the pressure and Coriolis forces. The latter is also responsible for the difficulty of computing the local time derivatives of the velocity components from the equations of motion, because the accelerations resulting from the imbalance between those forces are small.

2.07. In much the same way, the almost complete balance between mechanical forces makes it difficult to estimate the vertical component of velocity. To demonstrate this, we make use of the continuity equation and the condition for hydrostatic equilibrium, to obtain the following expression for the material derivative of pressure

$$\frac{dp}{dt} = \mathbf{V} \cdot \nabla p - g\int_z^\infty \nabla \cdot \rho\mathbf{V}\, dz.$$

An alternative expression can be found by combining the continuity equation with the first law of thermodynamics for adiabatic processes,

$$\frac{dp}{dt} = -\gamma p\left(\nabla \cdot \mathbf{V} + \frac{\partial w}{\partial z}\right).$$

Finally, by equating the two independent expressions for the total derivative of pressure, we obtain a formula expressing the vertical component of velocity in terms of space derivatives only.

$$w = \mathbf{V}_h \cdot \nabla h - \int_h^z \nabla \cdot \mathbf{V}\, dz - \kappa \int_h^z p^{-1}\mathbf{V} \cdot \nabla p\, dz + \kappa g \int_h^z p^{-1} \int_z^\infty \nabla \cdot \rho \mathbf{V}\, dz\, dz.$$

Viewed in the light of previous remarks, concerning the computability of the horizontal momentum divergence, the first and third integrals are evidently small, but extremely sensitive to error. It is, of course, a direct consequence of approximate Eq. (1) that the second integral is small when the atmosphere is nearly in geostrophic balance. Finally, it follows that the vertical component of velocity over flat terrain, is likewise computed as the small difference between individually large terms.

2.08. To summarize the foregoing arguments, both the horizontal momentum divergence and the vertical component of velocity are necessarily small in an atmosphere which is almost in mechanical equilibrium. For the very reason that they are small, the errors incurred by computing them from the primitive hydrodynamical equations are of the same general order of magnitude as the quantities themselves. The above statements imply that the local time derivative of density and the resulting local variations of pressure are also small, and that the computed values of those quantities are sensitive to small errors.

2.09. Viewing Richardson's experiment in the light of these facts, it is almost inevitable that the local time derivatives computed from the primitive equations should have contained large percentage errors, for it is observed that the atmosphere (when considered in the large) is very close to a locally maintained state of mechanical equilibrium. At first glance it might appear that this difficulty would be present in any method of integrating the hydrodynamical equations, numerical or otherwise. As will be shown later, however, this is fortunately not the case. In fact, those features of the atmosphere's meteorological behavior which make the prediction problem difficult are exactly those which truly characterize it. Furthermore, the very smallness of deviations from the state of complete mechanical equilibrium can be turned to advantage in specializing the general problem.

2.10. It is pertinent to note here that the difficulty might be obviated by inventing new physical variables whose local time derivatives are independent of the magnitude of the external force. With respect to those variables, the atmosphere would behave as if it were unaware that it is actually near the state of mechanical equilibrium. The theorems of angular momentum conservation are particularly suggestive in this connection.

DIFFICULTIES INHERENT IN MATHEMATICAL METHOD

2.11. The third class of errors is of an entirely different nature, since it arises from the very method by which one chooses to solve the hydrodynamical equations. To demonstrate the reality of this purely mathematical phenomenon, let us return to Richardson's experiment. Because it dealt with primitive equations which were essentially unmodified, it is implicit in those equations that they possess solutions corresponding, say, to sound waves. It can therefore be stated at the outset that, in order to integrate the complete equations numerically, one must at least be able to integrate the system of equations governing the propagation of sound waves in that manner. It is well known, of course, that sound waves are governed by a system of first-order equations of the following type

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0 \tag{2}$$

$$\frac{\partial p}{dt} + \gamma p \frac{\partial u}{\partial x} = 0. \tag{3}$$

These equations, together with the initial values of u and p, completely determine the solution at all times in the future.

2.12. The application of Richardson's method to the solution of this system is quite straightforward. Let us consider the values of u and p at points $(i\Delta x, j\Delta t)$ in the (x, t) plane, spaced at regular intervals of Δt in the t-direction and Δx apart in the x-direction (see Fig. 1). Since we are given the values $u(i\Delta x, 0)$ and $p(i\Delta x, 0)$ at some arbitrarily chosen time-origin, it is therefore possible to compute the values $u(i\Delta x, j\Delta t)$ and $p(i\Delta x, j\Delta t)$ at all later times from the following finite-difference equations

$$2\rho\Delta x\{u[i\Delta x, j\Delta t] - u[i\Delta x, (j-1)\Delta t]\} + \Delta t\{p[(i+1)\Delta x, (j-1)\Delta t] - p[(i-1)\Delta x, (j-1)\Delta t]\} = 0$$
$$2\Delta x\{p[i\Delta x, j\Delta t] - p[i\Delta x, (j-1)\Delta t]\} + \gamma p\Delta t\{u[(i+1)\Delta x, (j-1)\Delta t] - u[(i-1)\Delta x, (j-1)\Delta t]\} = 0.$$

These equations, which were obtained simply by replacing the differential quotients in Eqs. (2) and (3) with the corresponding ratios of finite differences, are essentially recursion formulas. Setting j equal to one, for example, these equations enable us to calculate $u(i\Delta x, \Delta t)$ and $p(i\Delta x, \Delta t)$ directly from the given initial values $u(i\Delta x, 0)$ and $p(i\Delta x, 0)$, whence the process can be repeated indefinitely.

2.13. In discussing the errors of this method, however, it is actually simpler to deal with an equivalent system, in which only one of the variables appears explicitly. This is arrived at by cross-differentiating Eqs. (2) and (3) to eliminate u, whence

$$\frac{\partial^2 p}{\partial t^2} = \gamma p\rho^{-1}\frac{\partial^2 p}{\partial x^2}. \tag{4}$$

Since this equation is now of the second order with respect to time, both p and its local time derivative must be known initially to determine the solution. According to the original conditions of the problem, the initial values of p itself are known. The local time derivative of p is evidently given in terms of the initial values of u by Eq. (3). Equation (4) is the familiar one-dimensional wave equation, a hyperbolic equation whose properties and solutions have already been studied exhaustively. It is well known, for instance, that its solutions correspond to sound waves traveling at speeds $\pm(\gamma p\rho^{-1})^{1/2}$ in the x-direction.

2.14. Finite-difference methods for solving this and similar hyperbolic equations have been discussed by Courant, Friedrichs and Lewy (1928). In much the same way as outlined earlier, they consider the values of p at a network of discrete points, spaced Δx apart along the length-axis and Δt apart along the time-axis, and develop a recursion formula corresponding to Eq. (4) in order to compute the values of p at all points from its initial values. To summarize their results, they find that making the intervals Δx and Δt infinitesimally small is not sufficient to insure that the approximate solution will converge on the exact solution. Also, the finite interval of time Δt must always be chosen equal to or less than the finite increment of length Δx divided by the natural wave speed. Thus,

$$\Delta t \leqslant (\gamma p\rho^{-1})^{-1/2}\,\Delta x.$$

If this condition is not satisfied, the computed wave solutions will grow to an indefinitely large amplitude. The exact solution, on the other hand, indicates that the waves will actually be propagated without any essential change in form. Moreover, the equations are incapable of distinguishing between real and spurious variations in the initial values, so that small errors can presumably be amplified to the point of completely obscuring the true solution. An elementary demonstration of such effects is given in Appendix 1.

2.15. The meteorological implications of this result are rather devastating. Since solutions corresponding to sound waves are implicit in the unmodified primitive equations, and because the equations themselves are incapable of distinguishing between an error in the initial data and a physically real disturbance, the most direct and obvious form of the finite-difference method will amplify the "sound" waves until they finally obscure the large-scale, slowly moving disturbances that are of primary interest. In

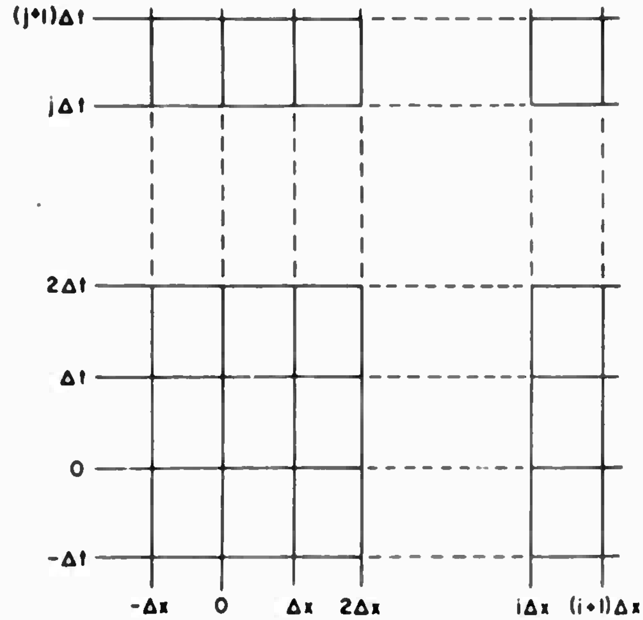

Fig. 1. Network of finite differences.

a manner of speaking, the noise level will rise so high that the weak meteorological signals will become unintelligible.

2.16. With regard to straightforward numerical integration of the unmodified hydrodynamical equations, we are apparently faced with a trilemma, represented by the following three alternatives. We must either resign ourselves to committing considerable error or, second, satisfy the Courant-Friedrichs-Lewy condition for computational stability or, third, modify the basic method to eliminate the source of instability.

2.17. The first alternative, of course, is intolerable. The second requires that the time intervals separating successive stages of the integration must be less than the time it takes a sound wave to travel the distance between two adjacent points in the space grid. It would be illusory to make the distance between adjacent grid-points less than the distance between adjacent observation stations, but it is equally undesirable to lose what little resolving power does exist. The distance between grid-points should be comparable, therefore, with the distance between observation stations, i.e., on the order of a hundred miles, rather than ten miles or a thousand.

2.18. This implies that the interval between successive stages of integration must be on the order of ten minutes or less, and that the number of stages required to produce one 24-hr prediction would be one hundred or more. The computations involved in one such prediction would be a staggering task, at least an order of magnitude greater than can be undertaken with the facilities and resources available at present. On economic grounds alone, the second alternative is not satisfactory. Moreover, one instinctively feels that the requirements for computational stability provide more time-resolution than is necessary to predict the course of the slowly moving meteorological disturbances.

2.19. With reference to the third alternative, it should be mentioned that there are methods, recently developed by von Neumann, for eliminating the source of error instability in the basic method of finite differences. In general, these methods remove the errors of simple extrapolation by "centering" all differences on one point and by attaching greater weight to some approximations than to others. As might be expected, however, the advantages of these methods are bought at a price, and give rise to other disadvantages and difficulties, for example, that of inverting a matrix of large order, applying a sort of Green's function to the initial data, or possibly that of overstabilizing the solution In summary, none of the three alternatives is completely satisfactory.

DISCUSSION OF DIFFICULTIES WITH REFERENCE TO SPECIAL SYSTEMS

2.20. To point up the essential difficulties of the problem and to suggest a way out of them, we shall consider a hydrodynamical system somewhat simpler to deal with than the atmosphere, but whose behavior is in certain crucial respects quite similar. Let us temporarily suppose, for the sake of argument, that we are interested in predicting the elevation of the ocean's free surface. By way of orientation, this is equivalent to predicting the pressure at some fixed level beneath the surface. For simplicity, we shall also suppose that the ocean is of uniform depth and that we are concerned only with small deviations from the state of rest. Finally, to simplify matters further, it will be assumed that the flow velocity depends only on the east-west coordinate and time. This system has been studied by Sverdrup (1927), Rossby (1938) and others, and has been used for purposes of analogy by Charney (1949). The differential equations governing its motions are well known. They are

$$\frac{\partial u}{\partial t} - \lambda v + g\frac{\partial h}{\partial x} = 0 \tag{5}$$

$$\frac{\partial v}{\partial t} + \lambda u + g\frac{\partial h}{\partial y} = 0 \tag{6}$$

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0. \tag{7}$$

Viewing the motions of the system in the large, all of the difficulties that have been discussed previously must be present in the problem of predicting its behavior by the most direct and obvious means, i.e., by numerically integrating the primitive equations that govern it.

2.21. To prepare the way for later development, the eliminations will be carried out in two stages, first by eliminating u to obtain two equations in h and v, and finally by eliminating v. By cross-differentiating Eqs. (5) and (6) and making use of Eq. (7), we obtain the vorticity equation

$$\frac{\partial^2 v}{\partial x \partial t} + \beta v = \lambda H^{-1}\frac{\partial h}{\partial t}\cdot \tag{8}$$

Likewise, eliminating u between Eqs. (5) and (7), we arrive at an independent equation in h and v.

$$gH\frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial t^2} = \lambda H\frac{\partial v}{\partial x}\cdot \tag{9}$$

In passing it might be noted that, if the earth were not rotating, λ would be zero and the motions would be governed by the simple wave equation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2}\cdot \tag{10}$$

The solutions of this equation correspond to the "shallow-water" gravity waves traveling at speeds $\pm(gH)^{1/2}$. On the other hand, if the motions are purely horizontal, the vorticity equation reduces to a telegrapher's equation, whose solutions correspond to the long Rossby waves.

$$\frac{\partial^2 v}{\partial x \partial t} + \beta v = 0.$$

The latter always travel toward the west (relative to the medium) at the speed $\beta\alpha^{-2}$. Proceeding with the eliminations, we now differentiate Eq. (8) once more with respect to x and substitute for $\frac{\partial v}{\partial x}$ from Eq. (9), to obtain a single equation in h.

$$c_g^2 \frac{\partial^4 h}{\partial x^3 \partial t} - \frac{\partial^4 h}{\partial x \partial t^3} + \beta c_g^2 \frac{\partial^2 h}{\partial x^2} - \beta \frac{\partial^2 h}{\partial t^2} - \lambda^2 \frac{\partial^2 h}{\partial x \partial t} = 0. \tag{11}$$

This equation is the basis for further discussion of the motions of the system.

2.22. We turn next to the problem of estimating the relative orders of magnitude of each of the terms in Eq. (11). Because the governing equation is linear, the motions corresponding to various types of solutions coexist without interaction, whence it is permissible to consider each type of motion separately. To give a rough description of each type, we now ascribe to it a characteristic "half wavelength," a measure of the distance between successive pronounced maxima and minima; a characteristic "half period," a measure of the time interval between successive maxima and minima passing a fixed point; and, finally, a characteristic "amplitude," which measures the difference in height between adjacent pronounced maxima and minima. Since the terms in Eq. (11) are only estimated to the correct order of magnitude, we may approximate all derivatives by ratios of characteristic numbers.

$$\left|\frac{\partial h}{\partial x}\right| \simeq A_h L^{-1} \qquad \left|\frac{\partial h}{\partial t}\right| \simeq A_h T^{-1}$$

and, in general,

$$\left|\frac{\partial^{m+q} h}{\partial x^m \partial t^q}\right| \simeq A_h L^{-m} T^{-q}.$$

Actually, it is simpler to compare estimates if they are expressed in terms of a characteristic "phase speed" and a characteristic "double frequency," defined by the "half wavelength" and "half period" as follows

$$c = LT^{-1} \qquad f = T^{-1}.$$

The relative magnitudes of the terms in Eq. (11) are displayed below, each estimate appearing beneath the corresponding term in the equation.

$c_g^2 \frac{\partial^4 h}{\partial x^3 \partial t}$	$\frac{\partial^4 h}{\partial x \partial t^3}$	$\beta c_g^2 \frac{\partial^2 h}{\partial x^2}$	$\beta \frac{\partial^2 h}{\partial t^2}$	$\lambda^2 \frac{\partial^2 h}{\partial x \partial t}$
$\left(\frac{c_g}{c}\right)^2$	1	$\left(\frac{\beta c_g}{\lambda^2}\right)\left(\frac{c_g}{c}\right)\left(\frac{\lambda}{f}\right)^2$	$\left(\frac{\beta c_g}{\lambda^2}\right)\left(\frac{c}{c_g}\right)\left(\frac{\lambda}{f}\right)^2$	$\left(\frac{\lambda}{f}\right)^2$

The state of motion is evidently characterized by the values of three nondimensional parameters, one of which depends only on the properties of the medium and two of which depend on the type of motion in question.

2.23. We may distinguish two types of motion, each characterized by an extreme value of one of the free parameters. For example, if the characteristic frequency of the motion is much greater than the frequency of the earth's rotation, *and if c is independent of f*, then the last three terms of Eq. (11) are much less than the first two. In that case Eq. (11) reduces to the wave equation (Eq. (10)). The phase speed of the "shallow-water" gravity waves is independent of their frequency, so that the previously stated condition on the phase speed is fulfilled a posteriori. On the other hand, if the characteristic phase speed is much less than the speed of "shallow-water" waves, *and whether c depends on f or not*, then the second term in Eq. (11) is much less than the first and the fourth is much less than the third. In this case the governing equation reduces to

$$\frac{\partial^3 h}{\partial x^2 \partial t} + \beta \frac{\partial h}{\partial x} - \lambda^2 c_g^{-2} \frac{\partial h}{\partial t} = 0. \qquad (12)$$

Fundamental solutions of this equation correspond to waves of the Rossby type, traveling toward the west at the speed $-\beta(\alpha^2 + \lambda^2 c_g^{-2})^{-1}$.

2.24. Let it now be supposed that the elevation of the ocean's free surface represents the combined effects of two distinct and superposable types of motion, one characterized by the fact that its frequency is much larger than the frequency of the earth's rotation, and the other by the fact that its phase speed is much less than that of "shallow-water" gravity waves. Moreover, to strengthen the analogy between ocean and atmosphere, we must imagine that the height amplitude of the slowly moving disturbance is considerably greater than that of the high-frequency disturbance, a situation which is rather unusual. There are evidently two ways to go about predicting the elevations of the sea surface in this case. The most direct method would be to solve Eqs. (5), (6) and (7) by stepwise numerical integration, subject to known initial conditions. An exactly equivalent scheme is to deal with the two types of motion separately, integrating the eq[illegible]s that govern each one without reference to the other and later superposing the solutions. This is the [illegible]t where the difficulty arises. Because the high-frequency disturbances are really gravity waves, they are governed by the hyperbolic wave equation (Eq. (10)) and, unless the Courant-Friedrichs-Lewy condition is satisfied, the simple method of finite differences will amplify the computed gra[illegible]ity wave solutions to the point that they will obscure the one which is really dominant.

2.25. There are two ways out of this difficulty. First, Eq. (10) might be solved by some more stable method, possibly by exact analytic methods. Second, it might not be necessary to deal with the gravity waves at all. Let us suppose that the initial values of the surface elevation have somehow been separated into two superposable components, one due to the slowly moving disturbances and the other due to the gravity waves. We now investigate the error incurred by applying the equation for the slowly moving disturbances to the total initial values of surface elevation, *whether it is due to the Rossby type of wave or to gravity waves*. Evidently the only source of error lies in the fact that the initial disturbances which are actually manifestations of gravity waves will be propagated at speeds differing from the correct one by an amount dependent on their wavelength and, in general, will be propagated too slowly. As specified earlier, however, the amplitude of the gravity waves is much less than that of the slowly moving disturbances, whence the percentage error in applying the equation for Rossby type waves to the complete initial conditions is not very great. This would be desirable for the very reason that the Rossby waves do travel so slowly. Even if it were necessary to satisfy some condition for computational stability, the required time resolution would be much less. At this point it is important to note that *the pressure amplitudes of sound and gravity waves in the atmosphere are one or two orders of magnitude less than that of the large-scale, slowly moving weather disturbances.*

2.26. Although it is certainly improper to extend the ocean-atmosphere analogy to all aspects of each system, it is interesting and perhaps legitimate to interpret certain of these results in the light of observed facts about the atmosphere. It is observed that the large-scale disturbances in the mean flow move very slowly and, even more significant, generally move in only one direction relative to the medium, i.e., toward the west.* The latter fact is very suggestive. If these disturbances are governed by a special form of some differential equation, similar to Eq. (11), it is quite clear that such an equation must be of the first order with respect to time. Referring to the previous estimates of the relative magnitudes of terms in Eq. (11), it is seen that the governing equation will contain terms that are no higher than the first order with respect to

* Speed of movement, as used here, refers to the phase speed or speed of individual extrema, and must not be identified with the rate of energy propagation, which, through dispersion effects, can be much greater than the phase speed.

time if and only if the phase speed is much less than the speed of gravity waves. In extenso, therefore, it is proposed that the essential fact—the property of large-scale atmospheric disturbances which distinguishes them from all other types of motion—is not really that their scale is large nor that their frequency is small, but that they move so slowly relative to the medium.

2.27. It has already been shown that introducing such information explicitly into the governing equation leads to an approximate equation from which solutions corresponding to certain types of motion are excluded. In a manner of speaking, the high-frequency noise has been filtered out by making systematic use of the approximations which characterize the large-scale disturbances. Owing to their peculiar nature, approximations of the type

$$c \ll c_g \qquad\qquad \lambda \ll f$$

will be called "filtering approximations," after Charney (1948).

2.28. Although the foregoing analysis provides a clear indication of the features which distinguish the large-scale oceanic disturbances, it is difficult to see how this method can be extended to cover atmospheric disturbances of finite amplitude and, in particular, how one can derive a single nonlinear equation which can be subjected to dimensional analysis. The difficulty lies in carrying out the eliminations under such genera conditions. A fact that is especially significant in this connection, and which was first pointed out by Charney (1947), is that Eq. (12) could have been obtained by introducing the so-called "geostrophic approximation"

$$\lambda v \simeq g \partial h / \partial x$$

directly into the vorticity equation (Eq. (8)). This suggests that the "geostrophic approximation" may be equivalent to the "filtering approximation," if it is applied only in the vorticity equation. Although it would be difficult to demonstrate under more general conditions, it can be shown that this equivalence is valid in the present case. Approximating the derivatives in Eq. (5) by ratios of characteristic numbers yields the following estimates

$$\left|\frac{\partial u}{\partial t}\right| \sim A_u T^{-1} \qquad\qquad \left|\frac{\partial h}{\partial x}\right| \sim A_h L^{-1}.$$

Moreover, Eq. (7) provides an independent relation between A_u and A_h.

$$A_h T^{-1} \sim H A_u L^{-1}.$$

The relative magnitudes of the terms in Eq. (5) are displayed below, each estimate appearing beneath the corresponding term of the equation.

$\frac{\partial u}{\partial t}$	λv	$g \frac{\partial h}{\partial x}$
$\left(\frac{c}{c_g}\right)^2$	$\frac{L\lambda v}{gA_h}$	1

It therefore appears that, if the characteristic phase speed is much less than the speed of gravity waves, the first term is much smaller than the third and, consequently, smaller than the second. This implies that the winds (i.e., ocean currents) associated with the slowly moving Rossby waves are typically geostrophic. Although this result cannot be regarded as holding under all circumstances, at least it contains a clue as to the pattern that a more general development should follow. As will be shown later, the introduction of the geostrophic approximation into the vorticity equation is sufficient to exclude the solutions corresponding to high-speed sound and gravity waves. The remaining question is whether or not it is more than sufficient

2.29. The results of this section will later be extended to apply to atmospheric disturbances, by developing a similar "scale" theory for the adiabatic flow of a compressible gas. As before, the method of development will consist first in eliminating all but one of the dependent variables (pressure) to obtain a single equation; second, in assigning characteristic numbers to describe each type of motion; third, in expressing estimates of the relative magnitudes of the terms in the governing equation in terms of a minimum number of nondimensional characteristic parameters; finally, in discovering what type of motion corresponds to an extreme value of each of the characteristic parameters.

THE MODELS OF ROSSBY AND CHARNEY

2.30. The assumptions adopted by Rossby (1939) to demonstrate the mathematical existence of long waves, have precisely the effect of "filtering out" the sound and gravity waves. Because he dealt with the motion of a homogeneous and, by implication, incompressible fluid, the medium was incapable of propagating sound waves. Secondly, Rossby assumed that the large-scale motions of the atmosphere are essentially horizontal, whence there could exist no gravity waves. It is likewise clear that the hydrodynamical equations, when applied to the purely horizontal flow of a homogeneous nonviscous fluid, will have no solutions corresponding to the excluded types of motion. By default, therefore, all that remain are the long Rossby waves.

2.31. Aside from the fact that they provide no very deep insight into the essential physical nature and distinguishing features of large-scale atmospheric disturbances, the above assumptions are stated rather baldly and without adequate justification. Very considerable advances toward justifying Rossby's end result (if not his assumptions) and otherwise toward circumventing the difficulties of the problem have been made in the past few years by Charney (1948, 1949), Charney and Eliassen (1949) and Charney, Fjortoft and von Neumann (1950).

2.32. In a manner similar to that outlined earlier, Charney (1948) has introduced the notions of characteristic lengths, periods and amplitudes of the velocity, pressure and density disturbances. By approximating the derivatives in the *unreduced primitive equations* (rather than in a single reduced equation) as ratios of characteristic numbers, Charney has shown that it is typical of large-scale disturbances that the vertical motions associated with them are small, that the winds are almost geostrophic, and that the atmosphere is very nearly in hydrostatic equilibrium. To indicate how this information is to be incorporated into the hydrodynamical equations and to provide a basis of concreteness for future discussions, the main points of a development due to Charney and Eliassen (1949) will now be presented.

2.33. To begin with, it is assumed that the horizontal accelerations resulting from vertical motion are negligible. In that event, the vector equation of horizontal motion assumes the simple form

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + \mathbf{K}\times(\zeta+\lambda)\mathbf{V} + \rho^{-1}\nabla p = 0. \qquad (13)$$

Most of the discussion will be centered around the vorticity equation, obtained by applying the operator $\nabla \times (\quad)$ to Eq. (13).

$$\frac{\partial \zeta}{\partial t} + \mathbf{V}\cdot\nabla(\zeta+\lambda) + (\zeta+\lambda)\nabla\cdot\mathbf{V} + \rho^{-2}\nabla p \times \nabla\rho = 0.$$

We now introduce the geostrophic approximation

$$\lambda\mathbf{V} \simeq \mathbf{K}\times\rho^{-1}\nabla p$$

into the solenoidal term, and neglect ζ (where it appears undifferentiated) in comparison with λ.

$$\rho\left[\frac{\partial\zeta}{\partial t} + \mathbf{V}\cdot\nabla(\zeta + \lambda)\right] + \lambda\rho\nabla\cdot\mathbf{V} + \lambda\mathbf{V}\cdot\nabla\rho = 0.$$

Finally, combining terms according to the rules for partial differentiation of vectors,

$$\rho\frac{d}{dt}(\zeta + \lambda) + \lambda\nabla\cdot\rho\mathbf{V} = 0. \tag{14}$$

2.34. We have now reached a crucial point in the argument. Equation (14) may be regarded as a means of computing the local time derivative of vorticity, provided all other terms can be computed accurately from observed initial values. It has already been shown that the horizontal momentum divergence associated with large-scale disturbances is necessarily and actually small. However, owing to the fact that it is to be computed as the small difference between individually large terms, it is subject to large percentage errors. The conclusion is inescapable. It is actually better to set the horizontal momentum divergence exactly equal to zero than it is to compute it directly from the initial data as defined.

2.35. It should be mentioned in passing that, because the horizontal momentum divergence of the large-scale flow is much smaller than the advective changes of angular momentum, the vertical component of absolute vorticity is essentially conserved, whence Rossby's final conclusion is substantially correct. Moreover, if the absolute vorticity is actually conserved, *the local time derivative of vorticity is not given as the small difference between large terms*. The reasons for this will appear later.

2.36. Returning to the main theme of this development, it is still possible to form an accurate estimate of the vorticity-generating effects of momentum divergence. To show this, Eq. (14) is integrated vertically with respect to height from the ground surface to an infinite height above the earth.

$$\int_0^{p_h}\frac{d}{dt}(\zeta + \lambda)\,dp + \lambda g\int_h^\infty \nabla\cdot\rho\mathbf{V}\,dz = 0. \tag{15}$$

An independent expression for the second integral can be obtained from the so-called "tendency equation"

$$\frac{\partial p_h}{\partial t} = g\rho_h\mathbf{V}_h\cdot\nabla h - g\int_h^\infty \nabla\cdot\rho\mathbf{V}\,dz. \tag{16}$$

Finally, eliminating the integrated momentum divergence between Eqs. (15) and (16)

$$\int_0^{p_h}\frac{d}{dt}(\zeta + \lambda)\,dp + \lambda g\rho_h\mathbf{V}_h\cdot\nabla h - \lambda\frac{\partial p_h}{\partial t} = 0. \tag{17}$$

The degree to which absolute vorticity is not conserved is given, therefore, in the mean, in terms of quantities that can be accurately computed. One is tempted to conclude from this, as does Charney, that the tendency equation is not to be regarded as a means of computing the pressure tendency (Bjerknes and Holmboe (1944)), but as a means of estimating the integrated effect of horizontal momentum divergence.

2.37. Charney and Eliassen next establish a correspondence between the motions of the real atmosphere and those of a fictitious "equivalent-barotropic" atmosphere, by assuming that the wind direction (though not the wind speed) is independent of height. To be exact, they assume that the winds at all levels have the same direction as the density-weighted mean wind $\overline{\mathbf{V}}$.

$$\mathbf{V} = A(p)\overline{\mathbf{V}}(x, y, t) \tag{18}$$

where the operator $\overline{(\quad)}$ is defined by

$$\overline{(\quad)} = p_h^{-1}\int_0^{p_h}(\quad)\,dp.$$

This restriction, together with several other minor approximations, makes it possible to invert the order of differentiation and integration in the first term of Eq. (17). We find, for example, that

$$\zeta \simeq A(p)\bar{\zeta} \qquad (\bar{\zeta} = \mathbf{K} \cdot \nabla \times \bar{\mathbf{V}}),$$

whence

$$\int_0^{p_h} \frac{\partial \zeta}{\partial t}\, dp \simeq \frac{\partial \bar{\zeta}}{\partial t} \int_0^{p_h} A\, dp,$$

$$\int_0^{p_h} \mathbf{V} \cdot \nabla \zeta\, dp \simeq \bar{\mathbf{V}} \cdot \nabla \bar{\zeta} \int_0^{p_h} A^2\, dp$$

and

$$\int_0^{p_h} \mathbf{V} \cdot \nabla \lambda\, dp \simeq \beta \bar{v} \int_0^{p_h} A\, dp.$$

Integrating both sides of Eq. (18) with respect to p, between the limits 0 and p_h, we also note that

$$\int_0^{p_h} A\, dp = p_h.$$

Finally, the local and advective changes of absolute vorticity are collected to obtain a new equation, similar in form to the unintegrated vorticity equation, but referring to integrated values of the original dependent variables.

$$\frac{\partial \bar{\zeta}}{\partial t} + \overline{A^2}\bar{\mathbf{V}} \cdot \nabla \bar{\zeta} + \beta \bar{v} + \lambda g R^{-1} T_h^{-1} \mathbf{V}_h \cdot \nabla h - \lambda p_h^{-1} \frac{\partial p_h}{\partial t} = 0. \tag{19}$$

Evidently, Eq. (19) governs the motions of a fictitious two-dimensional atmosphere, in which the flow velocity is the density-weighted vertical average of the winds observed in the real atmosphere.

2.38. Although it is not exactly permissible to do so, one is tempted to think of $\bar{\mathbf{V}}$ as the actual wind at some one height, and to conceive of Eq. (19) as applying to the horizontal motions within a surface of such points. This surface, which is located at altitudes where the observed winds equal the vertically integrated mean winds, is known as the "equivalent-barotropic level." It is a matter of experience that it is a nearly level surface and does not ascend or descend much from day to day. It is generally located somewhere around the 500- or 600-mb level, roughly coinciding with the so-called "level of nondivergence."

2.39. With this interpretation, Charney and Eliassen next apply Eq. (19) to horizontal motions at the equivalent-barotropic level. At this point they introduce the "filtering approximation," substituting the geostrophic wind for the true wind, wherever it enters undifferentiated or wherever it is used to compute vorticity.

$$\bar{\mathbf{V}} \simeq \mathbf{K} \times g\lambda^{-1}\nabla z \qquad \bar{\zeta} \simeq g\lambda^{-1}\nabla^2 z. \tag{20}$$

This is permissible, of course, because the horizontal momentum divergence has been eliminated between the vorticity and tendency equations, whence there is no further need to compute it. Moreover, the information that was lost in treating the divergence as an eliminant must be resupplied by introducing some sort of stream function. Subject to one further restriction, specifying the connection between local pressure changes at different levels, the meteorologically significant motions are found to be governed by a single equation involving only one dependent variable—the height of a surface of constant pressure at the equivalent-barotropic level. To eliminate surface pressure from Eq. (19) Charney and Eliassen originally assumed that the height tendency is the same at all levels, later remarking that it would be more reasonable to relate the tendencies at the surface and at the equivalent-barotropic level by a factor of proportionality equal to the ratio of the wind speeds at those levels. Taken together with the hydrostatic condition, this assumption requires that

$$\frac{\partial p_h}{\partial t} \simeq g\rho_h A(p_h) \frac{\partial z}{\partial t}. \tag{21}$$

Inserting the relations expressed in Eqs. (20) and (21) into Eq. (19), we finally obtain

$$\frac{\partial}{\partial t} \nabla^2 z + g\lambda^{-1}\overline{A^2} J(z, \nabla^2 z) + \beta \frac{\partial z}{\partial x} - \lambda^2 c_n^{-2} A(p_h) \frac{\partial z}{\partial t} + g\lambda c_n^{-2} A(p_h) J(z, h) = 0. \tag{22}$$

However aptly it may describe the behavior of large-scale pressure disturbances, Eq. (22) is nonlinear and must therefore be solved by numerical methods, as opposed to analytic methods.

