TRANSLATION

TURBULENT BOUNDARY LAYER OF A COMPRESSIBLE GAS

By

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FOREIGN TECHNOLOGY DIVISION

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

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В в	*В в*	V, v	Т т	*Т т*	T, t
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Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
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Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

ANNOTATION

In the book the theory of a turbulent boundary layer of a compressible gas is set forth or based on a study of the relative change of the coefficients of friction and heat transfer with growth of the M number, the heat transfer factor $\Delta\psi$ and the wall permeability factor b. The existence is shown of a limiting law, corresponding to very large Re numbers and almost full of self-simulation of relative changes of the coefficients of friction and heat transfer. On this basis simple engineering methods are presented for solving the basic problems of friction and heat transfer in turbulent flow of a gas past a solid body.

The theoretical conclusions are compared with experimental data.

The book is designed for scientists, engineers of aerodynamics and thermophysics, students of senior courses in these specialties and may be used as a guide for practical calculations in design offices.

PREFACE

In nature turbulent flows are the most wide-spread, starting with the flow of water in a stream and finishing with the motion of interstellar gas. Flow of operating media in machines and apparatuses of contemporary technology in the overwhelming majority of cases is turbulent. During high speed flight, in combustion chambers and the nozzles of motors, in nuclear reactors, in high-forced gas equipment and so forth streamlining of solid bodies of different shapes by a turbulent stream of compressible gas takes place. In addition, flow is accompanied by intense heat transfer or mass transfer of some sort of matter.

As is known, aerodynamic drag and heat and mass transfer are determined by the conditions of the transfer of momentum, heat, and matter in the boundary layer of gas or liquid forming near a streamlined surface. In a stream of compressible gas, all phenomena are complicated by innate compressibility and by changes of temperature, connected with high glow velocities. In this connection, the great attention, which has been allotted in the past few years to the problem of the boundary layer of a compressible gas is understandable. However, if the theory of a laminar boundary layer can be considered as basically completed, then with respect to the turbulent boundary layer

the position to now, remains far from satisfactory.

The semi-empirical theories of near-wall turbulence of Prandtl-Karman and Taylor to a certain degree permitted, explanation of the existence of a logarithmic velocity profile in an isothermal fluid current with weak pressure gradients and impenetrable surface. Extension of this theory to nonisothermal flow with zero pressure gradients was carried out in the works of F. I. Frank' and V. V. Voyshel', A. A. Dorodnitsyn, R. Deisler, L. Ye. Kalikhman, V. M. Lyevlev, E. Van Driest, W. Dorrance and F. Dore, V. P. Motulevich, Yu. V. Lapin, et al. In the works of K. K. Fedyaevskiy, V. Stsablevskiy, L. G. Loytsyanskiy, et al. Certain problems are considered of the flow of an incompressible liquid in the presence of a pressure gradient in the flow region, not close to the point of separation of the boundary layer. In addition, it is necessary to accept a series of additional, physically insufficiently valid, and sometimes contradictory assumptions. As a result calculations by these methods substantially differ among themselves, which was noted repeatedly in the literature. Therefore, it is not surprising, that in engineering practice purely empirical methods of calculation of friction and heat transfer in the turbulent boundary layer of a compressible gas, based on the introduction of some "governing" temperature, have obtained wide circulation. It is necessary to note that in the semi-empirical methods of calculation it is also necessary to introduce a "governing" temperature for calculation of physical properties of gas in a viscous sublayer. The problem of the parameters of separation of the turbulent boundary layer remains theoretically nearly unexplored.

This monograph is devoted to discussion of the limiting law of change of the coefficient of friction in a turbulent boundary layer

under nonisothermal conditions, transverse flow of matter, and pressure gradient as established by the authors. This law, derived for $Re \to \infty$, in the general form does not depend on empirical constants of turbulence and is not connected with any special type of semi-empirical theories.

The known fact of the weak influence of the Reynolds number on the relative change of the coefficients of friction and heat transfer in connection with nonisothermal conditions and transverse flow of matter permits extending the limiting law to flows with finite Re numbers, with good accuracy. As a result we managed to construct relatively simple methods of solving the integral equations for the momentum and energy of a turbulent bounday layer.

Theoretical results are compared with a large number of diverse experimental data.

It is possible to propose that the possibilities of the new method are by no means exhausted by the problems, considered in this monograph and will be still developed, supplementing the existing theory of a turbulent boundary layer.

It is assumed that the reader is acquainted with the principles of aerodynamics and the theory of heat and mass transfer.

The authors will be very grateful for all critical remarks, which are raised by readers in connection with the material considered below.

List of Basic Cyrillic Symbols

кг	=	kg	
м	=	m	
ккал	=	kcal	
кгм	=	kg.m	
ст	=	wall	= wall
сек	=	sec	= second
град	=	deg	= degree
т	=	t	= thermal or thickness
ч	=	hr	= hour
гр	=	b	= boundary
кр	=	cr	= critical
т	=	t	= thermal turbulent

Basic Symbols

$A = \frac{1}{427}\left[\frac{ккал}{кг.м}\right]$ - thermal equivalent of work;

$a = \frac{w_{ст}}{w_{\infty}}$ - relative speed of blowing (in pumping) of matter into the boundary layer through the wall;

$a\left[\frac{м^2}{сек}\right]$ - thermal diffusivity coefficient;

$\tilde{a}\left[\frac{м}{сек}\right]$ - speed of sound;

$b = \frac{2\rho_{ст} w_{ст}}{c_{f_0}\rho_0 w_{\infty}}$ - wall permeability factor;

$b_1 = \frac{2\rho_{cm} w_{cm}}{c_f \rho_\infty w_\infty}$ - wall permeability factor, referred to the real value of the friction coefficient;

c_f - local friction coefficient

c_F - average friction coefficient;

c_{f0}, c_{F0} - friction coefficients during streamlining of a flat, impenetrable plate to an unlimited isothermal flow;

$c_p \left[\frac{\text{ккал}}{\text{кг} \cdot \text{град}}\right]$ - specific heat at constant pressure;

$c_v \left[\frac{\text{ккал}}{\text{кг} \cdot \text{град}}\right]$ - specific heat at constant volume;

D [м] - diameter;

$D \left[\frac{м^2}{сек}\right]$ - diffusion coefficient;

F [$м^2$] - surface area;

$f = \frac{\delta^{**}}{w_\infty} \cdot \frac{dw_\infty}{dx}$ - shape parameter;

$G \left[\frac{кг}{сек}\right]$ - mass flow rate;

$g \left[\frac{м}{сек^2}\right]$ - acceleration of gravity;

$H = \frac{\delta^*}{\delta^{**}}$ - shape parameter, the ratio of the depths of displacement to loss of momentum;

$i \left[\frac{\text{ккал}}{кг}\right]$ - specific enthalpy;

$j \left[\frac{кг \cdot сек}{м^3}\right]$ - pulsational component of mass velocity (flow + rate);

$j_1 \left[\frac{кг \cdot сек}{м^3}\right]$ - mass velocity of transverse flow of matter through a permeable wall;

$K = \frac{r}{c \Delta T}$ - criterion of phase transition;

$k = \frac{c_p}{c_v}$ - index of the adiabatic curve;

l [м] - linear dimension;

l [м] - length of the path of hydrodynamic mixing;

$l_т$ [м] - length of the path of thermal mixing;

L [м] - full length of the body;

m - exponent;

$Nu = \frac{\alpha l}{\lambda}$ - Nusselt number;

n - exponent;

$M = \frac{w}{a}$ - Mach — Mayevskiy number;

$Pr = \frac{\nu}{a}$ - Prandtl number;

$p \left[\frac{кг}{м^2}\right]$ - pressure;

$Q \left[\frac{ккал}{ч}\right]$ - heat flow;

$q \left[\frac{ккал}{м^2 \cdot ч}\right]$ - heat flux;

$R \left[\frac{кг \cdot м}{кг \cdot град}\right]$ - gas constant;

R [м] - radius;

$Re = \frac{wl}{\nu}$ - Reynolds number;

$Re^{**} = \frac{w_0 \delta^{**}}{\nu_0}$ - characteristic Reynolds number of boundary layer;

r - temperature recovery coefficient;

$r \left[\frac{ккал}{кг}\right]$ - latent heat of flow transition;

$St = \frac{\alpha}{c_p \gamma w_0}$ - Stanton number;

T [°K] - absolute temperature;

T^* [°K] - inhibition temperature;

$T^*_{ст}$ [°K] - adiabatic wall temperature;

T^+ [°K] - calculated temperature, determined by formula (2.45);

t [°C] - temperature on the centigrade scale;

$U = \frac{w_0}{\sqrt{\frac{2gc_p}{A}T_0^*}}$ - ratio speed of undisturbed flow to the maximum possible speed of flow;

$u = \frac{w_x}{\sqrt{\frac{2gc_p}{A}T_0^*}}$ - the same for the local value of velocity component w_x;

V [$м^3$] - volume;

$V\left[\frac{м^3}{сек}\right]$ - volume flow + rate;

$\vec{V}\left[\frac{м}{сек}\right]$ - pulsational vector component of velocity;

$v\left[\frac{м^3}{кг}\right]$ - specific volume;

$v_* = \sqrt{\frac{\tau_{ст}}{\rho_0}}\left[\frac{м}{сек}\right]$ - dynamic velocity;

$w\left[\frac{м}{сек}\right]$ - flow velocity;

$w_0\left[\frac{м}{сек}\right]$ - velocity outside the dynamic boundary layer;

$w_1\left[\frac{м}{сек}\right]$ - velocity on the calculated boundary of turbulent nucleus and viscous underlayer;

$w_0' = \frac{dw_0}{dx}$ - longitudinal gradient of velocity outside the boundary layer;

x - coordinate, directed along the flow according to the outline of the contour;

y - coordinate, normal to the streamlined surface;

y_1 - calculated thickness of the viscous underlayer;

$\alpha \left[\frac{ккал}{м^2 \cdot ч \cdot град}\right]$ - heat transfer coefficient;

β° - angle;

β - compressibility factor in the expression for turbulent tangential stresses;

$\Gamma = \frac{2}{c_{f_0}} f$ - form parameter;

$\gamma = K\rho \left[\frac{кг}{м^3}\right]$ - density;

Δ - sign of difference;

$\Delta\vartheta = \frac{\Delta T}{T_\infty}$ - heat transfer factor;

$\delta \left[\frac{кг \cdot сек^2}{м^4}\right]$ - pulsational component of density;

δ [м] - thickness of the dynamic boundary layer;

δ_T [м] - thickness of the thermal boundary layer;

δ^* [м] - thickness of displacement;

δ^{**} [м] - thickness of loss of momentum;

δ^{***}_T [м] - thickness of loss of energy;

ε - coefficient nonsimilarity temperature and velocity fields;

ε_D - coefficient nonsimilarity of concentration and velocity fields;

ζ - coefficient of aerodynamic drag during flow in a pipe;

$\eta = \frac{v_* y}{\nu}$ - dimensionless distance from wall, expressed in the form of "local Reynolds number;"

$\eta_{11} = \frac{v_* y_1}{\nu}$ - dimensionless thickness of the viscous underlayer;

$\theta[°K]$ - pulsational component of temperature;

Φ - dimensionless temperature (see formula [2.46]);

$\Lambda = \frac{\delta}{\tau_{ст}} \cdot \frac{dp}{dx}$ - form parameter;

$\Lambda_0 = \Psi_\Lambda \Lambda$ - form parameter, referred to c_{fo};

$\lambda \left[\frac{ккал}{м \cdot ч \cdot град}\right]$ - coefficient of thermal conductiuity;

$\mu \left[\frac{кг \cdot сек}{м^2}\right]$ - coefficient of dynamic viscosity;

$\nu \left[\frac{м^2}{сек}\right]$ - coefficient kinematic viscosity;

$\xi = \frac{y}{\delta}$ - dimensionless distance from wall;

$\rho \left[\frac{кг \cdot сек^2}{м^4}\right]$ - density of the medium;

$\tau \left[\frac{кг}{м^2}\right]$ - tangential stress;

$\bar{\tau} = \frac{\tau}{\tau_{ст}}$ - relative magnitude of tangential stress in the thickness of the boundary layer;

$\varphi = \frac{w_x}{v_0}$ - dimensionless velocity;

$\Psi_f = \left(\frac{c_f}{c_{f0}}\right)_{Re^{**}}$ - relative change of coefficient of friction at Re^{**} = = idem;

$\Psi_{St} = \left(\frac{St}{St_0}\right)_{Re_T^{**}}$ - relative change of the Stanton number at $Re_T{}^{**}$ = idem;

$\psi = \frac{T_{ст}}{T_0}$ - temperature factor;

$\psi^* = \frac{T^*_{ст}}{T_0}$ - kinetic temperature factor;

$\Omega[м^2]$ - cross sectional area;

$w = \frac{w_x}{w_0}$ - dimensionless velocity;

Indices:

$_{гр}$ = b = boundary;

$_{кр}$ = cr = critical;

$_{т}$ = t = thermal, turbulent;

$_{ст}$ = wall = wall;

0 - scale point, parameters outside boundary layer;

00 - conditions for inhibition parameters;

1 - parameters of matter, introduced into the boundary layer at the intersection of the wall;

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Turbulent Boundary Layer of a Compressible Gas. Publishing House of the Siberian Branch of the Academy of Sciences of the USSR. Novosibirsk, 1962
Pages: Cover — 180.

C H A P T E R I

BASIC EQUATIONS OF A TURBULENT BOUNDARY LAYER

1.1. Equations of Motion and Thermal Conductivity of a Plane Boundary Layer of Gas

List of Designations Appearing in Cyrillic

ст = wall = wall

кг = kg

м = m

A stream of fluid forms near a streamlined surface a dynamic boundary layer, i.e., a region, in which the velocity of the fluid changes from the velocity at the wall (for a nonrarefied gas it is equal to zero) to a magnitude very close to the speed of an undistrubed flow — w_0. In the presence of heat transfer and diffusion thermal and diffusion boundary layers appear. In a thermal boundary layer the temperature in practice changes from T_{wall} to T_0; in a diffusion boundary layer the concentration at the diffusing substance changes from ρ''_{wall} to ρ''_0. Strictly speaking, for a dynamic boundary layer the boundary conditions take the form;

$$|y = 0,\ w_x = 0;\ y = \infty,\ w_x = w_0|. \qquad (1.1)$$

Due to a sharp change of speed in direct proximity to the wall, the exact conditions of (1.1) can be replaced by the approximate;

$$[y = 0,\ w_x = 0;\ y = \delta,\ w_x = (1 - \varepsilon)\, w_\infty]. \tag{1.2}$$

where ε is a prescribed small magnitude.

In that sense we are speaking of a boundary layer of finite thickness δ.

In experiments the value of ε can coincide with the sensitivity of the measuring instrument.

In Fig. 1 is given a diagram of a boundary layer on a certain curvilinear surface.

In a plane boundary layer in the absence of significant transverse forces (for instance, centrifugal) the following conditions are fulfilled:

$$\left.\begin{aligned} \frac{\partial p}{\partial y} &\ll \frac{\partial p}{\partial x}; \\ \frac{\partial f_i}{\partial x} &\ll \frac{\partial f_i}{\partial y}; \\ \frac{\partial^2 f_i}{\partial x^2} &\ll \frac{\partial^2 f_i}{\partial y^2}, \end{aligned}\right\} \tag{1.3}$$

where for f_i are understood w_x, T, and ρ.

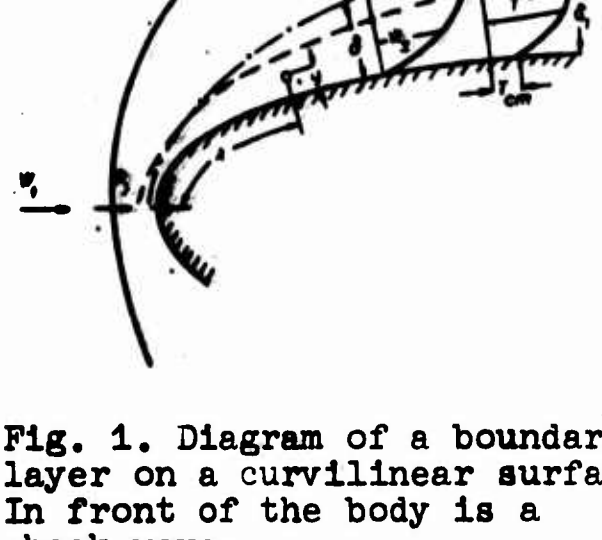

Fig. 1. Diagram of a boundary layer on a curvilinear surface. In front of the body is a shock wave.

In connection with this, the equations of thermal conductivity, motion, and continuity for a stationary, plane boundary layer of compressible gas on an impenetrable surface have the form,

$$\frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + A\mu\left(\frac{\partial w_x}{\partial y}\right)^2 = g\rho\left(w_x \frac{\partial i}{\partial x} + w_y \frac{\partial i}{\partial y}\right) - A w_x \frac{dp}{dx}; \tag{1.4}$$

$$-\frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial w_x}{\partial y}\right)=\rho\left(w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}\right); \tag{1.5}$$

$$\frac{\partial(\rho w_x)}{\partial x}+\frac{\partial(\rho w_y)}{\partial y}=0. \tag{1.6}$$

At $c_p = \text{const}$ $\frac{\partial i}{\partial x} = c_p \frac{\partial T}{\partial x}$, what is practically true for a uniform gas.

To these equations one should join the equation of state, determining the magnitude of the density of the gas ρ, and temperature functions of the coefficients of thermal conductivity and viscosity.

For a gas, obeying the Clapeyron – Mendleyev equation, we have

$$\rho=\frac{p}{gRT}. \tag{1.7}$$

The coefficients of thermal conductivity and viscosity are related to the specific heat via the Prandtl number,

$$Pr=\frac{g\mu c_p}{\lambda}. \tag{1.8}$$

For undissociated gases the values of Pr and c_p change little with temperature and pressure. Therefore, for such a medium practically one may assume that

$$\frac{\mu}{\lambda}=\text{const}. \tag{1.9}$$

The magnitude of Pr depends chiefly on the number of atoms in the molecules of the gases, and its order is given in Table 1.1.

Table 1.1. Order of Pr number for undissociated gases.

Atoms	1	2	3	≥4
Pr number.....	0.66	0.75	0.84	1

Outside the boundary layer frictional forces do not appear ($\frac{\partial w_x}{\partial y} = 0$) and for steady-state flow

$$-\frac{dp}{dx} = \rho_\infty w_\infty \frac{dw_\infty}{dx}. \qquad (1.10)$$

Substituting in equation (1.4) the value of $\frac{dp}{dx}$ from equation (1.5), we obtain the thermal conductivity equation for a plane boundary layer in the form of M. F. Shirokov (at c_p = const):

$$\frac{\partial}{\partial y}\left\{\lambda \frac{\partial}{\partial y}\left[T^* + (Pr - 1)\frac{A w_x^2}{2g c_p}\right]\right\} = g c_p \rho \left(w_x \frac{\partial T^*}{\partial x} + w_y \frac{\partial T^*}{\partial y}\right). \qquad (1.11)$$

Here

$$T^* = T + \frac{A w_x^2}{2g c_p} \qquad (1.12)$$

is the temperature of stagnation.

1.2. Turbulent Viscosity and Thermal Conductivity in a Plane Boundary Layer

The real (actual) instantaneous characteristics of turbulent flow effect at every point disordered oscillations around a certain mean value. Thus, for the flow velocity we have

$$\vec{w} = \vec{\bar{w}} + \vec{V}, \qquad (1.13)$$

where $\vec{\bar{w}}$ — vector of the averaged velocity at a given point of flow,

$\vec{V}$ — vector of pulsational component, giving the deviation of the true velocity, at a given moment of time, from the averaged value.

In a compressible gas the flow velocity, pressure, temperature, density, and flow + rate of the medium pulsate. The corresponding equation of motion of a plane steady state boundary layer has the form

$$-\frac{d\bar{p}}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial w_x}{\partial y} - \overline{\rho V_y V_x}\right) = \overline{\rho w_x} \frac{\partial \bar{w}_x}{\partial x} + \overline{\rho w_y} \frac{\partial \bar{w}_x}{\partial y}. \qquad (1.14)$$

Here a line above a letter signifies averaging during a time fairly large as compared to the period of pulsation.

In turn, the averaged product of the vector components of pulsation of the flow + rate j_y and vector velocity pulsation V_x are

$$\overline{j_y V_x} = \bar{\rho}\,\overline{V_x V_y} + \bar{w}_y \overline{V_x \delta} + \overline{V_x V_y \delta}, \qquad (1.15)$$

where δ is the density pulsation.

By comparing in equation (1.14) the members $\mu \frac{\partial \bar{w}_x}{\partial y}$ and $\overline{j_y V_x}$, it is possible to arrive at the conclusion that the latter value can be considered as some tangential stress, appearing in the averaged stream under the influence of turbulent pulsations.

In a gas, obeying equation (1.7).

$$\delta = \rho - \bar{\rho} = -\bar{\rho}\frac{\theta}{\bar{T}} \qquad (1.16)$$

and correspondingly

$$\tau_T = -\bar{\rho}\,\overline{V_x V_y} + \frac{\bar{\rho}\bar{w}_y}{\bar{T}}\overline{\theta V_x} + \frac{\bar{\rho}}{\bar{T}}\overline{\theta V_x V_y}. \qquad (1.17)$$

where θ is the temperature pulsation.

The value

$$\mu_T = \frac{\tau_T}{\frac{\partial \bar{w}_x}{\partial y}} \qquad (1.18)$$

is called the coefficient of turbulent viscosity. This value is a complex function of velocity and temperature.