2.40. In "A Numerical Method for Predicting Perturbations," Charney and Eliassen (1949) go on to consider solutions of a linear equation related to Eq. (22). For the sake of simplicity, they further restrict themselves to flow in which the vorticity is due mainly to the curvature of the streamlines, rather than to shear across the flow. In that case, small deviations from uniform west-east flow are governed by the following "one-dimensional" perturbation equation

$$\frac{\partial^3 z}{\partial x^2 \partial t} + \overline{A^2} U \frac{\partial^3 z}{\partial x^3} + \beta \frac{\partial z}{\partial x} - \lambda^2 c_n^{-2} A(p_h) \frac{\partial z}{\partial t} = 0. \tag{23}$$

Because we are primarily interested in the free oscillations of the atmosphere—i.e., the transient disturbances—the term arising from vertical motion at the ground has been omitted. The latter at worst creates a forced oscillation, to be superposed over the dominant free oscillations.

2.41. It should be noted that Eq. (23) is of the same general type as Eq. (12), whose solutions correspond to Rossby waves in an ocean. In fact, it can be verified directly that Eq. (23) possesses no solutions corresponding to sound and gravity waves. The frequency equation for wave solutions has only one root, corresponding to a dispersive system of waves traveling toward the east at speeds

$$c = \frac{\overline{A^2} U \alpha^2 - \beta}{\alpha^2 + \lambda^2 c_n^{-2} A(p_h)}.$$

This result is in good qualitative accord with the observed fact. It is probably safe to say that the nonlinear Eq. (22) also has no solutions corresponding to sound and gravity waves, for it is unlikely that additional *continuous** solutions would be admitted by the sole reason of its nonlinearity. Retracing our way through this development, it appears that the high-frequency disturbances have been "filtered out" by imposing two special conditions. First, sound wave solutions are evidently excluded by treating the atmosphere as if it were exactly in hydrostatic equilibrium. In a manner of speaking, the pressure changes at different levels are so rigidly coupled together that they can be brought about only by changes in the effective depth of the atmosphere. Second, the external gravity waves have been excluded in the process of substituting geostrophic winds into the vorticity equation, a device which was discussed earlier at considerable length.

2.42. To review our position briefly, the development of Charney and Eliassen leads to a single governing equation that is free of high-frequency "noise," i.e., those solutions which, aside from the analytical difficulties involved, are awkward from the standpoint of solving the equation numerically. It appears, therefore, that one of the fundamental difficulties, namely, that of satisfying an inconveniently strong condition for computational stability, has been evaded completely.

* Under certain conditions, the nonlinearity of equations does permit special solutions, such as shock waves. These, however, are essentially discontinuous solutions.

2.43. It should also be noted that the second major difficulty has been overcome during the course of the development. Turning back to Eq. (19), we see that it expresses the local time derivative of the mean vorticity in terms of quantities that can be computed accurately, in the sense that the local derivative is not invariably given as the small difference between individually large terms. In principle, therefore, one can predict the mean vorticity by extrapolating its instantaneous local change a short time into the future. The remaining difficulty is the purely mathematical problem of reconstructing the velocity distribution from a known distribution of vorticity, in order to regenerate the initial conditions.

2.44. In connection with the latter problem, it is worth noting that the velocity distribution is completely determined by the knowledge of both the vorticity and velocity divergence. As indicated earlier, the horizontal momentum divergence associated with the large-scale disturbances is characteristically small, owing to the fact that the winds are almost in geostrophic balance.

$$\frac{\partial}{\partial x}(\rho u) \simeq -\frac{\partial}{\partial y}(\rho v).$$

Integrating vertically from the ground surface (now assumed flat) to an infinite height, we find that

$$\frac{\partial}{\partial x}(p_h \bar{u}) \simeq -\frac{\partial}{\partial y}(p_h \bar{v})$$

and

$$p_h \frac{\partial \bar{u}}{\partial x} + \bar{\mathbf{V}} \cdot \nabla p_h \simeq -p_h \frac{\partial \bar{v}}{\partial y}.$$

According to the conditions of the problem, however, $\bar{\mathbf{V}}$ is very nearly perpendicular to ∇p_h, whence

$$\frac{\partial \bar{u}}{\partial x} \simeq -\frac{\partial \bar{v}}{\partial y}.$$

This relation implies the existence of a stream function ψ, such that

$$\bar{u} \simeq -\frac{\partial \psi}{\partial y} \qquad \text{and} \qquad \bar{v} \simeq \frac{\partial \psi}{\partial x}.$$

Since the vorticity of the mean flow can be predicted with fair accuracy from a conservation equation, we may regard it as known at some time in the future. The problem of regenerating initial conditions is then reduced to that of solving the system

$$\mathbf{K} \cdot \nabla \times \bar{\mathbf{V}} = F(x, y)$$

$$\nabla \cdot \bar{\mathbf{V}} = 0 \qquad \text{or} \qquad \bar{\mathbf{V}} = \mathbf{K} \times \nabla \psi$$

where $F(x, y)$ is the known distribution of vorticity. Combining these equations to obtain a single equation in one unknown, we arrive at a well-known equation of the Poisson type.

$$\nabla^2 \psi = F(x, y).$$

It is interesting to note here that $g\lambda^{-1}z$ plays the role of a stream function. This demonstrates the physical and mathematical equivalence of the condition of geostrophic balance and the almost complete compensation between the separate components of momentum divergence. It also provides additional justification for introducing the geostrophic wind into the vorticity equation.

2.45. From the foregoing treatment and from previous discussions of the filtering approximation, it appears that the development of a suitable prognostic equation—one which is free of major computational and analytical difficulties—should be centered around a vorticity equation in some form. Apart from historical reasons, there is an obvious, but heretofore undiscussed, purpose in regarding the vorticity or angular momentum as the fundamental variable. It is simply this: The pressure force, one of the two external forces acting to bring about relative accelerations, is a potential vector. Thus the equation which results from applying the curl operator to the force equation is *independent of the magnitude of the pressure force*, for the form of the vorticity equation remains unaltered by the addition of any other potential force whatever. The physical interpretation of this fact is that, so far as its vorticity-generating processes are concerned, the atmosphere behaves as if it were not actually near the state of complete mechanical equilibrium. One should expect, therefore, that the difficulties due to quasi-geostrophic and quasi-hydrostatic conditions would not be present in the vorticity equation.

2.46. Although the development of Charney and Eliassen appears sufficient to meet the fundamental difficulties of constructing meteorologically significant solutions of the hydrodynamical equations, and although it provides a pattern for further development, the treatment is still not general enough to insure that the special restrictions which are sufficient to make the problem truly meteorological are altogether necessary. In any case, there are several points at which the theory could bear generalization. In the first place, no matter how small the vertical motions (associated with the large-scale disturbances) might be, there is still some doubt that they might not be effective in producing substantial changes in vorticity by advection from one level to another. In other words, although the motions themselves are small, the vertical gradients of velocity and vorticity are frequently large. Apart from such major objections, there have been introduced, as needed, a number of unstated minor approximations which although they probably do not significantly affect the form and accuracy of the final result, are rather unpalatable and cast some doubt on the general validity of the theory. Some of these approximations have entered at several points in the same way, whence it is possible that they are compensating and actually unnecessary. In any event, it would be desirable to postpone the introduction of special approximations until as late in the development as possible.

2.47. There are certain features of the Charney-Eliassen development which it is desirable and perhaps necessary to retain. For example, there are obvious advantages to be gained by dealing with the motion of a fictitious "two-dimensional" atmosphere which, at least in a mathematical sense, is equivalent to the actual atmosphere. Aside from the convenience of doing so, there are also strong physical reasons for treating the problem in this way. To illustrate this, let us consider the behavior of an atmosphere in which no energy is received from outside sources. As suggested earlier by Charney (1948), the behavior of large-scale pressure disturbances in such an atmosphere is evidently governed by a pair of equations expressing the conservation of entropy and potential vorticity,

$$\frac{d}{dt}\left[(\zeta + \lambda)\rho^{-1}\frac{\partial\theta}{\partial z}\right] = 0 \tag{24}$$

$$\frac{d\theta}{dt} = 0, \tag{25}$$

together with the conditions for geostrophic and hydrostatic equilibrium. These may be regarded as two independent equations in p and w. If we attempt to deal with the three-dimensional motions in complete generality, we shall be faced with two equally unsatisfactory alternatives. Either the vertical component of velocity must be computed from the equation

$$w = -\left(\frac{\partial\theta}{\partial z}\right)^{-1}\left(\frac{\partial\theta}{\partial t} + \mathbf{V}\cdot\nabla\theta\right),$$

or it must be eliminated between Eqs. (24) and (25), in which it enters linearly. Now, the troposphere is actually so near neutral static stability that the errors in estimating the vertical derivative of potential temperature are of the same general order of magnitude as the derivative itself. Considering the first alternative, therefore, we conclude that the computations of w would be extremely sensitive to small errors in estimating the static stability. In the second case, the coefficient of the term of highest order would contain the static stability as a factor, whence the solution of the pressure equation would also be quite sensitive to small errors in estimating the vertical derivative of potential temperature from observed initial data. This suggests that our knowledge of the state of the atmosphere is not sufficiently accurate to allow us to deal with its three-dimensional motion in complete generality, and that the effects of vertical motion must be treated in a highly implicit manner, without direct reference to the vertical motions themselves. What is significant is that this can be accomplished by integrating out the vertical coordinate, in much the same way as Charney and Eliassen have done.

2.48. The remainder of this report deals with an attempt to generalize the theory of large-scale pressure disturbances in the atmosphere, following a scheme of development very similar in broad outline to that of Charney and Eliassen. In general, and insofar as it is feasible, special assumptions and approximations will be postponed until as late in the development as possible, so that one can see more clearly what they really entail. We shall then discuss the relative merits and disadvantages of several methods for solving the prognostic equation, finally presenting an improved method for solving the "two-dimensional" vorticity equation. The theory is supported by a comparison of actually observed pressure changes with the corresponding predicted changes, based on solutions of the two-dimensional linearized vorticity equation.

2.49. Before undertaking the development of a prognostic equation, the problem of classifying the various kinds of atmospheric motion will be considered, with a view to isolating those features of the large-scale slowly moving disturbances which distinguish them from all other types. As mentioned earlier, Charney (1948) has developed a "scale theory" to deal with exactly the latter problem, and has succeeded in demonstrating the mutual equivalence of the filtering approximation for large-scale motions with several manifestations of quasi-equilibrium conditions. However, owing to the fact that he has introduced characteristic numbers into the unreduced primitive equations, his method is incapable of simultaneously revealing the approximations which characterize every type of motion. We therefore proceed directly to the development of a somewhat different scale theory, designed along the same general lines as the one discussed earlier in this section.

3.00 A SCALE THEORY AND THE NATURE OF THE FILTERING APPROXIMATIONS

3.01. The first step in the development of the scale theory, as outlined in the preceding section, is to eliminate all but one of the dependent variables between the hydrodynamical equations, later introducing characteristic time and length scales to describe each type of motion. We therefore start with the Eulerian equations of motion and continuity

$$\frac{d\mathbf{V}}{dt} + \mathbf{K} \times \lambda \mathbf{V} + \rho^{-1} \nabla p = 0 \tag{26}$$

$$\frac{\partial p}{\partial z} + g\rho = 0 \tag{27}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} + \frac{\partial}{\partial z}(\rho w) = 0 \tag{28}$$

taken together with a suitable energy equation. We shall suppose that no energy is being added to the system, whence the thermodynamic processes are adiabatic.

$$\frac{d\theta}{dt} = 0 \tag{29}$$

where $\theta = p^{\kappa}\rho^{-1}$ and $\kappa = C_V C_P^{-1}$.
It should also be noted that the complete vertical equation of motion has been replaced by the hydrostatic equation (Eq. (27)) at the very outset.

3.02. The two major assumptions implicit in the above equations deserve some comment. To assume that no energy is being supplied from external sources is simply to accept the existence of an initial distribution of energy, without regard to the manner in which it was established, and to describe the processes by which that energy is adiabatically redistributed. By the same token, this restriction prevents us from penetrating to "first causes" or to the mechanism by which disturbances are originated. At first glance, therefore, it might appear that the assumption of "no added energy" would not permit the development of "new" disturbances. However, Kuo (1949) has shown that the latent instability of a locally undisturbed state is sufficient to bring about development of large-scale disturbances after they have been initiated. In view of this fact, and because the changes of eddy energy involved are so tremendous, it seems unlikely that all—or even a major part—of the energy of a developing disturbance is derived from external sources—i.e., from the initial impulses of energy required to set it off. It seems probable, rather, that most of the kinetic energy of the disturbance is derived from an *already established* distribution of energy. This, of course, really begs the question, for the energy from external sources is certainly instrumental in establishing an inherently unstable state. The remaining question is how fast external sources of energy bring about changes in the configuration of flow on a very large scale. It is actually observed that the structure of the mean or general circulation does not change markedly from week to week, whereas new disturbances quite frequently develop in the course of a day or two. It therefore seems reasonable to suppose that the rapid development of disturbances is due mainly to the adiabatic adjustment of a distribution of energy already existing. For this reason, and because disturbing influences are always present, it is probably sufficient to assume that the thermodynamic processes are adiabatic, if one is concerned with predicting the course of events over only a few days.

3.03. Having stipulated that no energy is being received from outside sources, it is only consistent to require that kinetic and potential energy not be degraded into molecular motion through the action of dissipative forces. Otherwise, of course, the atmosphere would slowly run down until all its energy were transformed into heat. Accordingly, the forces due to molecular viscosity have been omitted from the equations of motion. They are quite small, in fact, compared with the observed pressure and gravitational forces. This is not to say, however, that the Reynolds stresses due to disturbances on a scale smaller than the mesh size of the observing network are also negligible. In estimating the effect of small-scale eddy stresses, two points must be considered. The first, which has already been discussed, is that the energy of disturbances of various scales generally decreases with decreasing scale. Second, attention will be confined to a vertically integrated mean value of velocity. Because the energy of very small scale disturbances is apparently concentrated in a rather shallow boundary layer, such disturbances make only a negligible contribution to the eddy stresses of the mean wind. Eddy stresses are therefore omitted from the equations of motion, which are now assumed to apply "in the large."

A QUASI-LAGRANGIAN COORDINATE SYSTEM

3.04. With the foregoing rationalization, we return to the classical equations of hydrodynamics. To simplify the problem of carrying out the eliminations, we shall next develop differentiation formulas for a coordinate system which appears to be the most convenient and natural to the problem. Because the potential temperature (or entropy) is conserved, it is natural to regard it as a Lagrangian coordinate identifying a material surface. Moreover, the hydrostatic equation introduces a fundamental asymmetry among the space coordinates, in that dependence on the vertical coordinate is different from dependence on either of the horizontal coordinates. The vector equation of horizontal motion is, of course, independent of the horizontal coordinate system. This suggests that we might adopt θ as an independent variable to represent the vertical coordinate, regarding the height z of an isentropic surface as a dependent variable. This leads to a variant of the quasi-Lagrangian coordinate systems first proposed by Starr (1945), in which one of the coordinates is Lagrangian and the rest are Eulerian.

3.05. Applying the partial differentiation formulas for a change of independent variable, the derivatives of any dependent variable ϕ taken with respect to the old coordinates (x, y, z, t) become, with respect to the new coordinates (x, y, θ, t),

$$\left(\frac{\partial \phi}{\partial x}\right)_z = \left(\frac{\partial \phi}{\partial x}\right)_\theta + \left(\frac{\partial \phi}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial x}\right)_z$$

$$\left(\frac{\partial \phi}{\partial y}\right)_z = \left(\frac{\partial \phi}{\partial y}\right)_\theta + \left(\frac{\partial \phi}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial y}\right)_z$$

$$\left(\frac{\partial \phi}{\partial t}\right)_z = \left(\frac{\partial \phi}{\partial t}\right)_\theta + \left(\frac{\partial \phi}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial t}\right)_z$$

$$\left(\frac{\partial \phi}{\partial z}\right) = \left(\frac{\partial \phi}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial z}\right).$$

The subscripts indicate which variable has been held fixed in the process of differentiation. The total derivative then assumes the form

$$\frac{d\phi}{dt} = \left(\frac{\partial \phi}{\partial t}\right)_\theta + \mathbf{V} \cdot \nabla_\theta \phi + \frac{\partial \phi}{\partial \theta}\frac{d\theta}{dt}.$$

According to our assumption, however, the processes are adiabatic, whence the material derivative takes on the simple "two-dimensional" form

$$\frac{d\phi}{dt} = \left(\frac{\partial \phi}{\partial t}\right)_\theta + \mathbf{V} \cdot \nabla_\theta \phi. \qquad (30)$$

Similarly, we express the horizontal vector gradient in terms of the new coordinates.

$$\nabla_z \phi = \nabla_\theta \phi + \frac{\partial \phi}{\partial \theta} \nabla_z \theta.$$

In particular, introducing the condition for hydrostatic equilibrium, the horizontal vector gradient of pressure is given by

$$\nabla_z p = \nabla_\theta p + \left(\frac{\partial p}{\partial z}\right)\left(\frac{\partial z}{\partial \theta}\right) \nabla_z \theta = \nabla_\theta p - g\rho \frac{\partial z}{\partial \theta} \nabla_z \theta. \qquad (31)$$

Moreover, making use of the definition of the slope of an isentropic surface in the x and y directions, Eq. (31) reduces to

$$\nabla_z p = \nabla_\theta p + g\rho\nabla_\theta z.$$

It must be re-emphasized that the dependent variable z is now the height of an *isentropic* surface. Since ρ is a function of θ and p,

$$\begin{aligned}\rho^{-1}\nabla_z p &= \theta p^{-\kappa}\nabla_\theta p + g\nabla_\theta z \\ &= \theta(1-\kappa)^{-1}\nabla_\theta p^{1-\kappa} + g\nabla_\theta z \\ &= \nabla_\theta[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}]\,. \qquad (32)\end{aligned}$$

This expression shows that the acceleration due to the pressure force is a potential vector, i.e., that the integral of its tangential component, taken around a closed curve in an isentropic surface, vanishes identically.

3.06. To complete the preliminary development, we also express the horizontal divergence in terms of the new coordinates. Applying the differentiation formulas to the horizontal components of velocity,

$$\nabla_z \cdot \mathbf{V} = \nabla_\theta \cdot \mathbf{V} + \frac{\partial \mathbf{V}}{\partial \theta} \cdot \nabla_z \theta. \qquad (33)$$

The total divergence, however, also contains the term $\partial w/\partial z$. In terms of the new variables,

$$w = \left(\frac{\partial z}{\partial t}\right)_\theta + \mathbf{V} \cdot \nabla_\theta z$$

whence, by direct differentiation,

$$\begin{aligned}\frac{\partial w}{\partial z} &= \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}\left[\left(\frac{\partial z}{\partial t}\right)_\theta + \mathbf{V}\cdot\nabla_\theta z\right] \\ &= \frac{\partial \theta}{\partial z}\left[\frac{d}{dt}\left(\frac{\partial z}{\partial \theta}\right) + \frac{\partial \mathbf{V}}{\partial \theta}\cdot\nabla_\theta z\right] \\ &= \frac{d}{dt}\left(\ln\frac{\partial z}{\partial \theta}\right) - \frac{\partial \mathbf{V}}{\partial \theta}\cdot\nabla_z\theta \qquad (34)\end{aligned}$$

Adding Eqs. (33) and (34), the total divergence takes the simple form

$$\nabla_z \cdot \mathbf{V} + \frac{\partial w}{\partial z} = \nabla_\theta \cdot \mathbf{V} + \frac{d}{dt}\left(\ln\frac{\partial z}{\partial \theta}\right). \qquad (35)$$

The equation of continuity (Eq. (28)) can be written as

$$\frac{d}{dt}(\ln\rho) + \nabla_z \cdot \mathbf{V} + \frac{\partial w}{\partial z} = 0$$

which, with a substitution from Eq. (35), reduces to

$$\frac{d}{dt}\left(\ln\rho\frac{\partial z}{\partial \theta}\right) + \nabla_\theta \cdot \mathbf{V} = 0. \qquad (36)$$

Since all differentiations with respect to x, y and t with z held fixed have now been expressed in terms of those with θ held fixed, the subscripts will be dropped.

3.07. To summarize the results of the preceding development, we shall simply list the new hydrodynamical equations, expressing all derivatives as differentiations with respect to the quasi-Lagrangian coordinates and expanding total derivatives as the sum of local and advective derivatives.

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \mathbf{K} \times \lambda \mathbf{V} + \nabla[gz + \theta(1 - \kappa)^{-1} p^{1-\kappa}] = 0 \tag{37}$$

$$\frac{\partial}{\partial t}\left(\ln \frac{\partial p}{\partial \theta}\right) + \mathbf{V} \cdot \nabla\left(\ln \frac{\partial p}{\partial \theta}\right) + \nabla \cdot \mathbf{V} = 0 \tag{38}$$

$$(1 - \kappa)^{-1} p^{1-\kappa} - \frac{\partial}{\partial \theta}[gz + \theta(1 - \kappa)^{-1} p^{1-\kappa}] = 0. \tag{39}$$

Equation (37) was obtained by substituting Eq. (32) into Eq. (26), the vector equation of horizontal motion. Equation (38) combines Eqs. (27) and (36) and Eq. (39) was obtained by introducing the definition of θ into the hydrostatic equation (27). Since the vector equation of horizontal motion actually consists of two independent scalar equations, Eqs. (37), (38) and (39) constitute a complete system of equations involving the derivatives of the four dependent variables u, v, p and z.

3.08. The new equations are similar in form to those that would be obtained by omitting the vertical advection terms from the original Eulerian equations. In a manner of speaking, therefore—because the vertical component of velocity does not appear explicitly—the quasi-Lagrangian equations refer to a kind of "two-dimensional" motion. The effects of vertical motion are evidently implicit in the peculiar coordinate system we have chosen. At any rate, the vertical component of velocity has been eliminated effectively from all the equations.

THE PERTURBATION EQUATION FOR SURFACE PRESSURE

3.09. The remaining eliminations will be simplified by considering small deviations from the state of rest, in which the undisturbed values of p and z are necessarily independent of x, y and t. If the amplitude of the disturbance is chosen small enough, the nonlinear advective derivatives become negligible in comparison with the linear local time derivatives, whence all total time derivatives may simply be replaced by partial time derivatives. The horizontal component of velocity will be assumed constant along the intersections of the isentropic surfaces with the planes x = constant. This has the effect of forcing all disturbances to travel in the same direction, thereby permitting a direct comparison of their characteristic phase speeds and frequencies. Introducing these two restrictions into Eqs. (37), (38) and (39) yields a set of linear perturbation equations.

$$\frac{\partial u}{\partial t} - \lambda v + \frac{\partial}{\partial x}[gz + \theta(1 - \kappa)^{-1} p^{1-\kappa}] = 0 \tag{40}$$

$$\frac{\partial v}{\partial t} + \lambda u + \frac{\partial}{\partial y}[gz + \theta(1 - \kappa)^{-1} p^{1-\kappa}] = 0 \tag{41}$$

$$(1 - \kappa)^{-1} p^{1-\kappa} - \frac{\partial}{\partial \theta}[gz + \theta(1 - \kappa)^{-1} p^{1-\kappa}] = 0 \tag{42}$$

$$\frac{\partial}{\partial t}\left(\ln \frac{\partial p}{\partial \theta}\right) + \frac{\partial u}{\partial x} = 0. \tag{43}$$

We next cross-differentiate Eqs. (40) and (41) to get the vorticity equation, bearing in mind that u and v are independent of y, and making use of Eq. (43) to eliminate u.

$$\frac{\partial^2 v}{\partial x\, \partial t} + \beta v - \lambda \frac{\partial}{\partial t}\left(\ln \frac{\partial p}{\partial \theta}\right) = 0. \tag{44}$$

Moreover, by differentiating Eq. (40) with respect to x and Eq. (43) with respect to t, u can be eliminated to obtain a completely independent equation in v, p and z

$$\frac{\partial^2}{\partial x^2}[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}] - \frac{\partial^2}{\partial t^2}\left(\ln\frac{\partial p}{\partial \theta}\right) - \lambda\frac{\partial v}{\partial x} = 0. \tag{45}$$

Finally, v is eliminated by differentiating Eq. (44) once more with respect to x and substituting from Eq. (45).

$$\frac{\partial^4}{\partial x^3\,\partial t}[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}] - \frac{\partial^4}{\partial x\,\partial t^3}\left(\ln\frac{\partial p}{\partial \theta}\right) + \beta\frac{\partial^2}{\partial x^2}[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}] - \beta\frac{\partial^2}{\partial t^2}\left(\ln\frac{\partial p}{\partial \theta}\right)$$
$$-\lambda^2\frac{\partial^2}{\partial x\,\partial t}\left(\ln\frac{\partial p}{\partial \theta}\right) = 0. \tag{46}$$

It remains to eliminate either p or z between Eqs. (46) and (42). Since z appears explicitly only in the expression

$$gz + \theta(1-\kappa)^{-1}p^{1-\kappa},$$

we differentiate Eq. (46) with respect to θ and substitute from Eq. (42).

$$(1-\kappa)^{-1}\frac{\partial^4}{\partial x^3\,\partial t}(p^{1-\kappa}) - \frac{\partial^5}{\partial x\,\partial\theta\,\partial t^3}\left(\ln\frac{\partial p}{\partial \theta}\right) + \beta(1-\kappa)^{-1}\frac{\partial^2}{\partial x^2}(p^{1-\kappa}) - \beta\frac{\partial^3}{\partial\theta\,\partial t^2}\left(\ln\frac{\partial p}{\partial \theta}\right)$$
$$-\lambda^2\frac{\partial^3}{\partial x\,\partial\theta\,\partial t}\left(\ln\frac{\partial p}{\partial \theta}\right) = 0. \tag{47}$$

The above equation involves only one dependent variable, namely, the pressure at the point (x, y, θ, t).

3.10. Inasmuch as it contains no restrictions as to the barotropy, waveform, phase speed, or stability of the disturbances, Eq. (47) is of considerable interest in itself. However, it is not within the scope of this report to solve it and, because it is so general, it is difficult to interpret it in terms of what is already known about the physically possible types of atmospheric motion. Since Eq. (47) is too general for our purposes, we shall revert to an earlier stage in the development, introducing such specializations as are necessary to exclude all but neutrally stable disturbances. Products of perturbation quantities may be neglected, whence

$$\frac{\partial^{m+q}}{\partial x^m\,\partial t^q}[\theta(1-\kappa)^{-1}p^{1-\kappa}] = \rho^{-1}\frac{\partial^{m+q}}{\partial x^m\,\partial t^q}(p).$$

For the same reason

$$\frac{\partial^{m+q}}{\partial x^m\,\partial t^q}\left(\ln\frac{\partial p}{\partial \theta}\right) = \kappa p^{-1}\frac{\partial^{m+q}}{\partial x^m\,\partial t^q}(p) + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}\left[\frac{\partial^{m+q}}{\partial x^m\,\partial t^q}(z)\right].$$

Substituting these expressions in Eq. (46), we find that its coefficients can be manipulated to give it the form

$$\gamma p\left[g\frac{\partial^4 z}{\partial x^3\,\partial t} - \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}\left(\frac{\partial^4 z}{\partial x\,\partial t^3}\right)\right] + \left[c_s^2\frac{\partial^4 p}{\partial x^3\,\partial t} - \frac{\partial^4 p}{\partial x\,\partial t^3}\right] + \beta\gamma p\left[g\frac{\partial^2 z}{\partial x^2} - \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}\left(\frac{\partial^2 z}{\partial t^2}\right)\right]$$
$$+\beta\left[c_s^2\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2}\right] - \lambda^2\gamma p\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}\left(\frac{\partial^2 z}{\partial x\,\partial t}\right) - \lambda^2\frac{\partial^2 p}{\partial x\,\partial t} = 0 \tag{48}$$

where $c_s^2 = \gamma p\rho^{-1}$.

At this point it is easy to verify that Eq. (48) has solutions corresponding to all the known types of atmospheric motion. If, for example, the earth were not rotating and if the atmosphere were in purely horizontal motion, then only two terms of Eq. (48) would remain

$$c_s^2\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0.$$

This is the familiar one-dimensional wave equation whose solutions correspond to sound waves traveling at speeds $\pm(\gamma p\rho^{-1})^{1/2}$ in the x-direction. Similarly, if the earth were not rotating and if the atmosphere were incompressible (or, more aptly, uncompressed), Eq. (48) would reduce to

$$g\frac{\partial^2 z}{\partial x^2} - \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}\left(\frac{\partial^2 z}{\partial t^2}\right) = 0. \tag{49}$$

To permit a rough interpretation of this equation, let us apply it at the height H, where the vertical displacements of the isentropic (or material) surfaces are greatest, assuming at the same time that the underlying terrain is flat. Thus the potential temperature must have been initially constant along the ground surface, and must conserve that constant value at all later times. Therefore the derivatives of z, with respect to x and t, vanish at the ground and, approximately,

$$\frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}\left(\frac{\partial^2 z}{\partial t^2}\right) = \frac{\partial}{\partial z}\left(\frac{\partial^2 z}{\partial t^2}\right) \simeq \frac{1}{H}\frac{\partial^2 z}{\partial t^2},$$

from which Eq. (49) takes the form of the wave equation

$$gH\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0.$$

The solutions of this equation correspond to gravity waves traveling at speeds $\pm(gH)^{1/2}$ in the x-direction. If the motion were purely horizontal and if the atmosphere were incompressible (or uncompressed), Eq. (44) would reduce to

$$\frac{\partial^2 v}{\partial x\,\partial t} + \beta v = 0,$$

which is the familiar equation for the Rossby waves.

3.11. To deal simultaneously with all types of motion, however, all but one dependent variable must be eliminated from Eq. (48). To do so, Eq. (48) will first be applied to conditions at the ground level. The ground surface will be assumed flat, inasmuch as forced oscillations induced by irregular terrain are not a primary concern. The potential temperature at the surface must have been constant initially, in the undisturbed state, and must remain constant at all times thereafter. When this restriction is introduced into Eq. (48), certain terms vanish at the lower boundary.

$$c_s^2\frac{\partial^4 p_0}{\partial x^3\,\partial t} - \frac{\partial^4 p_0}{\partial x\,\partial t^3} - \gamma p_0\frac{\partial}{\partial z}\left(\frac{\partial^4 z}{\partial x\,\partial t^3}\right)_0 + \beta\left(c_s^2\frac{\partial^2 p_0}{\partial x^2} - \frac{\partial^2 p_0}{\partial t^2}\right) - \beta\gamma p_0\frac{\partial}{\partial z}\left(\frac{\partial^2 z}{\partial t^2}\right)_0 - \lambda^2\frac{\partial^2 p_0}{\partial x\,\partial t}$$
$$- \lambda^2\gamma p_0\frac{\partial}{\partial z}\left(\frac{\partial^2 z}{\partial x\,\partial t}\right)_0 = 0. \tag{50}$$

The subscript zero refers to conditions at the ground surface. We next consider the motions of an atmosphere consisting of two isentropic layers, separated by a material surface of discontinuity at height H—which might be thought of as corresponding to the tropopause. It is a curious coincidence that the maximum amplitudes of both the large-scale slowly moving disturbances and of internal gravity waves (whose maximum amplitude must lie near a discontinuity of density) are attained somewhere around the tropopause. For this reason, and because only the relative orders of magnitude of the terms in Eq. (50) are to be estimated, it is legitimate to introduce the following approximation

$$\frac{\partial}{\partial z}\left(\frac{\partial^{m+q} z}{\partial x^m\,\partial t^q}\right) \simeq \frac{1}{H}\frac{\partial^{m+q} H}{\partial x^m\,\partial t^q}.$$

The relation between the derivatives of H and p_0 is provided by Eq. (27), the condition for hydrostatic equilibrium. Requiring that the pressure be continuous across the interface,

$$g\theta_2^{-1}(\theta_2 - \theta_1)\frac{\partial H}{\partial t} = \rho_0^{-1}\frac{\partial p_0}{\partial t},$$

in which θ_2 and θ_1 are the values of θ above and below the discontinuity. The derivatives of z at the ground are then given by

$$\frac{\partial}{\partial z}\left(\frac{\partial^{m+q} z}{\partial x^m\,\partial t^q}\right) \simeq \rho_0^{-1}c_g^{-2}\frac{\partial^{m+q} p_0}{\partial x^m\,\partial t^q},$$

where c_g is $[gH\theta_2^{-1}(\theta_2 - \theta_1)]^{1/2}$, the speed of internal gravity waves traveling along the surface of discontinuity. Finally, introducing the above approximation into Eq. (50), we arrive at an equation which involves only the surface pressure.

$$\frac{\partial^4 p_0}{\partial x^3\,\partial t} - (c_s^{-2} + c_g^{-2})\frac{\partial^4 p_0}{\partial x\,\partial t^3} + \beta\frac{\partial^2 p_0}{\partial x^2} - \beta(c_s^{-2} + c_g^{-2})\frac{\partial^2 p_0}{\partial t^2} - \lambda^2(c_s^{-2} + c_g^{-2})\frac{\partial^2 p_0}{\partial x\,\partial t} = 0. \qquad (51)$$

This equation provides the basis for further discussion of the scale theory. Equation (51) is of the same general form as Eq. (11), which applies to the elevation of the sea surface, and has almost identical coefficients if c_g^{-2} is replaced by $(c_g^{-2} + c_s^{-2})$.

THE SCALE THEORY

3.12. As before, we ascribe to each type of motion a characteristic wavelength and period. It will then be possible to approximate the derivatives in Eq. (51) by ratios of characteristic numbers and to express these estimates in terms of a characteristic phase speed and frequency. The relative magnitudes of the terms in Eq. (51) are displayed below, each estimate appearing beneath the corresponding term in the equation.

$\frac{\partial^4 p_0}{\partial x^3\,\partial t}$	$c_m^{-2}\frac{\partial^4 p_0}{\partial x\,\partial t^3}$	$\beta\frac{\partial^2 p_0}{\partial x^2}$	$\beta c_m^{-2}\frac{\partial^2 p_0}{\partial t^2}$	$\lambda^2 c_m^{-2}\frac{\partial^2 p_0}{\partial x\,\partial t}$
$\left(\frac{c_m}{c}\right)^2$	1	$\left(\frac{c_m}{c}\right)\left(\frac{c_m\beta}{\lambda^2}\right)\left(\frac{\lambda}{f}\right)^2$	$\left(\frac{c}{c_m}\right)\left(\frac{c_m\beta}{\lambda^2}\right)\left(\frac{\lambda}{f}\right)^2$	$\left(\frac{\lambda}{f}\right)^2$

Note: $c_m^{-2} = c_s^{-2} + c_g^{-2}$

Thus the state of motion is again characterized by the values of three nondimensional parameters, one of which depends only on such quantities as the gravitational constant, the gas constants, the effective depth of the atmosphere, the absolute angular speed and radius of the earth, and the thermodynamic structure of the undisturbed state. The remaining two depend on the type of motion one chooses to consider.

3.13. In the atmosphere, as in the ocean, there are evidently two distinct classes of motion, one distinguished by the fact that its characteristic frequency is much greater than the frequency of the earth's rotation, and the other by the fact that its characteristic phase speed is much less than that of either sound waves or internal gravity waves. For example, if $f \gg \lambda$ (and if c is independent of f), then the last three terms of Eq. (51) are much less than the first two and Eq. (51) reduces to the hyperbolic wave equation

$$c_m^2\frac{\partial^2 p_0}{\partial x^2} - \frac{\partial^2 p_0}{\partial t^2} = 0. \qquad (52)$$

The solutions of this equation correspond to modified gravity waves, for, strictly speaking, the sound waves have been excluded by imposing the condition of hydrostatic equilibrium. On the other hand, if $c \ll c_m$ (and whether or not c depends on f), then the second term of Eq. (51) is much less than the first and the fourth is much less than the third. In the latter case, the motions are governed by a general equation of which Rossby's is a special form

$$\frac{\partial^3 p_0}{\partial x^2\,\partial t} + \beta\frac{\partial p_0}{\partial x} - \lambda^2(c_s^{-2} + c_g^{-2})\frac{\partial p_0}{\partial t} = 0. \tag{53}$$

It can be easily verified that this equation has no solutions corresponding to sound or gravity waves, simply by noting that its wave solutions are propagated in only one direction. It is also noteworthy that, as c is made smaller and smaller in comparison with c_m, the third term of Eq. (53) likewise becomes smaller and smaller in comparison with the second. If the characteristic phase speed is very small, therefore, the motions are actually governed by a telegrapher's equation, identical with Rossby's equation for long waves

$$\frac{\partial^2 p_0}{\partial x\,\partial t} + \beta p_0 = 0.$$

This implies that, the slower the movement of the disturbances relative to the medium, the more nearly is the absolute vorticity conserved.

3.14. This analysis does not provide a clearcut distinction between the sound and gravity waves because they cannot, in general, be distinguished solely on the basis of their respective phase speeds or frequencies. However, Eq. (52) correctly yields pure gravity waves in the limiting case of complete incompressibility. As the bulk modulus approaches infinity, the speed of sound also approaches an infinitely great value. In the limiting case of incompressibility, the speed of sound therefore becomes infinite and Eq. (52) reduces to

$$c_g^2\frac{\partial^2 p_0}{\partial x^2} - \frac{\partial^2 p_0}{\partial t^2} = 0,$$

which is the equation for pure internal gravity waves.

3.15. In view of the comments of paragraph 2.26, concerning necessary conditions for unidirectional propagation, and because large-scale atmospheric disturbances actually are observed to move very slowly relative to the medium, it is reasonable to conclude that the large-scale motions are governed by an equation of the same general form as Eq. (53), and that they are distinguished from all other types of motion by the very fact that they do move so slowly. It is difficult, however, to see how one can make direct use of the filtering approximation in dealing with a more general set of equations. By *simultaneously assuming quasi-horizontal motion* and *substituting the geostrophic wind in the vorticity equation* (*Eq.* (44)), we can obtain an equation which is identical to Eq. (53), except in minor respects

$$\frac{\partial^3 p_0}{\partial x^2\,\partial t} + \beta\frac{\partial p_0}{\partial x} - \lambda^2 c_s^{-2}\frac{\partial p_0}{\partial t} = 0.$$

Following the procedures outlined in paragraph 2.28, it can be shown that the so-called geostrophic approximation is exactly equivalent to the filtering approximation, $c \ll c_m$. Approximating the derivatives of u and p in Eqs. (40) and (44) by ratios of characteristic numbers, and requiring that the motion be quasi-horizontal, one obtains the following magnitude estimates of each term:

$\frac{\partial u}{\partial t}$	λv	$\rho^{-1}\frac{\partial p}{\partial x}$	(54)
$A_u T^{-1}$	λv	$\rho^{-1} A_p L^{-1}$	

$\gamma^{-1}p^{-1}\dfrac{\partial p}{\partial t}$	$\dfrac{\partial u}{\partial x}$
$\gamma^{-1}p^{-1}A_pT^{-1}$	A_uL^{-1}

Combining these estimates, we find that the relative magnitudes of the terms in Eq. (40) are

$\dfrac{\partial u}{\partial t}$	λv	$\rho^{-1}\dfrac{\partial p}{\partial x}$
$\left(\dfrac{c}{c_s}\right)^2$	$\lambda v\rho LA_p^{-1}$	1

Thus, if c is much less than c_m (and still less than c_s), then the first term of Eq. (54) is much less than the third. This is equivalent to saying that the winds are quasi-geostrophic. In the general development to follow, therefore, it is permissible—and, in fact, necessary—to introduce the geostrophic winds into more general forms of the vorticity equation. This, as will be shown a posteriori, is at least sufficient to exclude solutions corresponding to the irrelevant "nonmeteorological" types of motion, leaving an equation which applies only to the large-scale slowly moving disturbances of pressure.

3.16. Although the foregoing discussion has been confined to small deviations from the state of rest, and although several artificial constraints on the geometry of the motions have been introduced, the physical system is self-consistent and contains the essential mechanisms by which all types of disturbances are propagated. For this reason, and because the physical character of the system is not radically altered by its nonlinearity, it is reasonable to expect that the qualitative results of this analysis will apply under less restrictive conditions, i.e., to the general nonlinear equations. We shall therefore proceed to the development of a vorticity equation which holds under most actually observed conditions.

4.00 THE VORTICITY EQUATION FOR ADIABATIC FLOW

4.01. There are sound physical and mathematical reasons for developing the theory of large-scale disturbances around some form of the vorticity equation. We shall therefore attempt to derive a vorticity equation which holds under very general conditions, subject to the sole restriction that no energy is supplied from external sources. The equation will then be specialized to conform to actually observed types of flow. The physical basis for this development lies in the quasi-Lagrangian equations of motion and continuity, previously derived in Section 3.00. The material derivative of the horizontal velocity component can be decomposed by applying the following vector identity:

$$\mathbf{V}\cdot\nabla_\theta\mathbf{V} \equiv \nabla_\theta(\mathbf{V}\cdot\mathbf{V}/2) + \mathbf{K}\times\zeta\mathbf{V},$$

where $\zeta = \mathbf{K}\cdot\nabla_\theta\times\mathbf{V}$. Accordingly, expanding the material derivative as the sum of the local time derivative and the advective derivative,

$$\frac{d\mathbf{V}}{dt} \equiv \left(\frac{\partial\mathbf{V}}{\partial t}\right)_\theta + \nabla_\theta\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + \mathbf{K}\times\zeta\mathbf{V}.$$

Substituting this expression into Eq. (37), the vector equation of horizontal motion,

$$\frac{\partial\mathbf{V}}{\partial t} + \nabla\left(\frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) + \mathbf{K}\times(\zeta+\lambda)\mathbf{V} + \nabla[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}] = 0. \qquad (55)$$

We next apply the vector operator $\nabla_\theta\times(\quad)$ to Eq. (55) to obtain the vorticity equation; noting that the second and fourth terms are potential vectors and therefore do not contribute to the curl.

$$\partial\zeta/\partial t + \nabla\cdot(\zeta+\lambda)\mathbf{V} = 0.$$

For later convenience, the absolute vorticity $Z = \zeta + \lambda$ is introduced, so that

$$\partial Z/\partial t + \nabla \cdot Z\mathbf{V} = 0. \tag{56}$$

This equation can also be written as

$$dZ/dt + Z\nabla \cdot \mathbf{V} = 0. \tag{57}$$

The vorticity equation in this form simply states that the absolute vorticity of barotropic, nondivergent flow is conserved. However, this type of flow is too special for present purposes, and an independent expression for the velocity divergence is needed. The necessary information is contained in Eq. (38), the equation of continuity, which may be written as

$$\frac{d}{dt}\left(\frac{\partial p}{\partial \theta}\right) + \frac{\partial p}{\partial \theta} \nabla \cdot \mathbf{V} = 0. \tag{58}$$

Finally, eliminating the velocity divergence between Eqs. (57) and (58), we obtain a variant of the so-called "potential vorticity" equation

$$\frac{d}{dt}\left(Z \frac{\partial \theta}{\partial p}\right) = 0. \tag{59}$$

This result is similar, but not quite identical to Rossby's theorem of potential vorticity (1940).

4.02. In passing, it should be mentioned that Eq. (59), taken together with the conditions for geostrophic and hydrostatic equilibrium, involves only one dependent variable—namely, pressure as a function of x, y, θ and t. The potential vorticity equation may, therefore, be regarded as an acceptable prognostic equation in the sense that it does not suffer from the fundamental difficulties discussed in Section 2.00. The remaining difficulty is that the Lagrangian coordinate θ is really a function of x, y, z and t, and in order to locate the coordinate surfaces one is forced to compute the vertical component of velocity in one way or another. This question has already been discussed in paragraph 2.47. It was concluded that, because the static stability of the troposphere is actually quite small, any estimate of the vertical component of velocity is critically sensitive to small errors in the initial data. Therefore one cannot (with the present data) deal with the general three-dimensional motion of the atmosphere, and must resort to a mathematical device much like that adopted by Charney and Eliassen (1949) for somewhat different reasons. Vertical dependence will be eliminated by the simple expedient of integrating the vorticity equation through the entire vertical extent of the atmosphere, so that the resulting equation will refer to vertically integrated values of the original variables. The latter, of course, depend only on x, y and t and apply to a fictitious horizontal motion which, in a purely mathematical sense, is two-dimensional.

THE MEAN VORTICITY EQUATION

4.03. To carry out this scheme for integrating out the vertical coordinate, Eq. (56) is multiplied by $\partial p/\partial \theta$. Applying the rules for partial differentiation of products and inverting the order of differentiation where permissible,

$$\frac{\partial}{\partial t}\left(Z \frac{\partial p}{\partial \theta}\right) + \nabla \cdot Z \frac{\partial p}{\partial \theta} \mathbf{V} = Z\left[\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right) + \mathbf{V} \cdot \nabla\left(\frac{\partial p}{\partial \theta}\right)\right] = Z \frac{\partial}{\partial \theta}\left(\frac{dp}{dt}\right) - Z\nabla p \cdot \frac{\partial \mathbf{V}}{\partial \theta}. \tag{60}$$

We next integrate Eq. (60) with respect to θ, from the value of θ *at the ground surface* to a *constant* value of θ which occurs at some very great height d:

$$\int_{\theta_h}^{\theta_d} \frac{\partial}{\partial t}\left(Z \frac{\partial p}{\partial \theta}\right) d\theta + \int_{\theta_h}^{\theta_d} \nabla \cdot Z \frac{\partial p}{\partial \theta} \mathbf{V}\, d\theta = \int_{\theta_h}^{\theta_d} Z \frac{\partial}{\partial \theta}\left(\frac{dp}{dt}\right) d\theta - \int_{\theta_h}^{\theta_d} Z\nabla p \cdot \frac{\partial \mathbf{V}}{\partial \theta}\, d\theta. \tag{61}$$

According to the rules for differentiating definite integrals with a variable limit of integration,

$$\frac{\partial}{\partial t}\int_{\theta_h}^{\theta_d} Z\frac{\partial p}{\partial \theta}\,d\theta \equiv \int_{\theta_h}^{\theta_d}\frac{\partial}{\partial t}\left(Z\frac{\partial p}{\partial \theta}\right)d\theta - Z_h\left(\frac{\partial p}{\partial \theta}\right)_h\frac{\partial \theta_h}{\partial t}$$

and, similarly,

$$\nabla\cdot\int_{\theta_h}^{\theta_d} Z\mathbf{V}\frac{\partial p}{\partial \theta}\,d\theta \equiv \int_{\theta_h}^{\theta_d}\nabla\cdot Z\frac{\partial p}{\partial \theta}\mathbf{V}\,d\theta - Z_h\left(\frac{\partial p}{\partial \theta}\right)_h\mathbf{V}_h\cdot\nabla\theta_h.$$

On introducing the above expressions, the left-hand side of Eq. (61) reduces to

$$\frac{\partial}{\partial t}\int_{\theta_h}^{\theta_d} Z\frac{\partial p}{d\theta}\,d\theta + \nabla\cdot\int_{\theta_h}^{\theta_d} Z\mathbf{V}\frac{\partial p}{\partial \theta}\,d\theta + Z_h\left(\frac{\partial p}{\partial \theta}\right)_h\frac{d\theta_h}{dt}.$$

However, because the potential temperature is conserved, the third term vanishes. After changing the limits and variable of integration, the left-hand side then assumes the form

$$\frac{\partial}{\partial t}\int_{p_h}^{p_d} Z\,dp + \nabla\cdot\int_{p_h}^{p_d} Z\mathbf{V}\,dp.$$

We now choose d large enough that p_d is effectively zero.

4.04. At this point, it is convenient to introduce a mean horizontal velocity $\overline{\mathbf{V}}$ and mean absolute vorticity $\overline{Z}$, defined as follows:

$$(\overline{\quad}) = p_h^{-1}\int_0^{p_h}(\quad)\,dp.$$

At the same time $\mathbf{V}'$ and Z', are defined as deviations from the mean values $\overline{\mathbf{V}}$ and $\overline{Z}$. The left-hand side of Eq. (61) can now be expressed in terms of the mean values and the deviations from the means. After interchanging the limits of integration, the left-hand side of Eq. (61) is

$$-\frac{\partial}{\partial t}(p_h\overline{Z}) - \nabla\int_0^{p_h}(\overline{Z}+Z')(\overline{\mathbf{V}}+\mathbf{V}')\,dp = -\frac{\partial}{\partial t}(p_h\overline{Z}) - \nabla\cdot p_h\overline{Z}\overline{\mathbf{V}} - \nabla\cdot\int_0^{p_h}Z'\mathbf{V}'\,dp$$

$$= -p_h\left(\frac{\partial\overline{Z}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\overline{Z}\right) - \overline{Z}\left(\frac{\partial p_h}{\partial t}+\nabla\cdot p_h\overline{\mathbf{V}}\right) - \nabla\cdot\int_0^{p_h}Z'\mathbf{V}'\,dp.$$

In summary, the left-hand side of Eq. (61) is replaced by the above expression:

$$p_h\left(\frac{\partial\overline{Z}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\overline{Z}\right) + \overline{Z}\left(\frac{\partial p_h}{\partial t}+\nabla\cdot p_h\overline{\mathbf{V}}\right) + \nabla\cdot\int_0^{p_h}Z'\mathbf{V}'\,dp = \int_{\theta_h}^{\theta_d}Z\nabla p\cdot\frac{\partial\mathbf{V}}{\partial\theta}\,d\theta - \int_{\theta_h}^{\theta_d}Z\frac{\partial}{\partial\theta}\left(\frac{dp}{dt}\right)d\theta. \tag{62}$$

4.05. In exactly the same way, the continuity equation is integrated with respect to θ, between the limits θ_h and θ_d, using a form of the equation obtainable from Eq. (58)

$$\int_{\theta_h}^{\theta_d}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\theta}\right)d\theta + \int_{\theta_h}^{\theta_d}\nabla\cdot\frac{\partial p}{\partial\theta}\mathbf{V}\,d\theta = 0. \tag{63}$$

As before, we apply the rules for differentiating definite integrals with a variable limit of integration, bearing in mind that θ_d is a constant,

$$\frac{\partial}{\partial t}\int_{\theta_h}^{\theta_d}\frac{\partial p}{\partial\theta}\,d\theta = \int_{\theta_h}^{\theta_d}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\theta}\right)d\theta - \left(\frac{\partial p}{\partial\theta}\right)_h\frac{\partial\theta_h}{\partial t}$$

$$\nabla\cdot\int_{\theta_h}^{\theta_d}\frac{\partial p}{\partial\theta}\mathbf{V}\,d\theta = \int_{\theta_h}^{\theta_d}\nabla\cdot\frac{\partial p}{\partial\theta}\mathbf{V}\,d\theta - \left(\frac{\partial p}{\partial\theta}\right)_h\mathbf{V}_h\cdot\nabla\theta_h.$$

Substituting these expressions into Eq. (63),

$$\frac{\partial}{\partial t}\int_{\theta_h}^{\theta_d}\frac{\partial p}{\partial\theta}\,d\theta+\nabla\cdot\int_{\theta_h}^{\theta_d}\mathbf{V}\,\frac{\partial p}{\partial\theta}\,d\theta+\left(\frac{\partial p}{\partial\theta}\right)_h\frac{d\theta_h}{dt}=0.$$

After changing the limits and variable of integration, and also noting that θ_h is conserved,

$$\frac{\partial}{\partial t}\int_{p_h}^{p_d}dp+\nabla\cdot\int_{p_h}^{p_d}\mathbf{V}\,dp=0.$$

We now choose d large enough so that p_d is effectively zero, whence, according to the definition of $\overline{\mathbf{V}}$,

$$\frac{\partial p_h}{\partial t}+\nabla\cdot p_h\overline{\mathbf{V}}=0. \tag{64}$$

This is the continuity equation expressed in terms of the integrated variables, which depend only on x, y and t. Substituting from Eq. (64) into Eq. (62), the vorticity equation now reduces to

$$p_h\left(\frac{\partial\overline{Z}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\overline{Z}\right)+\nabla\cdot\int_0^{p_h}Z'\mathbf{V}'\,dp=\int_{\theta_h}^{\theta_d}Z\,\nabla p\cdot\frac{\partial\mathbf{V}}{\partial\theta}\,d\theta-\int_{\theta_h}^{\theta_d}Z\,\frac{\partial}{\partial\theta}\left(\frac{dp}{dt}\right)d\theta. \tag{65}$$

Finally, in order to put the vorticity equation into a more recognizable form, the second integral on the right-hand side of Eq. (65) can be integrated by parts to obtain

$$p_h\left(\frac{\partial\overline{Z}}{\partial t}+\overline{\mathbf{V}}\cdot\nabla\overline{Z}\right)-Z_h\frac{dp_h}{dt}=\int_{\theta_h}^{\theta_d}Z\nabla p\cdot\frac{\partial\mathbf{V}}{\partial\theta}\,d\theta+\int_{\theta_h}^{\theta_d}\frac{dp}{dt}\,\frac{\partial Z}{\partial\theta}\,d\theta-\nabla\cdot\int_0^{p_h}Z'\mathbf{V}'\,dp. \tag{66}$$

This is the form in which the vorticity equation for adiabatic flow will be considered. It has been arranged in this particular way to emphasize the "barotropic" aspects of the large-scale flow and to isolate the effects of baroclinity.

4.06. In the case of barotropic flow, for example, the velocity and, in consequence, the vorticity are essentially independent of height. Under those conditions, all the terms on the right-hand side of Eq. (66) vanish, and the vorticity equation assumes the simple form

$$\frac{d}{dt}\,(p_h^{-1}Z)=0.$$

This, of course, is exactly Rossby's vorticity equation for barotropic flow. We may therefore regard Eq. (66) as a direct extension of the barotropic vorticity equation, in that it is split up in such a way as to isolate the purely baroclinic effects on the vorticity-generating mechanism from those which are essentially barotropic. That is to say, the terms on the left-hand side of Eq. (66) have the same general form as the terms in the barotropic vorticity equation, and the terms on the right-hand side are non-zero only if the flow is baroclinic. The terms on the right-hand side therefore represent the purely baroclinic effects.

4.07. Thus far no concessions have been made except to make use of the adiabatic law and the condition for hydrostatic equilibrium. The next stage in the development is to introduce such specializations into the "baroclinic" terms on the right-hand side of Eq. (66) as are necessary to express them in terms of the vertically integrated mean variables, at the same time conforming to the observed facts as closely as possible. As justification for approximating the baroclinic terms on the right-hand side of Eq. (66), it should be noted that the atmosphere is very nearly barotropic. Therefore, those terms are rather small to begin with, and one has considerable latitude in approximating baroclinic terms, without danger of losing any of the essential features of the vorticity-generating mechanism.

4.08. Since, in dealing with the purely baroclinic effects one is concerned mainly with estimating the vertical variations of velocity, it is quite natural to invoke the so-called "thermal wind equation." Because all large-scale motion is characteristically geostrophic, that relationship applies equally well to barotropic and baroclinic flow. To express the thermal wind equation in terms of the quasi-Lagrangian coordinates, we begin with a geostrophic equation obtained by omitting the acceleration terms from Eq. (37)

$$\mathbf{V} = \mathbf{K} \times \lambda^{-1}\nabla[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}]. \tag{67}$$

Differentiating Eq. (67) with respect to θ, and substituting from the hydrostatic equation (Eq. (39)), the thermal wind equation is obtained in the following form:

$$\begin{aligned} \frac{\partial \mathbf{V}}{\partial \theta} &= \mathbf{K} \times \lambda^{-1}\nabla \frac{\partial}{\partial \theta}[gz + \theta(1-\kappa)^{-1}p^{1-\kappa}] \\ &= \mathbf{K} \times \lambda^{-1}(1-\kappa)^{-1}\nabla p^{1-\kappa} \\ &= \mathbf{K} \times \lambda^{-1}p^{-\kappa}\nabla p. \end{aligned} \tag{68}$$

An immediate consequence of Eq. (68) is that the first integral on the right-hand side of Eq. (66) vanishes, whether the flow is baroclinic or not

$$\nabla p \cdot \frac{\partial \mathbf{V}}{\partial \theta} = \mathbf{K} \cdot \lambda^{-1}p^{-\kappa}(\nabla p \times \nabla p) = 0.$$

Turning to the integrand of the second integral on the right-hand side of Eq. (66), we observe that the advective part of the material derivative of pressure is

$$\mathbf{V} \cdot \nabla p.$$

According to the thermal wind equation (Eq. (68)), the advective derivative of pressure is

$$\begin{aligned} \mathbf{V} \cdot \nabla p &= \mathbf{K} \cdot \lambda p^{\kappa}\left(\mathbf{V} \times \frac{\partial \mathbf{V}}{\partial \theta}\right) \\ &= \lambda p^{\kappa} V^2 \frac{\partial \chi}{\partial \theta}, \end{aligned}$$

where χ is the angle between the wind vector and some geographically fixed horizontal line. If the wind direction is independent of height, therefore, the advective derivative of pressure $\mathbf{V} \cdot \nabla p$ vanishes.

4.09. The latter deserves some further comment, for the conditions under which the advective derivative of pressure vanishes are not too far from the actually observed flow conditions associated with large-scale disturbances. That is to say, the configuration of the streamlines of *large-scale* flow displays the same general shape, phase and amplitude at all levels. Stated in still another way, the lines of constant temperature on a surface of constant pressure coincide fairly well with the contours of the pressure surface. There are, however, more deep-seated reasons for believing that the nonvariability of wind direction with height is a characteristic of the very large-scale disturbances. To show this, we shall consider the adiabatic equation, temporarily reverting to Eulerian coordinates

$$\left(\frac{\partial \theta}{\partial t}\right)_z + \mathbf{V} \cdot \nabla_z\theta + w\frac{\partial \theta}{\partial z} = 0. \tag{69}$$

The thermal wind equation is also written in terms of Eulerian coordinates to obtain an alternative expression for the horizontal advection of potential temperature

$$\mathbf{V} \cdot \nabla_z\theta = -g^{-1}\lambda\theta V^2 \frac{\partial \chi}{\partial z}.$$

Substituting this expression into Eq. (69), the adiabatic equation then has the form

$$\frac{\partial \chi}{\partial z} = g\lambda^{-1}\theta^{-1}V^{-2}\left[\left(\frac{\partial \theta}{\partial t}\right)_s + w\frac{\partial \theta}{\partial z}\right]. \tag{70}$$

Now, it has already been noted that the vertical component of velocity and the local changes of pressure and density are characteristically small for the very large-scale disturbances. Because the potential temperature is a function of pressure and density alone, it follows that the local variations of potential temperature are correspondingly small. Thus the entire right-hand side (and therefore the left-hand side) of Eq. (70) can be regarded as small if we are considering the *very large-scale* disturbances. In a rather roundabout way this argument shows that the vertical variations of wind direction associated with large-scale disturbances are characteristically small. From this point onward, therefore, it will be required that the wind direction (though not the wind speed) not vary with height, in much the same way as Charney and Eliassen (1949) have specified.

4.10. A simple physical interpretation of the above assumption is most easily provided by considering the ways in which vorticity can be created *within a fixed level*. It is a consequence of our assumption, of course, that no vorticity can be created through the action of solenoids, because density is essentially a function of pressure within a fixed level. On the other hand, no restriction has been placed on the vertical variability of wind speed, so that it is still possible to "create" vorticity within a fixed level by advection of thermal vorticity. It must be reemphasized that this applies only to large-scale disturbances.

4.11. Considering Eq. (66), in view of the assumption discussed above, it will be noted that the first integral on the right-hand side vanishes in any case, and that the advective part of the material derivative of pressure (which enters into the integrand of the second integral) vanishes under the special conditions we have assumed

$$p_h\left(\frac{\partial \bar{Z}}{\partial t} + \bar{\mathbf{V}}\cdot\nabla\bar{Z}\right) - Z_h\frac{dp_h}{dt} = -\int_0^{p_h}\frac{\partial p}{\partial t}\frac{\partial Z}{\partial p}\,dp - \nabla\cdot\int_0^{p_h} Z'\mathbf{V}'\,dp. \tag{71}$$

The present concern is to estimate the contribution of the terms on the right-hand side of Eq. (71) and to express them in terms of the integrated variables. In the first place, the local derivative of pressure generally has the same sign and same general order of magnitude at all levels, whereas the vertical derivative of absolute vorticity usually changes sign at about the level of the tropopause, taking on large positive (or negative) values above that level and somewhat smaller negative (or positive) values below it. Thus, the integrand of the first integral on the right-hand side of Eq. (71) tends to contain positive and negative contributions in approximately equal degree. For reasons discussed in paragraph 4.07 and because its integrand has oscillatory properties even in markedly baroclinic flow, the first integral on the right-hand side of Eq. (71) will be omitted.

4.12. We next consider the second term on the right-hand side of Eq. (71). Since we have assumed that the wind direction is independent of height, the wind vector at any level is a scalar multiple of the wind vector at any other level. For convenience, we therefore write

$$\mathbf{V} = (1 + A_V)\bar{\mathbf{V}}$$

whence

$$\mathbf{V}' = A_V\bar{\mathbf{V}}.$$

Similarly,

$$Z' = \zeta' = A_\zeta\bar{\zeta}.$$

Introducing these definitions into the term in question,

$$\nabla \cdot \int_0^{p_h} \zeta' \mathbf{V}' \, dp = \nabla \cdot \bar{\zeta}\bar{\mathbf{V}} \int_0^{p_h} A_v A_\zeta \, dp$$
$$= \nabla \cdot (\tau - 1) p_h \bar{\zeta}\bar{\mathbf{V}},$$

where $\tau = 1 + \overline{A_v A_\zeta}$. The quantity τ is assumed to be a "slowly-varying" function which depends on the distribution of velocity and vorticity above each point on the ground surface. In fact, it is actually observed that the limits between which τ varies are quite narrow. It is almost always greater than 1.0 and rarely exceeds 1.3, varying from its maximum and minimum values over the "characteristic half-wavelength" of the large-scale disturbances. Therefore, τ may be treated as a constant with respect to horizontal differentiation, whence the second term on the right-hand side of Eq. (71) is

$$-(\tau - 1) \nabla \cdot p_h \bar{\zeta}\bar{\mathbf{V}}.$$

Finally, expanding Eq. (71) into local and advective derivatives, and introducing the above expression on the right-hand side,

$$p_h \left(\frac{\partial \bar{\zeta}}{\partial t} + \tau \bar{\mathbf{V}} \cdot \nabla \bar{\zeta} + \beta \bar{v} \right) - [(\tau - 1)\bar{\zeta} + Z_h] \frac{\partial p_h}{\partial t} - Z_h \mathbf{V}_h \cdot \nabla p_h = 0. \tag{72}$$

This is the mean vorticity equation, which applies to the "two-dimensional" vertically integrated variables p_h, $\bar{\zeta}$ and $\bar{\mathbf{V}}$.

4.13. By way of orientation, it should be noted that the values of A_v and A_ζ are both zero if the velocity does not vary with height, whence the value of τ for barotropic flow is unity. In that case Eq. (72) correctly reduces to the vorticity equation for barotropic flow

$$p_h \frac{dZ}{dt} - Z \frac{dp_h}{dt} = 0.$$

Equation (72) is therefore to be regarded as a generalization of the barotropic vorticity equation in the sense that it applies to a special but commonly observed type of baroclinic flow in which the wind direction (but not the wind speed) is very nearly independent of height.

4.14. Before discussing the way in which Eq. (72) will be used to formulate a suitable prognostic equation, it is appropriate to add a few general remarks about the ultimate validity of the approximations introduced to obtain the mean vorticity equation. The terms which have been selected as representing the largest effects of baroclinity are actually rather small in comparison with either of those involving the local derivative or the advective derivative of mean absolute vorticity. It is therefore safe to say that the terms approximated were already small. Moreover, because the entire left-hand side of Eq. (66)—including the largest terms—was arrived at *without such approximations*, Eq. (72) might be expected to describe the large-scale motions of the atmosphere with a fairly high degree of accuracy.

5.00 THE PROGNOSTIC EQUATION

5.01. An equation will now be developed which, with suitable approximations, involves only one dependent variable. The solution of such an equation, subject to observed initial conditions, will constitute a verifiable prediction, by which the general validity of the theory can be tested. We shall take as a starting point the mean vorticity equation (72), first investigating the last term. Because the lower boundary is fixed, the advective derivative of the surface pressure can be separated into two parts

$$\mathbf{V}_h \cdot \nabla p_h = \mathbf{V}_h \cdot (\nabla_s p)_h - g\rho_h \mathbf{V}_h \cdot \nabla h.$$

However, since the winds are very nearly geostrophic, $\mathbf{V}_h$ is almost perpendicular to $\nabla_s p$ at the surface. This simply shows that the "advection" of surface pressure is due mainly to variations in the height of the terrain. The latter, in fact, is the major part played by irregular terrain in the generation of mean vorticity. In all terms of Eq. (72) *except the last*, therefore, it will be assumed that the lower boundary is a flat surface located at height zero. Equation (72) then reduces to

$$p_0\left(\frac{\partial\bar{\zeta}}{\partial t} + \tau\bar{\mathbf{V}} \cdot \nabla\bar{\zeta} + \beta\bar{v}\right) - [(\tau - 1)\bar{\zeta} + Z_0]\frac{\partial p_0}{\partial t} + g\rho_0 Z_0 \mathbf{V}_0 \cdot \nabla h = 0, \tag{73}$$

where p_0 is the pressure reduced to sea level by means of the hydrostatic equation.

5.02. The above assumption also simplifies the problem of estimating $\bar{\zeta}$ in terms of $\bar{\mathbf{V}}$, whose definition is now

$$\bar{\mathbf{V}} = p_0^{-1}\int_0^{p_0} \mathbf{V}\, dp.$$

It will next be shown that, to within the accuracy of the observations,

$$\bar{\zeta} = \mathbf{K} \cdot \nabla \times \bar{\mathbf{V}}.$$

To begin with, we apply the operator $\nabla \times (\quad)$ to $\bar{\mathbf{V}}$, expanding the derivatives according to the rules for differentiating definite integrals with variable limits of integration

$$\begin{aligned}\nabla \times \bar{\mathbf{V}} &= p_0^{-1}\nabla \times \int_0^{p_0} \mathbf{V}\, dp + p_0^{-1}\bar{\mathbf{V}} \times \nabla p_0 \\ &= p_0^{-1}\int_0^{p_0} \nabla_p \times \mathbf{V}\, dp - p_0^{-1}\mathbf{V}_0 \times \nabla p_0 + p_0^{-1}\bar{\mathbf{V}} \times \nabla p_0.\end{aligned} \tag{74}$$

The derivatives of $\mathbf{V}$, with p held fixed can be related to those with θ held fixed, as follows

$$\nabla_p \times \mathbf{V} = \nabla_\theta \times \mathbf{V} + g^{-1}\rho^{-1}\frac{\partial \mathbf{V}}{\partial z} \times \nabla_\theta p.$$

Substituting this expression for $\nabla_p \times \mathbf{V}$ in the integral on the right-hand side of Eq. (74), we find

$$\nabla \times \bar{\mathbf{V}} = \bar{\zeta}\mathbf{K} + g^{-1}p_0^{-1}\int_0^{p_0} \rho^{-1}\frac{\partial \mathbf{V}}{\partial z} \times \nabla_\theta p\, dp - p_0^{-1}\mathbf{V}_0 \times \nabla p_0 + p_0^{-1}\bar{\mathbf{V}} \times \nabla p_0. \tag{75}$$

For purposes of estimating the last three terms on the right-hand side of Eq. (75), we introduce two provisional approximations, provisional in the sense that, if they later lead to a strong inequality between the last three terms and the single term on the left-hand side, they are justified a posteriori. Tentatively replacing ∇p_0 in the last two terms by the corresponding geostrophic pressure gradient, and substituting for $\nabla_\theta p$ in the integral on the right-hand side the value given by the thermal wind equation (Eq. (68)), Eq. (75) then reduces to

$$\mathbf{K} \cdot \nabla \times \bar{\mathbf{V}} = \bar{\zeta} - \lambda p_0^{-1}\int_0^{p_0}\left(g^{-1}\theta\frac{\partial z}{\partial \theta}\right)\left(\frac{\partial \mathbf{V}}{\partial z} \cdot \frac{\partial \mathbf{V}}{\partial z}\right) dp + \lambda c_n^{-2}\mathbf{V}_0 \cdot \mathbf{V}_0 - \lambda c_n^{-2}\mathbf{V}_0 \cdot \bar{\mathbf{V}}, \tag{76}$$

where c_n is the Newtonian speed of sound at sea level. Now, because the Mach number of atmospheric flow is of the order of 0.1, the last two terms on the right-hand side of Eq. (76) are two orders of magnitude less than λ. On the other hand, it is observed that the range of variability of the left-hand side is of the *same* order of magnitude as λ. The last two terms are therefore less than the instrumental error in measuring $\mathbf{K} \cdot \nabla \times \bar{\mathbf{V}}$ directly, and can be omitted. In exactly the same way, it can be shown that the integral on the right-hand side of Eq. (76) is negligible, for its integrand is of the same order of magnitude as the square of the Mach number. The conclusion, therefore, is that, for all intents and purposes,

$$\bar{\zeta} = \mathbf{K} \cdot \nabla \times \overline{\mathbf{V}}. \tag{77}$$

Inasmuch as it reduces the number of dependent variables by one, this is an extremely important result.