The idea of the coefficient of thermal conductivity λ_t is developed in an analogous manner.

In the method proposed the values μ_t and λ_t are not used directly and only their ratio is important

$$Pr_T = \frac{c_p \mu_T \rho}{\lambda_T}; \qquad (1.19)$$

called the turbulent Prandtl number.

This value near a solid wall is close to unity.

1.3. Integral Equations of a Boundary Layer

The equation of motion can be written recorded in the form

$$\rho_0 w_0 \frac{dw_0}{dx} + \frac{\partial \tau}{\partial y} = \rho w_x \frac{\partial w_x}{\partial x} + \rho w_y \frac{\partial w_x}{\partial y}. \qquad (1.20)$$

true for laminar and turbulent flows, if corresponding values of variables are introduced into it.

Integration of this equation with respect to y from 0 to δ gives

$$\rho_0 w_0 \frac{dw_0}{dx}\delta + \tau_{y=\delta} - \tau_{y=0} = \int_0^\delta \left(\rho w_x \frac{\partial w_x}{\partial x} + \rho w_y \frac{\partial w_x}{\partial y}\right) dy. \qquad (1.21)$$

On the outer boundary of the layer, by definition $\tau = 0$ and $w_x = w_0$.

On the surface of a streamlined body

$$\tau = \tau_{cr},\ w_x = 0,\ \rho w_y = j_1.$$

Here

$$j_1 = \rho_{cr} w_{cr} \qquad (1.22)$$

is mass flow velocity of matter through a surface. This flow may be as a result of a change of the aggregate state (evaporation, condensation), porosity of the wall (blast of gas into the boundary layer or pumping from the boundary layer), chemical reactions.

Subsequently, a surface, on which $j_1 \neq 0$, we will call permeable.

Taking into account these boundary conditions and the equation of continuity, it is possible to transform equation (1.21) to the form

$$\frac{d\delta^{**}}{dx} + \frac{\dot{w}_0}{w_0}\delta^{**}(2+H) + \frac{\dot{\rho}_0}{\rho_0}\delta^{**} - \frac{j_1}{\rho_0 w_0} = \frac{\tau_{cr}}{\rho_0 w_0^2}, \qquad (1.23)$$

where

$$H = \frac{\delta^*}{\delta^{**}} \text{ and } \dot{\rho}_0 = \frac{d\rho_0}{dx}.$$

In this equation, called the Karman momentum equation

$$\delta^* = \int_0^{\infty \cdot \delta} \left(1 - \frac{\rho w_x}{\rho_0 w_0}\right) dy \qquad (1.24)$$

is the thickness of displacement,

$$\delta^{**} = \int_0^{\infty \cdot \delta} \frac{\rho w_x}{\rho_0 w_0}\left(1 - \frac{w_x}{w_0}\right) dy \qquad (1.25)$$

is the thickness of loss of momentum.

The upper limit of integration $y = \infty$ corresponds to the theory of an asymptotic boundary layer; the limit $y = \delta$ corresponds to the theory of a layer of finite thickness.

Due to a sharp change of speed in the interval from 0 to δ, δ^* and δ^{**} have the same value for both upper limits, i.e., there are certain "internal" linear scales, the one and the same within the theory of an asymptotic layer, and in the theory of a layer of finite thickness.

The analogous integration of the thermal conductivity equation leads to the equation of the energy of a boundary layer (without consideration of radiation and internal sources),

$$\frac{d\delta_T^{**}}{dx} + \left[\frac{w_0'}{w_0} - \frac{(\Delta T)'}{\Delta T} + \frac{\rho_0'}{\rho_0}\right]\delta_T^{**} - \frac{c_{p1} j_1}{c_{p0} \rho_0 w_0} = \frac{q_{cT}}{g c_{p0} \rho_0 w_0 \Delta T}, \qquad (1.26)$$

where $(\Delta T)' = (d/dx)(\Delta T)$, ΔT is the temperature difference,

c_{p1} is the specific heat of the gas, introduced through a permeable surface,

c_{p0} is the specific heat of the basic gas.

The quantity

$$\delta_T^{**} = \int_0^{\infty \cdot \delta} \frac{\rho w_x}{\rho_0 w_0}\left(1 - \frac{i}{i_0}\right) dy \qquad (1.27)$$

is called the thickness of loss of energy.

The coefficient of friction is

$$c_f = \frac{2\tau_{cr}}{\rho_\infty w_\infty^2}, \tag{1.28}$$

and the coefficient of heat transfer is

$$\alpha = \frac{q_{cr}}{\Delta T}. \tag{1.29}$$

The dimensionless forms of the heat coefficient transfer (Stanton and Nusselt numbers) have the form,

$$St = \frac{\alpha}{c_p \rho_0 w_\infty}; \tag{1.30}$$

$$Nu = \frac{\alpha l_0}{\lambda_0}, \tag{1.31}$$

where l_0 is a characteristic linear dimension.

In integral (1.27) $ is a certain dimensionless temperature. At Pr = 1 and dp/dx = 0, the value of $ is exactly equal to the ratio $T_{wall}-T^*/T_{wall}-T_0^*$, where T* is the value of the inhibitian temperature at a given point and T_0^* is the value of the stagnation temperature outside the boundary layer.

The temperature difference is determined by the expression

$$\Delta T = T_{cr} - T_{cr}^*, \tag{1.32}$$

where T_{wall}^* is the adiabatic temperature of the wall, i.e., that temperature, which is established on a given surface, if the latter is completely heat-insulated and $q_{wall} = 0$.

In general it is assumed that

$$T_{cr}^* = T_0 + r\frac{A w_0^2}{2g c_p}, \tag{1.33}$$

where r is the temperature recovery factor.

In a turbulent boundary layer

$$r \approx Pr^{1/3}, \tag{1.34}$$

i.e., for gases close to or equal to unity.

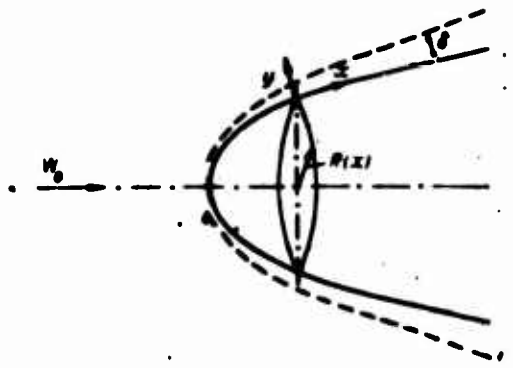

Fig. 2. Diagram of a boundary layer on an axially symmetric body.

In Fig. 2 is given a diagram of a boundary layer on an axially symmetric body. Because of the small thickness of the boundary layer relative to the radius of curvature R_x, the equations of motion and thermal conductivity do not change their form as compared to Planar flow. The form of the continuity equation changes, taking the form:

$$\frac{\partial(\rho w_x R_x)}{\partial x} + \frac{\partial(\rho w_y R_x)}{\partial y} = 0. \tag{1.36}$$

In connection with this the equations of momentum and energy are somewhat modified; namely,

$$\frac{d\delta^{**}}{dx} + \frac{w_0'}{w_0}\delta^{**}(2+H) + \left(\frac{\rho_0'}{\rho_0} + \frac{R_x'}{R_x}\right)\delta^{**} - \frac{j_1}{\rho_0 w_0} = \frac{\tau_{cr}}{\rho_0 w_0^2}; \tag{1.37}$$

$$\frac{d\delta_1^{**}}{dx} + \left(\frac{w_0'}{w_0} + \frac{\Delta T'}{\Delta T} + \frac{\rho_0'}{\rho_0} + \frac{R_x'}{R_x}\right)\delta_1^{**} - \frac{c_{p1} j_1}{c_{p0}\rho_0 w_0} = \frac{q_{cr}}{g c_{p0}\rho_0 w_0 \Delta T}. \tag{1.38}$$

Here

$$\delta^* = \int_0^{w_0 \cdot \delta} \left(1 - \frac{\rho w_x}{\rho_0 w_0}\right)\left(1 \pm \frac{y}{R_x}\cos\beta\right) dy; \tag{1.39}$$

$$\delta^{**} = \int_0^{w_0 \cdot \delta} \frac{\rho w_x}{\rho_0 w_0}\left(1 - \frac{w_x}{w_0}\right)\left(1 \pm \frac{y}{R_x}\cos\beta\right) dy; \tag{1.40}$$

$$\delta_1^{**} = \int_0^{w_0 \cdot \delta} \frac{\rho w_x}{\rho_0 w_0}(1-\theta)\left(1 \pm \frac{y}{R_x}\cos\beta\right) dy. \tag{1.41}$$

1.4. Characteristic Reynolds Number for a Boundary Layer

The trivial determination of the Reynolds number has the form:

$$Re = \frac{w_0 l_0}{\nu_0}, \tag{1.42}$$

where w_0 is characteristic velocity;

l_0 the characteristic linear dimension of a streamlined body;

ν_0 the kinematic viscosity at a characteristic point of the system. However, in boundary layer theory such a determination of a basic hydrodynamic criterion is ineffective. Actually, from the momentum equation it is clear that such a very important integral characteristic of the interaction of flow with a body; as the coefficient of friction c_f, is connected not with the length x, but with certain "internal" dimensions of the boundary layer — δ^{**} and δ^*. Besides the structure of the momentum equation shows that the first of these quantities is the most important.

The universality of the linear characteristics δ^*, δ^{**} and δ_t^{**} becomes especially evident, if we remember that their determination is not connected with the idea of a layer of finite thickness and the change of the upper limit in integrals (1.24), (1.25), and (1.27) from δ to ∞ does nct change their value. In connection with this characteristic Reynolds number for a dynamic boundary layer it is expedient to construct in the form of the quantity:

$$Re^{**} = \frac{w_0 \delta^{**}}{\nu_0} \tag{1.43}$$

or

$$Re^{*} = \frac{w_0 \delta^{*}}{\nu_0}. \tag{1.44}$$

In an isothermal turbulent boundary layer far from the point of breakaway, the ratio δ^*/δ^{**} is almost constant (see formula (3.13)),

i.e., in this case the numbers Re** and Re* differ only by a constant factor.

For a thermal boundary layer we have correspondingly

$$Re_t^{**} = \frac{w_0 \delta_t^{**}}{\nu_0}. \tag{1.45}$$

The dominating role of the quantity δ** in the momentum equation and consideration of the expedient uniformity of the characteristics of dynamic and thermal boundary layers force us to select as basic modifications of the Reynolds the expressions (1.43) and (1.45). Regarding selection of the quantity ν_0, it is more convenient to relate to conditions outside the boundary layer. Such a determination is convenient from a computation point of view, inasmuch as the parameters of undisturbed flow usually are known.

The relation between the Reynolds number

$$Re_x = \frac{w_0 x}{\nu_0} \tag{1.46}$$

and the numbers Re**, Re_t** is established via the equations of momentum and energy, if the following relations are known:

$$c_f = f_1(Re^{**}) \tag{1.47}$$

and

$$St = f_2(Re_t^{**}). \tag{1.48}$$

These relations are called, respectively, the law of resistance and law of heat transfer.

Definition of these laws is a basic problem of boundary layer theory.

1.5. Similarity of the Velocity and Temperature Fields

The equations of motion (1.5) and thermal conductivity (1.11) become identical relative to the quantities w_x and T* upon fulfillment of the conditions:

$$\left\{\frac{dp}{dx}=0;\quad Pr=1\right\}. \tag{1.49}$$

But identity of the differential equations signifies identity of their integrals during similarity of boundary conditions.

Consequently, during pressureless streamlining of a solid body by a polyatomic gas (Pr = 1) and for similarly defined boundary conditions the longitudinal flow velocity and stagnation temperature fields are similar, i.e.,

$$\frac{T_{ст}-T^{*}}{T_{ст}-T_0^{*}}=\frac{w_x}{w_0}. \tag{1.50}$$

These conditions are exactly fulfilled during longitudinal flow by an unbounded stream of a polyatomic, planar gas with a constant surface temperature.

Similarity the velocity and temperature fields at Pr = 1 means that between the heat flux q and the tangential stresses τ there exists the relationship

$$\frac{q}{\tau}=\frac{\lambda\frac{\partial t}{\partial y}}{\mu\frac{\partial w_x}{\partial y}}=\frac{gC_p}{w_0}(T_{ст}-T_0^{*}), \tag{1.51}$$

or

$$St=\frac{cf}{2}. \tag{1.52}$$

C H A P T E R II

LAWS OF RESISTANCE AND HEAT TRANSFER

Definitions of Cyrillic Items in Order of Appearance

т = t = thermal, turbulent

ст = wall = wall

кр = cr = critical

2.1. Tangential Stresses in a Planar Boundary Layer of a Compressible Gas

Molecular friction is commensurate with turbulent only in the thin near-wall layer. This region of the turbulent boundary layer is called the viscous underlayer.

The relative thickness of the viscous underlayer

$$\xi_1 = \frac{y_1}{\delta} \tag{2.1}$$

in a flow of an incompressible liquid is 1-3%, and in a flow of a compressible gas can reach 10-15%.

In the remaining part of the turbulent boundary layer, the so-called nucleus,

$$\mu_т \gg \mu$$

and

$$\tau = \tau_т = -\bar{\rho}\,\overline{V_x V_y}(1-\beta), \tag{2.2}$$

where

$$\beta = \frac{\bar{v}_y \overline{V_x \theta} + \overline{V_x V_y}\,\bar{\theta}}{\overline{V_x V_y}\,\bar{T}} \tag{2.3}$$

– a coefficient, taking into account the influence of density pulsations on momentum transfer.

Below the sign of the average values of ρ, w, and T will be omitted.

As can be seen from formula (2.2), the fundamental problem of the theory of turbulence is the connection between the average products of the vector components of the pulsational component of the flow velocity and the average velocity. This relation is expressed very generally by the Prandtl formula,

$$\overline{V_x V_y} = \left(l \frac{\partial w_x}{\partial y}\right)^2, \tag{2.4}$$

where l is a certain proportionality factor, having the dimension of length. Usually the quantity l is called the length of the mixing path.

Formula (2.4) follows from dimensional analysis, if we assume that in the region of significant changes of the average velocity, momentum is proportional to the derivative $\partial w_x/\partial y$.

Introducing the value $\overline{V_x V_y}$ from (2.4) in (2.2) we can write that, in a turbulent nucleus

$$\tau = \tau_t = \rho\left(l \frac{\partial w_x}{\partial y}\right)^2 (1 - \beta). \tag{2.5}$$

This expression leads to the form

$$\frac{c_f}{2} = \frac{\rho(1-\beta)}{\rho_0 w_0^2 \tilde{\tau}}\left(l \frac{\partial w_x}{\partial y}\right)^2, \tag{2.6}$$

where $\tilde{\tau} = \frac{\tau}{\tau_{wall}}$ is the distributive law of tangential stresses along the thickness of the boundary layer.

2.2. Law of Resistance

From equation (2.6) it follows that

$$\frac{c_f}{2} = \left(\frac{\int_0^1 \sqrt{\frac{\rho}{\rho_0 \tilde{\tau}}}\, d\bar{w}}{\int_0^{\tilde{\delta}} \frac{dy}{l\sqrt{1-\beta}}}\right)^2. \tag{2.7}$$

where y_1 is the thickness of the viscous underlayer, i.e., the coordinate of the lower boundary of the turbulent nucleus of flow.

$\omega_1 = \frac{w_1}{w_0}$ is the dimensionless velocity at this boundary.

Equation (2.7) is written in a form, not connected with any sort of theory concerning a boundary layer of finite thickness. The transition of the terms of the theory of a boundary layer of finite thickness is accomplished according to equation

$$\int_{y_1}^{\infty} \frac{dy}{\dots\sqrt{1-\dots}} \approx \int_{\xi_1}^{1} \frac{d\xi}{\frac{1}{\delta}\dots\sqrt{1-\dots}}, \qquad (2.8)$$

where δ is the thickness of the dynamic boundary layer;

$\xi = \frac{y}{\delta}$ is the dimensionless distance from the wall;

$\xi_1 = \frac{y_1}{\delta}$ is the dimensionless thickness of the viscous underlayer.

The upper integral of equation (2.7) has the same limits in the theory of an asymptotic boundary layer and in the theory of a layer of finite thickness.

Let us introduce the relationship

$$\psi = \left(\frac{c_f}{c_{f0}}\right)_{Re^{**}}; \qquad (2.9)$$

where c_f is the value of the local friction coefficient at given conditions of streamlining of a body;

c_{f0} is the value of the friction coefficient at an isothermal, gradient less streamlining of an impenetrable wall, i.e., during isothermal streamlining of a planar impenetrable plate by an unbounded flow.

Comparison the friction coefficients is carried out for identical values of the characteristic Reynolds number of the boundary layer, because integration in (2.8) is carried out with respect to the thickness of the latter.

Multiplying both parts of equation (2.6) by the quantity $\tilde{\tau}_0$, it is possible to write:

$$\psi = \left(\frac{1}{Z}\int_{\omega_1}^{1} \sqrt{\frac{\tilde{\rho}\tilde{\tau}_0}{\dots}}\, d\omega\right)^2 \qquad (2.10)$$

where

$$Z=\sqrt{\frac{c_{f0}}{2}}\int_{y_1}^{\delta}\sqrt{\frac{\tilde{\tau}_0}{1-\beta}}\cdot\frac{dy}{l}. \qquad (2.11)$$

In the formulas $\tilde{\tau}_0$ is the distributive law of the quantity $\frac{\tau}{\tau_{wall}}$ along the thickness of the isothermal boundary layer at an impenetrable plate, with streamlining of an unbounded unlimited flow (i.e. for $\frac{dp}{dx} = 0$).

From the given formulas it is clear that for establishment of law of resistance it is necessary to know the laws, determining the quantities l, y_1, τ, $\frac{\rho}{\rho_0}$, and β. The quantity w_1 can be computed by calculation of molecular friction and molecular thermal conduction, if the quantity y_1 and the relation $\mu(T)$ are known.

In the contemporary semi-empirical theories equation (2.7) is calculated on the assumption that $\beta = 0$, $\tau = \tau_{wall}$, $l \sim y$, or which is practically the very same,

$$l\sim\left(\frac{\partial w_x}{\partial y}\Big/\frac{\partial^2 w_x}{\partial y^2}\right),$$

and the dimensionless thickness of the viscous underlayer,

$$\eta_1=\frac{v_* y_1}{\nu}$$

maintains the value, found experimentally for isothermal flow on a plate, if the viscosity is related to a certain "determining" temperature. It is clear that boundary layer theories for a compressible gas based on such assumptions cannot lead to sufficiently reliable results.

However, equation (2.10) obtains special properties at $Re \to \infty$, to which attention has not been paid until now. We will now turn to an examination of these properties.

2.3. Values of the Quantities β, ω_1 and ξ_1 at $Re \to \infty$

We will show that at $Re \to \infty$ the quantities β, ω_1 and ξ_1 approach

zero.

Equation (2.4) assumes the existence of a correlation between the vector components of the pulsotional portion of the velocity of the form:

$$\overline{V_x^2} \sim \overline{V_y^2} \sim \left(\frac{\partial w_x}{\partial y}\right)^2.$$

Analogously, for temperature pulsation it is possible to assume the relationship:

$$V_y\theta \sim l^2 \frac{\partial w_x}{\partial y} \cdot \frac{\partial T}{\partial y}.$$

From (2.2) it follows that

$$l\frac{\partial w_x}{\partial y} = \sqrt{\overline{V_x V_y}} \approx v_* \sqrt{\frac{\rho_0}{\rho}\tilde{\tau}}, \qquad (2.12)$$

where $v_* = \sqrt{\frac{\tau_{cr}}{\rho_0}}$ is the dynamic velocity.

With accuracy up to the coefficients, taking into account the influence of density pulsation, the relationship between the heat flux and tangential stress in a planar turbulent boundary layer has the form

$$\frac{q}{\tau} = g\, c_p \frac{\overline{V_y\theta}}{\overline{V_x V_y}}.$$

In the degree of similarity temperature and velocity fields condition (1.51) is fulfilled, i.e.,

$$\frac{\overline{V_y\theta}}{\overline{V_x V_y}} \approx \frac{\Delta T}{w_0}. \qquad (2.13)$$

Introducing these relationships in formula (2.3), we can write that in order

$$\beta \approx \left(\frac{w_y}{w_0} + \frac{v_*}{w_0}\sqrt{\frac{\rho_w}{\rho}\tilde{\tau}}\right)\frac{\Delta T}{T}. \qquad (2.14)$$

The transverse velocity component at an impenetrable wall in succession is equal to the quantity:

$$w_0 \frac{d\delta}{dx} \approx w_0 \frac{\delta}{\delta^{**}} \cdot \frac{d\delta^{**}}{dx} \approx w_0 \frac{\delta}{\delta^{**}} \cdot \frac{c_f}{2},$$

i.e., in this case

$$\beta \approx \left(\frac{\delta c_f}{2\delta^{**}} + \sqrt{\frac{\rho_0}{2\rho}\tilde{c}_f}\right)\frac{\Delta T}{T}. \qquad (2.15)$$

Let us consider flow with "disappearing viscosity," i.e., with $\mu \to 0$. In this case $\mathbf{Re} \to \infty$, and the friction coefficient $c_f \to 0$.

Correspondingly, the coefficient β also approaches zero. At a permeable wall the maximum additional component of the velocity component w_y occurs at $y = 0$ and, as is shown in Chapter IV, is equal to $w_{wall} \leq \frac{c_{f0}\rho_0}{2\rho_{wall}} b_{cr} w_0$, where b_{cr} is a finite quantity. Consequently, at $c_f \to 0$, the quantity w_{wall} also approaches zero.