A SIMPLE ELIMINATION SCHEME

5.03. To indicate how the remaining eliminations can be carried out, let us suppose that we are dealing with flow over perfectly flat terrain, in which case Eq. (73) can be written as

$$\frac{\partial \bar{\zeta}}{\partial t} + \tau \overline{\mathbf{V}} \cdot \nabla \bar{\zeta} + \beta \bar{v} - p_0^{-1}[(\tau - 1)\bar{\zeta} + \zeta_0 + \lambda] \frac{\partial p_0}{\partial t} = 0. \tag{78}$$

Now, the bracketed factor in the last term of Eq. (78) has the same order of magnitude as λ, whereas the remaining factor—the percentage local change in sea level pressure—is observed to be of the order of one percent per day. Thus the last term of Eq. (78) is at least one and possibly two orders of magnitude less than (say) the second term, whose range of variability is about λ per day. Without serious loss of accuracy, it might therefore be assumed that the large-scale disturbances are governed by the simple equation

$$\frac{\partial \bar{\zeta}}{\partial t} + \tau \overline{\mathbf{V}} \cdot \nabla \bar{\zeta} + \beta \bar{v} = 0. \tag{79}$$

This equation, which can also be regarded as a special form of the Charney-Eliassen equation (Eq. (19)), simply states that the absolute vorticity of the mean flow is conserved to within a fair degree of accuracy under actual conditions, and is exactly conserved if τ is unity. Since τ is generally no greater than 1.3, one would be tempted to regard it as exactly 1.0. However, the actual value of τ has an important effect on the speed of wave disturbances, as will be evident later.

5.04. Although the mean vorticity can be expressed in terms of other variables by the use of Eq. (77), Eq. (79) still contains two dependent variables, namely, the components of mean horizontal velocity. At this point, according to the original plan of development, the velocity would be replaced by the geostrophic velocity. This would have the twofold effect of excluding the solutions corresponding to external gravity waves and of expressing both velocity components in terms of derivatives of a single variable. An alternative (but exactly equivalent) scheme can be developed around the continuity equation. Noting that the local variation of sea level pressure or—which is the same thing—the integrated momentum divergence is quite small, we may regard Eq. (64) as a condition for the existence of a stream function

$$\nabla \cdot p_0 \overline{\mathbf{V}} = 0$$

from which

$$p_0 \nabla \cdot \overline{\mathbf{V}} + \overline{\mathbf{V}} \cdot \nabla p_0 = 0.$$

However, because the winds are quasi-geostrophic,

$$p_0 \nabla \cdot \overline{\mathbf{V}} + \mathbf{K} \cdot \lambda \rho_0 \overline{\mathbf{V}} \times \mathbf{V}_0 = 0.$$

It has been stipulated that the wind direction does not vary with height, so that $\overline{\mathbf{V}} \times \mathbf{V}_0$ vanishes and

$$\nabla \cdot \overline{\mathbf{V}} = 0.$$

This condition is evidently satisfied if

$$\overline{\mathbf{V}} = \mathbf{K} \times \nabla \psi.$$

where ψ is the stream function. The mean vorticity then takes the form

$$\bar{\zeta} = \nabla^2\psi.$$

Substituting these results into Eq. (79), we finally obtain an equation which involves only one dependent variable ψ.

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \tau J(\psi, \nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0. \tag{80}$$

Analytic solutions of this nonlinear equation have been studied by Craig (1945), Neamtan (1946), Thompson (1948) and Machta (1949), and numerical methods for solving it have been developed by Charney, Fjörtoft and von Neumann (1950).

5.05. It should be noted in passing that the stream function is determined only to within an arbitrary constant of integration. However, because it is sufficient to know only the derivatives of ψ in order to regenerate the initial conditions, it is not really necessary to determine the arbitrary constant. Charney has skirted this difficulty by applying Eq. (79) to conditions at the "equivalent-barotropic level," so his "stream function" is the height of a surface of constant pressure located near that level.

5.06. The few solutions of Eq. (80) that have actually been constructed are in good qualitative accord with what is observed, correctly predicting the general direction and speed of large-scale disturbances. Quite aside from the quality of numerical results, however, there are rather obvious philosophical objections to the theory on which they are based. For example, some meteorologists have complained that, because this simple theory does not afford any mechanism for creating vorticity, it cannot provide for the development of "new" disturbances. In short, if one accepts the theory, one must simply accept the existence of already developed disturbances without regard to their origin. Two remarks should be attached to this viewpoint. In the first place, many of the developments which the meteorologist regards as "new" may, in fact, be due to pure dispersion effects in an essentially barotropic medium. In the second report referred to in the foreword some evidence will be advanced to support this possibility. The second point is that even the simple theory summarized in Eq. (79) does not preclude advection of thermal vorticity, by which vorticity can be "created" *within a fixed level.* This is reflected in the fact that τ is in general different from unity, whence the theory does not require complete conservation of absolute vorticity. This question is largely a matter of conjecture, however, and for this reason — if only to meet some of the fundamental objections to a theory of complete vorticity conservation — a more general theory will be developed around Eq. (73). The latter, of course, contains several mechanisms by which vorticity can be generated, namely, by the divergence associated with large-scale flow over perfectly flat terrain, by the divergence enforced by irregular terrain, and by the advection of thermal vorticity.

INTRODUCTION OF THE FILTERING APPROXIMATION

5.07. Returning to the problem of eliminating all but one of the variables in Eq. (73), there are evidently two alternative methods of approach. As outlined in paragraph 5.04, one possibility is to derive an equation which applies to a single *integrated* variable and to regard the basic problem as one of predicting that variable, starting with its known initial values. Because it involves no qualitative interpretation, this approach is most satisfying to one's mathematical instincts. Having solved the problem in that form, however, we should then be faced with the practical difficulty of interpreting the solution, to get a rough idea of the flow at some given reference level. The other alternative, which is the one adopted by Charney and Eliassen, is to interpret the mean vorticity equation as applying to the actually observed motions at the "equivalent barotropic level," i.e., the level at which the observed wind speed equals the speed of the density-weighted mean wind. Since it apparently makes no difference where the burden of interpretation is placed the latter point of view will be adopted, because there are other advantages in doing so. From this point onward Eq. (73) will be treated as if it applied to the flow observed on a surface of constant pressure, located somewhere near the "equivalent-barotropic level"

$$p_0\left(\frac{\partial \zeta}{\partial t} + \tau \mathbf{V} \cdot \nabla \zeta + \beta v\right) - [(\tau - 1)\zeta + Z_0]\frac{\partial p_0}{\partial t} + g\rho_0 Z_0 \mathbf{V}_0 \cdot \nabla h = 0. \tag{81}$$

Since we shall deal exclusively with conditions at the equivalent-barotropic level from now on, the variables at that level will be denoted by unbarred quantities. As a consequence of Eq. (77), A_ζ is very nearly equal to A_v,† and the vorticity at the reference level is the curl of the horizontal velocity at that same level. Thus Eq. (81), regarded as a quasi-linear equation, involves the derivatives of only three variables, u, v and p_0.

5.08. In the course of developing the scale theory, it was shown that to introduce the geostrophic approximation into the vorticity equation, at the same time requiring that the motions be quasi-horizontal, is equivalent to introducing the filtering approximation into a single reduced equation. Since the vertical coordinate has been integrated out, the mean vorticity equation already applies to purely horizontal motion, however fictitious it might be, and it remains only to replace the "true" velocity by the corresponding geostrophic velocity in Eq. (81)

$$\mathbf{V} = \mathbf{K} \times g\lambda^{-1}\nabla z.$$

Because the quasi-Lagrangian variables will not be discussed further, z will be used to represent the height of a constant pressure surface. The "geostrophic" vorticity, expressed in terms of the contour height, is then

$$\begin{aligned}\mathbf{K} \cdot \nabla \times \mathbf{V} &= \nabla \cdot g\lambda^{-1}\nabla z \\ &= g\lambda^{-1}\nabla^2 z + \beta\lambda^{-1}u.\end{aligned}$$

Now, the order of magnitude of the second term on the right-hand side is given by the number of times an imaginary point, traveling at a speed of $2\pi u$ along the equator, will completely circle the earth in one day. The left-hand side, of course, is of the order of ten radians per day, and the second term on the right-hand side is at least one and generally two orders of magnitude less than the first

$$\zeta \simeq g\lambda^{-1}\nabla^2 z.$$

In similar fashion, it can be shown that λ may be regarded as a constant with respect to all other differentiations required by Eq. (81). Substituting the above expressions into the vorticity Eq. (81),

$$\frac{\partial}{\partial t}\nabla^2 z + \tau g\lambda^{-1} J(z, \nabla^2 z) + \beta\frac{\partial z}{\partial x} - \lambda g^{-1} p_0^{-1}[\lambda + \zeta_0 + (\tau - 1)\zeta]\frac{\partial p_0}{\partial t} + \lambda c_n^{-2}(\lambda + \zeta_0)\mathbf{V}_0 \cdot \nabla h = 0. \tag{82}$$

Again regarding it as quasi-linear, this equation still contains derivatives of two dependent variables, z and p_0. Although there is obviously some relation between local changes in sea-level pressure and changes in the height of a surface of constant pressure at some higher level, it evidently requires information which the integrated equations cannot furnish, namely, a knowledge of the density changes throughout the layer below the equivalent-barotropic level. In this connection, it is perhaps more fortunate than significant that derivatives of p_0 enter into the quasi-linear form of Eq. (82) only in the fourth term, which, as was pointed out earlier, is much smaller than (say) the second term and might even be omitted altogether. With the assurance that the final result cannot be seriously affected by doing so, we therefore approximate the fourth term of Eq. (82), expressing it in terms of derivatives of the contour height z.

† This carries the further implication that $\tau = 1 + \overline{A_v^2}$, whence $\tau \geq 1$.

5.09. The connection between the sea-level pressure tendency and the height tendency at the equivalent-barotropic level will be provided simply by stating that the pressure disturbances at different levels are "rigidly coupled together," in that the horizontal direction of movement and speed of the large-scale disturbances are independent of height. This appears to be a very reasonable assumption, for it is observed that large-scale disturbances maintain their identity over long periods of time, traveling along for several days without essential change in vertical structure. Stated in mathematical terms, the relation between p_0 and p is then

$$\frac{\partial p_0}{\partial s}\left(\frac{\partial p}{\partial t}\right)_z = \left(\frac{\partial p}{\partial s}\right)_z \frac{\partial p_0}{\partial t},$$

where s is the coordinate along curves which are, let us say, locally orthogonal to the isobars on a surface of constant height. Making use of the condition for geostrophic equilibrium,

$$\frac{\partial p_0}{\partial t} = \rho_0 \rho^{-1} V_0 V^{-1} \left(\frac{\partial p}{\partial t}\right)_z$$

and finally, introducing the hydrostatic condition,

$$\frac{\partial p_0}{\partial t} = g\rho_0 V_0 V^{-1} \frac{\partial z}{\partial t}.$$

Substituting this result into Eq. (82) yields an equation which, in its quasi-linear form, involves only one dependent variable.

$$\frac{\partial}{\partial t}\nabla^2 z + \tau g \lambda^{-1} J(z, \nabla^2 z) + \beta \frac{\partial z}{\partial x} - \lambda c_n^{-2} V_0 V^{-1}[\lambda + \zeta_0 + (\tau - 1)\zeta] \frac{\partial z}{\partial t}$$
$$+ \lambda c_n^{-2} V_0 V^{-1}[\lambda + \zeta_0]\mathbf{V} \cdot \nabla h = 0. \tag{83}$$

This is the prognostic equation, whose solutions may be regarded as predictions of the mean flow conditions, integrated pressure-wise throughout the entire depth of the atmosphere, or of the flow actually occurring at the equivalent-barotropic level.

5.10. At this juncture it is appropriate to review the development of Eq. (83) with regard to the previously discussed difficulties of the general problem. In the first place, the prognostic equation refers to the height of a constant pressure surface (or, in other words, the pressure at a surface of constant height). Of all the physical variables, this is the one least sensitive to disturbances whose scale is smaller than the mesh size of the observation network. Moreover, the equation applies to motions which are representative of the vertically integrated mean motions, in that the equivalent-barotropic level is the "center of momentum" of the atmosphere. It will later be shown that the solutions of Eq. (83) may be interpreted as horizontally integrated mean values of the initial data, whence they evidently satisfy all of the original requirements on the representativeness of "statistics" formed from incomplete observations of the state of the atmosphere. With regard to the difficulties discussed in Section 2.00, it should be noted that Eq. (83) is essentially a vorticity equation, which expresses the local time derivative of vorticity (the Laplacian derivative of z) in terms of computable quantities—i.e., computable in the sense that the local derivative is not invariably given as the small difference between individually large terms. The other major difficulty, that of attaining time-resolution sufficient to continue solutions corresponding to sound and gravity waves, has been met by introducing the filtering approximation to exclude the "high-speed" solutions. In this connection, it is relevant to note that Eq. (83) is of the same general form as Eq. (53), which was obtained by introducing the approximation $c \ll c_m$ directly into an equation containing all types of motion. Finally, it is important to realize that the prognostic equation deals with a species of "two-dimensional" motion. That is to say, integrating out the vertical coordinate obviates the difficulty of computing the vertical component of velocity from observed initial data. In summary, Eq. (83) appears to have none of the obviously undesirable features outlined in the previous discussion of known difficulties.

5.11. It should also be recognized that Eq. (83) differs only in minor respects from the Charney-Eliassen equation (22). The most significant difference, perhaps, lies in the fact that Charney and Eliassen later assumed that the slowly varying function τ is equal to one. This has the general effect of making the eastward progress of the disturbances too slow, by an amount $(\tau - 1)U$. It has frequently been observed, of course, that the eastward movements predicted by the Rossby "trough-formula" are too small (Namias and Clapp, (1944)). The real point, however, is that Eq. (83) was arrived at by a different and, in many ways, more attractive route. The fact that the two independently derived equations do agree is simply added evidence that both are essentially correct.

The next concern, of course, is to extract observable consequences from Eq. (83) or, in other words, to solve it subject to given initial conditions. Inasmuch as the method of solution presented here is rather unusual (at any rate quite different from that proposed by Charney, Fjortoft, and von Neumann), considerable attention will be given to the details of the method, as well as to details of the final solution.

6.00 METHODS FOR SOLVING THE PROGNOSTIC EQUATION

6.01. The simplest nontrivial form of Eq. (83) will be solved as a preliminary to the discussion of methods for solving the general prognostic equation. In particular, we shall consider small deviation from a uniform west-east flow over perfectly flat terrain. For the sake of simplicity, it will be required also that the velocity disturbance be independent of y, in which case the linearized prognostic equation takes on the one-dimensional form,

$$\frac{\partial^3 z}{\partial x^2\,\partial t} + \tau U\frac{\partial^3 z}{\partial x^3} + \beta\frac{\partial z}{\partial x} - \lambda^2 c_n^{-2}\frac{\partial z}{\partial t} = 0. \qquad (84)$$

In fact, as has been noted several times earlier, the fourth term of this equation is, in general, much less than either of the first two, whence the essential features of the large-scale motions will not be lost by restricting attention to the simple equation for Rossby waves

$$\frac{\partial^2 z}{\partial x\,\partial t} + \tau U\frac{\partial^2 z}{\partial x^2} + \beta z = 0. \qquad (85)$$

This is an equation of the telegrapher's type, and the boundary and initial conditions necessary and sufficient to determine its solutions are well known. What is evidently required are the data along the semi-infinite line $t = 0$, representing the initial conditions, taken together with the data along a line $t = mx + b\,(m \neq 0)$ in the (x, t) plane. The external constraints on the problem, on the other hand, are such that information is provided only up to $t = 0$. On the face of it, therefore, the solution is not uniquely determined by the information actually available, namely, the initial data at $t = 0$. In passing, however, it should be noted that, in the limiting case when m approaches zero, the curves along which the boundary data must be known do approach the infinite line $t = 0$.

6.02. Despite the apparent indeterminacy involved in regarding the solution of Eq. (85) as an initial value problem, we continue with the formal development of solutions which satisfy previously specified initial conditions. A fundamental set of wave solutions is given by the function

$$z = a\cos\alpha(\xi - x + ct), \qquad (86)$$

where $c = \tau U - \beta\alpha^{-2}$. Since both the amplitude factor a and the phase angle ξ are arbitrary, the functions of Eq. (86) form a complete set of solutions. The solution of the initial value problem is regarded as the superposition of a continuous spectrum of such wave functions, corresponding to a continuous sequence of wave numbers α. Since Eq. (85) is linear, we may sum up the solutions for any or all values of the wave number to obtain more general solutions, later adjusting the arbitrary constants a and ξ to fit the initial values. At this point, of course, it would be quite natural to pass over immediately to the Fourier integral

$$\frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} z(\xi, 0) \cos \alpha\left(\xi - x + \tau U t - \frac{\beta t}{\alpha^2}\right) d\xi$$

which is simply a sum of solutions of type (86), ostensibly satisfying the initial conditions. That is to say, when $t = 0$, the above integral appears to reduce to Fourier's representation of the function $z(x, 0)$ on an infinite interval. The character of this integral undergoes a complete change, however, when t is set exactly equal to zero. We shall therefore resort to a device for eliminating the irregular behavior of solutions near $t = 0$, first considering trigonometric solutions whose wavelengths are submultiples of a fixed length L.

$$z = \begin{cases} a_n \sin \dfrac{n\pi}{L}(x - c_n t) \\[2ex] b_n \cos \dfrac{n\pi}{L}(x - c_n t) \end{cases}$$

where $c_n = \tau U - \beta L^2 \pi^{-2} n^{-2}$. The complete solution of Eq. (85), according to the principle of superposition, is then

$$z = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}(x - c_n t) + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi}{L}(x - c_n t). \tag{87}$$

It remains to determine the constants a_n and b_n. Now, instead of determining the arbitrary constants directly from the initial values $z(x, 0) = F(x)$, they will be fixed by inverting the equation for $\partial z / \partial x$ at $t = 0$.

$$\frac{\pi}{L}\sum_{n=1}^{\infty} n a_n \cos \frac{n\pi x}{L} - \frac{\pi}{L}\sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{L} = \frac{dF}{dx}.$$

Making use of the orthogonality properties of the trigonometric functions,

$$a_n = \frac{2}{n\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dF}{d\xi} \cos \frac{n\pi\xi}{L}\, d\xi$$

$$b_n = -\frac{2}{n\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dF}{d\xi} \sin \frac{n\pi\xi}{L}\, d\xi.$$

Substituting these expressions for a_n and b_n in Eq. (87) and interchanging the order of summation and integration, gives the Fourier series solution expanded on the interval $\left(-\frac{L}{2}, \frac{L}{2}\right)$.

$$z(x, t) = 2\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dF}{d\xi}\left[\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi}{L}(x - \xi - c_n t)\right] d\xi.$$

Finally, because there is no natural periodicity, we let L become infinitely large. Passing directly to the limit, yields the Fourier integral solution of Eq. (85).

$$z(x, t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{dF}{d\xi}\, d\xi \int_0^\infty \frac{1}{\alpha} \sin \alpha(x - \xi - c_\alpha t)\, d\alpha.$$

"equivalent barotropic level," i.e., the level at which the observed wind speed equals the speed of the density-weighted mean wind. Since it apparently makes no difference where the burden of interpretation is placed the latter point of view will be adopted, because there are other advantages in doing so. From this point onward Eq. (73) will be treated as if it applied to the flow observed on a surface of constant pressure, located somewhere near the "equivalent-barotropic level"

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This is an equation of the telegrapher's type, and the boundary and initial conditions necessary and sufficient to determine its solutions are well known. What is evidently required are the data along the semi-infinite line $t = 0$, representing the initial conditions, taken together with the data along a line $t = mx + b (m \neq 0)$ in the (x, t) plane. The external constraints on the problem, on the other hand, are such that information is provided only up to $t = 0$. On the face of it, therefore, the solution is not uniquely determined by the information actually available, namely, the initial data at $t = 0$. In passing, however, it should be noted that, in the limiting case when m approaches zero, the curves along which the boundary data must be known do approach the infinite line $t = 0$.

6.02. Despite the apparent indeterminacy involved in regarding the solution of Eq. (85) as an initial value problem, we continue with the formal development of solutions which satisfy previously specified initial conditions. A fundamental set of wave solutions is given by the function

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where $c = \tau U - \beta\alpha^{-2}$. Since both the amplitude factor a and the phase angle ξ are arbitrary, the functions of Eq. (86) form a complete set of solutions. The solution of the initial value problem is regarded as the superposition of a continuous spectrum of such wave functions, corresponding to a continuous sequence of wave numbers α. Since Eq. (85) is linear, we may sum up the solutions for any or all values of the wave number to obtain more general solutions, later adjusting the arbitrary constants a and ξ to fit the initial values. At this point, of course, it would be quite natural to pass over immediately to the Fourier integral

$$\frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^{\infty} z(\xi, 0)\cos\alpha\left(\xi - x + \tau U t - \frac{\beta t}{\alpha^2}\right) d\xi$$

which is simply a sum of solutions of type (86), ostensibly satisfying the initial conditions. That is to say, when $t = 0$, the above integral appears to reduce to Fourier's representation of the function $z(x, 0)$ on an infinite interval. The character of this integral undergoes a complete change, however, when t is set exactly equal to zero. We shall therefore resort to a device for eliminating the irregular behavior of solutions near $t = 0$, first considering trigonometric solutions whose wavelengths are submultiples of a fixed length L.

$$z = \begin{cases} a_n \sin \dfrac{n\pi}{L}(x - c_n t) \\ b_n \cos \dfrac{n\pi}{L}(x - c_n t) \end{cases}$$

where $c_n = \tau U - \beta L^2 \pi^{-2} n^{-2}$. The complete solution of Eq. (85), according to the principle of superposition, is then

$$z = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}(x - c_n t) + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi}{L}(x - c_n t). \tag{87}$$

It remains to determine the constants a_n and b_n. Now, instead of determining the arbitrary constants directly from the initial values $z(x, 0) = F(x)$, they will be fixed by inverting the equation for $\partial z/\partial x$ at $t = 0$.

$$\frac{\pi}{L}\sum_{n=1}^{\infty} n a_n \cos\frac{n\pi x}{L} - \frac{\pi}{L}\sum_{n=1}^{\infty} n b_n \sin\frac{n\pi x}{L} = \frac{dF}{dx}.$$

Making use of the orthogonality properties of the trigonometric functions,

$$a_n = \frac{2}{n\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dF}{d\xi}\cos\frac{n\pi\xi}{L}\, d\xi$$

$$b_n = -\frac{2}{n\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dF}{d\xi}\sin\frac{n\pi\xi}{L}\, d\xi.$$

Substituting these expressions for a_n and b_n in Eq. (87) and interchanging the order of summation and integration, gives the Fourier series solution expanded on the interval $\left(-\frac{L}{2}, \frac{L}{2}\right)$.

$$z(x, t) = 2\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dF}{d\xi}\left[\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi}{L}(x - \xi - c_n t)\right] d\xi.$$

Finally, because there is no natural periodicity, we let L become infinitely large. Passing directly to the limit, yields the Fourier integral solution of Eq. (85).

$$z(x, t) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{dF}{d\xi}\, d\xi \int_0^\infty \frac{1}{\alpha}\sin\alpha(x - \xi - c_\alpha t)\, d\alpha.$$

Up to this point the location of the origin has been left unspecified, and it may therefore be shifted in such a way that the dependent variable always applies at the point $(\tau Ut, t)$ to the east of the origin

$$z(\tau Ut, t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{dF}{d\xi} J(\xi)\, d\xi, \tag{88}$$

where the "kernel function" J is

$$\int_0^{\infty} \sin\left(\frac{\beta t}{\alpha} - \alpha\xi\right)\frac{d\alpha}{\alpha}\cdot$$

As it turns out, the kernel J is a well-known (Watson, (1922)) integral representation of the Bessel function of order zero with real argument.

$$J = \begin{cases} \frac{\pi}{2} J_0(2\sqrt{-\beta\xi t}) & (\xi \leqslant 0) \\ 0 & (\xi > 0). \end{cases}$$

Introducing these values of J into Eq. (88),

$$z(\tau Ut, t) = \int_{-\infty}^{0} \frac{dF}{d\xi} J_0(2\sqrt{-\beta\xi t})\, d\xi.$$

Finally, integrating by parts to obtain z directly in terms of its initial values,

$$z(\tau Ut, t) = z(0, 0) - \sqrt{\beta t}\int_{-\infty}^{0} z(\xi, 0)\, \frac{J_1(2\sqrt{-\beta\xi t})}{\sqrt{-\xi}}\, d\xi. \tag{89}$$

It is quite clear that this solution of Eq. (85) does satisfy the initial conditions.

6.03. It might be added that the foregoing procedure is exactly equivalent to breaking down the initial distribution of contour height into its Fourier spectrum, moving each wave component along at the phase speed corresponding to its wavelength, and finally superposing the displaced waves to obtain the distribution at some later time. The advantage of the Fourier series or Fourier integral methods is simply that they perform all those operations simultaneously and in a single step.

6.04 Viewed in the light of Eq (89), the indeterminacy of the initial value problem is only apparent, for the solution is completely determined by the values of z at some arbitrarily chosen initial moment. Although this demonstrates the uniqueness of suitably continuous solutions, there still remains the difficulty of calculating the height change as a semi-infinite integral—a process which, in view of the fact that $z(\xi, 0)$ is generally not analytic, must be carried out numerically by Simpson's rule or some other such method. We shall, therefore, investigate some of the properties of the integral on the right-hand side of Eq. (89), concentrating attention on the function

$$\frac{J_1(2\sqrt{-\beta\xi t})}{\sqrt{-\xi}} \qquad (\xi \leqslant 0).$$

This function plays the role of a Green's function or influence function, in that it measures the influence which a unit point disturbance, situated at $(\xi, 0)$, has on the local change in contour height at the origin. As shown in Fig. 2, the Green's function "dies out" rapidly as one proceeds away from the origin, approaching zero as a limit. It decreases so rapidly, in fact, that the integral can be truncated at some fairly great distance from the origin without seriously affecting the accuracy of the result. This suggests that the "effective domain of dependence"—i.e., the region over which data are required to compute the solution with a fixed degree of accuracy—is not infinite, but has a finite radius which depends on the period of the forecast and on

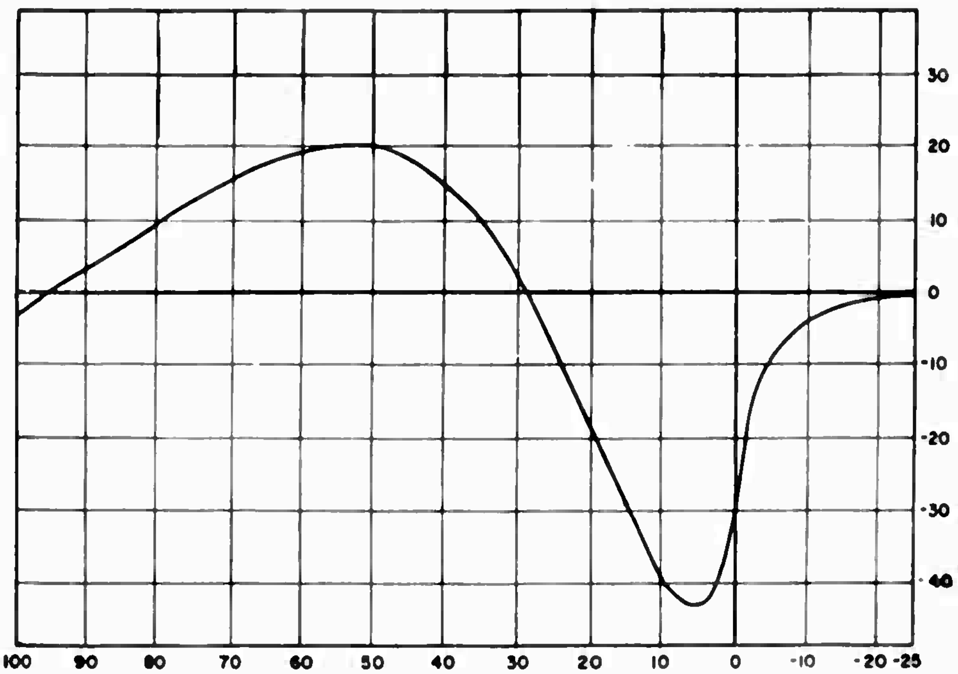


Fig. 2. The one-dimensional Green's function: (βt = 0.643).

the level of accuracy desired. It is significant that the radius of the effective domain of dependence also depends on β, the latitudinal variation of the Coriolis parameter. This result is not wholly unexpected, because β is a measure of the "restoring force" in the oscillating system and therefore controls the rate at which disturbances are propagated. It should be noted that this rapid decay of dependence on initial conditions remote from the origin is evidently due to destructive interference at great distances and is a property peculiar to dispersive wave systems.

6.05. As will be shown, it is typical of all solutions of the linearized prognostic equation that the local change in contour height can be expressed as an infinite line or surface integral, whose integrand can be naturally separated into two factors. The first factor is simply the initial value of the contour height. The second is essentially a Green's function, a function which is analytic, independent of the initial conditions, and which measures the influence of disturbances remote from the origin on the local change in contour height at the origin. In general, the Green's function for this initial value problem dies out quite rapidly as one proceeds away from the origin in any direction, limiting the "effective domain of dependence" to a radius which depends on the period of the forecast, a measure of the "restoring force" β, and the degree of accuracy required.

THE CHARNEY–VON NEUMANN METHOD

6.06. The more general problem of integrating the nonlinear Eq. (83), for lack of sufficiently powerful exact methods of analysis, must be solved by numerical methods. This question has also been discussed by Charney, Fjörtoft and von Neumann (1950), who have proposed a rather straightforward iterative scheme for solving a prognostic equation of the same general type as Eq. (83). The key to their method lies in regarding Eq. (83) as a nonhomogeneous linear equation in which the dependent variable is the height tendency. That is to say, all those terms in which local time derivatives do not appear explicitly are regarded as nonhomogeneous terms in the sense that they can be computed from the initial data and might be considered known for a short time after the initial moment. With this interpretation, Eq. (83) then takes the form

$$\nabla^2 \left(\frac{\partial z}{\partial t}\right)_i - \lambda c_n^{-2} V_0 V^{-1} [\lambda + \zeta_0 + (\tau - 1)\zeta] \left(\frac{\partial z}{\partial t}\right)_i = F_i(x, y), \qquad (90)$$

where $F_i(x, y)$ is the value of

$$-\left[\tau g\lambda^{-1}J(z, \nabla^2 z) + \beta\frac{\partial z}{\partial x} + \lambda c_n^{-2}V_0V^{-1}(\lambda + \zeta_0)\mathbf{V}\cdot\nabla h\right]$$

computed at the ith stage of the iteration. The coefficient of the second term of Eq. (90) will be treated as a slowly varying function. It has already been pointed out that the second term of Eq. (90) is much less than the first, whence we are quite safe in approximating its coefficient. Equation (90) becomes a linear equation with constant coefficients, the large nonlinear terms of Eq. (83) being lumped together in $F_i(x, y)$.

$$\nabla^2\left(\frac{\partial z}{\partial t}\right)_i - \mu^2\left(\frac{\partial z}{\partial t}\right)_i = F_i(x, y), \tag{91}$$

where $\mu^2 = \lambda^2 c_n^{-2}V_0V^{-1}$.

This is an equation of the Poisson type. The general properties of its solutions are well known and numerical methods for actually computing its solutions are developed to a high degree.