The equation of motion of a planar boundary layer in direct proximity to a wall has the form (since in this case $w_x \approx 0$).

$$-\frac{dp}{dx} + \frac{\partial \tau}{\partial y} = \rho w_y \frac{\partial w_x}{\partial y} \tag{2.16}$$

At $y < y_1 \ll \delta$ $\tau = \mu \frac{\partial w_x}{\partial y}$ and, in the first approximation, $\rho = \rho_{wall}$, $\mu = \mu_{wall}$, $w_y = w_{wall}$. Here for the quantity w_{wall} is understood the value of w_y at $y = 0$.

Integrating (2.16) we have

$$\tau - \tau_{cr} = \int_0^y \frac{\partial \tau}{\partial y} dy = \int_0^y \left(\frac{dp}{dx} + \rho_{cr} w_{cr} \frac{\partial w_x}{\partial y} \right) dy =$$

$$= \frac{dp}{dx} y + \rho_{cr} w_{cr} w_x. \tag{2.17}$$

Correspondingly, the velocity distribution in the nearest proximity of the wall is determined by the equation:

$$\mu_{cr} \frac{\partial w_x}{\partial y} = \tau_{cr} + \frac{dp}{dx} y + \rho_{cr} w_{cr} w_x. \tag{2.18}$$

By reducing this equation to the dimensionless form and taking into account dependency (1.10), we have

$$\frac{d\omega}{d\xi} = Re_{cr}^{**} \frac{\delta}{\delta^{**}} \left(\frac{c_f}{2} - \frac{\delta}{\delta^{**}} f \xi + \frac{\rho_{cr} w_{cr}}{\rho_0 w_0} \omega \right), \tag{2.19}$$

where

$$Re_{cr}^{**} = \frac{w_0 \delta^{**}}{\nu_{cr}}$$

the characteristic Reynolds number of the boundary layer, referred to the wall temperature.

The quantity

$$f = \frac{\delta^{**}}{w_0} \cdot \frac{dw_0}{dx} \tag{2.20}$$

can be considered as a measure "of aerodynamic curvature" of a streamlined surface and is called the form parameter.

During streamlining of an impenetrable surface $w_{wall} = 0$ and integration of equation (2.19) gives:

$$w = Re_{cr}^{**} \frac{\delta}{\delta^{**}} \left(\frac{c_f}{2} \xi - \frac{\delta f}{2\delta^{**}} \xi^2 \right). \tag{2.21}$$

The value of $\frac{\delta^{**}}{\delta}$ is finite for any Re numbers. The value of ω_1 by definition lies between 0 and 1. The coefficient of friction c_f is always inversely proportional to the **Re** number in a degree, less than 1 (see, for instance formula (3.24)).

Taking into account these circumstances and reducing equation (2.21) to the form:

$$c_f \xi_1 - \frac{\delta f}{\delta^{**}} \xi_1^2 = \frac{2\delta^{**} \omega_1}{\delta Re_{cr}^{**}}, \tag{2.22}$$

we note that at

$$Re \to \infty \qquad \xi_1 \to 0.$$

Thus, the thickness of the viscous underlayer decreases with growth of the **Re** number relatively faster than the thickness of the whole turbulent boundary layer.

At f = 0

$$\begin{aligned} Re_{sp\,1} &= \frac{w_1 y_1}{\nu_{cr}} = \omega_1 \xi_1, \\ Re_{cr}^{**} \frac{\delta}{\delta^{**}} &= \omega_1^2 \frac{2}{c_f}; \end{aligned} \tag{2.23}$$

at $f = f_{cr} < 0$, i.e., at the point of breakaway of the boundary layer, $c_f = 0$ and

$$Re_{sp\,1} = \frac{w_1 y_1}{\nu_{cr}} = \omega_1^{3/2} \sqrt{\frac{2Re_{cr}^{**}}{|f_{sp}|}}. \tag{2.24}$$

Since as the critical Reynolds number of the viscous underlayer is always finite then at $Re \to \infty$, $\omega_1 \to 0$.

An analogous result is also obtained for a permeable plate, i.e., in general:

$$\omega_1 |_{Re \to \infty} \to 0. \tag{2.25}$$

2.4. Limiting Law of Resistance

At $\mathbf{Re} \to \infty$, $\omega_1 \to 0$, $\xi_1 \to 0$, $\beta \to 0$, and according to formula (2.11)

$$Z \to \sqrt{\frac{c_{f\infty}}{2}} \int_0^1 \sqrt{\frac{\tilde{\tau}}{\tilde{\tau}_w}} \frac{\rho}{\rho_1} d\eta. \tag{2.26}$$

Let us expand the function $\Phi = Z \sqrt{\frac{c_{f\infty}}{2}}$ in a series by degrees of the perturbation factor and let us designate the sum of the terms from $i = 2$ to $i = \infty$ by $\Delta\Phi$. Then in accordance with form (2.11)

$$Z = Z_\infty + \Delta\Phi \sqrt{\frac{c_{f\infty}}{2}}. \tag{2.27}$$

The quantity $Z_0 = 1 - \omega_{10}$, which follows from (2.10), if in this equation are inserted $\rho = \rho_0$, $\tau = \tau_0$ and $\Psi = 1$.

Thus, it is possible to write that

$$Z = 1 - \omega_{10} + \Delta\Phi \sqrt{\frac{c_{f\infty}}{2}}. \tag{2.28}$$

At $\mathbf{Re} \to \infty$ $\omega_{10} \to 0$, $c_{f0} \to 0$. Consequently, if the function $\Delta\Phi$ at $\mathbf{Re} \to \infty$ remains finite or approaxhes infinity more weakly than $\sqrt{c_{f\infty}}$ approaches zero, then there occurs the condition:

$$Z_{Re \to \infty} = 1. \tag{2.29}$$

Thus, during certain conditions there exist certain limiting laws of the relative influence of nonisothermalness, compressibility, and other disturbing factors on the coefficient of friction in a turbulent boundary layer, determined by an integral of the form:

$$\int_0^1 \frac{d\eta}{\sqrt{\Psi \frac{\rho_w}{\rho} \cdot \frac{\tilde{\tau}}{\tilde{\tau}_w}}} = 1. \tag{2.30}$$

More detailed information about the properties of the limiting relative laws of friction in a turbulent boundary layer can be obtained, if one considers in a sufficiently general form the connection between the length of the path of mixing l and the coordinate y. Let us expand the function $\frac{l}{\delta}(\xi)$ in a series by degrees of the coordinate ξ:

$$\frac{1}{\delta} = \varkappa \sum_{i=1}^{n} \varkappa_i \xi^{i+1}. \qquad (2.31)$$

Experimental material shows that the quantity $\varkappa$ can be considered as some universal constant, but the sum of the terms $\varkappa_1 \xi^{i+1}$ is always finite, although the coefficients $\varkappa_1$ are also functions of the non-isothermalness, pressure gradient, transverse flow of the substance, and other disturbing factors.

In particular, during diffusion flow of an incompressible liquid on an impenetrable surface

$$0 < \sum \varkappa_i \xi^{i+1} < 1.$$

From experimental data it follows (see chap. III) that at $\mathbf{Re} \to \infty$

$$\sqrt{\frac{c_{f0}}{2}} \to \frac{\varkappa}{\ln Re^{**}}.$$

Introducing these relationships in (2.26) and expanding the function $\sqrt{\zeta}$ in a series by degrees of ξ, we find that at $\mathbf{Re} \to \infty$

$$Z \to \frac{\int_{\zeta=0}^{1} (1 + \Sigma \varkappa_i \xi^i) \frac{d\zeta}{\zeta}}{\ln Re^{**}} \to \frac{-\ln \xi_1}{\ln Re^{**}}. \qquad (2.32)$$

From the data, given in chapter III, it follows that on an impenetrable surface at $\frac{dp}{dx} = 0$ the relative thickness of the viscous underlayer is related to the Reynolds number by the relationship:

$$\xi_1 = \left| \frac{\eta_1 \delta^{**}}{Re^{**} \delta} \right| \sqrt{\Psi \frac{c_{f0}}{2}}.$$

In addition the values η_1, $\frac{\delta^{**}}{\delta}$, Ψ always are finite.

Putting this value at ξ_1, in formula (2.32), we find that in the considered case at $\mathbf{Re} \to \infty$:

$$Z \to 1 - \frac{\ln \ln Re^{**} - \ln \frac{\eta_1 \delta^{**}}{\varkappa \delta} + \frac{1}{2} \overline{\Psi}}{\ln Re^{**}} \to 1.$$

In Chapter V it is shown that during diffusion flow ($\frac{dp}{dx} > 0$) of an isothermal boundary layer on an impenetrable surface at the point of

breakaway of the boundary layer:

$$\xi_1 \approx \frac{\text{const}}{Re^{**0.5}}.$$

Putting the value ξ_1 in (2.32), we find that at $Re \to \infty$ $Z \to 0.65$.

Thus, in the general case the integral (2.32) is equal to a certain value Z_∞, not being a function of Reynolds number. Besides the parameter Z_∞ is exactly equal to unit or close to it.

2.5. Approximation of the Tangential Stresses Profile

From determination of a dynamic boundary layer of finite thickness we have the condition:

$$\begin{cases} \xi = 0, & \tau = \tau_{cr}; \\ \xi = 1, & \tau = 0. \end{cases} \tag{2.33}$$

From the condition of smoothness of the function $\tau(\xi)$ at the point $\xi = 1$ it follows that

$$\left| \frac{\partial \tau}{\partial \xi} \right|_{\xi=1} = 0. \tag{2.34}$$

In the region $\xi \approx 0$ with accuracy up to small quantities of the second order equation (2.17) should be satisfied.

Conditions (2.17), (2.33), and (2.34) are satisfied by the cubic parabola,

$$\bar{\tau} = 1 - 3\xi^2 + 2\xi^3 + (\Lambda\xi + b_1 \omega)(1-\xi)^2, \tag{2.35}$$

where

$$\Lambda = \frac{\delta}{\tau_{cr}} \cdot \frac{dp}{dx} = -\frac{2\delta}{c_f \delta^{**}} f; \tag{2.36}$$

$$b_1 = \frac{2\rho_{cr} w_{cr}}{c_f \rho_0 w_0}. \tag{2.37}$$

The first of these quantities is a certain modification of the form parameter. The second quantity characterizes the influence of the feeding or removal of a substance through the surface of a streamlined body. We will call it the parameter of wall permeability.

Subsequently we will deal again with several modifications of the form parameter and permeability parameter.

At gradiant less streamlining of an impenetrable wall according to the given approximation:

$$\tilde{\tau} = \tilde{\tau}_0 = 1 - 3\xi^2 + 2\xi^3. \tag{2.38}$$

Correspondingly

$$\frac{\tilde{\tau}}{\tilde{\tau}_0} = 1 + (\Lambda\xi + b_1\omega) f(\xi). \tag{2.39}$$

where

$$f(\xi) = (2\xi + 1)^{-1}.$$

The value of this function changes from 1 at $\xi = 0$ to $\frac{1}{3}$ at $\xi = 1$.

To formula (2.39) corresponds the relation

$$\Psi \frac{\tilde{\tau}}{\tilde{\tau}_0} = \Psi + (\Lambda_0\xi + b\omega) f(\xi). \tag{2.40}$$

where

$$\Lambda_0 = -\frac{2\delta}{c_{f_0} b^{m+1}} f; \tag{2.41}$$

$$b = \frac{2\rho_{cr} w_{cr}}{c_{f_0} \rho_0 w_0}. \tag{2.42}$$

2.6. Approximation of the Temperature Profiles

Inasmuch as pressure across boundary layer does not change, then in accordance with (1.7) the density in equation (2.2) at a uniform boundary layer can be expressed by the density of an undisturbed flow by formula

$$\rho = \rho_0 \frac{T_0}{T}. \tag{2.43}$$

Thus, for solution of equation (2.30) it is necessary to know the connection between the temperature and velocity fields. This relation was known previously only for the case of similarity of the velocity and inhibition temperature fields examined in section 1.5.

For gases the **Pr** numbers are equal to or differ little from unity, i.e., one of the basic conditions of the existence of such similarity is always fulfilled. Therefore, it is possible to put dependency

(1.50) in the base of the unknown relation. However, it is necessary to give it a form, taking into account the disturbance of similarity, due to the independent effect of the inhibition enthalpy $\frac{Ar_x^2}{2\beta}$ at $Pr \neq 1$, which is seen from the structure of equation (1.11).

Let us take the relationship

$$\frac{T_{cr} - T^+}{T_{cr} - T^*_{cr}} = \varepsilon(\xi) \frac{w_x}{w_0}, \tag{2.44}$$

where

$$T^+ = T + r(\xi) \frac{A w_x^2}{2g c_p} \tag{2.45}$$

The form of the functions $\varepsilon(\xi)$ and $r(\xi)$, in general, depends on the pressure gradient, nonisothermalness, and mass transfer.

At $\varepsilon = r = 1$, $T^+ = T^*$ and equation (2.44) transforms to (1.50), i.e., there occurs exact similarity of the velocity and stagnation temperature fields.

Let us require, that on the boundaries of the thermal layer the quantity

$$\vartheta = \frac{T_{cr} - T^+}{T_{cr} - T^*_{cr}} \tag{2.46}$$

satisfied the same conditions as the quantity ω on the boundaries of a dynamic layer. Then at $y = \delta_t$ $\vartheta = 1$, to which correspond the values $T^+ = T^*_{wall}$ and $r(\xi) = r$. In a turbulent boundary layer the quantity r has the order $Pr^{1/3}$, i.e., for gases close to unity. Therefore, without appreciable error it is possible to assume in all sections of a thermal boundary layer $r(\xi) \approx r$ and

$$T^+ \approx T + r \frac{A w_x^2}{2g c_p}. \tag{2.47}$$

The problem of the function $\varepsilon(\xi)$ is considered in section 2.7.

The value of the dynamic temperature can be connected with the dimensionless velocity

$$\omega = \frac{w_x}{w_0} \tag{2.48}$$

by the relationship

$$\frac{A w_x^2}{2g c_p T_0} = \frac{k-1}{2} M^2 \omega^2, \tag{2.49}$$

where

$$M = \frac{w_0}{\tilde{a}} \tag{2.50}$$

is the Mach – Mayevskiy number, referred to parameters outside the boundary layer; $k = \frac{c_p}{c_v}$ is the index of Poisson's adiabatic line.

From (2.44) and (2.47) it follows that

$$\frac{T}{T_0} = \psi - \Delta\psi 0 - (\psi^* - 1)\omega^2, \tag{2.51}$$

or

$$\frac{T}{T_0} = \psi - \Delta\psi \cdot \omega - (\psi^* - 1)\omega^2. \tag{2.52}$$

The quantity

$$\psi = \frac{T_{cr}}{T_0} \tag{2.53}$$

is called the temperature factor;

The quantity

$$\psi^* = \frac{T^*_{cr}}{T_0} = 1 + r\frac{k-1}{2} M^2 \tag{2.54}$$

is the kinetic temperature factor, determining the degree of aerodynamic heating of a body;

$$\Delta\psi = \psi - \psi^* \tag{2.55}$$

is the heat-transfer factor, since at $\Delta\psi = 0$ adiabatic streamlining of a body occurs;

At $\Delta\psi > 0$ a body gives up heat to a flow, at $\Delta\psi < 0$ a body receives heat from a flow.

2.7. Coefficient of Nonsimilarity of the Temperature and Velocity Fields

The velocity and temperature fields in the nucleus of a turbulent boundary layer are well approximated by exponential formulas of the form:

$$\left.\begin{aligned} \omega &= \left(\frac{y}{\delta}\right)^n; \\ \theta &= \left(\frac{y}{\delta_T}\right)^{n_T}. \end{aligned}\right\} \tag{2.56}$$

In flow regions, not close to the point of breakaway of the boundary layer (see chaps. IV and V), both exponents are small, i.e., the profiles are very complete.

From (2.56) it follows that

$$\theta = \omega^{\frac{1}{n}}; \tag{2.57}$$

where

$$n = \left(\frac{\delta}{\delta_t}\right)^{n_0}. \tag{2.58}$$

Far from the point of breakaway $n \approx n_0$, i.e., there exists a relative similarity of the temperature and velocity fields, expressed by the formula

$$\theta \approx \omega. \tag{2.59}$$

The ratio $\frac{\rho}{\rho_0}$ in a uniform boundary layer is determined by formula (2.43) and equation (2.51). In addition, two cases are distinguished: when the thickness of the dynamic boundary layer is less than the thickness of the thermal and when there is an inverse relationship.

In the first case $\delta < \delta_t$ and the process of heat transfer occurs in the whole thickness of the dynamic layer. Correspondingly, in the whole region $0 < y < \delta$ formula (2.52) is valid.

In the second case $\delta > \delta_t$, i.e., in the region $\delta_t < y < \delta$, heat transfer is absent and the temperature is determined by the condition $T^+ = \text{const}$. Correspondingly, in the region $0 < y < \delta_t$ formula (2.52) is valid, and in the region $\delta_t < y < \delta$ – formula

$$\frac{T}{T_\infty} = \psi^* - (\psi^* - 1)\omega^2. \tag{2.60}$$

2.8. Limiting Law of Heat Transfer

In the nucleus of a planar turbulent boundary layer the heat flux along the normal to the streamlined surface is determined by formula

$$q = q_t = -c_p \gamma \overline{V_y \theta}(1 - \beta_t), \tag{2.61}$$

where β_t is a coefficient, taking into account the influence of density

pulsations on turbulent heat transfer.

Analogously to (2.4) we can write that

$$\overline{V_y \theta} = l_T^2 \frac{\partial w_x}{\partial y} \cdot \frac{\partial T}{\partial y}. \tag{2.62}$$

In the general case $l_t \neq l$. From (2.57) it follows that

$$\frac{dw}{d\eta} = \frac{n}{n_T} \cdot \eta^{-\frac{n}{n_T}} \eta^{\frac{n-n_T}{n_T}}. \tag{2.63}$$

At c_p = const equation (2.61) is reduced to the form

$$St = \frac{\alpha}{c_p \rho_0 w_0} = \frac{2}{Re_0} \cdot \frac{1-\beta_T}{\tilde{q}} l_T^2 \frac{\partial w}{\partial y} \cdot \frac{\partial \theta}{\partial y}, \tag{2.64}$$

where $\tilde{q} = \frac{q}{q_{wall}}$.

In a region, not close to the point of breakaway $n \approx n_t$, and from (2.64) it follows that:

$$\psi_1 = \left(\frac{1}{Z_T} \int_{\eta_1}^{1} \sqrt{\frac{\rho \tilde{q}_0}{\rho_0 q_0}} d\eta \right)^2, \tag{2.65}$$

where

$$\psi_2 = \left(\frac{St}{St_0} \right)_{Re_T}; \tag{2.66}$$

$$Z_T = \sqrt{St_0} \int_{y_{1T}}^{\delta} \sqrt{\frac{\tilde{q}_0}{1-\beta_T}} \cdot \frac{dy}{l_T}. \tag{2.67}$$

Transition to a thermal boundary layer of finite thickness is carried out according to equation

$$\int_{y_{1T}}^{\delta} \sqrt{\frac{\tilde{q}_0}{1-\beta_T}} \cdot \frac{dy}{l_T} \approx \int_{\xi_{1T}}^{1} \sqrt{\frac{\tilde{q}_0}{1-\beta_T}} \cdot \frac{\delta_T}{l_T} d\xi_T, \tag{2.68}$$

where $\xi_t = \frac{y}{\delta_t}$.

In these equations St_0 is the value of the Stanton number for the conditions $\rho = \rho_0$, $\frac{dp}{dx} = 0$, $w_{wall} = 0$.

The quantities ψ_1, St_0, β_t and Z_t possess the properties of the quantities ω_1, c_{f0}, β and Z, i.e., at $Re \to \infty$, $St_0 \to 0$, $\psi_1 \to 0$, $\beta_t \to 0$,

$Z_+ \to 1$.

Correspondingly, at $\mathbf{Re} \to \infty$ equation (2.65) has a limiting solution of the form:

$$\Psi_s = \left(\int_0^{\xi_\tau} \sqrt{\frac{\xi_\tau}{a_{cr}\tilde{q}}}\, d\xi_\tau \right)^2. \qquad (2.69)$$

This equation is analogous to equation (2.39) for the limiting law of resistance.

2.9. Approximation of the Heat Flux Profile

From determination of the thermal boundary layer of finite thickness we have the conditions:

$$\left.\begin{aligned} \xi_\tau = 0, \quad q = q_{cr}; \\ \xi_\tau = 1, \quad q = q_0; \end{aligned}\right\} \qquad (2.70)$$

From the condition of smoothness of the function $q(\xi_\tau)$ at the point $\xi_\tau = 1$ it follows that:

$$\left.\frac{\partial q}{\partial \xi_\tau}\right|_{\xi_\tau = 1} = 0. \qquad (2.71)$$

The thermal conductivity equation near the wall ($w_x \approx 0$, $T \approx T^+$) has the form:

$$\frac{\partial q}{\partial y} = c_p \gamma_{cr} w_{cr} \frac{\partial T^+}{\partial y}. \qquad (2.72)$$

Integrating (2.72), we find that in the neighborhood of the wall

$$q = q_{cr} + c_p \gamma_{cr} w_{cr} T^+. \qquad (2.73)$$

These conditions are satisfied by the cubic parabola:

$$\tilde{q} = 1 - 3\xi_\tau^2 + 2\xi_\tau^3 + b_{1\tau}\vartheta(1 - \xi_\tau)^2. \qquad (2.74)$$

Correspondingly

$$\tilde{q}_0 = 1 - 3\xi_\tau^2 + 2\xi_\tau^3; \qquad (2.75)$$

$$\Psi_s \frac{\tilde{q}}{\tilde{q}_0} = \Psi_s + b_\tau \vartheta f(\xi_\tau), \qquad (2.76)$$

where

$$b_{1\tau} = \frac{\rho_{cr} w_{cr}}{St\, \rho_0 w_0}; \qquad (2.77)$$

$$b_\tau = \frac{\rho_{\tau\tau} w_{c\tau}}{St_0 \rho_0 w_0}; \qquad (2.78)$$

$$f(\xi_\tau) = (2\xi_\tau + 1)^{-1}. \qquad (2.79)$$

At $c_p \neq$ const the quantities b_{1t} and b_t should be multiplied by the ratio $\frac{c_{p\ all}}{c_{p0}}$.