6.07. To summarize the details of the Charney-von Neumann method, let us consider conditions at a single, arbitrarily chosen initial moment. Since all derivatives with respect to the horizontal coordinates can be computed from the initial data, it is possible to calculate F_1 as a function of x and y. The next step is to compute the instantaneous height tendency at the initial moment by inverting Eq. (91). There are a variety of ways in which the inversion can be carried out, among them the "relaxation" methods developed by Southwell (1946). Charney and von Neumann have chosen a variant of the latter. There are certain features of the relaxation method which make it difficult to insure rapid convergence. This stems from the fact that the solution of an elliptic type equation, for which the relaxation methods are designed, requires previous knowledge of the unknown function or its normal derivative on some closed curve. To meet this condition, Charney and von Neumann have been forced to assign artificial values of the height tendency around the boundary of a rather large region, surrounding the point (or area) at which the computed tendencies are to apply. They have assumed, in fact, that the height tendency vanishes at the boundaries, evidently reasoning that the solution in the central portion of the region is very nearly independent of conditions on a geographically remote boundary and, further, that the distribution of the height tendency around the boundary is essentially random or oscillatory. We shall return to this question later.

6.08. Granting that it is possible to compute the instantaneous height tendency in the manner outlined above, one can then extrapolate from the initial values of contour height to predict its value *at a short time later*. In this way the information available at the initial moment has been completely regenerated at a later time, making it possible to compute F_2, again to invert Eq. (91) and otherwise to repeat this process over and over again until the aggregate of short time intervals adds up to the required forecast period. The predictions made by this method, as presented in "Numerical Integration of the Barotropic Vorticity Equation" (Charney, Fjörtoft and von Neumann (1950)), are of course very encouraging as scientific results but are still not sufficiently accurate for practical purposes.

6.09. The drawback to the Charney-von Neumann method for computing the instantaneous height tendency lies in the fact that an inordinately large region of integration is required to assure convergence on the true solution. A rough measure of the error in assigning arbitrary boundary values is the value of the actual height tendency, integrated around the boundary curve, multiplied by a factor which weights the dependence of the solution on conditions at the boundary relative to its dependence on conditions near the origin. If the inversion were carried out by the method outlined in paragraph 6.02, for example, the relative weight to be attached to the boundary values would be the value of the Green's function on the boundary curve. Thus there are two elements that enter into the error estimate: first, the mean value of the actual height tendency on the boundary and, second, the radius of the "natural" domain of dependence *as fixed by the properties of the governing differential equation.* Taking up the first of these considerations, there is no way of assuring beforehand that the atmosphere will not conspire against the forecaster by producing boundary values of predominantly the same sign. In fact, if one is unfortunate enough to choose dimensions of the region of integration which are comparable with the characteristic wavelength of the large-scale disturbances, that state of affairs would occur quite frequently.

6.10. Turning to the second consideration, it is extremely important to realize that Eq. (91) yields no direct information about the manner in which large-scale disturbances are propagated. It is quite possible, of course, that this information is implicitly contained in the form of $F_i(x, y)$, but, even so, $F(x, y)$ has a distinctive form only after the initial stage of the iteration. That is to say, one can imagine any number of physical systems which are governed by an equation of the same form as Eq. (91) and for which the corresponding *initial* distribution $F_1(x, y)$ is the same. In short, the radius of the "natural" domain of dependence, as fixed by Eq. (91), is not determined by the rate at which disturbances are actually propagated. This conclusion is hard to reconcile with the results of paragraph 6.04, which indicate that the effective domain of dependence is actually quite small, owing to the peculiar way in which the phase speed depends on wavelength. The reason for the disparity is clear when one realizes that the Charney-von Neumann method cannot take advantage of dispersion effects—namely, destructive interference at great distances—simply because Eq. (91) does not explicitly contain the actual mechanism of wave propagation.

6.11. Another, and perhaps more satisfactory way of pointing up the shortcomings of the Charney-von Neumann method is to discuss the problem of inverting Eq. (91) from the standpoint of Green's method. The Green's function for the problem is a solution of the homogeneous part of Eq. (91) with a logarithmic singularity at the point at which the solution is desired. The Green's function is thus defined as a solution of

$$\nabla^2 G - \mu^2 G = 0.$$

We may also require that G depend only on the radial distance r from the singularity, whence

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} - \mu^2 G = 0.$$

The solutions of this equation are Bessel functions of order zero with imaginary argument. In particular, a solution which has a logarithmic singularity at $r = 0$ is $K_0(\mu r)$, whose asymptotic behavior is given by

$$K_0(\mu r) \simeq r^{-1/2}\, e^{-\mu r} \qquad (r \text{ large}).$$

Now the fact of the matter is that μ is small, so the Green's function does not decrease rapidly as one proceeds away from the origin. Accordingly, the natural domain of dependence for Eq. (91) is really quite large. Finally, it should be noted that Charney, Fjörtoft and von Neumann start out by assuming that $\mu = 0$. In this case the Green's function reduces to the logarithmic potential, *which contains no physical parameters at all.* For these reasons, because there is no way of assuring that the mean value of the height tendency on the boundary curve will be small and because the domain of dependence for Eq. (91) is not limited by the rate at which disturbances are actually propagated, the Charney-von Neumann method offers no way of insuring rapid convergence.

A NEW METHOD FOR INTEGRATING THE PROGNOSTIC EQUATION

6.12. There is, however, a rather obvious and direct way out of this difficulty. As suggested in paragraph 6.10, the trouble stems from the fact that *all* terms of Eq. (83) not involving local time derivatives—

and, in particular, the term $\beta\, \partial z/\partial x$ representing the "restoring force" on the system—have been regarded as known nonhomogeneous terms. The upshot, of course, was that the resulting equation contained no direct reference to the actual mechanism of propagation. Rather than to regard such terms as wholly nonhomogeneous, we shall separate their coefficients into two component parts: first, the mean value of each coefficient, integrated over a considerable geographical extent of the initial data; and second, the deviation from that mean. The latter, of course, gives rise to nonlinear terms which are often too large to justify linearizing Eq. (83) completely. On the other hand, the nonlinear residue terms are small enough that they can be regarded as nonhomogeneous, in the sense that they can be computed from the initial data and might be considered known for a short period of time after the initial moment. We therefore regard those nonlinear residue terms, *but only those terms*, as known nonhomogeneous terms in a linear equation. With this interpretation, the prognostic equation (Eq. (83)) then takes the form

$$\frac{\partial}{\partial t}\nabla^2 z + \tau U \frac{\partial}{\partial x}\nabla^2 z + \beta \frac{\partial z}{\partial x} - \mu^2 \frac{\partial z}{\partial t} + \mu^2 U \frac{\partial h}{\partial x} = N_i(x, y) = -u' \frac{\partial}{\partial x}\nabla^2 z - v \frac{\partial}{\partial y}\nabla^2 z - \mu^2 v \frac{\partial h}{\partial y}, \qquad (92)$$

where U is the mean value of u, taken over a considerable area, and u' is the deviation of u from that mean. As before, we regard the coefficient of the fourth term of Eq. (83) as a slowly varying function. The equation we are dealing with is therefore a nonhomogeneous linear equation with constant coefficients.

6.13. The method proposed for solving the nonlinear prognostic equation is the following. Beginning with the initial data, we first compute N_1 as a function of the coordinates x and y. Next, tentatively supposing that it is possible, we solve Eq. (92), subject to known initial conditions, to obtain a solution which is valid in the neighborhood of $t = 0$. We then continue the solution analytically to predict the contour height a short time after the initial moment and, finally, having generated a new set of initial conditions, compute N_2. This completes the first cycle in an iteration process, which can be repeated indefinitely until the total forecast period has reached the desired length.

6.14. The success of the method outlined above evidently hinges on whether or not Eq. (92) can be solved and, once the solution is attained, whether or not its convergence is assured. With regard to the latter, the results of our previous analysis of the difficulties inherent in the Charney-von Neumann method would lead us to suspect that the domain of dependence for Eq. (92) is naturally limited by the rate at which disturbances are propagated, if only for the simple reason that Eq. (92) does contain the term $\beta\partial z/\partial x$ which represents the "restoring force." This conjecture is confirmed by the results of the next section of this report, in which we shall present solutions of Eq. (92), the linear nonhomogeneous form of the prognostic equation. It will be shown that the radius of the effective domain of dependence, which is determined by the behavior of the Green's function for Eq. (92), is quite small for values of t of the order of one day, and, further, that convergence is assured if the region of integration covers the effective domain of dependence.

7.00 SOLUTION OF THE LINEAR PROGNOSTIC EQUATION

7.01. As a result of previous discussion of methods for solving the nonlinear prognostic Eq. (83), we have been led to consider the properties of the linear Eq. (92), which, for later convenience, is written in the form

$$\frac{\partial}{\partial t}\nabla^2 z + \tau U \frac{\partial}{\partial x}\nabla^2 z + \beta \frac{\partial z}{\partial x} - \mu^2 \frac{\partial z}{\partial t} = N_i(x, y) + M(x, y), \qquad (93)$$

where $M(x, y) = -\mu^2 U\, \partial h/\partial x$. It is convenient to think of $N_i(x, y)$ as representing the effects of nonlinearity due to finite deviations from a uniform west-east flow, and to think of $M(x, y)$ as the effect of irregular terrain. In many—probably in most—cases, in fact, the amplitude of the large-scale disturbances is small enough that it is permissible to disregard the effects of nonlinearity. In that event, we may set $N_i(x, y)$ equal to zero. Similarly, there is some reason to believe that the effect of irregular terrain on the propagation of large-scale disturbances is negligible, in which case $M(x, y)$ might be set equal to zero. For the present, however, the possible effects of nonlinearity and irregular terrain will be left an open question.

7.02. Because Eq. (93) is linear, its solution may be expressed as the sum of the solutions of three separate equations. That is to say,

$$z = z_F(x, y, t) + z_N(x, y) + z_M(x, y),$$

where the functions z_F, z_M, and z_N are defined as solutions of the equations

$$\left(\frac{\partial}{\partial t}\nabla^2 + \tau U\frac{\partial}{\partial x}\nabla^2 + \beta\frac{\partial}{\partial x} - \mu^2\frac{\partial}{\partial t}\right)z_F = 0 \tag{94}$$

$$\left(\tau U\frac{\partial}{\partial x}\nabla^2 + \beta\frac{\partial}{\partial x}\right)z_N = N_i(x, y) \tag{95}$$

$$\left(\tau U\frac{\partial}{\partial x}\nabla^2 + \beta\frac{\partial}{\partial x}\right)z_M = M(x, y). \tag{96}$$

The physical interpretation of this subdivision of solutions is simply that the actual height distribution is the superposition of one system of free oscillations and two systems of stationary forced oscillations. In other words, since Eq. (94) is homogeneous, its solution z_F corresponds to free oscillations and, similarly, because Eqs. (95) and (96) are nonhomogeneous, their solutions z_N and z_M correspond to forced oscillations. The solution z_F, of course, is associated with the large-scale transient disturbances, with which we are primarily concerned. In contrast, z_M may be identified with the semistationary "trough" of pressure observed in the horizontal flow to the lee of the mountain ranges in the western United States. The fact that the pressure-amplitude of the "lee trough" is generally observed to be somewhat less than that of the large-scale traveling disturbances is added evidence that we might safely set $M(x, y)$ and z_M equal to zero at the outset. For obvious reasons, there is no clearcut physical interpretation of the fictitious (and ever-changing) "forced oscillation" due to the effects of nonlinearity.

THE SOLUTION FOR LARGE-SCALE TRANSIENT DISTURBANCES

7.03. With the foregoing background, we shall proceed directly to the solution of Eqs. (94) and (96), simply noting that Eq. (95) is of exactly the same form as Eq. (96) and can be solved in much the same way. Because the free oscillations corresponding to the large-scale transient disturbances are of greatest intrinsic interest from the standpoint of prediction (and because they almost completely mask out the "lee wave") we shall first consider Eq. (94). The first step is to reduce the equation to the simplest terms possible.

7.04. On the face of it, Eq. (94) contains derivatives of the third order with respect to x. By a change of independent variable, however, it can be reduced to one which contains terms of no higher than the second order with respect to x, without altering the form of the remaining terms. In particular, we shall adopt a system of coordinates moving at the speed τU in the x-direction, whence the new coordinate x' is given by

$$x' = x + \tau U t.$$

With this change of variable, Eq. (94) reduces to

$$\frac{\partial}{\partial t}\nabla^2 z + \beta^*\frac{\partial z}{\partial x'} - \mu^2\frac{\partial z}{\partial t} = 0, \tag{94a}$$

where $\beta^* = \beta + \tau U\mu^2$. We shall now drop the primes, bearing in mind that the origin of the coordinate system is traveling at a speed τU toward the east.

7.05. There is evidently a choice of methods to be followed in solving Eq. (94a). One alternative, which has already been explored in paragraph 6.02, is simply to develop a fundamental set of wave solutions with two arbitrary phase angles, two wave numbers, and an arbitrary amplitude, later adjusting the constants in an infinite sum of such solutions to fit any given initial conditions. In general, such a procedure would lead to a double Fourier series or to a double Fourier integral. The difficulty with this method is that the kernel functions, which are given as definite integrals or infinite sums over all values of the wave numbers, are either extremely difficult or overwhelmingly tedious to evaluate. In fact, in the last analysis, the difficulty is not so much one of obtaining solutions of Eq. (94a) as it is of satisfying the initial conditions. The Laplace transform methods, on the other hand, introduce the initial conditions explicitly at the very outset, shifting the burden of difficulty to obtaining solutions of the transformed equation and to carrying out the inverse transformation. For this reason, and because there is a fundamental difference between the ways in which time dependence and position dependence enter the equations, we shall next apply the Laplace transform to Eq. (94a), replacing t by s, the variable of the transform.

$$\Phi(x, y, s) = L\{\phi(x, y, t)\} = \int_0^\infty \phi(x, y, t)e^{-st}\,dt.$$

Interchanging the order of integration with respect to t and differentiation with respect to x and y,

$$\nabla^2 L\left\{\frac{\partial z}{\partial t}\right\} + \beta^* \frac{\partial}{\partial x} L\{z\} - \mu^2 L\left\{\frac{\partial z}{\partial t}\right\} = 0.$$

We next make use of one of the fundamental operational properties of the transform. Integrating by parts,

$$L\left\{\frac{\partial z}{\partial t}\right\} = \int_0^\infty \left[\frac{\partial}{\partial t}(ze^{-st}) + sze^{-st}\right] dt$$

$$= -z(x, y, 0) + sL\{z\}.$$

Substituting this expression into the transformed equation, we obtain a nonhomogeneous equation whose dependent variable is the transform of z.

$$\left(s\nabla^2 + \beta^* \frac{\partial}{\partial x} - \mu^2 s\right) L\{z\} = \nabla^2 z_0 - \mu^2 z_0 = H(x, y). \tag{97}$$

It is worth emphasizing that the right-hand side of this equation *depends only on* z_0, the initial value of the contour height, and is therefore a known function of x and y.

7.06. Since the solution will be carried out by Green's method, and because there is otherwise some advantage in dealing with an equation whose homogeneous part is independent of the choice of coordinates, we shall next introduce a change of dependent variable, setting

$$L\{z\} = Ze^{-\beta^* x/2s}. \tag{98}$$

Thus, Eq. (97) reduces to

$$\nabla^2 Z - \nu^2 Z = s^{-1}e^{\beta^* x/2s}H(x, y) \tag{99}$$

where $\nu^2 = \mu^2 + (\beta^*/2s)^2$.

We now define the Green's function G for Eq. (99), first letting G be a solution of the homogeneous part.

$$\nabla^2 G - \nu^2 G = 0. \tag{100}$$

Next, multiplying Eq. (99) by G, Eq. (100) by Z, and subtracting Eq. (100) from Eq. (99),

$$G\nabla^2 Z - Z\nabla^2 G = s^{-1}e^{\beta^* x/2s}GH(x, y). \tag{101}$$

This is the form to which Green's theorem is most easily applied.

7.07. We now fix attention on some point at which the solution is desired. Since the origin of the coordinate system has been left unspecified, it is permissible to fix it at the point in question, so that the point for which the solution is to be computed has coordinates (0, 0). We consider next a closed path of integration in the (x, y) plane which is constructed as follows (Fig. 3). A small circle C is described around the origin, with the origin as its center. A somewhat larger circle Γ is also constructed around the origin, concentric with the smaller one. The annular region enclosed between the two circles will be called S. Finally, we make a "cut" through S adjoining C and Γ. The path of integration P is traversed by beginning at the outer end of the cut, proceeding all the way around Γ in the counterclockwise direction, up the left-hand edge of the cut, all the way around C in the clockwise direction, and back down the other side of the cut to the beginning point.

7.08. The next step is to integrate both sides of Eq. (101) over the area S.

$$\iint_S (G\nabla^2 Z - Z\nabla^2 G)\, d\xi\, d\eta = s^{-1} \iint_S e^{\beta^\bullet \xi/2^\bullet} G(\xi, \eta, s) H(\xi, \eta)\, d\xi\, d\eta \tag{102}$$

where ξ and η are variables of integration corresponding to x and y. Because pressure is continuous and has no singularities, Z is also continuous and has no singularities. Moreover, although it will later be specified that G has a logarithmic singularity at the origin, we require that it have no singularities in S, whence Green's theorem may be applied to the left-hand side of Eq. (102).

$$\oint_P \left(G\frac{\partial Z}{\partial n} - Z\frac{\partial G}{\partial n} \right)_P d\ell = s^{-1} \iint_S e^{\beta^\bullet \xi/2^\bullet} G(\xi, \eta, s) H(\xi, \eta)\, d\xi\, d\eta.$$

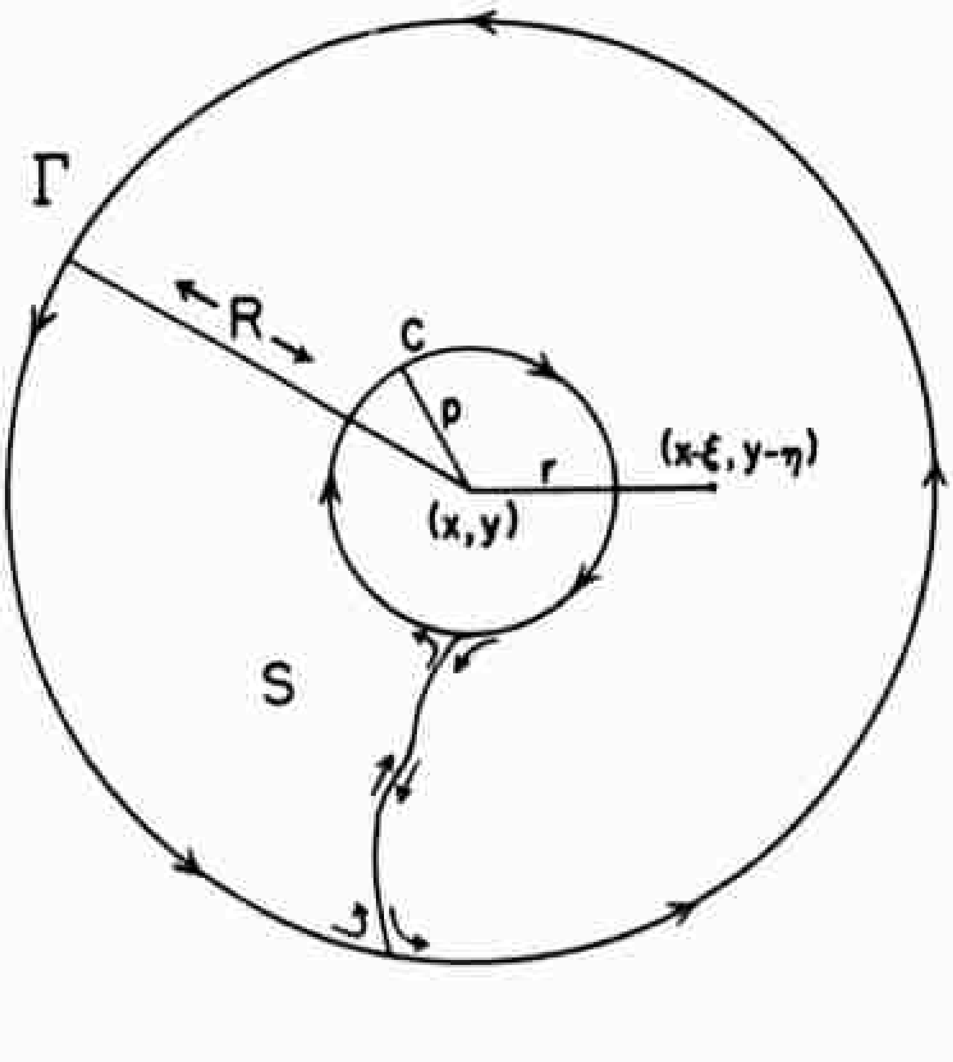


Fig. 3. Path of integration by Green's method.

The left-hand side of this equation can be expressed as the sum of integrals along the separate segments of the path P. Noting that the contribution along one side of the "cut" exactly cancels the contribution along the other side,

$$\oint_{\Gamma}\left(G\frac{\partial Z}{\partial r} - Z\frac{\partial G}{\partial r}\right)_{\Gamma} dl - \oint_{C}\left(G\frac{\partial Z}{\partial r} - Z\frac{\partial G}{\partial r}\right)_{C} dl = s^{-1}\iint_{S} e^{\beta^{*}\xi/2s}G(\xi,\eta,s)H(\xi,\eta)\,d\xi\,d\eta \tag{103}$$

where r is the radial distance from the origin. Equation (103) forms the basis for further discussion of a method for determining the contour height at the origin.

7.09. In principle, the right-hand side of Eq. (103) is known, for it involves only the initial data and certain analytic expressions which are independent of Z. Before estimating the integrals on the left-hand side of Eq. (103), however, we shall first discuss some of the properties of the Green's function G. Up to this time it has only been specified that G is a solution of

$$\nabla^2 G - \nu^2 G = 0.$$

We now require that G be a function of r only, whence the above equation reduces to an ordinary differential equation

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} - \nu^2 G = 0.$$

This will be recognized as the equation for Bessel functions of order zero with imaginary argument. There are two possible solutions, corresponding to Bessel functions of the first and second kind.

$$G = \begin{cases} K_0(\nu r) \\ I_0(\nu r). \end{cases}$$

We have already indicated that G will be required to have a logarithmic singularity at the origin, a condition which is satisfied by $K_0(\nu r)$, but not by $I_0(\nu r)$. The Green's function is, therefore, a zero-order Bessel function of the second kind with imaginary argument. As indicated in Fig. 4, the Green's function takes on an infinitely large value at the origin ($r = 0$), diminishing rapidly as one proceeds away from the origin.

7.10. Having completely specified the properties of the Green's function, we shall next estimate the integrals on the left-hand side of Eq. (103), letting the radius ρ of the small circle C approach zero and the radius R of the large circle Γ become indefinitely large. Focusing attention on the first integral on the left-hand side of Eq. (103), we note that it can be written as

$$\int_0^{2\pi}\left(G\frac{\partial Z}{\partial r} - Z\frac{\partial G}{\partial r}\right) R\,d\theta. \tag{104}$$

Since we are concerned only with estimating the value of this integral for large values of R, the Green's function may be replaced by the asymptotic expression for $K_0(\nu r)$

$$G \simeq \left(\frac{\pi}{2\nu r}\right)^{1/2} e^{-\nu r}. \qquad (r \text{ large})$$

Introducing the expression above, we obtain the following estimate of Eq. (104)

$$\left(\frac{\pi R}{2\nu}\right)^{1/2}\int_0^{2\pi}\left[\left(\frac{\partial L}{\partial r}\right)_R + L_R\left(\nu + \frac{1}{2R} + \frac{\beta^{*}\cos\theta}{2s}\right)\right] e^{-[\nu-(\beta^{*}\cos\theta/2s)]R}\,d\theta,$$

where $L = L\{z\}$, the transform of the contour height. Now, because the pressure is continuous and has no singularities, z and its derivatives are bounded and so also are $L\{z\}$ and its derivatives. Thus, the bracketed

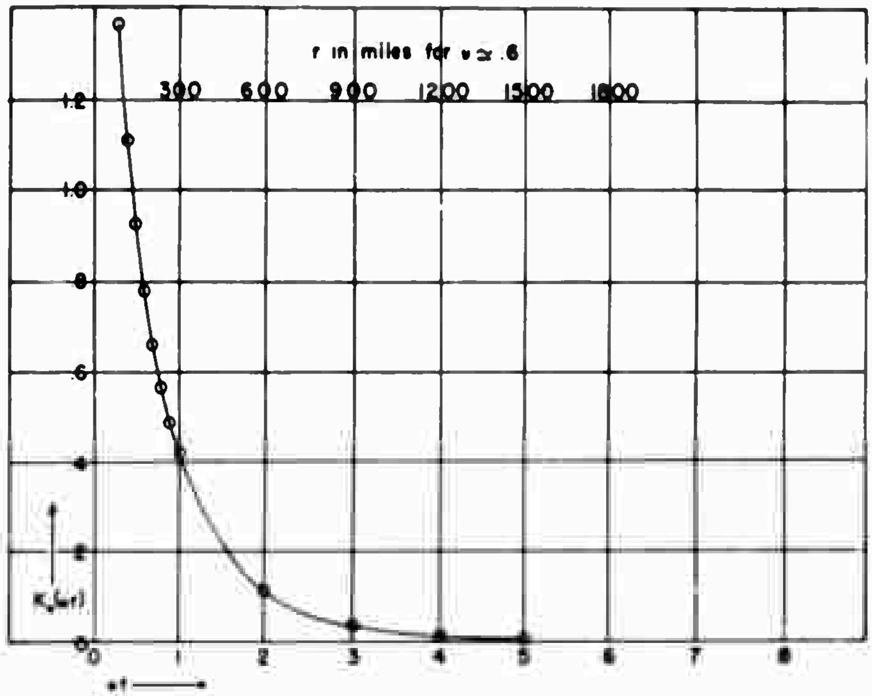


Fig. 4. Behavior of the function $K_0(\nu r)$.

factor in the integrand is finite, whence the behavior of Eq. (104) as R becomes infinite is dominated by the exponential factor

$$e^{-[\nu-(\beta^* \cos \theta/2s)]R}.$$

This is the decisive point in the argument. As originally defined, ν is given by

$$\nu^2 = \left(\frac{\beta^*}{2s}\right)^2 + \mu^2.$$

This definition, taken together with the fact that the cosine never exceeds unity, implies that

$$\nu > \frac{\beta^*}{2s} \geqslant \frac{\beta^* \cos \theta}{2s}$$

for all θ, provided μ is different from zero. If μ is non-zero, the exponent is always negative, with the result that the integral (104) converges to zero as R becomes infinitely large. Equation (103) therefore reduces to

$$\oint_C \left(Z\frac{\partial G}{\partial r} - G\frac{\partial Z}{\partial r}\right)_\rho dl = s^{-1}\iint_S e^{\beta^*\xi/2s} G(\xi, \eta, s) H(\xi, \eta)\, d\xi\, d\eta. \tag{105}$$

It is now understood that the region S covers the entire (ξ, η) plane, except for the area lying inside the small circle C.

7.11. It is interesting to note that the condition which insures that the boundary integral at infinity will vanish for all θ is that μ be different from zero. Interpreted in the light of the development of Eq. (92), this condition is tantamount to requiring that there be some divergence in the flow, *no matter how small*, simply because μ^2 is the coefficient of the term representing the effect of divergence. We therefore conclude that the effects of disturbances located at an infinite distance from the origin might be felt immediately, unless there were some divergence, however slight. This result is in accord with Yeh's statement (1949) concerning the maximum group velocity—namely, that the group velocity is finite for all wavelengths only if there is some divergence to generate and destroy vorticity systematically.

7.12. The limit of the integral on the left-hand side of Eq. (105) will next be evaluated as the radius ρ of the small circle C approaches zero. We begin by considering the integral

$$\oint_C G \frac{\partial Z}{\partial r}\, dl, \tag{106}$$

which may be rewritten as

$$\rho K_0(\nu\rho) \int_0^{2\pi} \left(\frac{\partial Z}{\partial r}\right)_\rho d\theta.$$

Because the singularity of $K_0(\nu r)$ at the origin is logarithmic, the factor $\rho K_0(\nu\rho)$ approaches a finite limit as ρ goes to zero. On the other hand, because Z is continuous at the origin, the integral factor becomes vanishingly small as ρ approaches zero. That is to say, the value of the normal derivative of Z at any point on C is, in the limit, equal but opposite in sign to the normal derivative at the point diametrically opposite. Accordingly, when the circle C is shrunk to a point at the origin, the integral (106) contributes nothing to the left-hand side of Eq. (105).

7.13. Finally, we consider the one remaining integral

$$\oint_C Z \frac{\partial G}{\partial r}\, dl,$$

which may be expressed as

$$\rho \left(\frac{dG}{dr}\right)_\rho \int_0^{2\pi} Z\rho\, d\theta.$$

Making use of the differentiation formulas for Bessel functions,

$$\rho \left(\frac{dG}{dr}\right)_\rho = \nu\rho K_1(\nu\rho)$$

whence

$$\lim_{\rho\to 0} \rho \left(\frac{dG}{dr}\right)_\rho = -1.$$

Moreover, because Z is continuous at the origin,

$$\lim_{\rho\to 0} \int_0^{2\pi} Z\rho\, d\theta = 2\pi Z(0, 0, s).$$

Substituting these values into Eq. (105), we obtain a formula expressing Z at the origin in terms of only the initial data and known analytic functions which are independent of Z

$$Z(0, 0, s) = -(2\pi s)^{-1} \iint_S e^{\beta^* \xi / 2\kappa} G(\xi, \eta, s) H(\xi, \eta)\, d\xi\, d\eta.$$

The right-hand side of this equation is therefore the formal solution of Eq. (99). The relation between Z and the Laplace transform of z is

$$Z = L\{z\} e^{\beta^* x / 2\kappa}.$$

Thus the solution for the Laplace transform of z *at any arbitrarily chosen origin* and, consequently, at any point (x, y) is given by the formula

$$L(0, 0, s) = -(2\pi s)^{-1} \iint_S e^{\beta^* \xi / 2\kappa} K_0(\nu r) H(\xi, \eta)\, d\xi\, d\eta. \tag{107}$$

The region S now covers the entire (ξ, η) plane.

7.14. The special properties of formula (107) merit some discussion. In much the same way as was the solution of Eq. (85), the solution for $L\{z\}$ is given as an integral (in this case a surface integral) over an infinite domain. Its integrand consists of two distinct factors, one depending only on the initial values of contour height, and the other representing the Green's function for the problem—a known analytic function independent of the initial data. The analogy between formula (107) and the results expressed in Eq. (89) can be carried further by noting that the infinite surface integral

$$\iint\limits_S e^{\beta^* \xi/2s} K_0(\nu r) H(\xi, \eta)\, d\xi\, d\eta$$

can be truncated at a sufficiently large finite radius R, without significantly altering the accuracy of the final results. Again making use of the asymptotic expression for $K_0(\nu r)$, the estimated maximum truncation error is

$$\pi \left(\frac{2\pi}{\nu}\right)^{1/2} |H|_{\max} \int_R^{\infty} r^{1/2} \exp\left[- \left(\sqrt{\left(\frac{\beta^*}{2s}\right)^2 + \mu^2} - \frac{\beta^*}{2s} \right) r \right] dr.$$

As before, the behavior of this estimate is dominated by the exponential factor, whose exponent is always negative. In other words, by choosing R sufficiently large, the error incurred by omitting contributions beyond a certain finite radius R can be made less than any previously assigned finite value, no matter how small. This result has direct bearing on a statement made by Ertel (1944), to the effect that the solution even a short time after the initial moment actually depends on the initial data over the entire domain, and that it is accordingly impossible to predict with complete accuracy. As it stands, this statement is quite true, but rather misleading. As we have just shown, the exact solution does depend on the initial values of contour height over an infinite domain. On the other hand, if one is willing to make small mistakes, it is possible to insure that they will not exceed a previously set value, by choosing the radius of the finite domain of integration large enough.

7.15. Returning to the solution of Eq. (94a), it remains only to carry out the inverse transformation of $L\{z\}$ to obtain z at any arbitrarily chosen origin. We next apply the inverse transformation to both sides of Eq. (107), interchanging the order of integration with respect to s and with respect to ξ and η.

$$z(0, 0, t) = -\frac{1}{2\pi} \iint\limits_S H(\xi, \eta) L^{-1}\{s^{-1} e^{\beta^* \xi/2s} K_0(\nu r)\}\, d\xi\, d\eta. \tag{108}$$

As before, the contour height is expressed as an infinite surface integral, whose integrand consists of a factor involving only the initial data multiplied by the Green's function for the problem.