At Pr = 1 and $\frac{\partial p}{\partial x} = 0$ $\delta = \delta_\tau$, $\xi = \xi_\tau$ and $\bar{q} = \tilde{\tau}$.

C H A P T E R III

LONGITUDINAL STREAMLINING OF AN IMPENETRABLE PLATE

Definitions of Cyrillic Items in Order of Appearance

ст = wall = wall

кр = cr = critical

т = t = thermal, turbulent

3.1. Isothermal Boundary Layer

In the chapter the streamlining of a flat plate by an unbounded flow of gas for the conditions

$$\frac{dp}{dx} = 0 \text{ and } T_{cr} = \text{const}$$

is considered.

A boundary layer for the variables w_0 and T_{wall} is considered in Chapter V.

A large quantity of experiments confirms for the region $\xi < 0.4$, the logarithmic distribution of velocities in a well-developed turbulent isothermal boundary layer is expressed by the Prandtl-Nikuradze formula:

$$\varphi = 5{,}5 + 2{,}5 \ln \eta. \qquad (3.1)$$

Here

$$\varphi = \frac{w_x}{v_*}; \quad \eta = \frac{v_* y}{\nu}; \quad v_* = \sqrt{\frac{\tau_{cr}}{\rho}} = w_0 \sqrt{\frac{c_f}{2}}.$$

The last quantity is called the dynamic velocity.

Formula (3.1) is not applicable at a large distance from the wall and because in it an unlimited increase in y leads to unlimited growth of w_x, whereas the latter quantity at $y \to \infty$ is equal to w_0. Thus, this formula should be used within the framework of the theory of a boundary layer of finite thickness.

Since in the region $\xi > 0.4$ in a turbulent boundary layer on an impenetrable plate $\omega > 0.9$, then the indicated circumstances do not introduce appreciable errors in the calculations of friction and heat transfer.

In Fig. 3 is given a graph of $\varphi(\eta)$, also including data for the region of the viscous underlayer. Distribution of velocities in it is determined by the expression

$$\varphi = \eta, \tag{3.2}$$

which corresponds to formula (2.21) at f = 0.

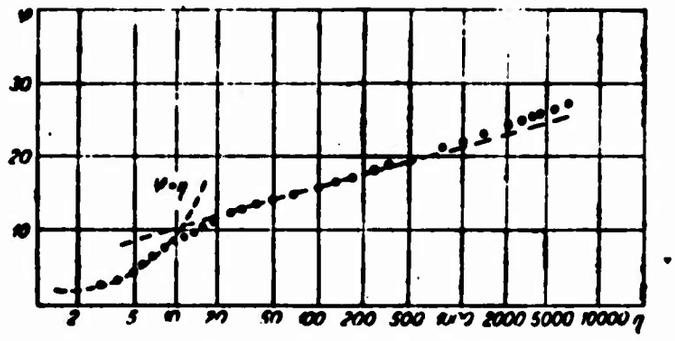

Fig. 3. Universal velocity profile on a flat plate.

Intersection of the lines, calculated by the formulas (3.1) and (3.2), gives the calculated thickness of the viscous underlayer in a two-layered diagram of isothermal turbulent flow $\eta_{10} = 11.6$. Dimensionless velocities on this boundary are equal to:

$$\varphi_{10} = 11{,}6; \tag{3.3}$$

$$w_{10} = 11{,}6 \sqrt{\frac{c_{f_0}}{2}}.$$

Such a diagram, which nominally divides the stream into a viscous underlayer, in which $\mu \gg \mu_t$ and a turbulent nucleus, in which $\mu \ll \mu_t$,

turns out to be fully acceptable for calculations of friction. It is also applicable for calculations of heat transfer for $Pr \approx 1$ numbers. At $y = \delta$ $w = w_0$, and from (3.1) it follows that

$$\sqrt{\frac{2}{c_{f0}}} = 5{,}5 + 2{,}5 \ln Re \sqrt{\frac{c_{f0}}{2}} \qquad (3.4)$$

where

$$Re = \frac{w_0 \delta}{\nu}.$$

Further, we have

$$\frac{w_0 - w_x}{v_*} = 2{,}5 \ln \frac{y}{\delta}; \qquad (3.5)$$

$$\frac{\delta^{**}}{\delta} = \int_0^1 \frac{w_x}{w_0}\left(1 - \frac{w_x}{w_0}\right) d\xi = 2{,}5\sqrt{\frac{c_{f0}}{2}} + 0{,}85 c_{f0}. \qquad (3.6)$$

Putting the value of δ^{**} in (3.4) instead of δ, we will obtain the law of resistance:

$$\sqrt{\frac{2}{c_{f0}}} = 5{,}5 + 2{,}5 \ln \frac{Re^{**}}{2{,}5 - 12{,}5\sqrt{\frac{c_{f0}}{2}}}. \qquad (3.7)$$

Calculations show that the quantity $\ln\left(2.5 - 12.5\sqrt{\frac{c_{f0}}{2}}\right)$ changes very slightly in a wide range of values of Re^{**} and in practice instead of expression (3.7), it is possible to use the significantly more convenient formula of T. Karman:

$$c_{f0} = \frac{2}{(2{,}5 \ln Re^{**} + 3{,}8)^2}. \qquad (3.8)$$

From (3.1) it follows that

$$\left(\frac{w_0 \delta}{\nu}\right)_0 = 0.111 \exp\sqrt{\frac{32}{c_{f0}}}. \qquad (3.9)$$

Hence

$$\xi_{10} = \varphi_{10}\left(\frac{\nu}{w_0 \delta}\right)_0 = 104 \exp\left(-\sqrt{\frac{32}{c_{f0}}}\right). \qquad (3.10)$$

The logarithmic velocity profile is an envelope of a family of exponential profiles

$$\varphi = A\eta^n. \qquad (3.11)$$

For many calculations the use of such an approximation of the velocity profile is very convenient.

By putting in (3.11) the values of $w_x = w_0$ and $y = \delta$, we find that

$$\frac{2}{c_{f0}} = (A\,Re_\delta^n)^{\frac{2}{1+n}}. \tag{3.12}$$

For the thickness of displacement and loss of momentum we obtain:

$$\left.\begin{aligned} \frac{\delta^*}{\delta} &= \frac{n}{1+n}; \\ \frac{\delta^{**}}{\delta} &= \frac{n}{(1+n)(1+2n)}; \\ H &= \frac{\delta^*}{\delta^{**}} = 1+2n. \end{aligned}\right\} \tag{3.13}$$

From this the exponential law of resistance follows:

$$c_{f0} = B\,Re^{**-m}, \tag{3.14}$$

where

$$\left.\begin{aligned} B &= 2A^{-\frac{2}{1+n}}\left[\frac{(1+n)(1+2n)}{n}\right]^{\frac{2n}{1+n}}; \\ m &= \frac{2n}{1+n}. \end{aligned}\right\} \tag{3.15}$$

For the thickness of the boundary layer and the relative thickness of the viscous underlayer we have:

$$\left(\frac{v_0\,\delta}{\nu}\right)_0 = A^{-\frac{1}{n}}\left(\frac{2}{c_{f0}}\right)^{\frac{n}{2}}; \tag{3.16}$$

$$\xi_{10} = 11{,}6\,A^{\frac{1}{n}}\left(\frac{c_{f0}}{2}\right)^{\frac{n}{2}}. \tag{3.17}$$

The momentum equation takes the form:

$$\frac{d\delta^{**}}{dx} = \frac{c_f}{2} \tag{3.18}$$

or

$$\frac{dRe^{**}}{dRe_x} = \frac{c_f}{2}. \tag{3.19}$$

Putting in (3.19) the value of c_f from (3.14) and considering boundary layer, developing turbulently from a certain cross section x_{cr}, we find that

$$Re^{**1+m} - Re_{**}^{1+m} = \frac{1-m}{2}B(Re_x - Re_{x\,кр}). \tag{3.20}$$

If $x_{cr} = 0$, i.e., the turbulent layer in practice starts at the leading edge of the plate,

$$Re^{**} = \left(\frac{1+m}{2} B Re_x\right)^{\frac{1}{1+m}}, \tag{3.21}$$

to which corresponds the value:

$$c_{f_0} = B_1 Re_x^{-m_1}, \tag{3.22}$$

where

$$\left.\begin{aligned} B_1 &= B^{\frac{1}{1+m}}\left(\frac{1+m}{2}\right)^{\frac{m}{1+m}}; \\ m_1 &= \frac{m}{1+m}. \end{aligned}\right\} \tag{3.23}$$

The values of the coefficients in the exponential laws of resistance and the velocity distributions are given in Table 3.1. In the same place are also given certain other quantities ensuing from these laws.

Table 3.1. Values of the coefficients in Formulas (3.14), (3.14), (3.22)

n	1/7	1/8	1/9	1/10
A	8,74	9,32	10,6	11,5
$\frac{\delta^{**}}{\delta}$	0,0975	0,0890	0,0818	0,0757
H	1,29	1,25	1,22	1,20
m	0,250	0,222	0,200	0,182
B	0,0252	0,0206	0,0190	0,0148
m_1	0,200	0,182	0,167	0,154
B_1	0,0576	0,0459	0,0362	0,0296

In practice in the region of Re^{**} numbers $< 10^4$, it is possible to use the formulas for the distributive law of velocities according to the exponent n = 1/7.

In the region $10^4 < Re^{**} < 10^6$ the Faulkner formula gives good results:

$$c_{f_0} = 0{,}0131\, Re^{**\,-\frac{1}{6}}. \tag{3.24}$$

For a turbulent layer, developing from a cross section x = 0, the formula

$$c_{f_0} = 0{,}0263\, Re_x^{-\frac{1}{7}} \tag{3.25}$$

corresponds to relationship (3.24). In Fig. 4 is given a comparison of the given formulas with experimental data.

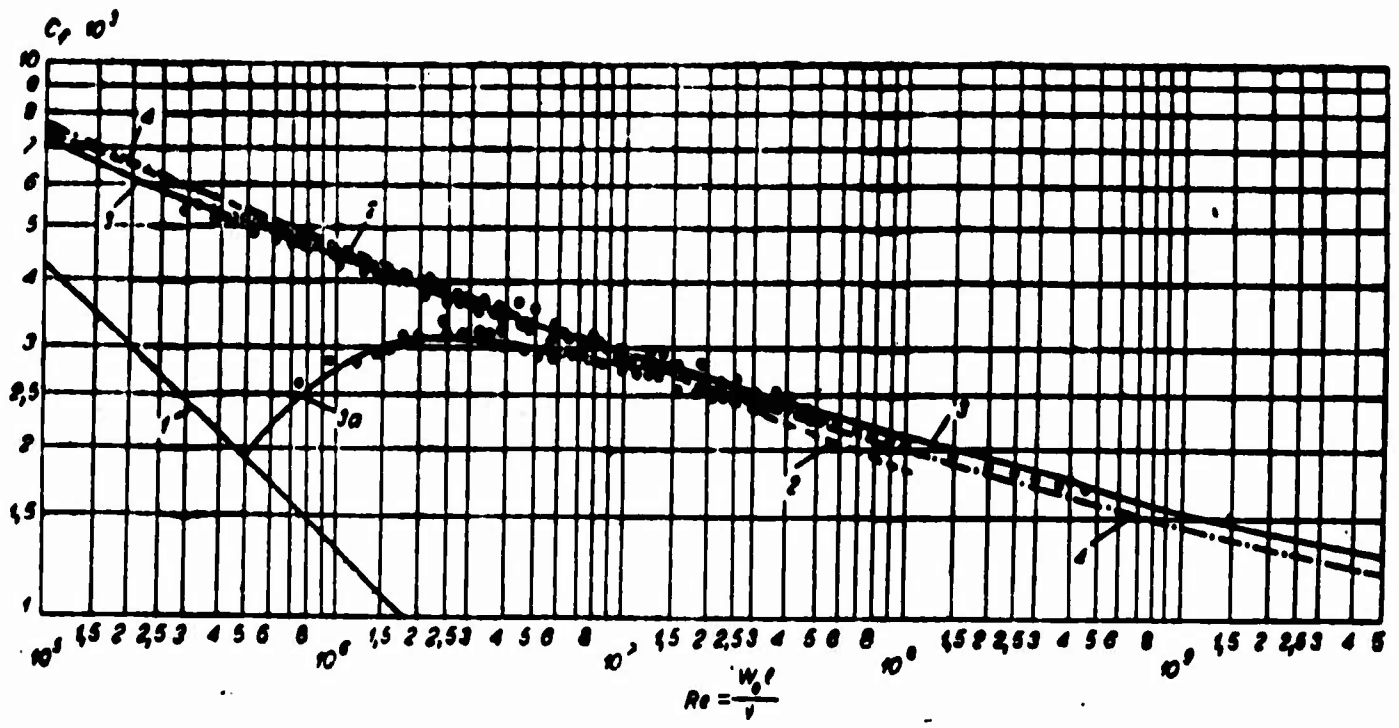

Fig. 4. Average coefficients of friction during isothermal streamlining of a flat plate: 1) Blasius' law (laminar conditions); 2) the Prandtl law; 3) the Prandtl-Schlichting law; 3a) transition region; 4) the Schultz-Grunov law. Measurements by Wiselsberger, Hebers, Frude, Kempf, Schenger.

3.2. Heat Transfer Coefficient for $\psi \simeq 1$

For polyatomic gases **Pr** ≈ 1 and in the case considered the dependency (1.52) is fulfilled.

Putting in it the value of c_f from (3.14) we have:

$$St_x = \frac{B}{2} Re^{**-m}. \tag{3.26}$$

For a plate, on which dynamic and thermal boundary layers are developing from a cross section x = 0, the dependences

$$St_0 = \frac{B_1}{2} Re_x^{-m_1} \tag{3.27}$$

or

$$Nu_0 = \frac{B_1}{2} Re_x^{1-m_1} \tag{3.28}$$

correspond to the formula (3.26).

For a region, in which we will apply distributive law of velocities by the degree n = 1/7, B_1 = 0.0576, and m_1 = 0.20. Correspondingly, at **Pr** = 1

$$Nu_0 = 0{,}0288\ Re_x^{0.8}. \tag{3.29}$$

In Fig. 5 are given the experimental data of B. S. Petukhov, A. A. Detlaf, and V. V. Kirillov on the local values of the Nusselt number during subsonic streamlining of a plate by air.

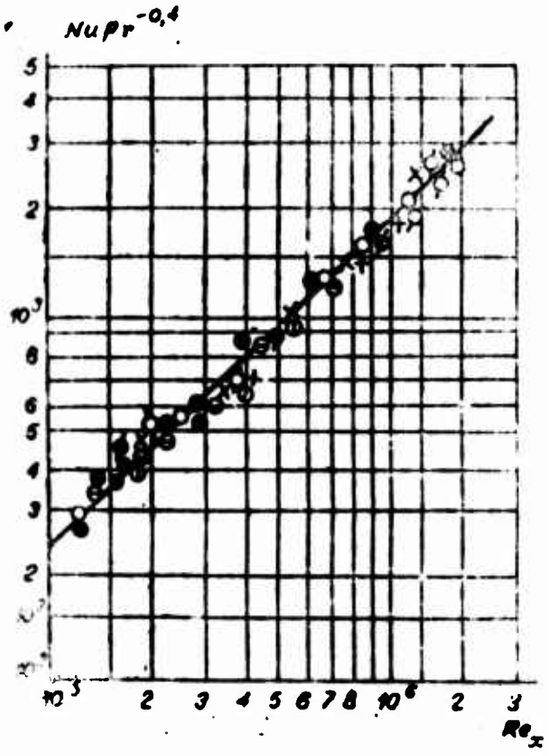

Fig. 5. Local values of Nu_x during subsonic streamlining of a plate by air (according to the experiments of B. S. Petukhov): ○ — w_0 = 260 m/sec; ⊙ — w_0 = 245 m/sec; ◇ — w_0 = [illegible] m/sec; × — w_0 = 168 m/sec; ⊗ — w_0 = 162 m/sec; ++ — w_0 = [illegible] m/sec; ⊖ — w_0 = 62.5 m/sec; ⊜ — w_0 = 42.3 m/sec; ⊕ — w_0 = 34.1 m/sec.

As can be seen, introduction of the factor $Pr^{0.4}$ puts these data on the line of formula (3.29). This circumstance is confirmed by the experiments of Emas, Frank, et al.

Therefore, for gases it is possible with great accuracy to postulate that

$$Nu_0 = \frac{B_1}{2} Pr^{0.4} Re_x^{1-m_1}, \quad (3.30)$$

to which corresponds the dependence

$$St_0 = \frac{B_1}{2} Pr^{-0.6} Re_x^{-m_1}. \quad (3.31)$$

At $(dp/dx) = 0$ and T_{wall} = const the energy equation takes the form:

$$\frac{dRe_t^{**}}{dRe_x} = St_0. \quad (3.32)$$

Putting in (3.32) the value of St_0 from (3.31), we find that for the conditions examined:

$$Re_t^{**} = \frac{B_1 Pr^{-0.6}}{2(1-m_1)} Re_x^{1-m_1}. \quad (3.33)$$

By replacing, according to this formula, Re_x in (3.31) by Re_t^{**}, we find that

$$St_0 = \frac{B}{2} Pr^{0.6(1-m)} \cdot Re_t^{-m}. \quad (3.34)$$

Inasmuch, as the experimental value of the exponent for the Prandtl number is determined for m = 0.25, and the number itself for

gases is close to unity, it is possible to consider with a sufficient degree of accuracy that

$$St_0 = \frac{B}{2} Pr^{-0.75} \cdot Re_1^{**-m}. \tag{3.35}$$

From formulas (3.35) and (3.14) it follows that

$$St_0 = \frac{c_{f_0}}{2} Pr^{-0.75} \left(\frac{\delta^{**}}{\delta_T^{**}}\right)^m. \tag{3.36}$$

Comparing formula (3.21) and (3.33) we find that

$$\frac{\delta^{**}}{\delta_T^{**}} = Pr^{0.6} \approx \frac{\delta}{\delta_T}. \tag{3.37}$$

Consequently, according to formula (2.58) the nonsimilarity coefficient for the velocity and temperature fields for a plate, completely covered by dynamic and thermal boundary layers, is equal to:

$$\varepsilon \approx Pr^{0.6m}. \tag{3.38}$$

By this formula for air $\varepsilon \approx 0.97$, i.e., it differs little from unity.

3.3. Law of Resistance for a Nonisothermal Boundary Layer

Let us integrate equation (2.10) for the case of nonisothermal streamlining of a flat, impenetrable plate by an unbounded flow of gas. Assuming $\tilde{\tau} = \tilde{\tau}_0$ and determining the ratio ρ/ρ_0 by formulas (2.43), (2.52), and (2.59), i.e., for conditions of relative similarity of the temperature and velocity fields, we will calculate the right integral of the equation.

For the case $\delta < \delta_t$ we obtain:

$$\psi = \frac{1}{(\psi^* - 1) Z^2} \left[\arcsin \frac{2(\psi^* - 1) + \varepsilon \Delta\psi}{\sqrt{4(\psi^* - 1)(\psi + \Delta\psi) + (\varepsilon\Delta\psi)^2}} - \right.$$
$$\left. - \arcsin \frac{2(\psi^* - 1)\omega_1 + \varepsilon\Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\varepsilon\Delta\psi)^2}} \right]_{\omega_1}^{1}. \tag{3.39}$$

In the case $\delta > \delta_t$

$$\int_{\omega_1}^{1} \sqrt{\frac{\rho}{\rho_0}}\, d\omega = \int_{\omega_1}^{\omega_T} \frac{d\omega}{\sqrt{\psi - \Delta\psi\varepsilon\omega - (\psi^* - 1)\omega^2}} + \int_{\omega_T}^{1} \frac{d\omega}{\sqrt{\psi^* - (\psi^* - 1)\omega^2}}, \tag{3.40}$$

where ω_t is the dimensionless velocity at the point $y = \delta_t$.

For the relative similarity $\omega_t = \varepsilon^{-1}$, because at the point $y = \delta_t$ $\Phi = 1$.

Carrying out the integration, we find that at $\delta > \delta_t$

$$\Psi = \frac{1}{(\psi^\circ - 1) Z^2}\left[\arcsin \frac{2(\psi^\circ - 1)\varepsilon^{-1} + \varepsilon\Delta\psi}{\sqrt{4(\psi^\circ - 1)(\psi^\circ + \Delta\psi) + (\varepsilon\Delta\psi)^2}} - \right.$$
$$- \arcsin \frac{2(\psi^\circ - 1)\varepsilon^{-1}\omega_1 + \varepsilon\Delta\psi}{\sqrt{4(\psi^\circ - 1)(\psi^\circ + \Delta\psi) + (\varepsilon\Delta\psi)^2}} + \qquad (3.41)$$
$$\left. + \arcsin \sqrt{\frac{\psi^\circ - 1}{\psi^\circ}} - \arcsin \sqrt{\frac{\psi^\circ - 1}{\varepsilon^2 \psi^\circ}}\right]^2.$$

It is necessary to consider that in formula (3.39) $\varepsilon < 1$, and in formula (3.41) $\varepsilon > 1$. These formulas can be used not only for theoretical analysis, but also for the experimental determination of the functions Z and ω_1.