7.16. We shall next discuss the properties of the Green's function, $I(\xi, \eta, t)$, where

$$I(\xi, \eta, t) = L^{-1}\{s^{-1} e^{\beta^* \xi/2s} K_0(\nu r)\}. \tag{109}$$

Having already insured that the boundary integral converges to zero, it should be noted that μ is actually quite small; in fact, for purposes of evaluating the I function, we shall let μ equal zero exactly, in which case Eq. (109) reduces to

$$\begin{aligned} I(\xi, \eta, t) &= L^{-1}\left\{ s^{-1} e^{\beta^* \xi/2s} K_0\left(\frac{\beta^* r}{2s}\right)\right\}. \\ &= L^{-1}\left\{ \left[s^{-1/2} e^{-(\beta^*/2s)(r-\xi)} \right] \left[s^{-1/2} e^{\beta^* r/2s} K_0\left(\frac{\beta^* r}{2s}\right)\right]\right\}. \end{aligned} \tag{110}$$

To carry out the inverse transformation, we make use of another of the fundamental operational properties of the Laplace transform, namely that the inverse transform of a product is given by

$$L^{-1}\{f_1 f_2\} = L^{-1}\{f_1\} * L^{-1}\{f_2\},$$

where ( ) * ( ) is the so-called "convolution operator." Identifying f_1 and f_2 with the factors on the right-hand side of Eq. (110),

$$f_1 = s^{-1/2} e^{-(\beta^*/2s)(r-\xi)}$$

$$f_2 = s^{-1/2} e^{\beta^* r/2s} K_0\left(\frac{\beta^* r}{2s}\right).$$

The inverse transforms of f_1 and f_2 are known (Churchill, (1944)). They are:

$$L^{-1}\{f_1\} = \frac{1}{\sqrt{\pi t}} \cos \sqrt{2\beta^*(r-\xi)t}$$

$$L^{-1}\{f_2\} = \frac{2}{\sqrt{\pi t}} K_0(2\sqrt{\beta^* r t}).$$

Finally, introducing these results into Eq. (110), we can express the Green's function $I(\xi, \eta, t)$ as a Faltung or convolution integral.

$$I(\xi, \eta, t) = \frac{2}{\pi} \int_0^t \frac{1}{\sqrt{\tau(t-\tau)}} \cos \sqrt{2\beta^*(r-\xi)(t-\tau)}\; K_0(2\sqrt{\beta^* r \tau})\, d\tau. \tag{111}$$

The remaining problem is to evaluate this definite integral for all values of the parameters ξ, η and t. By a long series of substitutions and changes of variable, it is possible to reduce the integral on the right-hand side of Eq. (111) to a less formidable and more easily computed form. However, because some of the procedures are rather involved, we shall relegate the details of the final reduction to Appendix II, presenting only the most important intermediate results below. A result which will later be used to simplify the numerical calculation of the solution is that

$$I(\xi, \eta, t) = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos(2\kappa\sigma \cos\phi) K_0(2\sigma \sin\phi)\, d\phi, \tag{112}$$

where $\kappa = \sin(\theta/2)$, $\sigma = \sqrt{\beta^* r t}$, and ϕ is simply a dummy variable of integration. By introducing an integral representation of the Bessel function in Eq. (112) and inverting the order of integration, we finally obtain the most compact form of the Green's function.*

$$I(\xi, \eta, t) = 2\int_0^{\infty} \frac{J_0(2\sigma\sqrt{\kappa^2 + z^2})}{\sqrt{1+z^2}}\, dz, \tag{113}$$

where, again, z is a dummy variable of integration. At this point it can easily be shown that $I(\xi, \eta, t)$, regarded as a function of the independent variables σ and κ, is a solution of

$$\frac{\partial^2 I}{\partial\sigma\, \partial\kappa} + 4\sigma\kappa I = 0. \tag{114}$$

The I function has actually been computed by numerically integrating Eq. (114) in the (σ, κ) plane, starting with known boundary values of I. The latter were found by evaluating Eq. (113) analytically for the special

* This form of the I function can be formally expanded in an infinite series of products of Bessel functions, according to the addition theorem of Graf (Watson, (1922)). However, because it is questionable that the series converges uniformly, we shall resort to numerical methods for evaluating Eq. (113).

values $\kappa = 0$ and $\sigma = \epsilon$. For example, the values $I(\sigma, 0)$ along the σ-axis are given by

$$2\int_0^\infty \frac{J_0(2\sigma z)}{\sqrt{1+z^2}}\,dz = 2I_0(\sigma)K_0(\sigma).$$

The method outlined above is simpler and far less laborious than evaluating Eq. (113) by finite sums, simply because the latter procedure must be carried out for each and every possible combination of the parameters σ and κ.

7.17. Some of the most striking properties of the Green's function, however, can be discovered simply by inspecting the behavior of the integral (113) as the parameter σ takes on successively larger values and as the parameter κ varies over the range zero to one. As σ increases indefinitely—i.e., as the distance from the origin becomes infinite—the value of the integral decreases rapidly to zero. It should also be noted that θ enters only in κ^2, so that

$$I(r, \theta, t) = I(r, -\theta, t).$$

Thus, the I function is symmetrical around the ξ-axis. It is not, however, symmetrical around the η-axis. The latter property, in fact, indicates that the Green's function does contain a mechanism for propagating the large-scale disturbances.

7.18. Although Eq. (108) is a perfectly legitimate solution of Eq. (95), it is inconvenient from the standpoint of actually carrying out the numerical computation of the solution from observed initial values. This comes about because the function $H(\xi, \eta)$ involves the Laplacian derivative of the initial contour height. In this connection, it is suggestive that part of the solution is given as a double integral of second derivatives of the contour height, multiplied by the Green's function. This suggests, in fact, that the right-hand side of Eq. (108) might be integrated by parts to obtain the solution in terms of the *undifferentiated* initial values. We therefore retrace our way through the development of the solution to Eq. (101), noting that its right-hand side can be rewritten as

$$s^{-1}[\nabla \cdot (e^{\beta^*\xi/2s}G\nabla z_0 - z_0\nabla e^{\beta^*\xi/2s}G) + z_0(\nabla^2 e^{\beta^*\xi/2s}G - \mu^2 e^{\beta^*\xi/2s}G)].$$

By introducing the definition of $G(\xi, \eta, s)$, the above expression can be reduced to

$$s^{-1}\nabla \cdot (e^{\beta^*\xi/2s}G\nabla z_0 - z_0\nabla e^{\beta^*\xi/2s}G) + \beta^* z_0\frac{\partial}{\partial\xi}(s^{-2}Ge^{\beta^*\xi/2s}). \tag{115}$$

To obtain an expression equivalent to the right-hand side of Eq. (108), we successively apply three operations to Eq. (115). First, Eq. (115) is to be integrated over the region S, and then evaluated as the radius R of the large circle Γ becomes infinite and as the radius ρ of the small circle C approaches zero. Second, the resulting limit must be divided by -2π. Finally, the inverse Laplace transform must be applied to obtain the new solution for the contour height.

7.19. Fixing attention on the first term of Eq. (115), we first integrate over the region S. Since neither G nor z_0 has singularities in S, the surface integral may be transformed into two line integrals by applying Gauss' theorem, whence the first term of Eq. (115) becomes

$$s^{-1}\int_0^{2\pi}\left(e^{\beta^*\xi/2s}G\frac{\partial z_0}{\partial r} - z_0\frac{\partial}{\partial r}e^{\beta^*\xi/2s}G\right)_R R\,d\theta$$
$$-s^{-1}\int_0^{2\pi}\left(e^{\beta^*\xi/2s}G\frac{\partial z_0}{\partial r} - z_0\frac{\partial}{\partial r}e^{\beta^*\xi/2s}G\right)_\rho \rho\,d\theta.$$

Following an argument identical to that outlined in paragraph 7.10, it can be shown that the first of these two integrals vanishes as R becomes infinite. Similarly, for reasons already given in paragraph 7.12, the integral

$$\int_0^{2\pi} \rho \left(e^{\beta^* \xi / 2s} G \frac{\partial z_0}{\partial r} \right)_\rho d\theta$$

vanishes as ρ approaches zero. The sole contribution from the first term of Eq. (115) therefore comes from the one remaining integral

$$s^{-1} \int_0^{2\pi} \left(z_0 \frac{\partial}{\partial r} e^{\beta^* \xi / 2s} G \right)_\rho \rho \, d\theta,$$

which, for small values of ρ, can be rewritten as

$$s^{-1} z(0, 0, 0) \rho \int_0^{2\pi} \left(\frac{\beta^* G \cos \theta}{2s} + \frac{dG}{dr} \right) d\theta.$$

Because the integral of the cosine vanishes, the limit as ρ goes to zero is

$$2\pi s^{-1} z(0, 0, 0) \lim_{\rho \to 0} \rho \left(\frac{dG}{dr} \right)_\rho = -2\pi s^{-1} z(0, 0, 0).$$

It remains to divide by -2π and apply the inverse transform. After carrying out those operations, we find that the contribution of the first term of Eq. (115) to the solution of Eq. (94a) is simply

$$L^{-1} \{ s^{-1} z(0, 0, 0) \} = z(0, 0, 0).$$

Finally, we apply the operations of integration and inverse transformation to the second term of Eq. (115)' summarizing our results in a new formula for the contour height at any arbitrarily chosen origin and, consequently, at any point (x, y).

$$z(0, 0, t) = z(0, 0, 0) - \frac{1}{2\pi} \iint_S z(\xi, \eta, 0) \frac{\partial}{\partial \xi} L^{-1} \{ \beta^* s^{-2} e^{\beta^* \xi / 2s} K_0(\nu r) \} \, d\xi \, d\eta. \tag{116}$$

As indicated earlier, the advantage of this formula is that it expresses the contour height in terms of the *undifferentiated* initial values.

7.20. As it stands, however, formula (116) gives the solution of Eq. (94a), whose independent variables are the coordinates in a system moving toward the east at a speed τU. To obtain the solution of the original Eq. (94), it remains only to shift the original coordinates at time t a distance $\tau U t$ forward to coincide with the new ones at time t. Simultaneously expressing the fact that the location of the origin is arbitrary, we may therefore write the solution of Eq. (94) in the following form:

$$\boxed{\begin{aligned} z_F(x + \tau U t, y, t) &= z_F(x, y, 0) \\ &\quad + \iint_S z_F(\xi - x, \eta - y, 0) T(\xi - x, \eta - y, t) \, d\xi \, d\eta \end{aligned}} \tag{117}$$

where the new "Green's function" $T(\xi - x, \eta - y, t)$ is

$$\boxed{T(\xi - x, \eta - y, t) = -\frac{1}{2\pi} \frac{\partial}{\partial \xi} L^{-1} \{ \beta^* s^{-2} e^{\beta^* (\xi - x)/2s} K_0(\nu r) \}} \tag{118}$$

and $r^2 = (\xi - x)^2 + (\eta - y)^2$.

The independent variables x, y and t now appear as parameters in the integral on the right-hand side of formula (117), ξ and η entering only as dummy variables of integration corresponding to x and y. Equation (117) represents the formal solution of the linear equation for large-scale transient disturbances, valid for any initial distribution of contour height.

7.21. The chief advantage and power of this method lie in the fact that the exact manner in which the prediction depends on the initial values is bound up in the behavior of a single analytic function, $T(\xi - x, \eta - y, t)$, which is independent of the initial values. One may think of the value of the Green's function at the point $(\xi - x, \eta - y)$ as the effect of a unit point disturbance located at that same point. With this interpretation, the integral on the right-hand side of Eq. (117) represents the integrated effect of an infinite number of point disturbances with strength $z(\xi - x, \eta - y, 0)$, distributed over the entire (x, y) plane. It is therefore possible to infer some of the general properties of the solution by examining only the behavior of the Green's function, without regard to the initial values. We shall next discuss the special properties of the T function, first evaluating it by the method outlined in paragraph 7.16. The definition of $T(\xi, \eta, t)$ given in Eq. (118) may be rewritten as follows

$$\begin{aligned} -2\pi T(\xi, \eta, t) &= \frac{\partial}{\partial \xi}\beta^* L^{-1}\left\{\left[s^{-1/2}e^{-(\beta^*/2s)(r-\xi)}\right]\left[s^{-1/2}e^{\beta^* r/2s}K_0\left(\frac{\beta^* r}{2s}\right)\right]\right\} \\ &= \frac{\partial}{\partial \xi}\beta^* L^{-1}\{f_1 f_2\}, \end{aligned} \tag{119}$$

where

$$f_1 = s^{-1/2}e^{-(\beta^*/2s)(r-\xi)}$$

$$f_2 = s^{-1/2}e^{\beta^* r/2s}K_0\left(\frac{\beta^* r}{2s}\right).$$

The inverse transforms of f_1 and f_2 are (Churchill, (1944)):

$$L^{-1}\{f_1\} = \frac{1}{\sqrt{\frac{\pi}{2}\beta^*(r-\xi)}}\sin\sqrt{2\beta^*(r-\xi)t}$$

$$L^{-1}\{f_2\} = \frac{2}{\sqrt{\pi t}}K_0(2\sqrt{\beta^* r t}).$$

Finally, the inverse transform of the product $f_1 f_2$ can be expressed in the form of a convolution integral

$$L^{-1}\{f_1 f_2\} = \frac{2}{\pi}\int_0^t \frac{1}{\sqrt{\frac{\beta^*(r-\xi)\tau}{2}}}\sin\sqrt{2\beta^*(r-\xi)(t-\tau)}\; K_0(2\sqrt{\beta^* r\tau})\, d\tau,$$

which, after some manipulation, can be rewritten as

$$L^{-1}\{f_1 f_2\} = \frac{4}{\pi\beta^* r\kappa}\int_0^{\frac{\pi}{2}} \sigma\cos\phi\sin(2\kappa\sigma\cos\phi)K_0(2\sigma\sin\phi)\,d\phi \tag{120}$$

where, as before, $\sigma = \sqrt{\beta^* r t}$ and $\kappa = \sin(\theta/2)$. We next return to Eq. (112), differentiating both sides with respect to κ.

$$\frac{\partial I}{\partial \kappa} = -\frac{8}{\pi}\int_0^{\frac{\pi}{2}} \sigma\cos\phi\sin(2\kappa\sigma\cos\phi)K_0(2\sigma\sin\phi)\,d\phi.$$

Substituting this result into Eq. (120),

$$L^{-1}\{f_1 f_2\} = -\frac{1}{2\beta^* r\kappa}\frac{\partial I}{\partial \kappa}.$$

Finally, replacing $L^{-1}\{f_1f_2\}$ in Eq. (119) by the above expression, we obtain $T(\xi, \eta, t)$ in terms of the function $I(\xi, \eta, t)$.

$$T(\xi, \eta, t) = \frac{1}{4\pi}\frac{\partial}{\partial\xi}\left(\frac{1}{r\kappa}\frac{\partial I}{\partial\kappa}\right). \tag{121}$$

It should be noted that $I(\xi, \eta, t)$ is actually a function of only two variables, σ and κ. Thus the I function can be tabulated once and for all by the method described in paragraph 7.16. The T function has been calculated from formula (121) simply by differentiating the previously tabulated values of $I(\xi, \eta, t)$. Since t enters in the Green's function only in the combination β^*t, it is simplest to compute $T(\xi, \eta, t)$ as $T(\xi, \eta; \beta^*t)$, a one-parameter family of functions of ξ and η, corresponding to various values of the single parameter β^*t. The T function has actually been tabulated for values of β^*t ranging from 0.5 to 1.1 radians per five degrees of latitude, at intervals of one-tenth units. A typical example of the results of these calculations is presented in Fig. 5. Although it is not indicated in the figure, it should be realized that the T function has a singularity at the origin, changing from negatively infinite values on the left-hand side of the origin to positively infinite values on the right, as one proceeds along the ξ-axis across the origin. Because the integral on the right-hand side of Eq. (117) must ultimately be evaluated by finite sums, we have simply replaced the value of T at the origin by its average value, integrated over a unit area surrounding the origin.

7.22. The most important aspects of the T function are illustrated in Fig. 5. The most striking feature is that it "dies out" rapidly as one proceeds away from the origin in any direction, decreasing to about one-tenth its maximum value at a radius of twenty degrees of latitude. In other words, for values of β in the middle latitudes and for a forecast period of one day, the radius of the effective domain of dependence is on the order of 1500 kilometers. It is also significant that the T function is not symmetrical around the η-axis, taking on predominantly negative values to the left of the origin and predominantly positive values on the right. As an example of the consequences of this property, let us suppose that we wish to compute the change in contour height resulting from an initial distribution which is symmetrical around the ξ-axis and exactly asymmetrical around the η-axis, the region of low pressure lying to the left of the origin and the area of high pressure to the right. Thus the value of the integral on the right-hand side of Eq. (117) would be positive, whence the contour height at time t and at the point $x = \tau Ut$ must be greater than it was initially at the origin. This implies that the disturbances must move westward *relative to an imaginary current of air moving at a speed τU toward the east*, a result which is in accord with the simple Rossby "trough" formula. The very asymmetry of the T function therefore contains the mechanism for propagating large-scale disturbances.

7.23. In concluding the discussion of prediction formulas for large-scale transient disturbances, it is appropriate merely to mention one other important property of the final solution. Formula (117) expresses the predicted contour height at any given point as a linear combination of its initial values at points in the immediate neighborhood, weighted according to their distance from the point in question. We may therefore interpret the predicted contour height as a sort of weighted mean of the initial data, integrated over the effective domain of dependence—a region which, for forecast periods on the order of a day or so, has linear dimensions comparable with the characteristic half-wavelength of the large-scale disturbances. Thus the solutions (117), computed by finite sums from discrete and widely separated values of the initial contour height, display the statistical stability of the mean of a finite but large sample. Owing to the observed "continuity" or "smoothness" of meteorological variables, the mean of the sample will closely approximate the true mean of the continuous distribution of contour height. In fact, if the errors in observing the initial values of contour height are not systematic, the percentage deviation from the true mean may actually be

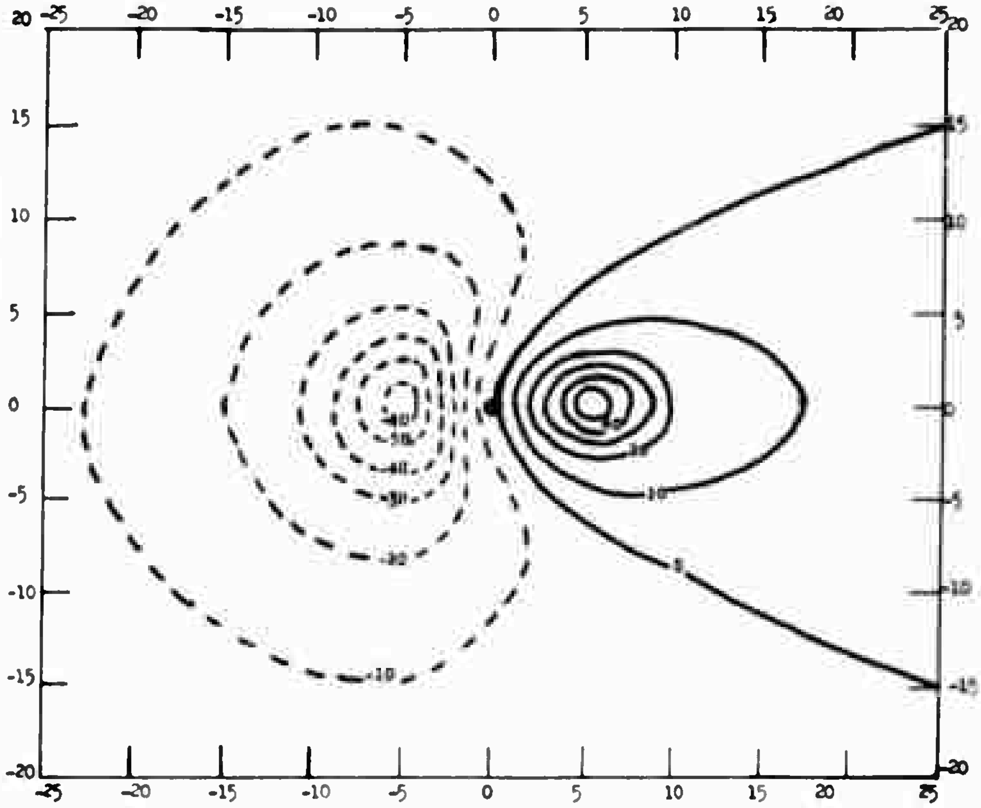


Fig. 5. The two-dimensional Green's function: $T(x - \xi, y - \eta, \beta t)$.

less than the ratio of the error in an individual observation to the true variability of the sample. Viewed in the light of previous remarks (paragraph 1.15), concerning the stability of statistics of the observed state, the values computed from formula (117) evidently satisfy the requirements imposed on the form of the solutions by the enforced inobservability of the atmosphere and by the inaccuracy of observations. In summary, because we are already dealing with the vertically integrated mean motions, it appears that the predictions computed from Eq. (117) are representative of conditions integrated over a volume whose horizontal dimensions are of the order of the characteristic half-wavelength of the large-scale disturbances, and whose vertical dimension is comparable with the entire vertical extent of the atmosphere.

THE SOLUTION FOR QUASI-STATIONARY FORCED OSCILLATIONS

7.24. We shall next discuss Eq. (96), whose solutions correspond to forced oscillations induced by irregularities of the underlying terrain. As mentioned earlier, the amplitude of the semipermanent "lee trough" is generally somewhat less than that of the superposed transient disturbances—enough smaller, in fact, that it is difficult to detect the "lee trough" at all, whether it is reconstructed from the mean flow averaged over a long period of time or from individual cases. For this reason, and because the semipermanent features of the large-scale flow are naturally of least concern from the standpoint of short range prediction, we shall merely outline the method for developing the formal solution of Eq. (96), finally presenting the solution in the form of a surface integral analogous to Eq. (117). We first note that each term of Eq. (96) has been differentiated with respect to x. A single integration with respect to x therefore reduces Eq. (96) to an equation of the second order.

$$\nabla^2 z + m^2 z = -\mu^2 \tau^{-1} h(x, y), \tag{122}$$

where $m^2 = \beta \tau^{-1} U^{-1}$.

7.25. At this point, it should be emphasized that the problem of solving the equation for transient disturbances is essentially different from that of solving the steady-state equation (Eq. (122)). The former, although it is superficially a problem of both boundary and initial values, turns out to be completely determined by the initial values alone. The solution of the steady-state equation, on the other hand, is by its very nature a boundary-value problem. In physical actuality, even the steady-state solution is completely determined by requiring only that the solution be everywhere continuous, because a "horizontal" surface is actually closed. In this respect, the locally Cartesian coordinate system we have adopted is highly artificial. To simulate the true state of affairs, we shall tentatively assume that the radius of the "effective domain of dependence" for Eq. (96) is no greater than, let us say, the width of the major oceans, later justifying this assumption a posteriori. Temporarily granting that it is valid, a consequence of this postulate is that the forced oscillation associated with any one land mass can be treated as if it were unmodified by the presence of other continents. It will therefore be required that the disturbance created by an isolated irregularity vanish everywhere over a semi-infinite region on the windward side of the irregularity. Owing to the fact that it introduces such boundary conditions explicitly, it is again natural to apply the Laplace transform, this time replacing x by the variable of the transform.

$$L(s,y) = L\{z(x,y)\} = \int_0^\infty z(x,y)e^{-sx}\,dx.$$

By making use of the operational properties of the transform, we find that Eq. (122) takes the form of a nonhomogeneous "ordinary" differential equation whose dependent variable is the transform of z.

$$\frac{\partial^2 L}{\partial y^2} + \kappa^2 L = -\mu^2\tau^{-1}H(s,y) + \left(\frac{\partial z}{\partial x}\right)_{x=0} + sz(0,y),$$

where $\kappa^2 = m^2 + s^2$ and $H(s,y) = L\{h(x,y)\}$.

Since the line $x = 0$ is assumed to lie entirely within the semi-infinite region where z vanishes, the above equation reduces to

$$\frac{\partial^2 L}{\partial y^2} + \kappa^2 L = -\mu^2\tau^{-1}H(s,y). \tag{123}$$

The remaining problems are to solve this equation for the Laplace transform of z and to carry out the inverse transformation to obtain z itself.

7.26. We next consider the Fourier integral

$$L(s,y) = -\frac{\mu^2}{\pi\tau}\int_0^\infty d\alpha\int_{-\infty}^\infty H(s,\eta)\,\frac{1}{\kappa^2-\alpha^2}\cos\alpha\,(y-\eta)\,d\eta. \tag{124}$$

Differentiating both sides of Eq. (124),

$$\frac{\partial^2 L}{\partial y^2} + \kappa^2 L = -\frac{\mu^2}{\pi\tau}\int_0^\infty d\alpha\int_{-\infty}^\infty H(s,\eta)\cos\alpha(y-\eta)\,d\eta. \tag{125}$$

According to Fourier's theorem, however, the integral on the right-hand side of Eq. (125) is simply the expansion of $H\,(s,y)$ on an infinite interval, whence

$$\frac{\partial^2 L}{\partial y^2} + \kappa^2 L = -\mu^2\tau^{-1}H(s,y).$$

Thus expression (124) is the solution of Eq. (123), the equation for the transform of z. To simplify matters, we invert the order of integration on the right-hand side of Eq. (124), writing the solution for the transform of z in the form

$$L(s, y) = -\frac{\mu^2}{\pi\tau}\int_{-\infty}^{\infty} H(s, \eta)K(s, y - \eta)\, d\eta, \tag{126}$$

where the kernel function K is

$$K(s, y - \eta) = \int_0^{\infty} \frac{1}{\kappa^2 - \alpha^2} \cos \alpha(y - \eta)\, d\alpha.$$

The integral K is well known and has been evaluated. Its value is

$$\int_0^{\infty} \frac{1}{\kappa^2 - \alpha^2} \cos \alpha(y - \eta)\, d\alpha = \frac{\pi}{2\kappa} \sin \kappa(y - \eta).$$

After substituting this expression for $K(s, y - \eta)$ in Eq. (126), we obtain the reduced Fourier integral for $L\{z\}$.

$$L(s, y) = L\{z\} = -\frac{\mu^2}{2\tau}\int_{-\infty}^{\infty} H(s, \eta)\left[\frac{1}{\kappa} \sin \kappa(y - \eta)\right] d\eta. \tag{127}$$

It remains to apply the inverse transform to Eq. (127) to find the solution for z itself. Interchanging the order of integration and transformation on the right side of Eq. (127),

$$z(x, y) = -\frac{\mu^2}{2\tau}\int_{-\infty}^{\infty} L^{-1}\left\{H(s, \eta)\left[\frac{1}{\kappa} \sin \kappa(y - \eta)\right]\right\} d\eta. \tag{128}$$

The inverse transform of the integrand can be written as the convolution of the inverse transforms of the separate factors in the product.

$$\begin{aligned} L^{-1}\left\{H(s, \eta)\left[\frac{1}{\kappa} \sin \kappa(y - \eta)\right]\right\} &= L^{-1}\{H(s, \eta)\} * L^{-1}\left\{\frac{1}{\kappa} \sin \kappa(y - \eta)\right\} \\ &= h(x, \eta) * L^{-1}\left\{\frac{1}{\kappa} \sin \kappa(y - \eta)\right\} \\ &= h(x, \eta) * \Phi(x, y - \eta). \end{aligned}$$

Finally, replacing the integrand of Eq. (128) by the convolution integral, we obtain the solution $z(x, y)$ in the form of a double integral.

$$\boxed{z_M(x, y) = -\frac{\mu^2}{2\tau}\int_{-\infty}^{\infty}\int_0^{x} h(\xi, \eta)\Phi(x - \xi, y - \eta)\, d\xi\, d\eta} \tag{129}$$

where

$$\boxed{\Phi(x, y - \eta) = L^{-1}\left\{\frac{1}{\sqrt{s^2 + m^2}} \sin \sqrt{s^2 + m^2}(y - \eta)\right\}.} \tag{130}$$

The expression on the right-hand side of Eq. (129) is the formal solution z_M of Eq. (96). Comparing it with Eq. (117), we see that formula (129) is very similar in form to the solution of the equation for transient disturbances, in that it expresses the solution as an infinite surface integral whose integrand contains two factors of essentially different kinds. The first factor, which is analogous to the initial value of z_F, is simply the height of the underlying terrain. The second is the Green's function for the problem, a function which is analytic and independent of the boundary data. The difference between the two solutions lies in the fact that the integration of Eq. (129) is to be carried out over the semi-infinite plane to the left of the line $\xi = x$, whereas the transient disturbance z_F depends on initial values over the entire plane. This implies that the height-disturbance due to the terrain-induced forced oscillation depends only on the character of the terrain to the windward side of the point in question, and is not affected by conditions on the leeward side.

7.27. We shall next investigate some of the properties of $\Phi(x, y - \eta)$, the Green's function for Eq. (96), with a view to estimating the radius of the effective domain of dependence. Up to this point it has only been indicated that the Green's function can be found by applying the inverse transformation to a known function of y and s, a process which is actually rather complicated. Some of the general properties of $\Phi(x, y - \eta)$, however, can be deduced by elementary methods, without carrying out the inverse transformation completely. We begin by considering the integral equation which defines Φ

$$\int_0^\infty \Phi(x, y - \eta) e^{-sx}\, dx = (s^2 + m^2)^{-1/2} \sin (y - \eta)(s^2 + m^2)^{1/2}.$$

By introducing new variables, this equation can also be written as

$$\int_0^\infty \Phi(t, u) e^{-\alpha t}\, dt = (\alpha^2 + 1)^{-1/2} \sin u(\alpha^2 + 1)^{1/2}, \tag{131}$$

where $t = mx$, $u = m(y - \eta)$, and $s = m\alpha$. This equation alone carries the implication that Φ involves x and $(y - \eta)$ only in the combinations mx and $m(y - \eta)$. The right-hand side of Eq. (131) may be expressed as

$$(\alpha^2 + 1)^{-1/2} \sin u(\alpha^2 + 1)^{1/2} = -\frac{1}{u\alpha} \frac{\partial}{\partial \alpha} \cos u(\alpha^2 + 1)^{1/2}. \tag{132}$$

Moreover, differentiating both sides of Eq. (131) with respect to u,

$$\int_0^\infty \frac{\partial \Phi}{\partial u} e^{-\alpha t}\, dt = \cos u(\alpha^2 + 1)^{1/2}.$$

Substituting the above expression for $\cos u(\alpha^2 + 1)^{1/2}$ on the right-hand side of Eq. (132),

$$\int_0^\infty u\Phi(t, u)\alpha e^{-\alpha t}\, dt = \int_0^\infty t \frac{\partial \Phi}{\partial u} e^{-\alpha t}\, dt. \tag{133}$$

The left-hand side of Eq. (133) can be integrated by parts as follows

$$\int_0^\infty u\Phi(t, u)\alpha e^{-\alpha t}\, dt = -\int_0^\infty u\Phi(t, u) \frac{\partial}{\partial t} e^{-\alpha t}\, dt$$

$$= -\Big[u\Phi(t, u) e^{-\alpha t}\Big]_0^\infty + \int_0^\infty u \frac{\partial \Phi}{\partial t} e^{-\alpha t}\, dt.$$

Introducing this result into Eq. (133) gives

$$\int_0^\infty \left(u \frac{\partial \Phi}{\partial t} - t \frac{\partial \Phi}{\partial u}\right) e^{-\alpha t}\, dt = u\Phi(0, u).$$

Finally, we differentiate the above equation with respect to α to obtain

$$\int_0^\infty \left(u \frac{\partial \Phi}{\partial t} - t \frac{\partial \Phi}{\partial u}\right) t e^{-\alpha t}\, dt = 0.$$

There is only one way in which this equation can hold for all u and α, and that is if

$$u \frac{\partial \Phi}{\partial t} - t \frac{\partial \Phi}{\partial u} = 0.$$

This is a linear partial differential equation of the first order, which can be solved by the method of Lagrange-Charpit. The characteristic curves are solutions of the following ordinary differential equation.

$$\frac{dt}{u} = -\frac{du}{t},$$

whence

$$\Phi = \Phi(t^2 + u^2) = \Phi\{m^2[x^2 + (y - \eta)^2]\} .$$

Thus the Green's function depends only on the radial distance from the point $(0, \eta)$.

7.22. To gain more detailed information about the general form of Φ, we return to Eq. (131), differentiating both sides twice with respect to u

$$\int_0^\infty \frac{\partial^2\Phi}{\partial u^2} e^{-\alpha t}\, dt = -(\alpha^2 + 1)^{1/2} \sin u(\alpha^2 + 1)^{1/2}$$

$$= -(\alpha^2 + 1)\int_0^\infty \Phi e^{-\alpha t}\, dt. \tag{134}$$

The right-hand side of Eq. (134) can be integrated by parts as follows

$$-(\alpha^2 + 1)\int_0^\infty \Phi e^{-\alpha t}\, dt = -\int_0^\infty \Phi e^{-\alpha t}\, dt - \int_0^\infty \Phi \frac{\partial^2}{\partial t^2} e^{-\alpha t}\, dt$$

$$= -\int_0^\infty \Phi e^{-\alpha t}\, dt - \int_0^\infty \frac{\partial^2\Phi}{\partial t^2} e^{-\alpha t}\, dt + \left[\alpha\Phi e^{-\alpha t} + \frac{\partial\Phi}{\partial t} e^{-\alpha t}\right]_0^\infty .$$

Substituting this result in Eq. (134) gives

$$\int_0^\infty \left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial t^2} + \Phi\right) e^{-\alpha t}\, dt = \alpha\Phi(0, u) + \left(\frac{\partial\Phi}{\partial t}\right)_{t=0} .$$

Finally, we differentiate the above equation twice with respect to α to obtain

$$\int_0^\infty \left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial t^2} + \Phi\right) t^2 e^{-\alpha t}\, dt = 0.$$

Again, this equation can hold for all u and α only if Φ is a solution of

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial t^2} + \Phi = 0. \tag{135}$$

It has already been shown. however, that Φ is a function of only one independent variable, $(t^2 + u^2)$. Introducing this information into Eq. (135), we find that Φ must satisfy the following linear ordinary differential equation:

$$\frac{d^2\Phi}{dz^2} + \frac{1}{z}\frac{d\Phi}{dz} + \Phi = 0$$

where $z^2 = t^2 + u^2$. This equation will be recognized as Bessel's equation of order zero, whose solutions are Bessel functions of order zero with real argument.

$$\Phi = \begin{cases} J_0(z) = J_0(\sqrt{t^2 + u^2}) = J_0[m\sqrt{x^2 + (y - \eta)^2}] \\ Y_0(z) = Y_0(\sqrt{t^2 + u^2}) = Y_0[m\sqrt{x^2 + (y - \eta)^2}]. \end{cases}$$

The Green's function is therefore a linear combination of such Bessel functions.*

$$\Phi(x - \xi, y - \eta) = AJ_0(mr) + BY_0(mr) \tag{136}$$

where $r^2 = (x - \xi)^2 + (y - \eta)^2$.

* The author has since discovered that $A = 0$ and $B = 1$. If anything, this result strengthens the arguments to follow, because the Bessel function of the second kind has a singularity at the origin. The latter implies that by far the largest contributions come from conditions near the origin, whence the effective domain of dependence is even smaller than that estimated from the behavior of the Bessel function of the first kind.

It should be noted that, in this case, the Green's function is symmetrical around the point (x, y).

7.29. The significance of these results lies in the behavior of $\Phi(mr)$ as r increases from zero to very large values. Both kinds of Bessel functions and, consequently, the Green's function itself decrease rapidly as r increases from zero to the first root of $J_0(mr) = 0$. Beyond the second root of $J_0(mr) = 0$, the behavior of the Bessel functions is closely approximated by their asymptotic expressions

$$J_0(mr) \simeq \left(\frac{2}{\pi mr}\right)^{1/2} \cos\left(mr - \frac{\pi}{4}\right)$$

$$Y_0(mr) \simeq \left(\frac{2}{\pi mr}\right)^{1/2} \sin\left(mr - \frac{\pi}{4}\right).$$

Thus as r becomes increasingly large, the Bessel functions oscillate around the value zero with very slowly decreasing amplitude. The full wavelength of the oscillation is approximately given by the asymptotic expressions as $2\pi m^{-1}$, which, for values of τU of the order of 20 meters per second, is about 4000 miles. This distance is comparable with the widths of the major continents and oceans. For this reason, owing to the oscillatory properties of the Green's function *beyond the second zero of* $J_0(mr) = 0$, the negative contributions to the integral (129) due to surface irregularities beyond that radius will tend to compensate the positive contributions. We may, therefore, take the radius of the effective domain of dependence to be somewhere between the second and third zeros of $J_0(mr) = 0$—i.e., comparable with the first "wavelength" of the Bessel functions. The first full cycle is completed at about $mr = 7$, whence the radius of the domain of dependence is of the order of three or four thousand miles. This result implies, for example, that the flow over the mountains of the western United States is not much affected by the presence of the Himalayas, a fact which was tentatively assumed in paragraph 7.25.

7.30. To summarize the results of this section, the solutions z_F and z_M—corresponding to large-scale transient disturbances and the forced oscillations due to irregularities of the underlying terrain—are presented in formulas (117) and (129). Both are given in terms of quantities which are initially known or computable. That is to say, the initial values are expressed in terms of the undifferentiated contour height, a quantity which is measured as a matter of routine, and the Green's functions can be evaluated in terms of already tabulated analytic functions. Finally, it has been pointed out that Eq. (95), representing the effects of nonlinearity, is of the same general form as Eq. (96). Although the solution of Eq. (95) requires a few modifications of the methods used to solve Eqs. (94) and (96), it can be carried out in much the same way. In short, the "linear" prognostic equation (Eq. (92)) can be solved completely

8.00 A METHOD OF NUMERICAL PREDICTION

8.01. Formulas (117) and (129) provide the basis for a rational system of predicting the height of an isobaric surface at the equivalent-barotropic level. This method we shall briefly outline below, leaving the detailed description of its application for the second report mentioned in the foreword. (a) Starting with the initial distribution of contour height, we first compute τ and the mean zonal component U of the horizontal velocity. (b) With this information, z_M can be computed from Eq. (129). (c) We next subtract z_M from z, the total initial contour height, to obtain $(z_F + z_N)$. (d) Using the latter distribution of contour height, we calculate the initial values of $N_i(x, y)$ and compute z_N by a method similar to that used in solving Eq. (96). (e) The initial values of z_F are obtained by subtracting z_N from the previously calculated values of $(z_F + z_N)$. (f) We next compute the predicted value of z_F from formula (117), continuing the solution only a short time beyond the initial moment. (g) Finally, the *total* contour height z at a time later than the initial moment is obtained by adding z_M, the initial value of z_N, and the predicted value of z_F. The predicted distribution of contour height at a time later than the initial moment is then regarded as a new set of initial data, whence the process outlined above can be repeated indefinitely, until the aggregate of short time intervals has reached the required total length.

8.02. To carry out this scheme completely would involve a tremendous number of calculations—so many, in fact, that it would be impossible to carry them out "by hand" in a time comparable with the length of the forecast period, and barely feasible to carry them out with the automatic computing machines available at the present time. Judging from the numerical experiments of Charney, Fjörtoft, and von Neumann (1950), the time interval between successive stages of the iteration should not be more than two or three hours, so that a twenty-four hour prediction would require on the order of ten iterations. In short, the application of this method to the *practical* problem of contour prognosis would still pose a very formidable computational problem, even if automatic facilities were available exclusively for that purpose.

8.03. It has already been noted, on the other hand, that z_M and z_N are frequently small in comparison with z and z_F. Under some conditions, therefore, it may be permissible to regard the observed irregularities in the initial distribution of contour height as due entirely to linear transient disturbances. From this point of view, the first five steps outlined in paragraph 8.10 simply add small corrections to the prediction that would result if formula (117) were applied directly to the observed initial values of contour height. The price one pays for those corrections, moreover, is a tenfold increase in the total number of computations, for the solution (117) depends explicitly on the length of the forecast period and can be carried out over a single time-stage. From a purely economic standpoint, therefore, it is desirable to apply formula (117) to the observed initial values of contour height, if only to establish the possibility that the resulting uncorrected predictions are significantly less accurate than the predictions which are corrected for the effects of nonlinearity. Unless the inclusion of nonlinear effects produces a signi icant increase in accuracy, the enormous added expense of including them would hardly be justified. With the passing remark that corrections for the terrain-induced forced oscillation can be included without materially increasing the total effort, we shall next turn to a discussion of the results of applying the solution for transient disturbances to observed (but uncorrected) initial values of contour height.

8.04. As the first step in testing the general validity of the theory, formula (117) was applied to two sets of initial data, which had already been exhaustively analyzed in connection with other studies. Aside from the fact that the network of observations was particularly dense at those times, there was no basis for selecting those two cases in preference to any others. When this report was begun, the twenty-four hour predictions computed from the two sets of initial data had just been completed.* The results of both computations are presented in the last part of this section, along with the sequence of events that was actually observed.

8.05. In order to give a brief description of the manner in which formula (117) has been applied to the problem of predicting the height of an isobaric surface at the equivalent-barotropic level, it is simplest to describe how all the quantities which enter into the formula—the initial values, the Green's function, and the coefficients of Eq. (94)—have been computed from the basic data, discussing each quantity separately. It is natural to begin with the initial values of z, which are most directly connected with physical measurements. The basic data are routine observations of wind and pressure, received via teletype from a network of U. S. Weather Bureau, U. S. Navy Aerological Service, Air Weather Service and Canadian weather stations over the continental U. S., the North Atlantic, the Western Pacific, Canada and Alaska. This information is transmitted in standard numerical code, giving the wind direction to the nearest value on a 36-point scale, the wind speed to the nearest five knots, and the height of certain selected isobaric surfaces to the nearest

* In the meanwhile, a series of twenty-four such predictions has been completed. The results of the latter test will be discussed in the second report referred to earlier.

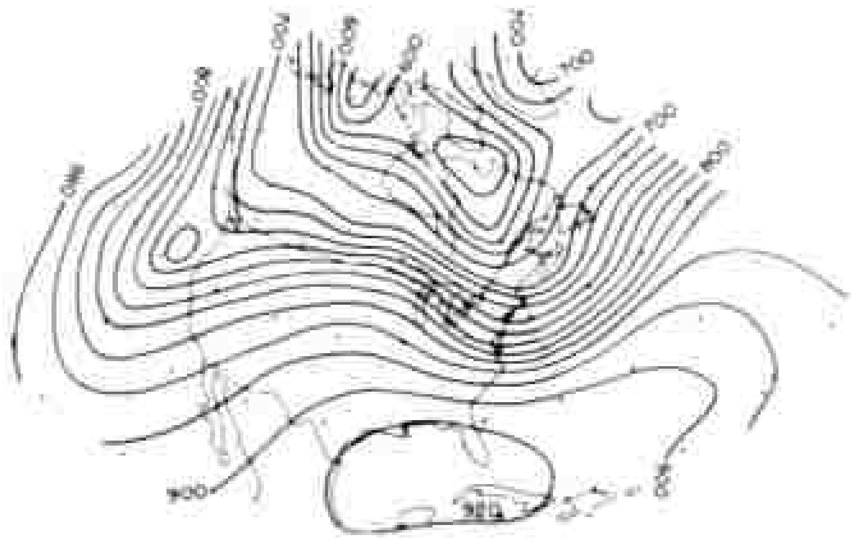
multiple of ten feet above mean sea level. The encoded pressure data, together with the wind velocity, were plotted on blank maps, each chart representing conditions on one of the standard surfaces of constant pressure—i.e., the 1000, 850, 700, 600, 500, 400, 300, 200 and 100 millibar surfaces. The wind direction within each pressure surface was represented graphically, by drawing in (with a protractor) lines with the directions of the wind vectors. The topography of each surface of constant pressure was then reconstructed by interpolating continuous contours at intervals of 100 feet. In regions where the data were quite dense, special care was taken not to violate direct measurements of height in favor of wind observations. On the other hand, wherever several adjacent observations of wind and pressure or of pressure alone appeared mutually inconsistent, the radio-balloon soundings were checked for internal consistency by using the hydrostatic equation. In regions where the data were quite sparse, a little more emphasis was placed on the wind observations—that is to say, the true wind was identified with the geostrophic wind, so that the direction of the contours and (to some extent) the spacing of the contours correspond to the direction and speed of the true wind. In addition, in regions where data are transmitted irregularly or where the network of observations is particularly loose, the topography of the pressure surfaces was partially inferred, by requiring spatial continuity in the horizontal and from one level to another, and by assuming temporal continuity over a series of 12-hour intervals. The first numerical experiment was based on four sets of contour charts, representing the topography of the standard pressure surfaces at the times 0300 hours Z (Greenwich) Time, 8 January 50; 1500Z, 8 January 50; 0300Z, 9 January 50; and 1500Z, 9 January 50. The topography of the 500-millibar surface at each of those times is displayed, for purposes of orientation, in Fig. 6, (a), (b), (c) and (d). The first two sets were regarded as initial data and the last two as verification data. The area covered by each analysis extended from 20° north latitude to 75° north latitude and from 40° west longitude to 140° west longitude. The density of the network of radiosonde, pibal, rawinsonde, and surface observation stations over that area is, of course, greater than that over any other region of the earth.

8.06. Preliminary estimates of the vertically integrated mean winds, computed at a number of points over the area of the data, showed that the equivalent-barotropic level lay at a mean pressure altitude of about 550 millibars. This height was considered sufficiently close to the 500-millibar surface to warrant applying formula (117) to conditions at the latter level, which is one of the so-called "standard" levels. The height of the 500-millibar surface at the intersections of meridians and latitude circles, spaced five degrees apart in either direction, was estimated by interpolating between continuous contours. At this point, it was realized that a perfectly legitimate (although degenerate) solution of Eq. (94) is the function

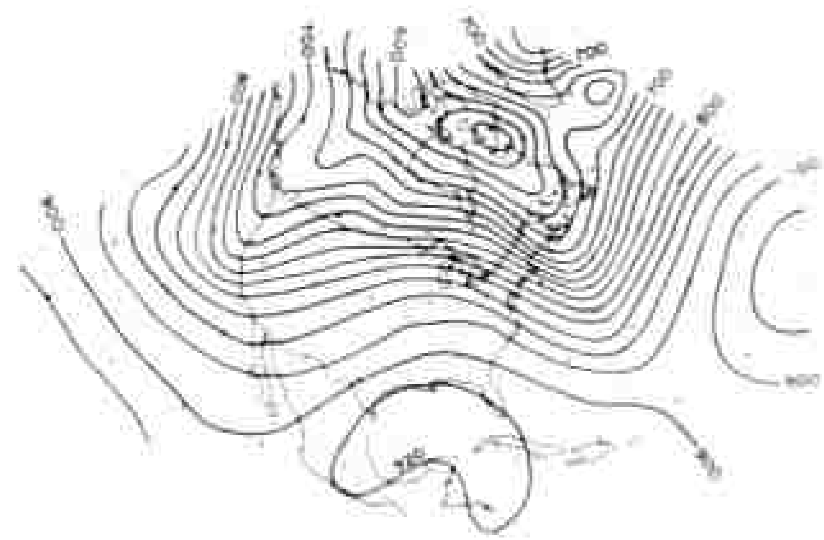
$$z = \bar{z} - \lambda g^{-1} U y,$$

where $\bar{z}$ is a constant, for convenience chosen equal to the mean height of the 500-millibar surface integrated over the entire extent of the data. This solution corresponds to zonal flow at a uniform speed U. It is also quite clear, for physical reasons, that the integral on the right-hand side of Eq. (117) must vanish in the case of uniform zonal flow. To insure that the solution would reduce properly in this special case, the height of a fictitious plane of constant pressure, whose mean height is $\bar{z}$ and whose slope is that required to maintain zonal flow with a uniform speed U, was subtracted from the previously tabulated height of the 500-millibar surface. The resulting differences at standard gridpoints (the intersections of meridians and latitude circles, spaced at five-degree intervals) were then recorded on punch cards with a locator index. The latter data are the initial values $z(x - \xi, y - \eta, 0)$ which enter into the integral on the right-hand side of formula (117).

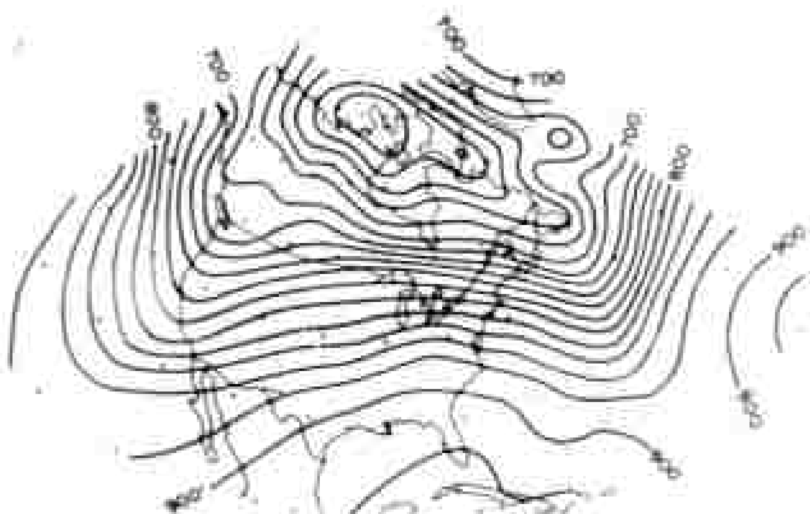
8.07. Because μ was set equal to zero in the course of evaluating the Green's function $T(x - \xi, y - \eta, t)$, only two of the coefficients in Eq. (94) appear in formula (117). These are the parameters τU and β. The latter depends only on the angular speed and radius of the earth and the geographical latitude of the point (x, y). The first factor τ in the one remaining parameter was computed as a mean of the values at the



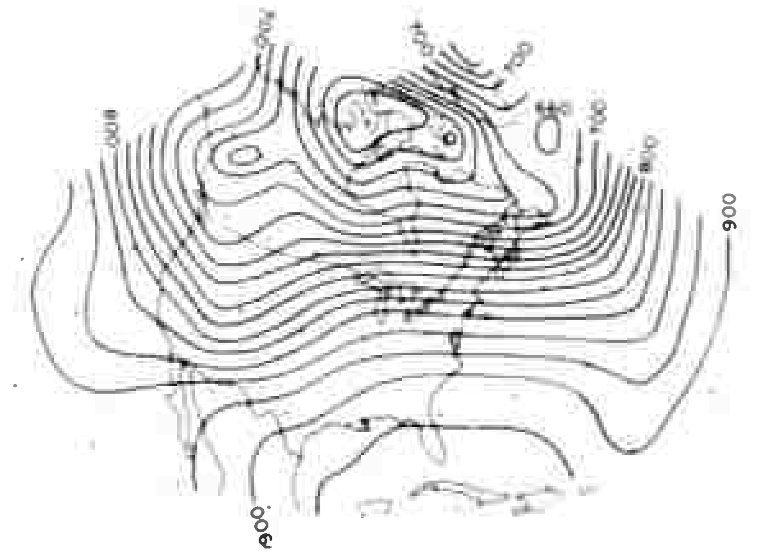
(a) 0300Z 8 JANUARY 1950



(b) 1500Z 8 JANUARY 1950



(c) 0300Z 9 JANUARY 1950



(d) 1500Z 9 JANUARY 1950

Fig. 6. Observed contours of the 500-millibar surface.

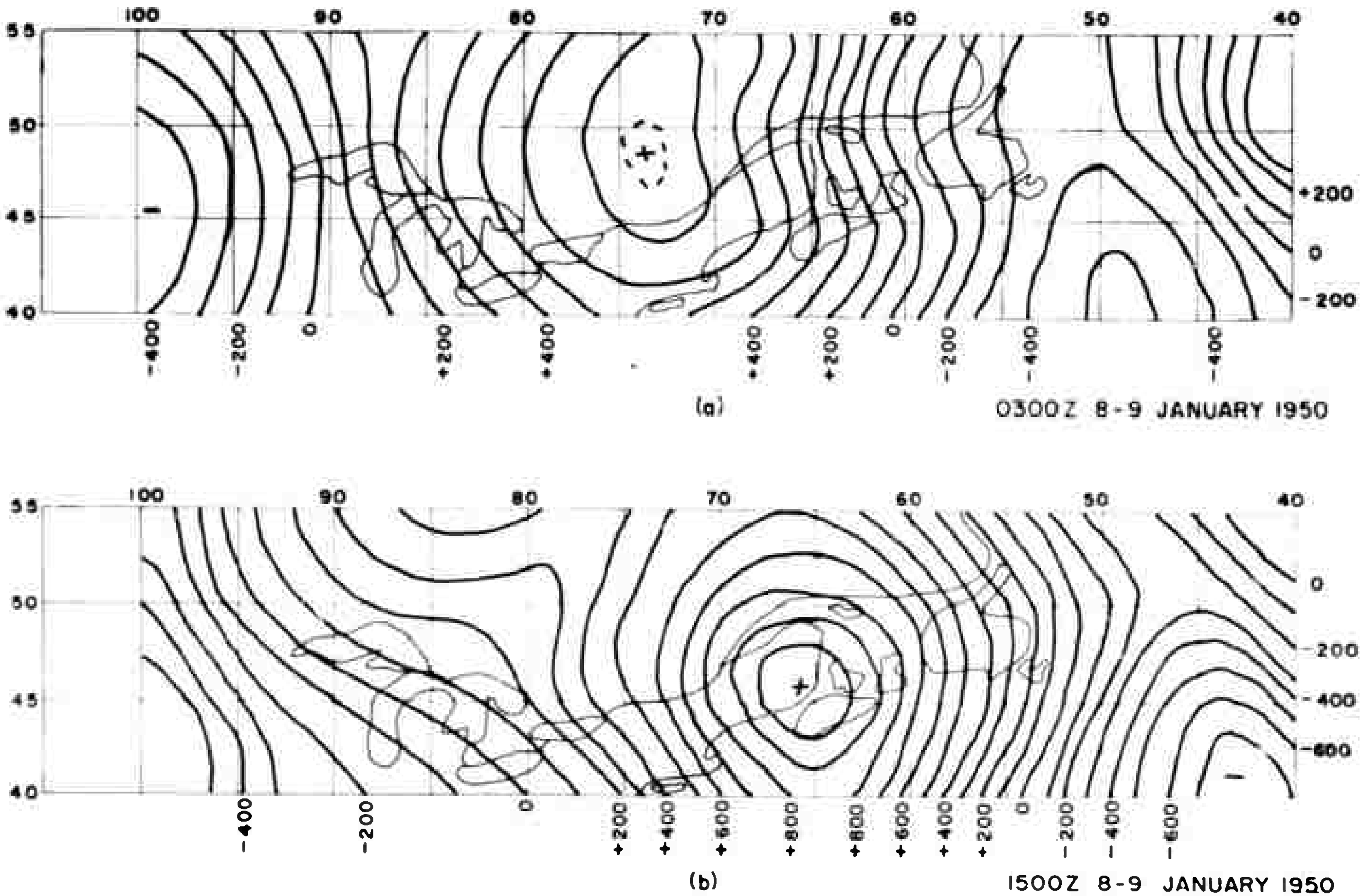


Fig. 7. Predicted 24-hour changes of 500-millibar contour heights.

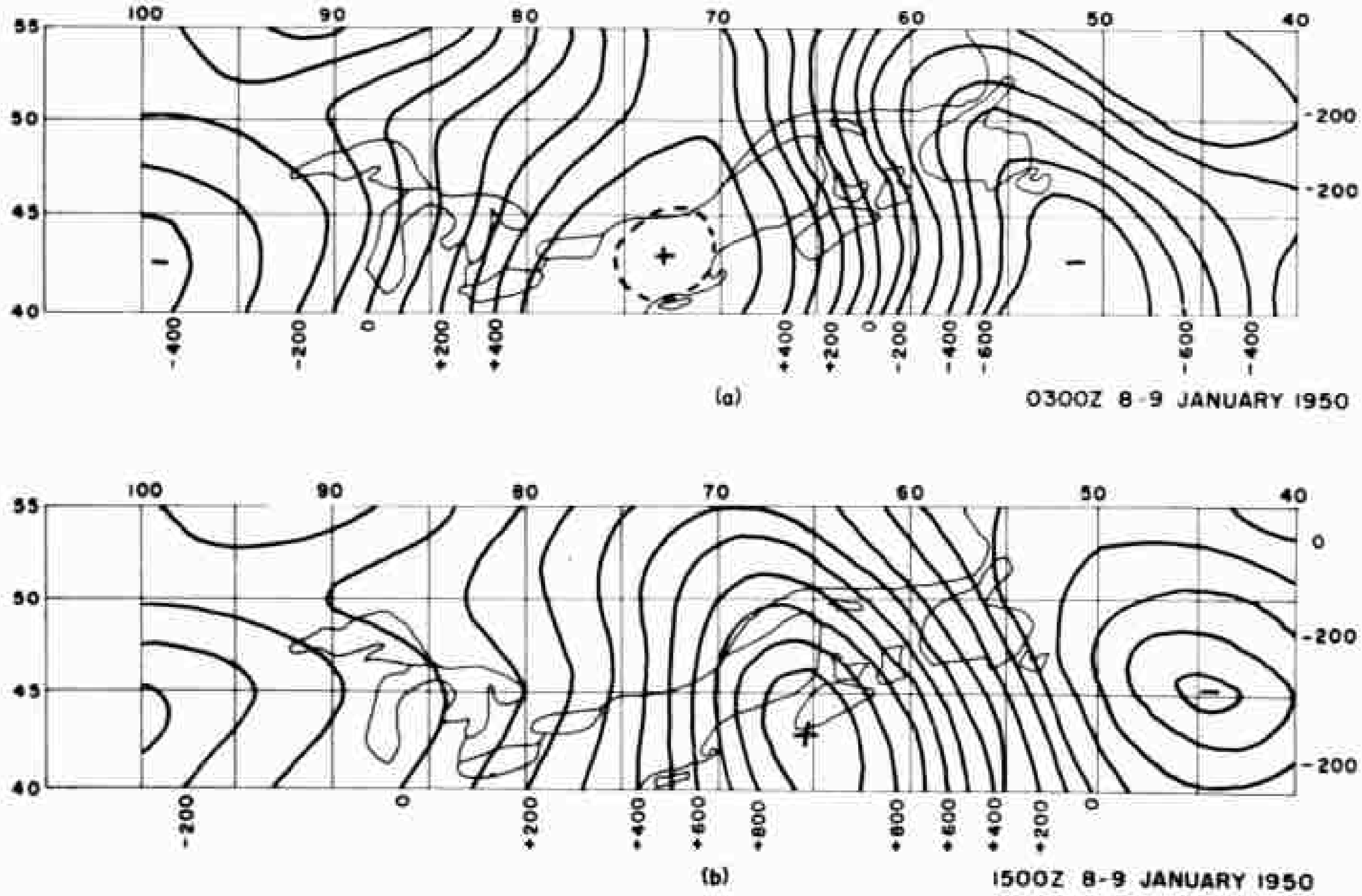


Fig. 8. Observed 24-hour changes of 500-millibar contour heights.

standard gridpoints, integrated over the entire area of the data. U was computed as a longitude-dependent mean value of the zonal component of the geostrophic wind at the standard gridpoints, integrated over intervals of 25 degrees of longitude and over the entire north-south extent of the data. Each "running" mean of the zonal wind was considered to apply at the center of its region of summation.

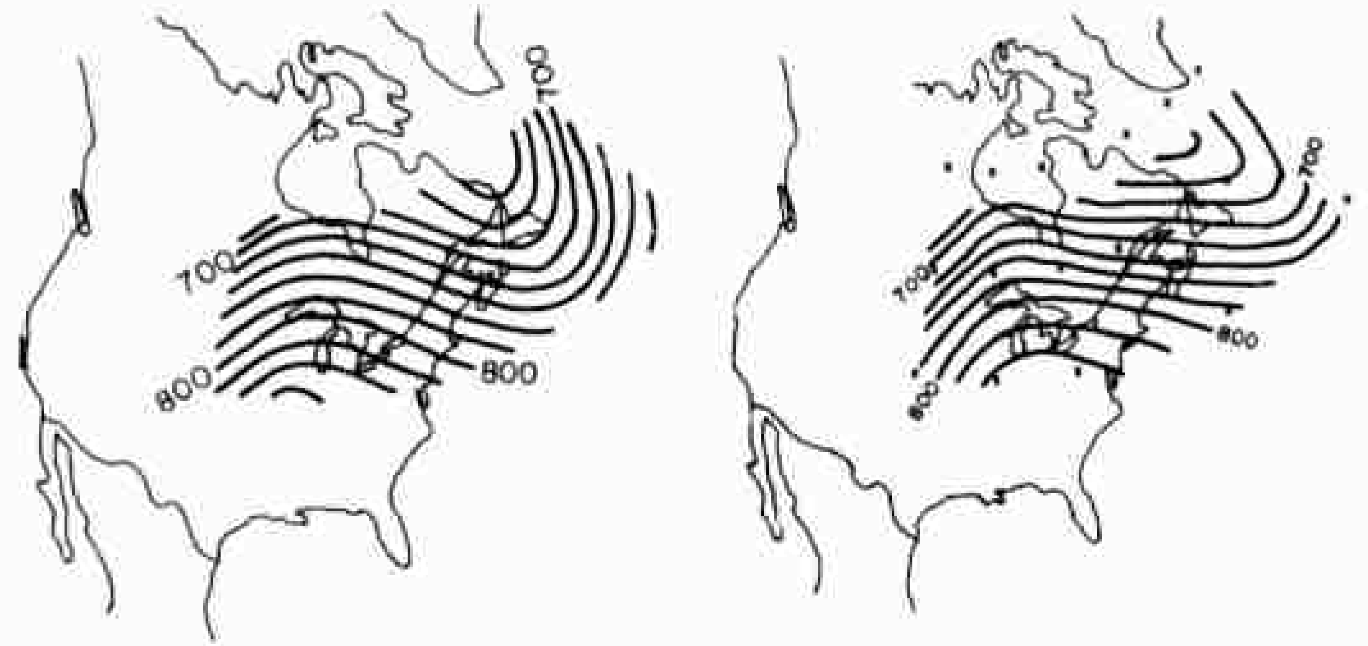
8.08. As indicated earlier, the Green's function $T(x - \xi, y - \eta, t)$ has been computed for a number of values of the single parameter βt, each corresponding to a combination of the latitude of the point (x, y) and the length t of the forecast period. The values of the T functions at gridpoints spaced five degrees of longitude apart in one direction and five degrees of latitude apart in the other were also recorded on punch cards, each deck of cards corresponding to a single value of the parameter βt. The area for which the T functions were recorded extends at least 20 degrees of latitude or longitude in any direction from the origin (x, y), approximately covering the effective domain of dependence.

8.09. The integral on the right-hand side of formula (117) was computed by finite sums for each gridpoint (x, y), ranging from 40° north latitude to 55° north latitude and from 60° west longitude to 120° west longitude. The first stage in computing the integral was to select the value of βt appropriate to the geographical latitude of the point (x, y). Two decks of cards, one bearing the coded values of the T function for the proper value of βt and the other bearing the initial values, were next arranged in such a way that the two cards with the same locator index $(x - \xi, y - \eta)$ were paired together. The combined deck was then passed through a Remington-Rand automatic multiplier, to form the products of paired values of the initial height-disturbance and the Green's function at the points $(x - \xi, y - \eta)$. The output of this operation is the sum of all such products, taken over the 81 points for which the T function was recorded—i.e., over the effective domain of dependence. That sum is an approximation to the integral at the point (x, y).

8.10. Two more operations were carried out before arriving at the predicted height of the 500-millibar surface. First, the computed value of the integral at the point (x, y) was added to the *total* initial height of the 500-millibar surface at the same point, to obtain the value that would apply if there were no mean zonal flow. Finally, the point (x, y) to which that value is attached was displaced a distance τUt toward the east. The resulting value of contour height is the one which, on the basis of formula (117), is predicted to occur at a time 24 hours later than the initial moment.

DISCUSSION OF RESULTS

8.11. The results of these computations are summarized in Fig. 7 (a) and (b), in the form of geographical distributions of the predicted change in the height of the 500-millibar surface, over the 24-hour periods ending at 0300Z, 9 January 50 and 1500Z, 9 January 50. These results are to be compared with the observed changes over the same periods, as presented in Fig. 8 (a) and (b). In both cases, the predicted positions of well-defined maxima and minima of height *change* are located within a distance of five degrees of latitude from their observed positions. With regard to the magnitude of the changes, it should be noted that the predicted height changes over the first period are systematically greater (more positive) than the observed changes—i.e., the predicted mean value of the contour height is greater than the actual mean value. In this same connection, it should also be noted that Eq. (94) is homogeneous and contains no term of order zero. This implies that the solution of Eq. (94) can predict the height change only to within the value of an arbitrary additive constant or, in other words, that this method is incapable of predicting a general rise in pressure over a limited area. If the area of the data extended over a much larger region, perhaps over an entire hemisphere, it would probably be safe to assume that the mean value of contour height integrated over that area does not vary from one day to the next. At the same time, it must be pointed out that we are not primarily interested in predicting the *absolute* height of the 500-millibar surface, but in predicting the



(a) 0300Z 9 JANUARY 1950 (b) 1500Z 9 JANUARY 1950

Fig. 9. Predicted contours of the 500-millibar surface.

circulation resulting from *variations* in the contour height. The latter depends only on derivatives of the contour height, and is therefore independent of the mean height. In order to estimate the accuracy with which this method can predict *relative* changes in height, we therefore focus attention on the relative strength of the maxima and minima. In the first case, the predicted difference between the maximum and minimum height changes is about 1600 feet, as compared with the observed difference of about 1450 feet. In the second case, the predicted relative amplitude of the height change was about 2100 feet, whereas the observed difference between the maxima and minima was around 1600 feet. In judging the accuracy of these predictions, it should be borne in mind that the probable error in the present system of pressure measurement is of the order of 1 millibar, which, at the 500-millibar level, corresponds to a height error of 50 feet. This is the irreducible minimum of error in the predicted values of contour height. Comparing errors against that standard, the predicted distributions of height change are in very good agreement with the observed distributions.

8.12. These same results are presented in another way in Fig. 9 (a) and (b), which are simply the distributions of contour height predicted to occur at 0300Z, 9 January 50 and at 1500Z, 9 January 50. Turning back to the sequence of contour charts in Fig. 6, one is most struck with the very slow movement of the "ridge" of high pressure over the central United States and the relatively rapid eastward movement of the trough off the east coast, with a resulting increase in wavelength. In the 24-hour period from 0300Z, 8 January 50 to 0300Z, 9 January 50, for example, the ridge line moved only about 7 degrees of longitude, from a mean longitude of 97° W to 90° W, while the trough line moved about 13 degrees of longitude, from 69° W to 56° W. In that 24-hour period, the half-wavelength increased from 28 degrees of longitude to 34 degrees of longitude. As indicated by Fig. 9 (a), the ridge line was actually predicted to move eastward a distance of 7 degrees of longitude, while the trough line was predicted to move very rapidly to longitude 50°, a distance 19 degrees of longitude east of its initial position. Accordingly, it was predicted that the half-wavelength would increase from 28 degrees of longitude to 40 degrees of longitude.

8.13. The second case is even more striking. For 24 hours after 1500Z, 8 January 50, the ridge of high pressure remained stationary at a mean longitude of 90° W, while the trough continued to move rapidly off the east coast. In the period from 1500Z, 8 January 50 to 1500Z, 9 January 50, the trough moved about 12 degrees of longitude, from a mean longitude of 63° W to 51° W. The half-wavelength increased by a distance of 12 degrees of longitude, from 27 degrees to 39 degrees of longitude. According to Fig. 9 (b), it was actually predicted that the ridge line would remain almost stationary over that period, moving only about 2 degrees of longitude from 90° W to 88° W. The trough, on the other hand, was predicted to move very rapidly to a mean longitude of 45° W, a distance of 18 degrees of longitude in 24 hours. The predicted increase in the half-wavelength was therefore 16 degrees of longitude, as compared with the actual increase of 12 degrees. Needless to say, it would be extremely difficult to predict such changes in wavelength by the extrapolation methods in common use, especially in view of the fact that the eastward movement of the ridge during the 12-hour period following 0300Z, 8 January 50 ceased abruptly after 1500Z, 8 January 50, the ridge line remaining stationary for the succeeding 24 hours.

8.14. In concluding the discussion of these results, it should be emphasized that the two cases presented here were the first two to be analyzed by these methods, and were not selected as the best examples from a larger number of cases. As it turned out later, the two described above were better than the average of 24 such predictions, but were still not among the exceptionally good examples.