3.4. Limiting Law of Resistance

Assuming in formulas (3.39) and (3.41) $\omega_1 = 0$ and $Z = 1$, we find that at $\mathbf{Re} \to \infty$:

a) at $\varepsilon < 1$

$$\Psi = \frac{1}{\psi^\circ - 1}\left[\arcsin \frac{2(\psi^\circ - 1) + \varepsilon\Delta\psi}{\sqrt{4(\psi^\circ - 1)(\psi^\circ + \Delta\psi) + (\varepsilon\Delta\psi)^2}} - \right.$$
$$\left. - \arcsin \frac{\varepsilon\Delta\psi}{\sqrt{4(\psi^\circ - 1)(\psi^\circ + \Delta\psi) + (\varepsilon\Delta\psi)^2}}\right]^2; \qquad (3.42)$$

b) at $\varepsilon > 1$

$$\Psi = \frac{1}{\psi^\circ - 1}\left[\arcsin \frac{2(\psi^\circ - 1)\varepsilon^{-1} + \varepsilon\Delta\psi}{\sqrt{4(\psi^\circ - 1)(\psi^\circ + \Delta\psi) + (\varepsilon\Delta\psi)^2}} - \right.$$
$$- \arcsin \frac{\varepsilon\Delta\psi}{\sqrt{4(\psi^\circ - 1)(\psi^\circ + \Delta\psi) + (\varepsilon\Delta\psi)^2}} + \qquad (3.43)$$
$$\left. + \arcsin \sqrt{\frac{\psi^\circ - 1}{\psi^\circ}} - \arcsin \sqrt{\frac{\psi^\circ - 1}{\varepsilon^2 \psi^\circ}}\right]^2.$$

Formulas (3.42) and (3.43) express the limiting laws of friction for a nonisothermal turbulent boundary layer on an impenetrable plate. They do not contain empirical "constants of turbulence" and are not connected with any special type of semiemperical theories of turbulence.

The quantity c_{f0} in the limiting law is defined independent of the method of its derivation, i.e., can be defined both from special theoretical considerations (for instance, proceeding from any sort of semiempirical theory of turbulence in isothermal flow), and directly from experimental data.

During adiabatic flow the limiting ratio c_f/c_{f0} does not depend on the degree of nonsimilarity of the temperature and velocity fields. This follows from formulas (2.52), (3.42), and (3.43), which at $\Delta\psi = 0$ give one and the same expression.

$$\Psi = \frac{\left(\arcsin\sqrt{\frac{\psi^* - 1}{\psi^*}}\right)^2}{\psi^* - 1}. \qquad (3.44)$$

For $\psi^* \to 1$, i.e., for subsonic flows, we obtain:

a) for $\varepsilon < 1$

$$\Psi = \left[\frac{2}{\sqrt{\psi} + \sqrt{\psi - (\psi - 1)\varepsilon}}\right]^2; \qquad (3.45)$$

b) for $\varepsilon > 1$

$$\Psi = \left[\frac{2}{\varepsilon\left(\sqrt{\psi} + 1\right)} + \frac{\varepsilon - 1}{\varepsilon}\right]^2; \qquad (3.46)$$

c) for $\varepsilon = 1$ and $\psi^* \approx 1$

$$\Psi = \left(\frac{2}{\sqrt{\psi} + 1}\right)^2. \qquad (3.47)$$

The formulas obtained show that the quantity ε affects the relative change of the friction coefficient due to nonisothermalness the most noticeably during subsonic velocities. The degree of this influence is seen from Table 3.2.

It is interesting to note that the influence of the nonsimilarity of the velocity and temperature fields on the quantity Ψ during cooling and heating of a gas is opposite and small.

In the majority of practically important cases the ratio δ/δ_t is located in the limits from 0.5 to 2, which allows one to assume

in the formulas for c_f/c_{f0} the quantity $\varepsilon = 1$. This circumstance essentially simplifies calculation of boundary layers.

Table 3.2. Values of $(c_f/c_{f0})_{Re^{**}}$ During Subsonic Velocities from the Limiting Formulas (3.45) and (3.46)

$\frac{\delta}{\delta_t}$	n	$\psi = 0{,}25$	0,5	1	2	3	4	5
0	0	4,00	2,00	1	0,50	0,33	0,25	0,25
0,25	0,71	2,65	2,35	1	0,62	0,45	0,35	0,29
0,5	0,90	1,88	1,45	1	0,65	0,50	0,41	0,35
0,8	0,97	1,81	1,41	1	0,67	0,52	0,43	0,37
1	1,00	1,78	1,38	1	0,69	0,54	0,45	0,38
2	1,10	1,69	1,33	1	0,71	0,58	0,47	0,43
5	1,26	1,59	1,29	1	0,75	0,63	0,54	0,49
10	1,39	1,54	1,86	1	0,77	0,65	0,58	0,54
∞	∞	1	1	1	1	1	1	1

[1]The quantity ε is defined by formula (2.58) at $n = 1/7$.

During supersonic flow the influence of ε decreases with the growth of ψ^*, i.e., the number M. In general, the quantity ε does not effect the limiting friction law during adiabatic flow.

In Fig. 6 is shown the dependence of Ψ on ψ^* and $\Delta\psi$, calculated by the formula:

$$\Psi = \frac{1}{\psi^* - 1}\left[\arcsin\frac{2(\psi^* - 1) + \Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}} - \arcsin\frac{\Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}}\right]. \tag{3.48}$$

This formula is from (3.42) and (3.43) at $\varepsilon = 1$.

As is seen, the function $\Psi(\psi^*; \Delta\psi)$ has a complicated character, whereby the degree of influence of the heat transfer factor $\Delta\psi$ decreases with growth of the M number. The upper limit for ψ^* is determined in this case by the fact that at $M \approx 8$-10 a noticeable dissociation of the gas begins due to high temperatures, developed during inhibition of flow in the boundary layer.

Cooling of a gas ($\Delta\psi < 0$) causes an increase of the resistance

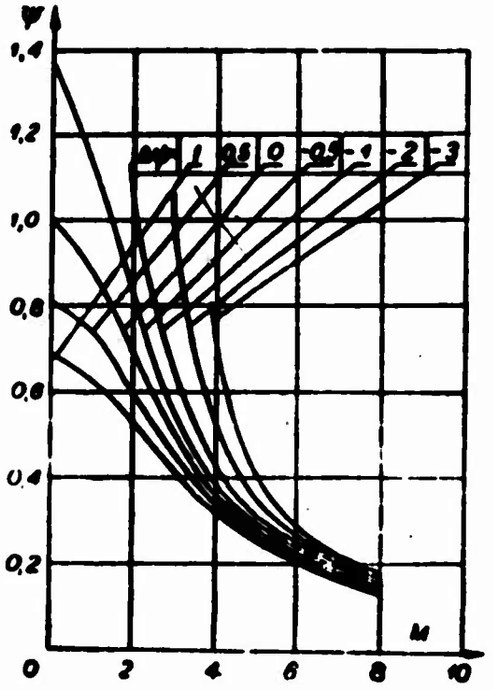

Fig. 6. Dependence of Ψ on ψ^* and $\Delta\psi$ from the limiting formula 3.48 (r = 0.9).

of friction of flow against the wall; heating of a gas ($\Delta\psi > 0$) lowers this resistance.

During subsonic flow $\psi^* \approx 1$ and, according to formula (3.47), the ratio c_f/c_{f0} at $\mathbf{Re} \to \infty$ depends only on the temperature factor ψ. The solution is asymmetric in relation to heating and cooling. During cooling of a gas the temperature factor ψ cannot be less than zero, to which corresponds at $\varepsilon = 1$ the limiting value $\Psi = 4$. In the region of heating the growth of ψ is unlimited (if the problem of dissociation is excluded) and the quantity Ψ can vary from 1 to 0.

In the neighborhood of $\psi = 1$ expansion of the right side of formula (3.47) gives

$$\Psi = \psi^{-0.5} = \frac{2}{\psi + 1}. \qquad (3.49)$$

The quantity

$$\frac{\psi + 1}{2} = \frac{T_{cr} + T_0}{2T_0}$$

is the dimensionless arithmethic mean temperature of the boundary layer.

The dependences (3.49) satisfactorily approximate the exact solution of (3.47) in the practically important range of values of Ψ from 0.5 to 3.

3.5. Limiting Law of Heat Transfer

If $\delta < \delta_t$, then in the region $0 < y < \delta$ distribution of the temperatures is determined by formula (2.51). In the region $y > \delta$

$\omega = 1$, i.e., at $\delta < y < \delta_t$

$$\frac{T}{T_0} = 1 + \Delta\psi(1 - \theta). \tag{3.50}$$

Besides the integral

$$\Psi_* = \left(\int_0^1 \sqrt{\frac{T_0}{T}}\, d\theta\right)^2. \tag{3.51}$$

following from (2.69) at $\tilde{q} = \tilde{q}_0$ and $\rho = \rho_0 \frac{T}{T_0}$, breaks up into two:

$$\int_0^1 \sqrt{\frac{T_0}{T}}\, d\theta = \int_0^{\varepsilon} \frac{d\theta}{\sqrt{\psi^* - \Delta\psi\theta - (\psi^* - 1)\left(\frac{\theta}{\varepsilon}\right)^2}} + \int_{\varepsilon}^1 \frac{d\theta}{\sqrt{1 + \Delta\psi(1 - \theta)}}. \tag{3.52}$$

Putting in (3.51) the corresponding expressions for T/T_0 and integrating, we obtain an expression for the limiting relationships of the Stanton numbers:

a) for $\delta < \delta_t (\varepsilon < 1)$

$$\Psi_* = \left\{\frac{\varepsilon}{\sqrt{\psi^* - 1}}\left[\arcsin \frac{2\frac{\psi^* - 1}{\varepsilon} + \Delta\psi}{\sqrt{4\frac{\psi^* - 1}{\varepsilon^2}(\psi^* + \Delta\psi) + (\Delta\psi)^2}} - \arcsin \frac{\Delta\psi}{\sqrt{4\frac{\psi^* - 1}{\varepsilon^2}(\psi^* + \Delta\psi) + (\Delta\psi)^2}}\right] + \frac{2}{\Delta\psi}\left[\sqrt{1 + \Delta\psi(1 - \varepsilon)} - \sqrt{1 + \Delta\psi}\right]\right\}^2; \tag{3.53}$$

b) for $\delta > \delta_t (\varepsilon > 1)$

$$\Psi_* = \frac{\varepsilon^2}{\psi^* - 1}\left[\arcsin \frac{2\frac{\psi^* - 1}{\varepsilon^2} + \Delta\psi}{\sqrt{4\frac{\psi^* - 1}{\varepsilon^2}(\psi^* + \Delta\psi) + (\Delta\psi)^2}} - \arcsin \frac{\Delta\psi}{\sqrt{4\frac{\psi^* - 1}{\varepsilon^2}(\psi^* + \Delta\psi) + (\Delta\psi)^2}}\right]^2. \tag{3.54}$$

During subsonic flows, when $\psi^* \simeq 1$, the limiting relationship of the Stanton numbers, in general, does not depend on

$$\psi_i = \left(\frac{2}{\sqrt{\psi}+1}\right)^2. \qquad (3.55)$$

The given analysis shows that during the majority of ratios of the thicknesses of the thermal and dynamic boundary layers met in practice in a region not close to the point of breakaway, it is possible to assume:

$$\psi_i = \psi. \qquad (3.56)$$

3.6. Comparison of the Limiting Law of Resistance with Experimental Data for Supersonic Flow

In Fig. 7 is given a comparison of calculations by the limiting formula (3.48) with experimental data, obtained at fairly large values of Re^{**}, and mainly at very high velocities and intense heat transfer.

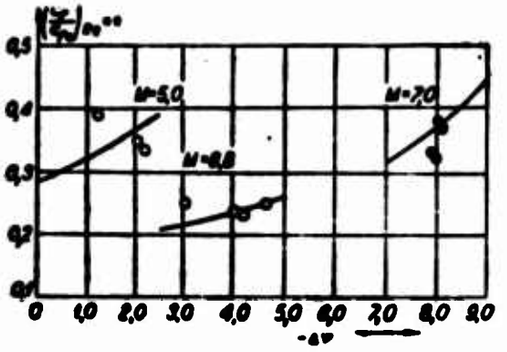

Fig. 7. Comparison of calculations by the limiting formula (3.48) with experimental data for large factors of heat transfer: O — experiments of Lobb, Winckler, and Persh; ⊕ — experiments of Sommer and Short.

Not only the qualitative, but also full satisfactory quantitative agreement of theory and experiment is clearly revealed.

Thus, the limiting law of the relative change of resistance of friction with nonisothermalness[1] sufficiently well describes real flows with finite Re numbers. This result agrees well with the known experimental fact of the weak influence of the Re number on the ratio c_f/c_{f0}. In this case it has an exclusively important value.[2]

Indeed, the limiting formula (3.48) is obtained from the general solution of (3.39) by means of conversion to limiting values of the

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[1]Subsequently, for brevity, the term "limiting law of resistance" will be used.

[2]Additional confirmation was obtained in the last work of Matting, Chapman, and Nyholm.

quantities Z and ω_1, which depend on the Reynolds number. But close agreement of the limiting ratios c_f/c_{f0} with the corresponding ratios for fully finite **Re** numbers means that the joint influence of the functions Z and ω_1 is small. Therefore, it is fully permissible, for calculation of the influence of the **Re** number on the relative change of the coefficient of friction due to the nonisothermalness of flow, to introduce in (3.39) the second known limiting value of these functions, corresponding to isothermal flow:

$$\left.\begin{aligned} Z_0 &= 1 - \varphi_{10}\sqrt{\frac{c_{f0}}{2}}; \\ \omega_{10} &= \varphi_{10}\sqrt{\frac{c_{f0}}{2}}. \end{aligned}\right| \qquad (3.57)$$

For a plate $\Phi_{10} = 11.6$.

With such a substitution we obtain the formula (at $\varepsilon < 1$):

$$\psi^* = \frac{1}{(\psi^* - 1)\left(1 - 8{,}2\sqrt{c_{f0}}\right)^2}\left[\arcsin\frac{2(\psi^* - 1) + \varepsilon\Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\varepsilon\Delta\psi)^2}} - \arcsin\frac{16{,}4(\psi^* - 1)\sqrt{c_{f0}} + \varepsilon\Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\varepsilon\Delta\psi)^2}}\right]^2. \qquad (3.58)$$

For adiabatic flow ($\Delta\psi = 0$) we have

$$\psi^* = \frac{1}{\psi^* - 1}\left(\frac{\arcsin\sqrt{\frac{\psi^* - 1}{\psi^*}} - \arcsin 8{,}2\sqrt{\frac{\psi^* - 1}{\psi^*}}\,c_{f0}}{1 - 8{,}2\sqrt{c_{f0}}}\right)^2. \qquad (3.59)$$

In Fig. 8 is given a comparison of a large number of experimental data with calculations by formula (3.58). Characteristics of the experiments are given in Table 3.3. Experimental data, obtained in the presence of heat transfer are reduced to adiabatic conditions by recalculation by formula (3.58). As can be seen, change of the $\mathbf{Re}^{**}$ number from its smallest value, at which a turbulent boundary layer can still exist to infinity leads to a change in the quantity ψ less than two times for **M** = 10. Essentially, the zone outlined by the

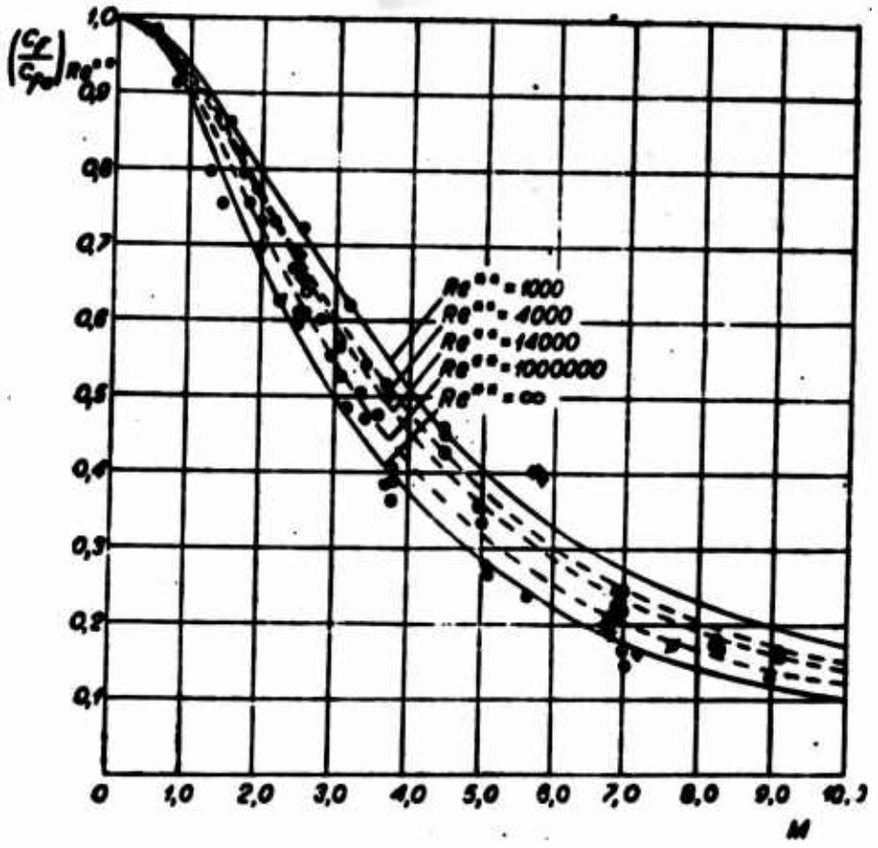

Fig. 8. Comparison of experimental data with formula (3.59) (designation see Table 3.3). Experimental data with heat transfer are reduced to heat-insulated conditions by equation (3.58).

theoretical dependences for $Re^{**} = 10^3$ and $Re^{**} = \infty$ embraces a zone of scattering of experimental points.

A fairly distinct tendency toward subdividing the experimental data according to Re^{**} numbers can be noted, which corresponds to theory.

Appearance of certain groups of points 10-20% from the theoretical value hardly can be considered significant, bearing in mind the whole complexity of carrying out the experiments in supersonic streams.

On the graph are placed also data on heat transfer. They are disposed together with data on aerodynamic drag. Thus, there is direct experimental confirmation of the validity of formula (3.56)

for a boundary layer of a gas in the range of values ε close to 1.

Table 3.3. Parameters of Experiments, from Which the Graph Fig. 8 is Constructed

a) Turbulent friction on a plate

Authors	Points (Fig. 8)	M	Re^{**}	$\frac{T_{wall}^*}{T_{wall}}$	Conditions of experiment	Method of determination of C_f
Coles	○	2,6	6600	1,0	Heat-insulated plate	Direct measurement with the aid of a float
	○	2,6	10200	1,0		
	●	3,7	4100	1,0		
	●	3,7	4560	1,0		
	◑	4,5	2900	1,0		
	◑	4,5	3470	1,0		
	◑	4,5	5240	1,0		
	●	4,5	6590	1,0		
Hill	◐	8,99	1245	0,448	Cooled wall of a conical nozzle	By the velocity gradients on the wall
	◐	9,04	1607	0,460		
	⊖	9,07	1908	0,471		
	⊘	9,10	2287	0,495		
	◑	8,22	2081	0,493		
	◑	8,35	2498	0,497		
	◑	8,27	2885	0,500		
	◑	8,29	3202	0,500		
	⊖	8,29	3451	0,502		
Korkegi	◇	5,747	2477	1,0	Heat-insulated plate	Direct measurement with the aid of a float
	◈	5,77	2780	1,0		
	◈	5,739	3629	1,0		
	◈	5,805	4040	1,0		
Lobb, Winckler, Persh	◔	4,93	5350	0,369	Cooled wall of a plane nozzle	By velocity gradients on the wall and via the Reynolds' analogy by measurements of heat fluxes in the investigated section
	◔	5,01	6480	0,718		
	◔	5,03	7950	0,575		
	◔	5,06	7370	0,535		
	◈	6,83	8550	0,613		
	◈	6,83	12540	0,508		
	◈	6,78	8400	0,513		
	◈	6,78	7980	0,457		
	◔	7,67	8410	0,465		

Table 3.3a Continued

Authors	Points (Fig. 8)	M	Re_{∞}	$\frac{T_{wall}^*}{T_{wall}}$	Conditions of experiment	Method of determination of C_f
Wilson		1,53		1,0	Heat-insulated plate	By changes of velocity profiles and momentum equation
		1,70		1,0		
		1,75		1,0		
		1,93		1,0		
		2,18		1,0		
Chapman, Kester		0,51		1,0	Streamlining of a heat-insulated cylinder in the longitudinal direction	Direct measurement of mean values
		0,81		1,0		
		1,99		1,0		
		2,49		1,0		
		2,95		1,0		
		3,36		1,0		
		3,60		1,0		
Lipman, Davan		0,42		1,0	Streamlining of a heat-insulated plate	Direct measurement of local values
		0,63		1,0		
		0,82		1,0		
		1,28		1,0		
		1,45		1,0		
Sommer, Short		2,81		0,400	Shot by a hollow cylinder in a wind tunnel toward the flow	By measurements of the velocity of flight of a model
		3,82		0,268		
		5,63		0,176		
		6,90		0,161		
		7,00		0,162		
		3,78		0,272		
		3,67		0,285		
Monahan		2,43		1,0		
Rubesin		2,55		1,0		By measurements of velocity profiles
Brinich		3,05		1,0	Cylindrical surface	By velocity profiles

b) Turbulent heat exchange on plate

Authors	Points (Fig. 8)	M	Conditions of experiment	Method of determination of St
Bradfield, Coursin		2,586	Streamlining of a cone	Measurement of heat transfer by a non-steady state method
		8,120		
		3,410		
Pappas		1,823	Plate with uniform pre-heating	By consumption of electric power and wall temperature
		2,290		

Table 3.3b Continued

Authors	Points (Fig. 8)	M	Conditions of experiment	Method of determination of St
Shulberg	●	2.00	Plate with uniform preheating	By consumption of heat transfer by a nonsteady state method
	●	2.50		
	●	3.09		
Slack, Fallis	✕	2.5	Data not cited	
	✕	2.5		

3.7. Plate with Initial Adiabatic Section

A diagram of the problem is shown in Fig. 9. A dynamic turbulent boundary layer is developed from the leading edge of plate. On the section with length x_0 heat transfer is absent. From section $x = x_0$ heat transfer begins between the plate and the gas. The forming thermal boundary layer is submerged in the dynamic layer, i.e., $\delta_t < \delta$.

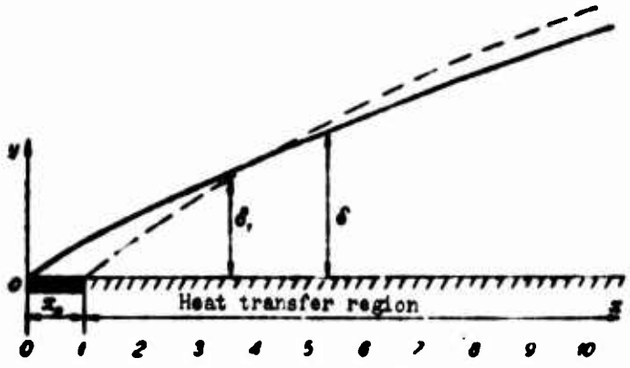

Fig. 9. Plate with initial heat-insulated section (Pr = 0.72).

Assuming Pr = 1, $\rho = \rho_0$, $\beta_t = 0$, $l_t = l$, $\tilde{q} = \tilde{q}_0$, and $\delta > \delta_t$, we bring equation (2.64) to the form:

$$\tilde{\tau}_w St_w = l^2 \frac{\partial u}{\partial y} \cdot \frac{\partial \theta}{\partial y}. \qquad (3.60)$$

On the other hand, for these conditions it follows from (2.6) that

$$l \frac{\partial u}{\partial y} = \sqrt{\frac{c_{p0}}{2} \tilde{\tau}_w}. \qquad (3.61)$$

Combining these equations and integrating, we obtain the dependence

$$St\int_{\frac{\delta_T}{\delta}}^{1}\sqrt{\frac{2\tau_0}{c_{f0}}}\,\frac{\delta}{l}\,d\xi = 1-\vartheta_{10}. \tag{3.62}$$

From (3.61) we have

$$\omega-\omega_{10}=\int_{\xi_0}^{\xi}\sqrt{\frac{c_{f0}}{2}\tilde{\tau}_0}\,\frac{\delta}{l}\,d\xi. \tag{3.63}$$

Consequently,

$$\frac{2St}{c_{f0}}(\omega_T-\omega_{10})=1-\vartheta_{10}, \tag{3.64}$$

where

$$\omega_T=\left(\frac{\delta_T}{\delta}\right)^n. \tag{3.65}$$

The dimensionless temperature difference in the viscous underlayer:

$$\vartheta_{10}=\frac{\alpha y_1}{\lambda}=St\,Pr\,\tilde{\tau}_{10}\sqrt{\frac{2}{c_{f0}}},$$

i.e.,

$$\vartheta_{10}=\frac{2St}{c_{f0}}Pr\,\omega_{10}. \tag{3.66}$$

Putting this value of ϑ_{10} in (3.64), we find that at **Pr** = 1:

$$\frac{2St}{c_{f0}}=\frac{1}{\omega_T}. \tag{3.67}$$

In a first approximation, for n = 1/7 and **Pr** = 1,

$$\left.\begin{aligned}\frac{dRe_T^{**}}{dRe_x}&=\frac{0{,}0129}{Re_T^{**0.25}};\\ \frac{dRe^{**}}{dRe_x}&=\frac{0{,}0129}{Re^{**0.25}}.\end{aligned}\right\} \tag{3.68}$$

Let us integrate the first of these equations, assuming that at $x = x_0$ $\delta_t = 0$. Let us integrate the second equation, assuming that at $x = 0$ $\delta = 0$. As a result we obtain relationship for the thicknesses of the boundary layers:

$$\frac{\delta_T}{\delta}=\left(\frac{x-x_0}{x}\right)^{0.8}. \tag{3.69}$$

Correspondingly,

$$\frac{2St}{c_{f0}} = \left(\frac{x}{x - x_0}\right)^{0.111}. \tag{3.70}$$

As can be seen from Fig. 10, this formula is well confirmed by experiment.

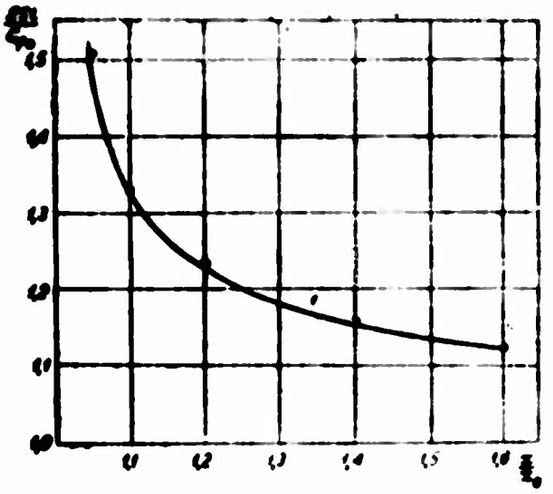

Fig. 10. Heat transfer to a plate with initial heat-insulated section: ——— calculation by equation (3.70); O — calculation by the equation, describing the experiments of W. Reynolds, W. Kays, and S. Kline.

Since in this case $Re_t^{**} = (\delta_t/\delta)Re^{**}$, then

$$St = \frac{0.0129}{Re_t^{**0.25}}\left(\frac{x - x_0}{x}\right)^{0.111} \tag{3.71}$$

or

$$\frac{St}{St_0} = \left(\frac{x - x_0}{x}\right)^{0.111}, \tag{3.72}$$

where

$$St_0 = 0{,}0129\, Re_t^{**\,0.25}.$$

A second approximation slightly refines this result.

The correction for isothermalness can be calculated by the formulas given earlier upon substitution in them of the value:

$$\varepsilon = \left[Pr^{0.6}\left(\frac{x}{x - x_0}\right)^{0.8}\right]^n. \tag{3.73}$$

Table 3.4. The Ratio St/St_0 by Formula (3.72)

$\frac{x}{x_0}$	1	1.00	1.25	1.50	1.75	2	3	∞
$\frac{St}{St_0}$	0	0,81	0,87	0,91	0,93	0,94	0,95	1

For quasiisothermal conditions it is possible to assume:

$$\frac{d\,Re_1^{0.8}}{d\,Re_x} = \frac{0.0129}{Pr^{0.75}\,Re_1^{0.25}}\left(\frac{x-x_0}{x}\right)^{0.8}. \qquad (3.74)$$

Hence, for a plate with a completely turbulent boundary layer:

$$Re_1^{0.8} = \frac{0.0155\,w_0}{Pr^{0.75}\,\nu}\int_{x_0}^{x}\left(\frac{x-x_0}{x}\right)^{0.8}\cdot dx. \qquad (3.75)$$

Correspondingly,

$$St = 0{,}0288\,Pr^{-0.6}\,Re_x^{-0.2}\,\varphi\left(\frac{x}{x_0}\right) \qquad (3.76)$$

or

$$Nu_x = 0.0288\,Pr^{0.4}\,Re_x^{0.8}\,\varphi\left(\frac{x}{x_0}\right). \qquad (3.77)$$

Here

$$\left.\begin{aligned}\varphi\left(\frac{x}{x_0}\right) &= \left[\frac{x_0}{x-x_0}\int_{x_0}^{x}\left(1-\frac{x_0}{x}\right)^{0.8}\cdot d\left(\frac{x}{x_0}\right)\right]^{-0.2};\\ Re_x &= \frac{w_0}{\nu}(x-x_0);\\ Nu_x &= \frac{\alpha}{\lambda}(x-x_0).\end{aligned}\right| \qquad (3.78)$$

The function $\varphi(x/x_0)$ is represented graphically in Fig. 11.

At $(x/x_0) > 1.50$ with an error not exceeding 8%, it is possible to assume:

$$Nu_x = 0{,}0288\,Pr^{0.4}\,Re_x^{0.8}, \qquad (3.79)$$

i.e., to calculate heat transfer by the usual formula with substitution in it of the true length of the heated section:

$$x_t = x - x_0. \qquad (3.80)$$

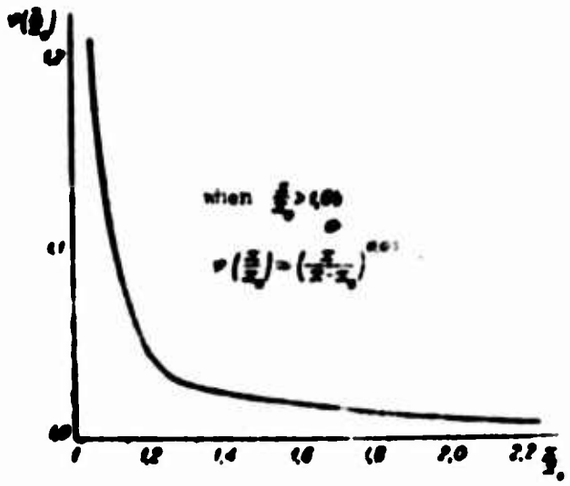

Fig. 11. The function $\varphi(x/x_0)$ in formula (3.78).

Formula (3.79) was offered by M. A. Mikheev on the basis of treatment of the experimental data (Fig. 12).

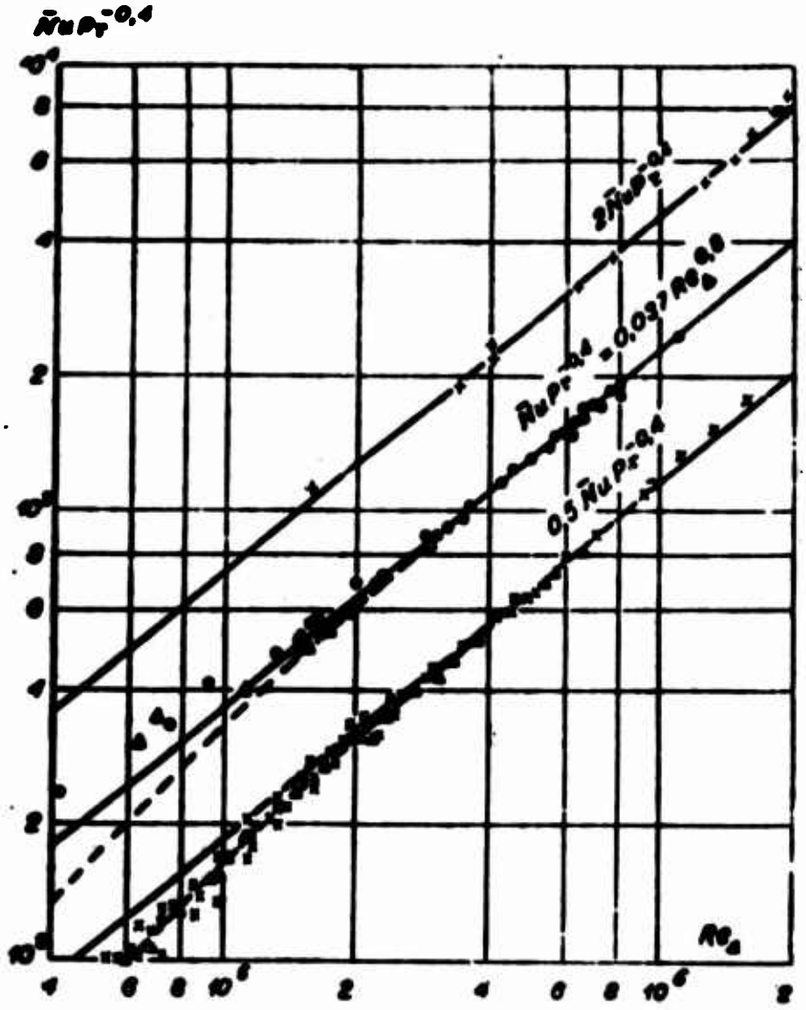

Fig. 12. Comparison of experimental data for heat transfer to a plate with formula (3.79). Dotted lines — calculation of the initial laminar section.

3.8. Solution of Equations of Momentum and Energy for a Nonisothermal Boundary Layer for $T_{wall} = const$

The analysis conducted showed that the difference in the thicknesses of the dynamic and thermal boundary layers slightly affects the relative change of the coefficient of friction in connection with the nonisothermalness of the flow. This important result allows in a majority of the practically interesting cases to assume in the expression for Ψ the quantity $\varepsilon = 1$.

Just as important is the fact of the weak influence on the Ψ Reynolds number. The latter allows introduction in the equations of momentum and energy the quantity Ψ, referred to the Re^{**} number, average in the considered section, or simply the limiting law, strictly self-simulating relative to the Re number.

Both these circumstances are the result of the small value of the exponent in formula (3.11), i.e., high population of the velocity profiles in turbulent flow.

Let us consider nonisothermal turbulent layers, developing from the leading edge of a plate at T = const. We have

$$\frac{dRe^{**}}{dRe_x} = \Psi \frac{c_{f0}}{2}; \tag{3.81}$$

$$\frac{dRe_t^{**}}{dRe_x} = \Psi_1 St_0. \tag{3.82}$$

Introducing in these expressions the values of c_{f0} and St_0 and carrying out integration for the conditions $\Psi = const$, $x = 0$, $Re^{**} = Re_t^{**} = 0$ we obtain

$$\left(\frac{c_f}{c_{f0}}\right)_{Re_x} = \left(\frac{St}{St_0}\right)_{Re_x} = \Psi^{\frac{1}{1+m}} \tag{3.83}$$

Thus, the ratio of the coefficients of friction and heat transfer for identical values of Re_x numbers more weakly depends on the nonisothermalness than for identical values of Re^{**} and Re_t^{**}.

In Fig. 13 is given a comparison of calculations by the theoretical formulas (3.58)-(3.83) with calculations according to a method, based on the semiempirical Prandtl-Karman theory. At the same place are placed experimental points. The known fact that, at large M numbers and intense heat transfer, the contemporary semiempirical theory does not give satisfactory results, is clearly confirmed.

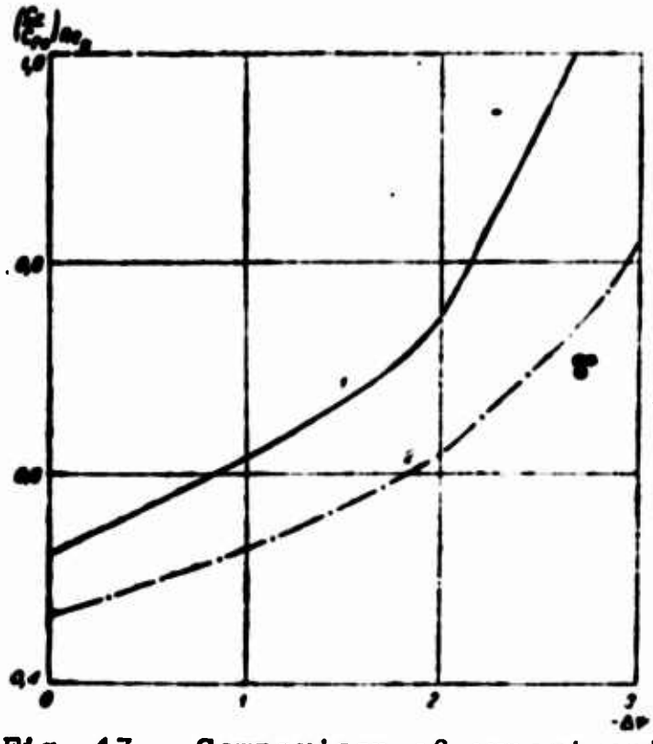

Fig. 13. Comparison of experimental data with calculations by various methods: 1) Van Driest's method; 2) by formula (3.83); ● — experiments of Sommer and Short.

C H A P T E R IV

LONGITUDINAL STREAMLINING OF A PERMEABLE PLATE

Definitions of Cyrillic Items in Order of Appearance

ст = wall = wall

кр = cr = critical

т = t = thermal, turbulent

изл = rad = radiated, radiation

отн = rel = ralation, relative

4.1. Formulation of the Problem

The problem of calculation of a boundary layer on a surface, penetrated by a flow of substance, has an extraordinarily important significance. Such processes arise during protection of parts of machines from the influence of high-temperature gas streams (so-called "porous cooling" of blades of gas turbines, the enclosing surfaces of combustion chambers, and so forth) during evaporation and condensation, the presence of chemical reactions in the flow and at the wall, hardening of a liquid, and fusing of solids.

Below is considered a turbulent boundary layer of a gas on a surface permeable at all points. If gas is injected into a boundary layer or is pulled from it, then the number of openings in the wall is

great, and their dimensions are small. Therefore, the temperature of the penetrating gas should be equal to the wall temperature. A diagram of the flow problem is shown in Fig. 14.

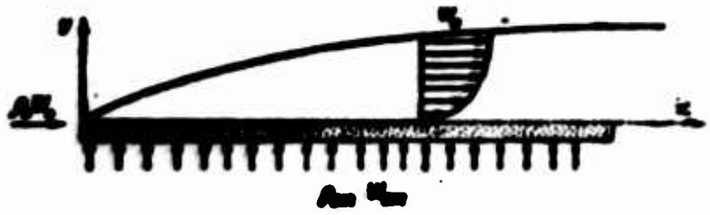

Fig. 14. Diagram of a Boundary Layer on a Permeable Plate.

Mass velocity of a stream of gas intersecting a wall,

$$j_1 = \rho_1 w_1. \qquad (4.1)$$

Inside the boundary layer this flow causes an additional component of the velocity vector w_y and gradually it is dispersed so that at the wall $j_1 = \rho_{wall} w_{wall}$, and in the region $y = \delta$ transverse flow of the injected substance is equal to zero.

Transverse flow of the substance causes the turbulent viscous underlayer. This circumstance favors application of the method of conversion to the limiting laws of friction and heat transfer, corresponding to $Re \to \infty$.

In connection with the above further research is built on the basis of equation (2.30).

Distribution of tangential stresses is determined by formula (2.40) at $\Lambda = 0$.

We have

$$\int_0^1 \frac{d\omega}{\sqrt{\left(\overline{\tau} + b \cdot f(\xi)\right) \frac{\rho_w}{\rho}}} = 1. \qquad (4.2)$$

This integral has a finite value, i.e., growth of the integrand should be limited. Since the quantities ω and $f(\xi)$ lie in the interval from 0 to 1, then changes of the integrand in this case are connected with the change of the permeability factor b. During injection of a gas through the wall into the boundary layer the quantity b is positive. Consequently, there should exist some limiting value of the permeability

factor $b = b_{cr}$, at which the value of Ψ turns out to be equal to zero. This phenomenon can be identified with separation of the boundary layer from the streamlined surface.[1]

The critical value of the permeability factor will be determined by equation

$$b_{cr} = \left(\int_0^1 \frac{d\eta}{\sqrt{\frac{h_0}{\rho} - f(\xi)}} \right)^2. \qquad (4.3)$$

which ensues from (4.2) at $\Psi = 0$.

Subsequently, we will be limited by solutions of equations (4.2) and (4.3) in an approximation, corresponding to the conditions $f(\xi) = 1$ and $\varepsilon = 1$.

Theoretical calculations, and the principal comparision with experimental data, show the full acceptability of these conditions for the majority of practically important cases.

Putting in (2.69) the value of $\Psi_1 \cdot \frac{\tilde{q}}{\tilde{q}_0}$ from formula (2.76), we obtain an expression for the limiting law of heat transfer on a permeable surface:

$$\int_0^1 \frac{d\eta}{\sqrt{|\Psi_1 + \varepsilon b_T \vartheta f(\xi)_T| \frac{h_0}{\rho}}} = 1. \qquad (4.4)$$

Since $f(\xi_t) = f(\xi)$, then solution of equations (4.2) and (4.4) at $\varepsilon = 1$ have the same form, i.e.,

$$\Psi_1(b_T) = \Psi(b). \qquad (4.5)$$

The gas introduced through the wall can differ from the substance of the main flow; therefore, we will distinguish uniform and nonuniform

---

[1]For greater detail see Chap. V.

boundary layers. A nonuniform boundary layer appears during injection of a gas, differing from the gas of the main flow. In this case the boundary layer consists of a mixture of gases, whereby at the wall the concentration of injected gas has the largest value, but at the external boundary of the layer is equal to zero.