9.00 SUMMARY, CONCLUSIONS AND OUTLOOK

9.01. In the eight preceding sections, we have discussed at considerable length the general problem of predicting the behavior of large-scale disturbances in the flow of the earth's atmosphere. It is a matter of experience that such disturbances, gradually growing and moving slowly eastward for periods of days or weeks before they finally decay, are directly associated with the more tangible aspects of weather. Because of the impossibility of observing the complete state of a physical system on so grand a scale, and because of other difficulties inherent in the general problem, we have been forced to retreat further and further toward successively more special problems, first restricting attention to the problem of predicting the mean local state of the atmosphere, and finally to that of predicting the behavior of large-scale disturbances of atmospheric pressure.

9.02. As we have seen, even the special problem of predicting the behavior of macroscopic pressure disturbances is far too general, for the complete hydrodynamical equations possess solutions corresponding to sound and gravity waves, which, from our limited point of view, are irrelevant and simply obscure the solutions in which we are primarily interested. In order to isolate the meteorological aspects of the problem and to reframe it in terms which make it explicitly meteorological— rather than acoustical or aerodynamical— a "scale theory" of atmospheric motions has been developed for classifying the various types of motion according to the values of certain nondimensional characteristic numbers. It has been shown, for example, that the large-scale "meteorological" disturbances are distinguished from all other types of motion by the fact that their characteristic phase speed (relative to the medium of propagation) is much less than the speed of sound and of high-speed internal gravity waves. By introducing the "filtering approximations"—i.e., the strong inequalities which characterize the large-scale motions—it is then possible to reduce the complete system of hydrodynamical equations to a single equation which governs only the large-scale motions, the extraneous solutions corresponding to sound and gravity waves having been excluded.

9.03. The development of a suitable prognostic equation is necessarily centered around one of the many forms of the vorticity equation. We have derived a vorticity equation which applies to the adiabatic flow of an ideal gas in quasi-hydrostatic equilibrium, introducing for convenience a system of quasi-Lagrangian coordinates in which the entropy replaces the vertical coordinate. In order to obviate the difficulty of computing the vertical component of velocity, we have completely eliminated vertical dependence, simply by integrating the vorticity equation through the entire depth of the atmosphere. The resulting equation applies to the density-weighted mean value of the horizontal component of velocity, integrated vertically from the surface to an infinite height. It therefore governs the motions of a fictitious two-dimensional fluid which, in a mathematical sense, is equivalent to the true atmosphere. In the course of developing the mean vorticity equation, it has been found convenient to consider a special, but frequently observed type of baroclinic flow—namely, flow in which the wind direction is independent of height. It has been shown, in fact, that the direction of adiabatic flow associated with the *very large-scale* disturbances cannot vary appreciably from one level to another.

9.04. Upon introducing the "filtering approximations," the mean vorticity equation reduces to a third-order partial differential equation, involving only one dependent variable—the pressure or, which is the same thing, the height of an isobaric surface. This equation—the so-called prognostic equation—is demonstrably free of the major difficulties inherent in the unreduced primitive equations, but has the considerable disadvantage of being nonlinear. An iterative scheme is therefore proposed for solving the nonlinear prognostic equation, based on the solutions of a succession of linear equations. In the course of developing this method, we have been led to consider the linear equations for large-scale transient disturbances and for the forced oscillations induced by irregularities of the underlying terrain. The general solutions of those equations have been expressed in terms of known initial values and the tabulated values of the appropriate Green's functions, which are analytic and independent of the initial data. These solutions form the basis for a rational system of prediction.

9.05. Finally, as a simple and rather sensitive test of the theory, the solutions corresponding to large-scale transient disturbances have been applied to observed initial conditions. The resulting predictions of the height of an isobaric surface near the equivalent-barotropic level are in good quantitative agreement with the observed facts. Although the number of cases presented here certainly does not justify an unqualified positive statement, these results indicate that the theory is essentially correct.* In this connection it should be pointed out that the predictions exhibited here are based on the solutions of linear equations. It is probably safe to say that the proposed iterative scheme for solving the nonlinear prognostic equation would yield better results.† The extent to which the inclusion of nonlinear effects improves the accuracy of prediction will be the subject of future studies.

9.06. With regard to practical applications, the most striking aspect of the theory is that it provides a general method for predicting the large-scale mean flow of the atmosphere. It will be shown in the second report that the general level of accuracy of predictions based on formula (117) is comparable with that attainable by a skilled forecaster, armed with techniques in current use in the field. It also appears, however, that the accuracy of contour predictions can be significantly improved by discriminate application of the method.

9.07. A second source of power lies in the fact that the application of this method, because of the very objectivity of a rational system, can be reduced to a routine and can therefore be carried out by machine methods. At first glance, it might appear that the procedure outlined in paragraph 8.05 necessarily contains

* Again, it must be acknowledged that this theory differs from the Charney-Eliassen theory principally in the method of development and in the method by which the prognostic equation has been solved. The prognostic equation itself differs only in minor respects.

† Of the 22 predictions completed since the beginning of this report, the worst few were computed from initial conditions in which the amplitudes of the disturbances obviously were not small.

some element of the subjective, in that the continuous contours of the isobaric surfaces are generally drawn in "by eye." However, since the only purpose which this operation serves is to facilitate the interpolation of contour heights at standard gridpoints, it can just as readily be carried out by machine. That is to say, the initial values at the gridpoints can be computed in terms of the initial values over stations in the immediate neighborhood of each gridpoint, from a numerical interpolation formula whose coefficients depend only on location and not on time. Because the coding and programming of the interpolations is an unvarying routine, this entire operation can be made a "built-in" function of a special machine, possibly a device of the analogue type. It is frequently argued that the analyst performs the additional functions of discovering gross errors and maintaining the internal consistency of the initial data. But it is certainly possible, by forcing the meteorologist to analyze his impressions, to establish acceptable standards of compatibility, deviations from which can be detected even by a machine. In any case, so long as the analyst can give adequate reasons for the things he does, his functions must be capable of some more or less objective description, whence the routine functions of data analysis can, in principle, be carried out by an automaton. In short, there are no obvious limitations—other than the purely engineering problem of modifying existing transcription equipment—to prevent carrying out all the operations of this method automatically, feeding data directly from the teletype equipment into a computing machine, with only the passive intervention of human hands.

9.08. Whether or not it is feasible, from the standpoints of logic and engineering, to predict the large-scale mean flow of the atmosphere by machine methods, it remains to discuss the economy of doing so. At the present time, by the efficient organization of human and machine effort, the computation of the predicted contour height over an area about the size of the United States can be carried out by formula (117) in a time comparable with the period of the forecast. If the entire process were made fully automatic, it could be carried out in a small fraction of the forecast period, even if standard production models of the "business-machine" type were the only ones available for this purpose. At the time of writing, in fact, the only automatic computing machines of standard make and proven reliability are those of the mechanical or electromechanical type.

9.09. To continue the discussion of the economy of machine methods, it is also pertinent to add that it would be neither necessary nor economical to install automatic forecasting equipment at each station. Because existing communications facilities provide data from a considerable area, permitting the prediction of conditions over a region of comparable size, it would be sufficient to maintain such facilities at only a few weather "centrals." Moreover, because it would require a highly trained team of specialists to operate and service the equipment, it would also be most economical to do so. The yearly cost of procuring, installing, continuously operating, and maintaining a facility of this type, amortized over the life of the equipment, would be comparable with the yearly salaries of ten professional forecasters.

9.10. It is also relevant to indicate the degree to which the normal activities of a weather service would be dislocated by the introduction of a few automatic forecasting units. As it stands, the method outlined in Section 8.00 requires no special data. It does, however, require that the winds be integrated through the entire vertical extent of the data, in order to locate the level at which the true wind equals the mean wind—i.e., to find the equivalent-barotropic level. Since those integrations comprise fully half the total computation time, the efficiency of the method would be considerably increased if those particular calculations could somehow be obviated. There is, in fact, a way of expressing the mean wind in terms of a single thermodynamic variable E, which bears the same relation to the mean geostrophic wind as the pressure p bears to the point value of the geostrophic wind. To show this, we introduce the geostrophic and hydrostatic relations into the definition of the density-weighted mean wind.

$$\bar{\mathbf{V}} = \frac{g}{p_0}\int_0^\infty \rho \mathbf{V}\, dz$$

$$= \mathbf{K} \times \frac{g}{\lambda p_0} \nabla \int_0^\infty p\, dz$$

Finally, making use of the equation of state,

$$\bar{\mathbf{V}} = \mathbf{K} \times \lambda^{-1} p_0^{-1} \nabla E$$

where*

$$E = \int_0^{p_0} RT\, dp.$$

Since it involves only the pressure and temperature, the new variable E can be evaluated directly from the original radiosonde record, in much the same way as the "height evaluation" is now carried out, and in a comparable time. If E were precomputed at the time when the "height evaluation" is normally carried out and transmitted as a matter of routine along with other meteorological variables, the components of the mean wind could later be computed simply by differentiating E, much as the pressure or contour height is now differentiated to obtain the geostrophic wind. In short, if the reliability of the method were proven sufficient to justify it, it might become desirable to adopt some slight changes of standard practice, to the extent of modifying existing methods of evaluating radiosonde records.†

10.00 CONCLUDING REMARKS

10.01. Although the physical nature of the atmosphere is not well enough understood to discuss it as if it were completely known, it is nevertheless hard to keep from speculating about the essential features of the physical system which determine its "meteorological" behavior. In fact, if a general theory of meteorological phenomena is ever to be evolved, it is desirable and perhaps necessary to hazard some guess as to how the "fundamental" but otherwise isolated problems of meteorology are related in physical fact and how their separate solutions are to be pieced together to form a single complete theory. Moreover, unless the entire problem is suddenly and completely resolved at a single stroke—which is unlikely—we shall eventually be forced into such speculation. At such time as two of the hierarchy of fundamental problems have been carried as far as their very isolation will allow, one must then consider the way in which the solution of one

* In view of the properties of its derivatives, the variable E appears to be an important meteorological quantity. It is most readily interpreted as a measure of the total pressure force acting on one side of a wall of unit width, extending from the surface to an infinite height, or of the total potential energy in a column of unit cross section and infinite height.

† It should also be noted that, if the initial values of E were precomputed and transmitted on a routine basis, there would be some point to redeveloping the theory from Eq. (81) onward along slightly different lines. It has already been shown that the mean geostrophic wind can be expressed in terms of derivatives of E:

$$\bar{\mathbf{V}} = \mathbf{K} \times \lambda^{-1} p_0^{-1} \nabla E.$$

Similarly, the mean vorticity is

$$\bar{\zeta} = \mathbf{K} \cdot \nabla \times \bar{\mathbf{V}} = \lambda^{-1} p_0^{-1} \nabla^2 E.$$

The introduction of these expressions into Eq. (81) leads to a new prognostic equation, identical in form to Eq. (83), but in which the contour height of an isobaric surface at the equivalent-barotropic level is now replaced by E. From that point onward, the development and solution of the prognostic equation follow the pattern outlined in Sections 6.00 and 7.00. The advantage of introducing the new variable E, aside from the obvious computational advantages of doing so, lies in the fact that it is no longer necessary to interpret the prognostic equation as applying at any one level. This obviates the need for locating the "equivalent-barotropic level"—a concept which, although convenient, is rather artificial.

problem affects the conditions of the other, whether the remaining problems have been solved or not. Now, it is certainly premature to anticipate the exact nature of the solutions to the fundamental problems of meteorology, but it is also true that to state a problem as a meaningful question is, in a certain sense, to reveal what one wishes and expects to attain. For these reasons, and in order to define more clearly the problem we have tried to solve, this report will be concluded with some speculative comments on the physical relationship between several of the fundamental and, by reason of their difficulty, isolated problems of meteorology. In substance, these remarks constitute the statement of one's viewpoint. They should therefore be received for what they are—as articles of meteorological faith.

10.02. To start with, it is probably safe to say that what makes the meteorological behavior of the atmosphere so distinctive lies in the peculiar way in which energy is supplied and withdrawn from the system, and in the way in which that energy is redistributed and dispersed within the system. The only thing that is particularly remarkable about the energy economy of the system is that the net rate at which energy per unit volume is gained and lost is evidently quite small when averaged over a volume with the horizontal dimensions of the characteristic wavelength of the large-scale disturbances and with the depth of the atmosphere. It is observed that the total heat energy of the atmosphere does not vary rapidly. Turning to the dynamical properties of the physical system itself, it also appears that the atmosphere reacts so rapidly to external impulses that energy from outside sources is redistributed and dispersed *on a large scale* almost as fast as it is fed in. This is manifested by the fact that the atmosphere never "blows up," creating surface pressures of several thousand millibars or, in other words, that there is never any great local concentration of energy. The features of the atmosphere which truly characterize its large-scale motions are a direct consequence of the postulated energy economy and energy-distributing properties of the system *provided* (1) that we consider the system *in the large* and (2) that the upper limit on the rate at which potential energy can be converted into kinetic energy and finally into the heat energy of molecular motion (or the rate at which large-scale disturbances can develop and die out) is not great. That is to say, if the atmosphere is capable of redistributing or dispersing energy on a large scale as fast as it receives it, then the system must always remain near the state of mechanical equilibrium. The latter, as we have already seen, is the key fact in the development of the theory of large-scale motions, from which virtually all other characteristics of the motions are deducible.

10.03. As a result of such considerations, one is led to conceive of the meteorological behavior of the atmosphere as a sequence of fairly distinct and isolatable physical processes.

(1) Because of local differences between incoming and outgoing radiation, the residual sources and sinks of heat energy induce weak pressure forces. The state ultimately attained as a result of rapid adjustment to those forces is one in which the pressure forces, due to the nonuniform distribution of heat sources, are almost exactly balanced by the Coriolis forces. This balance requires a mean or "general" circulation of the atmosphere, whose potential and kinetic energy is slowly built up to tremendous magnitudes over a long period of time. In general, this process tends to concentrate or localize the available energy of the general circulation in certain preferred regions of the atmosphere and it can continue (as an isolatable process) only as long as it does not create an inherently unstable situation. With modifications to be introduced later, the problem of giving a complete physical description of this process is the so-called "general circulation problem."

(2) Once a locally unstable situation has been set up as a result of the nonuniform distribution of energy sources, we then regard the general circulation as given and disregard all other effects of external sources of energy, except to postulate that there are frequent and perhaps random impulses of energy superposed on the undisturbed distribution of energy. In principle, the smallest impulse is sufficient to initiate a disturbance which, through adiabatic transformation of the potential and kinetic energy of the general circulation

into kinetic energy of the disturbance, will develop of its own accord. Since it is probable that the atmosphere does receive sudden and frequent impulses of energy, disturbances will tend to develop in regions where the local state is unstable and favorable to such development. In a manner of speaking, therefore, the only part which the energy from external sources plays in the development of disturbances is to set the stage for the development—i.e. to build up latent instability to the critical point, after which the energy of the disturbance is derived mainly from the energy of the general circulation by purely adiabatic processes. The problem of establishing the conditions under which small disturbances will develop spontaneously, without the further addition of energy from external sources, is the "stability problem."

(3) Because the conditions for instability are probably quite critical, and because the development of disturbances is evidently brought about by the juxtaposition of two independent (and not easily observed) sets of circumstances—namely, the condition for instability, taken together with the occurrence of an impulse sufficient to initiate the disturbance—the problem of predicting the initial development of disturbances is an inherently difficult one. The best one is likely to do, therefore, is to place the probability of development within certain broad limits determined by observability. If, on the other hand, the disturbance has already developed to a perceptible degree, there is some hope of predicting its further development as if it were brought about entirely by adiabatic readjustment of the existing energy distribution, because the change in the energy of the developing disturbance is much too great to be accounted for by nonadiabatic energy changes alone. On the other hand, as has been pointed out earlier, the period of full development is on the order of days, rather than of hours, whence the instability of the atmosphere is not so great that we are forced to deal with this problem in all cases. For purposes of predicting the course of events over two or three days, we might even treat the disturbance at each instant as if it were already fully developed, accepting its existence without inquiring into the manner in which it was created. The problem of predicting the future course of already developed or partially developed disturbances we shall call the "short-range forecasting problem." It might properly be termed the "propagation problem."

In passing, it should be noted that there is a serious consequence of accepting this view of the short-range forecasting problem. It is simply that there must be a rather low practical limit on forecasting accuracy. This limit is fixed partly by the detail in which one can observe both the conditions that are necessary for the development of new disturbances, but neither of which is in itself sufficient, and partly by the rate at which disturbances can develop spontaneously and without warning.

(4) Since disturbances are always present in the atmosphere, it is quite likely that they are instrumental in bringing about the redistribution of energy on a very large scale. In fact, as has been suggested by Priestly (1949), Starr (1948) and Bjerknes (1948), the horizontal eddy transport due to large-scale transient disturbances may be the dominant mechanism for redistributing the energy of the general circulation. This would bring us back full circle to the general circulation problem. One is thus led to think of the atmosphere as a gigantic feed-back mechanism, in which the structure of the general circulation (taken together with certain properties of the medium itself) controls the stability and propagation of the large-scale disturbances, and the disturbances in turn bring about the redistribution of energy necessary to build up and maintain the general circulation.

For this reason alone—i.e., because the conditions of one problem depend on the solution of a second problem which, in turn, depends on the solution of the first—the fundamental problems of meteorology are so inextricably bound together that it is apparently impossible to isolate them. As outlined in the foregoing discussion, however, it appears that there are certain points at which it is most natural and logical to separate them artificially, taking as external conditions for each problem *the observed solution* of the problem just preceding it in this hierarchy.

10.04. The problem that has been discussed here is, of course, only a small part of Problem 3. However tacitly it has been assumed, we have dealt with disturbances whose connection with agencies outside the system lies entirely in the past and whose energy in the future is withdrawn from the system itself. To this extent the conditions of the problem are unreal, but not so unreal as to make it impossible to predict the large-scale motions of the atmosphere over a period of a few days. Summarizing the situation, it appears that Problem 3 is on the way to solution.

10.05. With the gradual realization that the structure of the general circulation probably controls the growth of disturbances, increasing attention has been given to Problem 1. The recent results of Starr, Priestly, Bjerknes and Mintz (1949) hold out some hope of understanding how the energy from external sources can be redistributed on a macroscopic scale by the eddy transport of the large-scale disturbances. One must be aware, however, that this approach to the general circulation problem also postulates the existence of already developed disturbances. In short, even if Problem 1 were satisfactorily solved, the connecting link between Problems 1 and 3—the solution of the stability problem—would still be missing. In this and many other respects, therefore, both the connection between the general circulation and the large-scale transient disturbances and the key to a fairly complete understanding of the atmosphere's macroscopic behavior lie in the stability problem. Although certain limited aspects of this question have been studied very intensively, no theory has yet shown itself sufficiently general to apply to the large-scale disturbances of the atmosphere, and to draw together into one coherent whole the few isolated results that have been achieved in the past. The present point in the development of meteorological theory, therefore, appears an opportune time to review the general problem of atmospheric stability in the light of what is now known about the general circulation and the propagation of large-scale disturbances.

APPENDIX I

STABILITY OF SOLUTIONS OF HYPERBOLIC DIFFERENCE EQUATIONS

To illustrate the way in which errors can be amplified by the method of finite differences, we shall consider the simple case of periodic plane sound waves, reflected back and forth between two rigid walls parallel to the wave fronts. Such motions are governed by the hyperbolic wave equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

where c is the speed of sound, u is the component of particle velocity normal to the wave fronts and x is the perpendicular distance from one of the walls. Since the walls are rigid, the velocity of the particles in contact with them must vanish. That is to say,

$$u(0, t) = 0.$$

For convenience, the other wall will be fixed at $x = \pi$, so that

$$u(\pi, t) = 0.$$

It remains to specify initial conditions exactly sufficient to determine the solution. We shall imagine that initial velocities were imparted to the medium at various points, but that this state of motion was brought about so rapidly that the resulting divergence of mass could not immediately produce adiabatic changes in the initially uniform distribution of pressure. The latter clearly implies that there are no initial accelerations. We therefore take as initial conditions the statements that the initial velocities are known and that the initial acceleration is zero

$$u(x, 0) = \sin rx$$

$$\frac{\partial u}{\partial t}(x, 0) = 0,$$

where r is any integer equal to or greater than one.

The exact solution of this boundary- and initial-value problem is well known. It is

$$u(x, t) = \sin rx \cos rct.$$

The particle speed obviously vanishes at the walls at all times, and the accelerations are everywhere zero at the initial moment. By direct differentiation it can be verified that this function satisfies the wave equation. Moreover, because it also yields the correct initial velocities, it appears that the above expression is indeed the true solution of the problem.

We next turn to approximate solution of the same problem by the method of finite differences. This method consists in replacing the differential quotients in the wave equation by finite differences between the values of u at discrete points in the (x, t) plane. We begin by subdividing the space interval $(0, \pi)$ into N equal parts, each of length Δx. We also define discrete instants in time at intervals of Δt after the initial moment and consider *only those values of u which apply at the points whose coordinates are integer multiples of the intervals Δx and Δt.* The wave equation may now be written, in approximate form, as

$$k^2[u(n, m+1) - 2u(n, m) + u(n, m-1)] = u(n+1, m) - 2u(n, m) + u(n-1, m),$$

where $k = \Delta x/c\Delta t$ and the notation $u(n, m)$ is an abbreviation for $u(n\Delta x, m\Delta t)$. The quantities n and m are positive integers. The equation above is the so-called "difference equation" corresponding to the hyperbolic wave equation. With reference to the coordinate system of discrete points, the boundary conditions at the reflecting walls are now

$$u(0, m) = 0$$

and

$$u(N, m) = 0.$$

Similarly, if Δt is chosen very small, the initial conditions take the form

$$u(n, 0) = \sin \frac{rn\pi}{N}$$

$$u(n, 1) = \sin \frac{rn\pi}{N} \cdot$$

It is clear that the value of u at each of the discrete points (n, m) is completely and uniquely determined. That is to say, the difference equation expresses the value of u at the point $(n, m + 1)$ in terms of its values at the points $(n + 1, m)$, (n, m), $(n - 1, m)$ and $(n, m - 1)$. Thus, because the values at the points $(n, 0)$ and $(n, 1)$ are known, it is possible to compute the values at $(n, 2)$, and so on. It is important to note that, because the approximate solution is uniquely determined, any solution of the difference equation which satisfies the boundary and initial conditions must be identical with the one which is computed by the most obvious and direct means, namely, by repeated calculation.

It happens that the solution of the simple problem outlined above can be obtained in closed form. We first seek solutions of the "separated" type

$$u(n, m) = f(n)g(m).$$

Substituting this expression into the difference equation, and dividing both sides of the equation by $f(n)g(m)$,

$$\frac{k^2}{g(m)} [g(m + 1) - 2g(m) + g(m - 1)] = \frac{1}{f(n)} [f(n + 1) - 2f(n) + f(n - 1)].$$

The left-hand side of this equation depends only on the index m, while the right depends only on n. It therefore follows that each side is equal to a constant, e.g.

$$\frac{1}{f(n)} [f(n + 1) - 2f(n) + f(n - 1)] = -4 \sin^2 \frac{\alpha}{2},$$

where α is a conveniently chosen constant of separation. After some rearrangement, the above equation can also be written as

$$f(n + 1) - 2 \cos \alpha f(n) + f(n - 1) = 0.$$

We next set $f(n) = e^{i\beta n}$. The constant β is then fixed by the equation

$$e^{i\beta} - 2 \cos \alpha + e^{-i\beta} = 0$$

whence

$$\cos \beta = \cos \alpha.$$

Thus the admissible values of β are

$$\beta = \alpha \qquad\qquad \beta = -\alpha$$

and the corresponding solutions of the f-equation are

$$f(n) = ae^{i\alpha n} + be^{-i\alpha n} = A \sin \alpha n + B \cos \alpha n.$$

Thus far, no boundary and initial conditions have been imposed on $f(n)$ and $g(m)$. The conditions on $u(n, m)$ are evidently satisfied if

$$f(n) = \sin \frac{rn\pi}{N}$$

$$g(0) = 1 \qquad\qquad g(1) = 1.$$

The boundary conditions on $u(n, m)$ are satisfied because

$$f(0) = 0 \qquad\qquad f(N) = \sin r\pi = 0.$$

Moreover

$$u(n, 0) = f(n)g(0) = \sin \frac{rn\pi}{N}$$

$$u(n, 1) = f(n)g(1) = \sin \frac{rn\pi}{N} \cdot$$

The above condition on $f(n)$ is fulfilled if we take

$$A = 1, \qquad B = 0, \qquad \text{and} \qquad \alpha = r\pi/N.$$

It remains to fix $g(m)$ subject to the conditions $g(0) = 1$ and $g(1) = 1$. With the foregoing restriction on α, the difference equation which determines $g(m)$ is:

$$\frac{k^2}{g(m)}\,[g(m+1) - 2g(m) + g(m-1)] = -4\sin^2 \frac{r\pi}{2N}$$

which may be rewritten as

$$g(m+1) - 2\left[1 - \frac{2}{k^2}\sin^2 \frac{r\pi}{2N}\right] g(m) + g(m-1) = 0.$$

We now distinguish two types of solutions, according as k is chosen greater or less than unity. In the first case, the bracketed factor in the above equation is never greater than 1 nor less than -1, so that it is permissible to represent it as the cosine of some real angle

$$g(m+1) - 2\cos\phi g(m) + g(m-1) = 0,$$

where $\cos\phi = 1 - (2/k^2)\sin^2 (r\pi/2N)$. As before, we find solutions of the form

$$g(m) = C\sin\phi m + D\cos\phi m.$$

The side conditions on $g(m)$ are satisfied if

$$C = \frac{1 - \cos\phi}{\sin\phi} \qquad\qquad D = 1.$$

In the case when $k > 1$, the complete solution is therefore

$$u(n, m) = \sin \frac{rn\pi}{N}\left[\frac{\sin\phi m - \sin\phi(m-1)}{\sin\phi}\right] \cdot$$

It should be noted that, as N takes on increasingly large values, ϕ approaches $r\pi/kN$ and the solution converges on

$$\sin (rn\Delta x) \cos (rcm\Delta t).$$

This result is in agreement with the exact solution.

On the other hand, *if k is less than unity, there may exist some value of r for which the bracketed factor in the g-equation is less than -1.* In that case, the g-equation takes the form

$$g(m+1) + 2\mu g(m) + g(m-1) = 0,$$

where μ is a constant greater than one. To solve this equation, we seek solutions of the form

$$g(m) = p^m.$$

The constant p is then determined by the quadratic equation

$$p^2 + 2\mu p + 1 = 0,$$

whence the admissible values of p are

$$p_1 = -\mu + \sqrt{\mu^2 - 1}$$

$$p_2 = -\mu - \sqrt{\mu^2 - 1}$$

and the corresponding solutions of the g-equation are

$$g(m) = Rp_1{}^m + Sp_2{}^m.$$

The side conditions on $g(m)$ are satisfied if

$$R = \tfrac{1}{2}\left(1 + \frac{1+\mu}{\sqrt{\mu^2 - 1}}\right) \qquad S = \tfrac{1}{2}\left(1 - \frac{1+\mu}{\sqrt{\mu^2 - 1}}\right).$$

In the case when k is chosen less than unity, there is therefore some value of r for which the complete solution is of the following form:

$$u(n, m) = \sin\frac{rn\pi}{N}\,(Rp_1{}^m + Sp_2{}^m).$$

This is the crucial point in the development. The behavior of this type of solution clearly depends on the magnitudes and signs of the real constants p_1 and p_2. Both p_1 and p_2 are less than zero, but the absolute value of p_2 is greater than unity, whereas the absolute value of p_1 is less than unity. Accordingly, the term of the solution involving $p_1{}^m$ will approach zero as m increases indefinitely, while the magnitude of the term involving $p_2{}^m$ will grow exponentially as m increases. *Thus the magnitude of the solution $u(n, m)$ will increase without limit as m increases.* Moreover, because p_2 is negative, the sign of $g(m)$ changes with each increase in m, producing wild fluctuations of ever-increasing amplitude. This erratic behavior is called "computational instability."

Interpreting this result with regard to the uniqueness of the solution, it is clear that the process of repeated numerical calculation, if carried out without error, will lead to the closed forms derived above. For some values of r, the solution will be of the stable type whether or not k is chosen greater than unity. In practice, however, numerical calculations are at least subject to round-off errors. We now regard the errors of calculation at two successive stages of the iteration as a new set of initial conditions and assume that no calculation errors are committed beyond that point. Since the difference equation is linear, the computed solution will then be the superposition of the true solution and the solution which would be obtained by applying the difference equation to the initial errors, subject to the condition that the errors vanish at the boundaries. Now, because calculation errors are distributed in an almost random manner, there will in general exist some component of the error spectrum for which the "error" solution is unstable, *if k is less than unity*. In this case, regardless of the nature of the true solution, the amplitude of the computed solution will grow without limit.

If, on the other hand, k is chosen greater than unity, no component of the error spectrum will be amplified by the method of finite differences. In this case, no matter how the errors are distributed, the computed solution is stable in the sense that the errors will remain small if they were initially small. Therefore, in order to insure the stability of computed solutions of the hyperbolic difference equation, k must be chosen greater than unity. This implies that the increment of time Δt must be taken less than the ratio between the space increment Δx and the natural wave speed c.

APPENDIX II

TRANSFORMATION OF THE CONVOLUTION INTEGRAL $I(\xi, \eta, t)$

In accordance with the definition given in paragraph 7.16, we consider the integral

$$I(\xi, \eta, t) = \frac{2}{\pi}\int_0^t \frac{1}{\sqrt{\tau(t-\tau)}} \cos\sqrt{2\beta^*(r-\xi)(t-\tau)}\; K_0(2\sqrt{\beta^* r\tau})\, d\tau.$$

This expression can be considerably simplified by introducing a change in the variable of integration. Setting $\cdot = t \sin^2 \phi$

$$I(\xi, \eta, t) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} K_0(2\sigma\sin\phi)\cos(2\sigma\kappa\cos\phi)\, d\phi \tag{II-1}$$

vhere $\kappa = \sin(\theta/2)$ and $\sigma^2 = \beta^* rt$. We next make use of a simple integral representation of the Bessel unction, also involving a trigonometric function in the integrand (Watson, (1922)).

$$K_0(x) = \int_0^\infty \frac{\cos xz}{\sqrt{1+z^2}}\, dz,$$

here z is a dummy variable of integration. Substituting this expression into Eq. (II-1), and inverting he order of integration,

$$I(\xi, \eta, t) = \frac{4}{\pi}\int_0^\infty \frac{dz}{\sqrt{1+z^2}} \int_0^{\frac{\pi}{2}} \cos(2\sigma z\sin\phi)\cos(2\sigma\kappa\cos\phi)\, d\phi \tag{II-2}$$

/e now focus attention on the integral.

$$\int_0^{\frac{\pi}{2}} \cos(2\sigma z\sin\phi)\cos(2\sigma\kappa\cos\phi)\, d\phi$$

hich, after some manipulation of the trigonometric identities, can be put in the alternative forms

$$\tfrac{1}{2}\int_0^{\frac{\pi}{2}} [\cos(2\sigma z\sin\phi + 2\sigma\kappa\cos\phi) + \cos(2\sigma z\sin\phi - 2\sigma\kappa\cos\phi)]\, d\phi$$

$$\tfrac{1}{2}\int_0^{\frac{\pi}{2}} \cos[2\sigma\sqrt{\kappa^2+z^2}\sin(\phi+\psi)]\, d\phi + \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \cos[2\sigma\sqrt{\kappa^2+z^2}\sin(\phi-\psi)]\, d\phi,$$

iere $\psi = \arctan(\kappa/z)$. We have now reached a decisive turn in the argument. Bearing in mind that ϕ simply a dummy variable of integration, we let $\phi + \psi = \alpha$ in the first of the above integrals and let $- \phi = \alpha$ in the second. Introducing these changes of variables, the above expression can be written as

$$\tfrac{1}{2}\int_\psi^{\psi+\frac{\pi}{2}} \cos(2\sigma\sqrt{\kappa^2+z^2}\sin\alpha)\, d\alpha - \tfrac{1}{2}\int_\psi^{\psi-\frac{\pi}{2}} \cos(2\sigma\sqrt{\kappa^2+z^2}\sin\alpha)\, d\alpha$$

$$= \tfrac{1}{2}\int_\psi^{\psi+\frac{\pi}{2}} \cos(2\sigma\sqrt{\kappa^2+z^2}\sin\alpha)\, d\alpha + \tfrac{1}{2}\int_{\psi-\frac{\pi}{2}}^{\psi} \cos(2\sigma\sqrt{\kappa^2+z^2}\sin\alpha)\, d\alpha$$

d, finally, as

$$\tfrac{1}{2}\int_{\psi-\frac{\pi}{2}}^{\psi+\frac{\pi}{2}} \cos(2\sigma\sqrt{\kappa^2+z^2}\sin\alpha)\, d\alpha.$$

The integrand of this integral is not only an even function of α, but also is periodic with period π. The above expression, therefore, reduces to

$$\frac{1}{2}\int_0^{\pi} \cos\,(2\sigma\sqrt{\kappa^2 + z^2}\,\sin\alpha)\,d\alpha.$$

This integral is well known. It is, in fact, one of the many possible integral representations of the zero-order Bessel function of the first kind with real argument (Watson, (1922)).

$$\int_0^{\pi} \cos\,(2\sigma\sqrt{\kappa^2 + z^2}\,\sin\alpha)\,d\alpha = \pi J_0(2\sigma\sqrt{\kappa^2 + z^2}).$$

Finally, substituting this result into Eq. (II–2),

$$I(\xi, \eta, t) = 2\int_0^{\infty} \frac{J_0(2\sigma\sqrt{\kappa^2 + z^2})}{\sqrt{1 + z^2}}\,dz,$$

which is the form given in paragraph 7.16. Aside from compactness, its integrand has the advantage of possessing no singularities. That the integral converges can be shown by regarding it as an infinite alternating series of integrals, taken over the intervals between successive zeros of the Bessel function.

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