4.2. Law of Resistance for a Uniform, Isothermal Boundary Layer

Assuming in equations (4.2) and (4.3) $\rho = \rho_0$ and $f(\xi) = 1$, we find that

$$\Psi = (1 - 0{,}25\, b)^2; \tag{4.6}$$

$$b_{sep} = 4. \tag{4.7}$$

Correspondingly, formula (4.6) can be written in the form:

$$\Psi = \left(1 - \frac{b}{b_{sep}}\right)^2. \tag{4.8}$$

As will be shown below, this simple formula possesses great universality.

The moment in equation for gradientless streamlining of a permeable plate has the form:

$$\frac{d\delta^{**}}{dx} - \frac{\rho_{cr} w_{cr}}{\rho_0 w_0} = \frac{c_f}{2} \tag{4.9}$$

or

$$\frac{d\,Re^{**}}{d\,Re_x} = (b + \Psi)\frac{c_{f_0}}{2}. \tag{4.10}$$

Let us integrate equation (4.10), assuming that the turbulent boundary layer is developed from the leading edge of plate.

For the case of b = const we obtain:

$$\left(\frac{c_f}{c_{f_0}}\right)_{Re_x} = \Psi (b + \Psi)^{\frac{m}{m+1}}. \tag{4.11}$$

When the quantity Ψ is determined by (4.6),

$$\left(\frac{c_f}{c_{f0}}\right)_{Re_x} = \left(1 - \frac{b}{b_{cr}}\right)^2 \left(1 + \frac{b}{b_{cr}}\right)^{-\frac{2m}{1+m}}. \qquad (4.12)$$

Here m is the exponent in formula (3.14).

At $m = 0.25$ and $b_{cr} = 4$, from (4.12) it follows that

$$\left(\frac{c_f}{c_{f0}}\right)_{Re_x} = (1 - 0{,}25b)^2 (1 + 0{,}25b)^{-0.4}. \qquad (4.13)$$

For the case $\frac{\rho_{wall} w_{wall}}{\rho_0 w_0}$ = const the solution obtained is very awkward, but at $m = 0.25$ is well approximated by formula

$$\left(\frac{c_f}{c_{f0}}\right)_{Re_x} = (1 - 0{,}25b)^2 (1 + 0{,}25b)^{-0.5}. \qquad (4.14)$$

Formulas (4.13) and (4.14) agree with an error, not exceeding 2%.

In Fig. 15 is given a comparison of the values of $\frac{c_f}{c_{f0}}$, compared for identical values of Re^{**} and Re_x. The corresponding curves are close to each other.

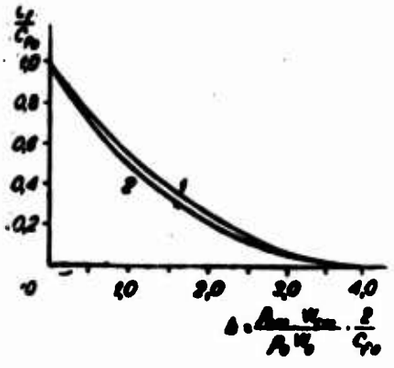

Fig. 15. Dependence of the coefficient of friction on the permeability factor:

1) $\frac{c_f}{c_{f0}}$ for Re^{**} = idem;

2) $\frac{c_f}{c_{f0}}$ for Re_x = idem.

In Fig. 16 is shown a comparison of the theoretical formula with experimental data. It is possible to ascertain full satisfaction of theory.

Due to the large slope of the relation $\Psi(b)$ in the pre-breakaway region, experimental determination of the point of breakaway is inaccurate.

In practice in experiments the condition of breakaway of the boundary layer for the case considered can be fixed already at values of b of the order of 3.

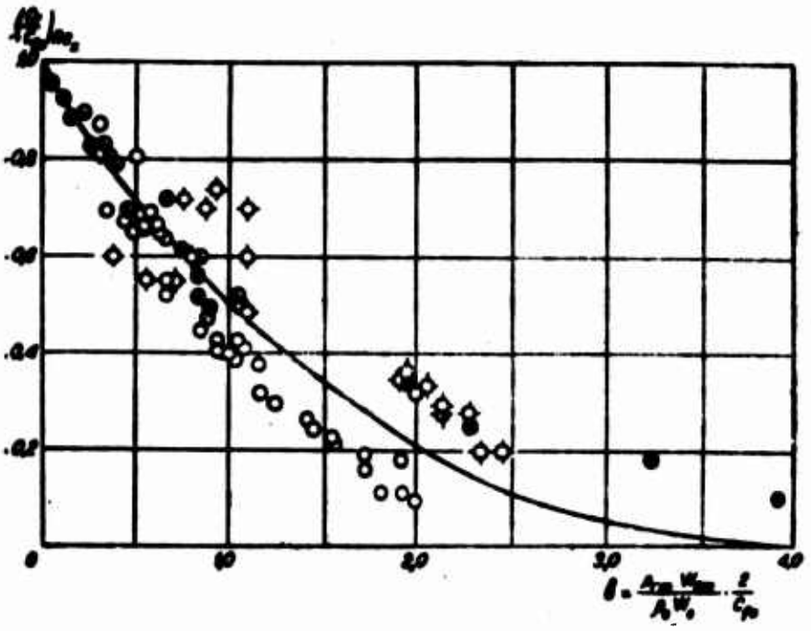

Fig. 16. Comparison of calculations by formula (4.14) with the experiments of Hacker, Mickley, Pappas, and Okuno.

O — experiments of Hacker $\left(\frac{c_f}{c_{f0}}\right)$;

⊕ — experiments of Mickley $\left(\frac{c_f}{c_{f0}}\right)$;

-⊙- — experiments of Mickley $\left(\frac{\alpha}{\alpha_0}\right)$;

● — experiments of Pappas and Okuno.

Since the quadratic term in formula (4.8) begins noticeably to have an effect only in the range of values of the permeability parameter, close to the critical, then for b, not very close to b_{cr},

$$\Psi \approx 1 - 2\frac{b}{b_{cr}}. \qquad (4.15)$$

The linear relation between the quantities Ψ and b was noted by several experimenters and is well explained by the theory expounded. At $b_{cr} = 4$ and $b \ll b_{cr}$

$$\Psi \approx 1 - 0{,}5\,b. \qquad (4.16)$$

4.3. Law of Heat Transfer for a Uniform, Quasiisothermal Boundary Layer

For small temperature differences, when $\psi \approx 1$, the physical properties of a gas in a boundary layer can be considered constant.

By formulas (4.5) and (4.8) the limiting law of heat transfer for a quasiisothermal boundary layer at a permeable wall has the form:

$$\varphi_1 = \left(1 - \frac{b_T}{b_{T.\,sp}}\right)^2. \tag{4.17}$$

In addition:

$$b_{T.\,sp} = b_{sp}. \tag{4.18}$$

In the presence of relative similarity of the velocity and temperature fields it is possible to apply formulas (3.36) and (3.37). Then

$$b_T \approx b\,Pr^{0.5}\,z^{-\frac{m}{n}}. \tag{4.19}$$

During development of both boundary layers from the leading edge of the plate condition (3.38) is fulfilled, and

$$b_T \approx b\,Pr^{0.5}. \tag{4.20}$$

4.4. Calculation of Cooling

Usually in the calculation of a boundary layer on a cooled wall distribution of the temperature $T_{wall}(x)$ and the initial temperature of the liquid coolant T_1 are assigned. It is necessary to determine the flow rate of this liquid through each cross section of the protected surface.

The quantity of heat, transmitted by the main flow to the wall, in a given section is

$$q_{cT} = \alpha\,(T_0 - T_{cT}) + q_{изл}. \tag{4.21}$$

where q_{rad} is the heat flow due to radiation.

Let us assume that part of this heat q_1 is transmitted through the wall, not in connection with introduction through it, of the cooling medium (for instance, by means of thermal conduction through structural metal). Then the quantity of heat, which the cooling medium should receive is:

$$q_2 = q_{ст} - q_1 = g c_{p1} j_1 (T_{ст} - T_1), \tag{4.22}$$

where $j_1 = \rho_1 w_1$ is the flow of the mass of cooling medium through the wall. Combining these equations, we find that

$$\Psi_s = \frac{b_т}{s} \cdot \frac{T_{ст} - T_1}{T_0 - T_{ст}}, \tag{4.23}$$

where

$$s = 1 \quad \frac{q_{нач.т} - q_1}{q_{ст}}.$$

On the other hand, for Ψ_s we have the dependence (4.17). Combining equations (4.17) and (4.23) and solving them relative to the value of the thermal permeability factor of the wall, we obtain the value

$$\frac{b_т}{b_{кр}} = 1 - 2\left[\frac{f_т}{s} - \sqrt{\frac{f_т}{s} + \left(\frac{f_т}{s}\right)^2}\right], \tag{4.24}$$

where

$$f_т = \frac{T_{ст} - T_1}{T_0 - T_{ст}}.$$

At

$$f_т = 0 \quad b_т = b_{кр}, \text{ when } f_т > 0 \quad b_т < b_{кр}.$$

4.5. Laws of Resistance and Heat Transfer for a Uniform, Nonisothermal, Subsonic Boundary Layer

In a uniform subsonic boundary layer of a gas $\varepsilon = 1$ according to formula (2.52)

$$\frac{p_0}{\rho} = \frac{T}{T_0} \approx \psi - (\gamma - 1)w. \tag{4.25}$$

Putting this value of $\frac{\rho_0}{\rho}$ in equation (4.2) and solving it at $f(\xi) = 1$, we find that:

a) at $\psi < 1$

$$\Psi = \frac{4}{(1-\psi)b_1}\left[\ln \frac{\sqrt{(1-\psi)(1+b_1)} + \sqrt{b_1}}{\sqrt{1-\psi} + \sqrt{b_1\psi}}\right]^2, \tag{4.26}$$

b) at $\psi > 1$

$$\Psi = \frac{4}{(\psi-1)b_1}\left[\operatorname{arc\,tg}\sqrt{\frac{b_1}{(\psi-1)(b_1+1)}} - \operatorname{arc\,tg}\sqrt{\frac{b_1\psi}{\psi-1}}\right]^2. \tag{4.27}$$

At $\psi = 1$ we have formula (4.6).

Correspondingly, the critical value of the parameter of permeability of the walls, calculated by equation (4.3) upon substitution in it of the value of $\frac{\rho_0}{\rho}$ from (4.25) and $f(\xi) = 1$, are determined by the formulas:

a) at $\psi < 1$

$$b_{cr} = \frac{1}{1-\psi}\left(\ln \frac{1+\sqrt{1-\psi}}{1-\sqrt{1-\psi}}\right)^2; \tag{4.28}$$

b) at $\psi > 1$

$$b_{cr} = \frac{1}{\psi-1}\left(\arccos\frac{2-\psi}{\psi}\right)^2. \tag{4.29}$$

Table 4.1. Values of b_{cr} by formulas (4.28) and (4.29).

ψ	0.25	0.50	0.75	1	2.00	4.00
b_{cr}	9,4	6,2	4,8	4	2,5	1,4
$\left(\frac{2w_{wall}}{c_{f_0}w_\infty}\right)_{cr}$	2,3	3,1	3,6	4	5,0	5,7

The relation $b_{cr}(\psi)$ is shown graphically in Fig. 17.

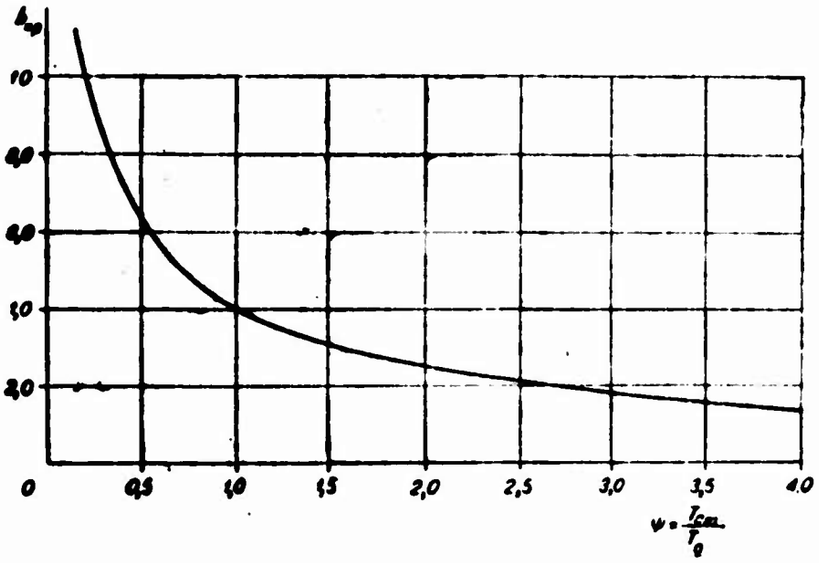

Fig. 17. Influence of heat transfer on the parameter of breakaway b_{cr} for a uniform boundary layer. ——— calculation by equation (4.28) for the case $\psi < 1$ and by equation (4.29) for the case $\psi > 1$.

It is interesting to note that in the region $\psi > 1$ the critical relationship of velocities changes significantly weaker than the critical relationship of mass flow rates. In the region $\psi < 1$ both parameters change almost equally, although there is observed a somewhat greater conservation of the ratio $\left(\frac{w_{wall}}{w_0}\right)_{cr}$.

For engineering practice the case $\psi < 1$ has the greatest significance since in this case introduction of a substance through the wall into the boundary layer protects the streamlined body from the thermal influence of the main flow of gas.

In Fig. 18 is given a comparison of calculations by formulas (4.26) and (4.27) with calculations by formula

$$\Psi = 4\left(\frac{1-\frac{b}{b_{w0}}}{\sqrt{\psi}+1}\right)^2. \qquad (4.30)$$

As can be seen, this simple combination of formulas (3.47) and (4.8) well approximates the exact solution.

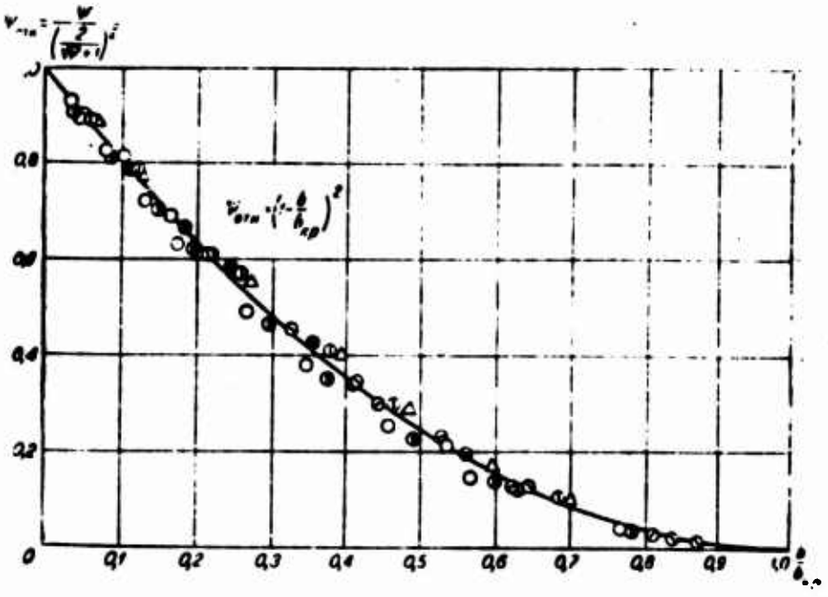

Fig. 18. Comparison of calculations by formulas (4.26) and (4.27) with formula (4.30) during determination of b_{cr} by formulas (4.28) and (4.29).

	Calculation by equation						
	(4.26)				(4.27)		
ψ	0,2	0,4	0,6	0,8	2	4	8
Designation	○	◑	◔	⊘	⊗	⊘	◿

In Fig. 19 is given a comparison of calculations by formula (4.30) for heat transfer on a plate and in the inlet section of a pipe with experimental data. In spite of a significant scattering of the experimental points, they are grouped in a mass around a theoretical line.

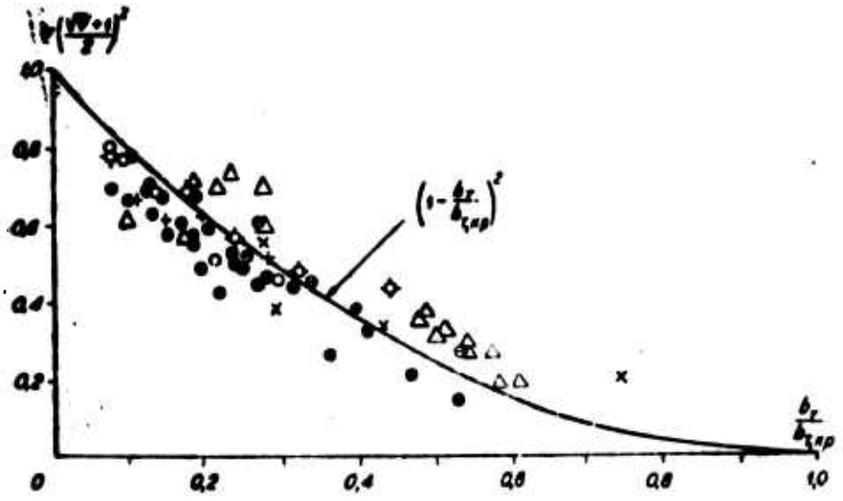

Fig. 19. Influence of injection of a gas on convective heat transfer. ———— calculation by equation (4.30), Δ – experiments of Mickley (plate); experiments of Friedman (pipe).

$Re_D \cdot 10^{-3}$	1.2	0.5	0.25	0.6	2.15	1.55	0.7	1.95	0.35	0.65
Designation	○	◌	×	⊣○⊢	⊙	+	●	◑	◌	◓

4.6. Law of Resistance for a Nonuniform, Isothermal Boundary Layer

During supply through a wall into a boundary layer of a foreign gas the process of diffusion arises. The partial density of the injected gas is changed from the value ρ_{wall} on a streamlined surface to zero in the region $y = \delta_D$, where δ_D is the thickness of the diffusion boundary layer. The field of concentrations can be connected with the velocity field by a formula of the type (2.44). For the flow, not close to the point of breakaway, it is possible to assume the relation[1]

$$\frac{\rho'_{cr} - \rho'}{\rho'_{cr}} = \varepsilon_D \dot{\omega}, \qquad (4.31)$$

[1]In the general case, for instance, during evaporation of water in humid air, $\rho_0 \neq 0$ and in the denominator of (4.31) one should write the difference $\rho_{wall} - \rho_0$.

where

$$\varepsilon_D = \left(\frac{\delta}{\delta_D}\right)^n. \tag{4.32}$$

Since in gases the diffusion Prandtl number is close to unity, then all the conclusions, made in chapters II and III about the weak influence of the degree of nonsimilarity of the temperature and velocity fields on the relative changes of the coefficients of friction and heat transfer, are also valid for the process of diffusion. Therefore, for a plate it is possible to consider the limiting law of friction, assuming $\varepsilon_D \approx \varepsilon \approx 1$.

The gas constant of the mixture is connected with the density of the mixture and partial density of the injected gas ρ' by a known relationship:

$$\frac{R}{R_0} = \frac{\rho'}{\rho}\left(\frac{R'}{R_0} - 1\right) + 1. \tag{4.33}$$

Here R, R', R_0 are, respectively, the gas constants of the mixture, injected gas, and basic gas.

On the other hand, the density of the mixture ρ is connected with the density of basic gas outside the boundary layer ρ_0 by the equation:[1]

$$\frac{\rho}{\rho_0} = \frac{R_0 T_0}{R T_0}, \tag{4.34}$$

i.e., under isothermal conditions

$$\frac{\rho}{\rho_0} = \frac{P_0}{R}. \tag{4.35}$$

Combining formulas (4.31), (4.33), and (4.35), we find that at $\varepsilon = 1$ and T = const

$$\frac{\rho}{\rho_0} \approx \frac{\rho_{CT}}{\rho_0} + \left(1 - \frac{\rho_{CT}}{\rho_0}\right)\omega. \tag{4.36}$$

---

[1] Since in the boundary layer $\frac{\partial p}{\partial y} = 0$.

Introducing this value of $\frac{\rho}{\rho_0}$ in equation (4.2), we find that:

a) at $R' > R_0$ (is injected with a small molecular weight $\mu' < \mu_0$)

$$\Psi = \frac{1}{\psi_1 \delta_1}\left[\sqrt{\psi_1(1+\delta_1)} - \frac{\psi_1 - 1 - \delta_1}{\sqrt{\delta_1(\psi_1 - 1)}} \ln \frac{\sqrt{\delta_1 \psi_1} + \sqrt{(\psi_1 - 1)(1+\delta_1)}}{\sqrt{\delta_1} + \sqrt{\psi_1 - 1}}\right]^2; \quad (4.37)$$

b) at $R' < R_0$ (gas is injected with a large molecular weight $\mu' > \mu_0$)

$$\Psi = \frac{1}{\psi_1 \delta_1^2}\left\{\sqrt{\psi_1(1+\delta_1)} - 1 - \frac{\psi_1 - 1 - \delta_1}{\sqrt{(1-\psi_1)\delta_1}} \times \right.$$
$$\left. \times \left[\operatorname{arc\,tg} \sqrt{\frac{(1-\psi_1)(1+\delta_1)}{\delta_1 \psi_1}} - \operatorname{arc\,tg} \sqrt{\frac{1-\psi}{\delta_1}}\right]\right\}^2, \quad (4.38)$$

where

$$\psi_1 = \frac{\rho_0}{\rho_{cr}}. \quad (4.39)$$

is a quantity to a certain degree analogous to the temperature factor ψ.

In a uniform flow of gas, obeying the Clapeyron-Mendeleev equation $\psi_1 = \psi$.

Definition of the parameter ψ_1 depends on the method of supply of the foreign gas to the boundary layer.

If a gas is introduced through a porous wall, then the flow of substance is

$$j_1 = \dot{\rho}_{cr} w_y - D_{cr}\left(\frac{\partial \rho'}{\partial y}\right)_{cr}. \quad (4.40)$$

where $D\left[\frac{m^2}{sec}\right]$ is the diffusion coefficient.

From (4.31) it follows that:

$$\left(\frac{\partial \rho'}{\partial y}\right)_{cr} = \varepsilon_D \frac{\dot{\rho}_{cr}}{w_0}\left(\frac{\partial w_x}{\partial y}\right)_{cr}. \quad (4.41)$$

On the other hand,

$$\left(\frac{\partial w_x}{\partial y}\right)_{cr} = \frac{c_f \rho_0 w_0^2}{2\mu_{cr}}. \tag{4.42}$$

Hence,

$$-D_{cr}\left(\frac{\partial \rho'}{\partial y}\right)_{cr} = \varepsilon_D Pr_D \frac{\rho'_{cr}}{\rho_{cr}} \rho_0 w_0 \frac{c_f}{2}, \tag{4.43}$$

where $Pr_D = \frac{\nu}{D}$ is the diffusion Prandtl number.

From these equations it follows that at $Pr_D \approx \varepsilon_D \approx 1$

$$\frac{\rho'_{cr}}{\rho_{cr}} = \frac{b_1}{1 + b_1}. \tag{4.44}$$

Taking into account (4.33), we determine the quantity

$$\psi_1 = \left[1 + \left(\frac{R'}{R_0} - 1\right)\frac{b_1}{1 + b_1}\right]. \tag{4.45}$$

In Fig. 20 is given a comparison of calculations by formulas (4.37), (4.38), (4.45) with experimental data. The theory well describes the results of the experiments with such heterogeneous pairs of media, as helium-air and freon-12-air.

At $b = b_{cr}$ separation of the boundary layer occurs and the surface of the wall is covered by a film of injected gas. Thus, in this case $\rho_{wall} = \rho'_{wall}$ and for the condition T = const

$$\frac{\rho}{\rho_0} \approx \frac{R_0}{R'} + \left(1 - \frac{R_0}{R'}\right)\omega. \tag{4.46}$$

Putting this expression in equation (4.3) and carrying out integration at $f(\xi) = 1$, we find that,

a) at $R' > R_0$

$$b_{sp} = \left(\frac{\frac{R_0}{R'}}{2\sqrt{1 - \frac{R_0}{R'}}} \ln \frac{1 + \sqrt{1 - \frac{R_0}{R'}}}{1 - \sqrt{1 - \frac{R_0}{R'}}}\right)^2; \tag{4.47}$$

b) at $R' < R_0$

$$b_{cr} = \left[1 - \frac{\frac{R_0}{R'}}{2\sqrt{\frac{R_0}{R'} - 1}} \arccos\left(2\frac{R'}{R_0} - 1\right)\right]^2. \qquad (4.48)$$

From formula (4.47) it follows that at $\frac{R_0}{R'} \to 0$ $b_{cr} \to 1$.

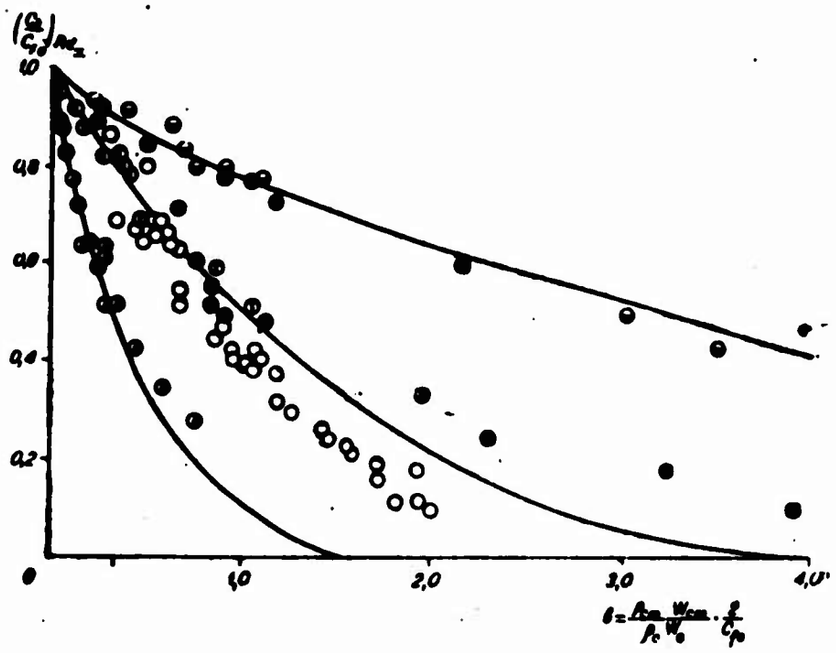

Fig. 20. Influence of injection of a foreign gas on the coefficient of friction on a plate. — — calculations by equations (4.6) — (air-air), (4.37) — (helium-air), and (4.38) — (Freon-12-air), O — experiments of Hacker (air-air), ⊗ — experiments of Mickley (air-air). Experiments of Pappas and Okuno: O — air-air, ◐ — helium-air, ⊖ — Freon-12-air.

In Fig. 21 is shown the dependence of the critical value of the parameter of wall permeability on the relation of the molecular weights of the injected gas and basic flow.

In Fig. 22 is given a comparison of the experimental data with calculation by formula (4.12) during determination of the quantity b_{cr} by formulas (4.47) and (4.48).

In that case, when the streamlined surface is covered by a film of

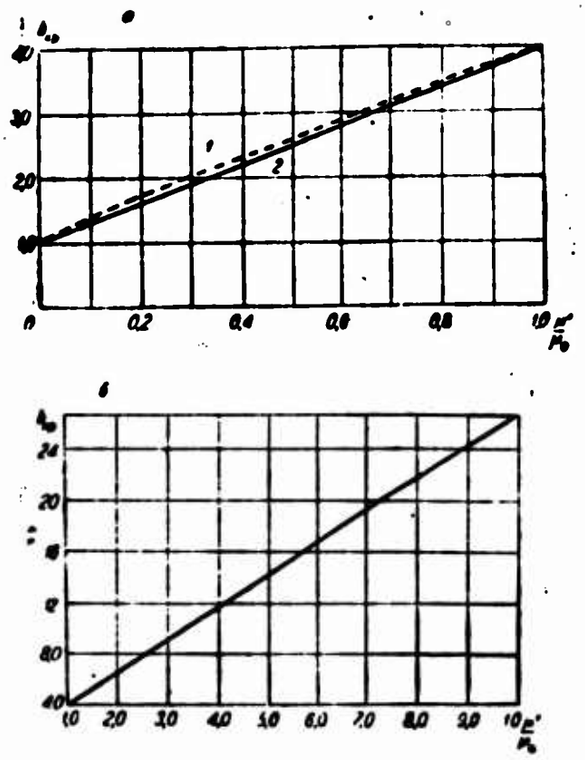

Fig. 21. Dependence of the critical parameter of injection b_{cr} on the ratio of the molecular weights ($\Psi = 1$): a) 1)--- by equation (4.47), 2)— by the equation

$$b_{кр} = 1 + 3\frac{\mu'}{\mu_0};$$

b)— by equation (4.48). In the range of $\frac{\mu'}{\mu_0}$ from 1 to 10 is the linear approximation is valid:

$$b_{кр} = 1{,}47 + 2{,}53\frac{\mu'}{\mu_0}.$$

liquid and evaporation occurs, the value of the density of the mixture at the wall is determined by the total pressure and partial pressure of the volatile liquid. The latter with a great degree of accuracy is equal to the saturated pressure at the evaporation temperature. In addition:

$$\psi_1 = \frac{\psi}{1 - \left(1 - \frac{R_0}{R_1}\right)\frac{p'_{ст}}{p_0}}. \quad (4.49)$$

where ρ_0 is the pressure outside the boundary layer.

For quasiisothermal conditions, when $\psi \approx 1$,

$$\psi_1 = \frac{1}{1 - \left(1 - \frac{R_0}{R'}\right)\frac{p'_{ст}}{p_0}}. \quad (4.50)$$

It is also possible to use these formulas for calculation of heat transfer during quasiisothermal conditions with replacement of the factor b by the factor

$$b_т = \frac{c_{p1} j_1}{c_{p0} \rho_0 w_0 St_0}. \quad (4.51)$$

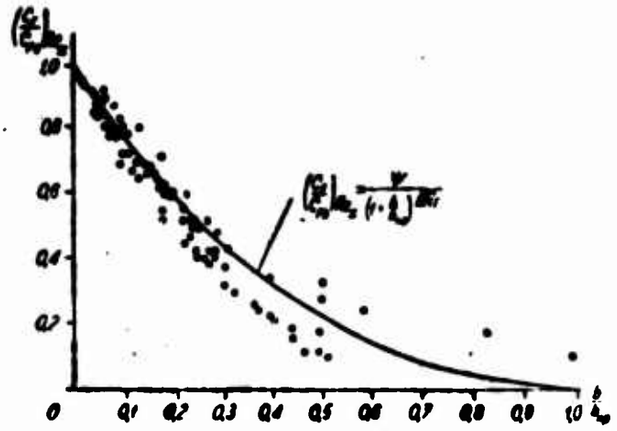

Fig. 22. Comparison of experimental data with formula (4.12), ○ — experiments of Hacker, ⊕ — experiments of Mickley. Experiments of Pappas and Okuno; ◑ — air-air, ◑ — helium-air, ◒ — Freon-12-air.

4.7. Law of Resistance for a Nonuniform, Nonisothermal, Subsonic Boundary Layer

During nonisothermal flow the intensity of the process of diffusion is more correctly determined by the difference of partial pressures. Let us assume that the following condition is fulfilled[1]

$$\frac{p'_{ст} - p'}{p'_{ст}} \approx \omega, \qquad (4.52)$$

which at T = const coincides with (4.31).

Taking into account (4.33), we obtain:

$$\left.\begin{aligned} \frac{R_w}{R} &= 1 - \frac{p'_{ст}}{p}(1-\omega)\left(1 - \frac{R'_w}{R'}\right); \\ \frac{\rho}{\rho_0} &= \frac{T_0}{T}\left[1 - \frac{p'_{ст}}{p}\left(1 - \frac{R'_w}{R'}\right)(1-\omega)\right]. \end{aligned}\right\} \qquad (4.53)$$

At the wall

$$\frac{\rho_{ст}}{\rho_0} = \frac{1}{\psi}\left[1 - \frac{p'_{ст}}{p}\left(1 - \frac{R_w}{R'}\right)\right]. \qquad (4.54)$$

Hence,

$$\frac{\rho}{\rho_0} = \frac{T_0}{T}\left[1 - \left(1 - \frac{\psi}{\psi_1}\right)(1-\omega)\right]. \qquad (4.55)$$

Determining the temperature by formula (2.52) at $\psi^* = 1$ and

---

[1] In the more general case in the denominator one should write the difference $p'_{wall} - p'_0$.

$\varepsilon \approx 1$, we have

$$\frac{\rho}{\rho_0} \approx \frac{\frac{\psi}{\psi_1} + \left(1 - \frac{\psi}{\psi_1}\right)\omega}{\psi + (1-\psi)\omega}. \tag{4.56}$$

In these formulas the factor ψ_1 is defined by (4.39). Introducing this value of $\frac{\rho}{\rho_0}$ in (4.2) and (4.3) and assuming $f(\xi) = 1$, we obtain the equation:

$$\psi = \left\{\int_0^1 \sqrt{\frac{\frac{\psi}{\psi_1} + \left(1 - \frac{\psi}{\psi_1}\right)\omega}{[\psi + (1-\psi)\omega](1 + b_1\omega)}}\, d\omega\right\}^2; \tag{4.57}$$

$$b_{cr} = \left\{\int_0^1 \sqrt{\frac{\frac{R_\omega}{R'} + \left(1 - \frac{R_0}{R'}\right)\omega}{[\psi + (1-\psi)\omega]\omega}}\, d\omega\right\}^2. \tag{4.58}$$

During derivation of equation (4.58) it is taken into account that at the point of breakaway $\rho_{wall} = \rho'_{wall}$ and $\psi_1 = \psi\frac{R'}{R_0}$.

In Fig. 23 are given the results of the numerical solution of equation (4.58) in the range of values of $\frac{R'}{R_0}$ from 0.1 to 1.0 and values of ψ from 0.2 to 2.0. Along the axis of ordinates on the graph is placed the ratio of the quantity b_{cr} at given values of ψ and $\frac{R'}{R_0}$ to the quantity b_{cr} at the same value of ψ and $\frac{R'}{R_0} = 1$ (i.e. b_{cr1} is calculated by formula (4.28) at $\psi < 1$ and by formula (4.29) at $\psi > 1$). These results are shown the graph by the points. The line given by these points is described by the formulas:

at $\mu' > \mu_0$

$$b_{cr} \approx b_{cr1}\left(0{,}37 + 0{,}67\frac{\mu'}{\mu_0}\right); \tag{4.59}$$

at $\mu' < \mu_0$

$$b_{cr} \approx b_{cr1}\left(0{,}25 + 0{,}75\frac{\mu'}{\mu_0}\right). \tag{4.60}$$

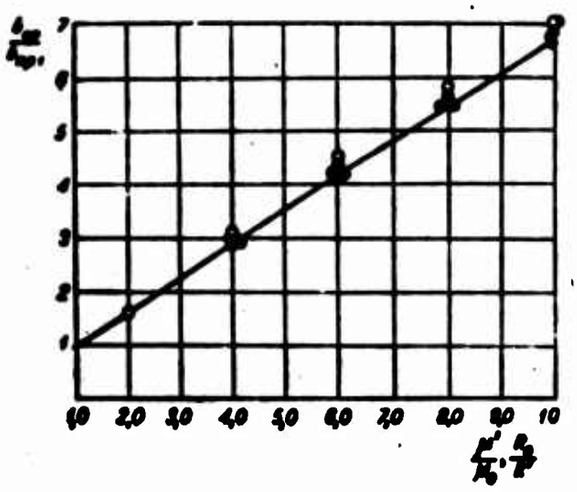

Fig. 4.23. Dependence of the critical parameter of injection the ratio of the molecular weights and the temperature factor.

In these and the other formulas μ' is the molecular weight of injected gas and μ_0 the molecular weight of the gas of the basic flow. For the case $\mu' \ll \mu_0$ the integral (4.58) is substantially simplified and for b_{cr} we have the formula:

$$b_{cr}\Big|_{\frac{\mu'}{\mu_0}\to 0} = \left(\frac{2}{\sqrt{\psi}+1}\right)^2. \quad (4.61)$$

Thus, the temperature factor affects b_{cr} in approximately the same manner as it does the coefficient of friction during streamlining of an impenetrable plate.

In Fig. 24 are given the results of the numerical solution of equation (4.57). Along the axis of ordinates is placed the ratio of Ψ (for given values of ψ and $\frac{\mu'}{\mu_0}$) to Ψ, calculated by formula (3.55). Along the abscissa is placed the ratio of b to b_{cr}, calculated by equation (4.58). Curve 1 corresponds to formula (4.8).

As can be seen, the exact calculation for the case of injection of freon-12 in a flow of air is near the approximate formula (4.30). In any case the divergences are of the same order as with the experimental data in Fig. 22. Exact calculation for the case of helium injection in a flow of air confirms satisfactoriness of the approximation of the influence of the temperature factor ψ, but deviates more significantly from approximation of the influence of the factor $\frac{b}{b_{cr}}$.

From the data of Fig. 22 on the injection helium into air, it

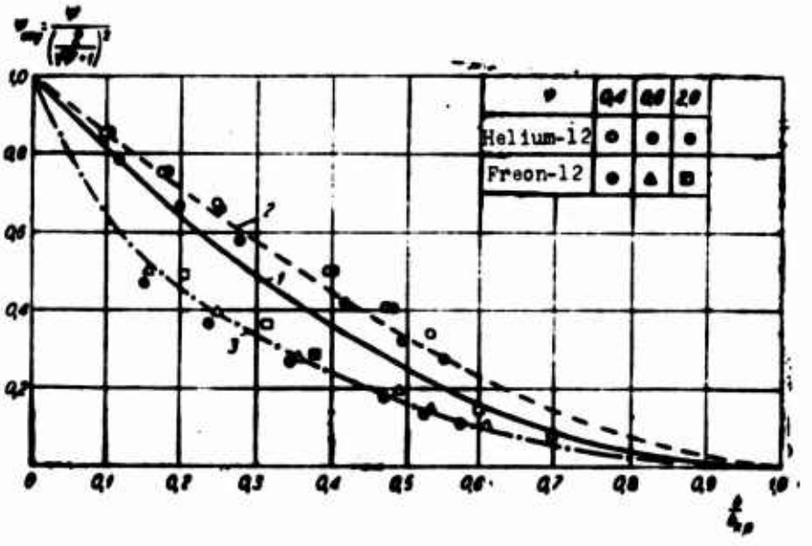

Fig. 24. Solution of equation (4.57). 1) calculation by equation (4.8); 2, 3) by experimental points.

is clear that this divergence decreases upon transition to a calculation by Re_x.

Considering this circumstance, and also the absence of experimental data for a nonisothermal nonuniform boundary layer, in a first approximation it is possible to assume the approximate relation (4.30).

C H A P T E R V

STREAMLINING OF A CURVED SURFACE

Definitions of Cyrillic Items in Order of Appearance

ст = wall = wall

кр = cr = critical

т = t = thermal, turbulent

5.1 Limiting Parameters of Breakaway of an Isothermal Boundary Layer on an Impenetrable Surface

During streamlining of a curved surface the flow velocity on the external boundary of a dynamic boundary layer changes along the streamlined contour and, consequently, $\frac{dp}{dx} \neq 0$.

Nozzle flow is distinguished, when $\frac{dp}{dx} < 0$, and divergent, when $\frac{dp}{dx} > 0$.

During nozzle flow, the stream is accelerated, the direction of motion of the liquid coincides with the direction of the pressure force, and the boundary layer is always stable in the sense that it is not detached from the streamlined surface.

During diffusion flow the stream is slowed down, pressure increases, and its action is directed toward the motion of the liquid.

The pressure gradient in the boundary layer is determined by formula (1.10), i.e., a change of pressure occurs in strict conformity with a change of veolcity w_0.

Inside the dynamic boundary layer flow is inhibited by friction and $w_x < w_0$. Therefore, the reserve of the kinetic energy of flow inside the boundary layer is insufficient for full surmounting of the action of the field of pressures directed toward it. As a result a positive pressure gradient evokes inside the boundary layer inhibition, and then stopping and reverse of the current of liquid near a streamlined body. This phenomenon is called separation of the boundary layer.

Beyond the point of breakaway a vortex motion of the liquid appears, accompanied by sharp growth of the resistance of pressure and still not yielding to theoretical calculation.

Fig. 25. Diagram of deformations of the velocity profile during flow of a boundary layer with a pressure gradient.

Schematically the sequence of deformations of the velocity profile in the region of divergent flow is shown in Fig. 25.

Inasmuch as separation of the boundary layer is characterized by conversion of the flow in direct proximity to the wall, i.e., there, where the flow is the most inhibited, then the point of breakaway is determined by the condition

$$\left(\frac{\partial w_x}{\partial y}\right)_{cr} = 0. \qquad (5.1)$$

Correspondingly, at the point of breakaway $c_f = 0$.

Actually, separation of the boundary layer does not occur at any exactly fixed point, but embraces a certain finite domain. In this region the velocity profiles are deformed from the mimimum stable form to a form, corresponding to condition (5.1). In addition, pulsations are possible, accompanied by oscillations of the point of breakaway within the limits of a certain region Δx.

Measurements, conducted by many researchers, show that during

nozzle flow the velocity profiles are more populated, and during diffusion flow are less populated than the velocity profile for the case $\frac{dp}{dx} = 0$.

For the point of breakaway of an isothermal boundary layer at an impenetrable wall we have the condition:

$$\{c_f = 0;\ \rho = \rho_w,\ \beta = 0;\ j_1 = 0\}. \tag{5.2}$$

Putting in equation (2.5) the value of the considered quantities, we obtain the equation, determining the velocity profiles at the point of breakaway of an isothermal turbulent boundary layer at an impenetrable wall:

$$\omega = \omega_{1.\,sp} + \int_{\xi_{1\,sp}}^{\xi} \left(\frac{\delta}{l}\right)_{sp} \sqrt{\frac{\tau_{sp}(\xi)}{\rho w_0^2}}\, d\xi. \tag{5.3}$$

The index "cr" indicates that the corresponding quantities are referred to a cross section, in which separation of the boundary layer appears.

Distribution of the tangential stresses across the boundary layer at an impenetrable wall in a sufficiently general form it is possible to write:

$$\tilde{\tau} = \frac{\tau}{\tau_{cr}} = \varphi_0(\xi) + \Lambda \xi \varphi_1(\xi). \tag{5.4}$$

During approximation by a cubic parabola in accoradance with (2.35)

$$\left.\begin{aligned} \varphi_0(\xi) &= 1 - 3\xi^2 + 2\xi^3; \\ \varphi_1(\xi) &= (1 - \xi)^2. \end{aligned}\right\} \tag{5.5}$$

and, taking into account that the form parameter Λ is determined by formula (2.36) and at the point of breakaway $\tau_{wall} = 0$, we have

$$\frac{\tau_{sp}}{\rho w_0^2} = \frac{\delta}{\rho w_0^2} \cdot \frac{dp}{dx} \xi \varphi_1(\xi) = \left(-f \frac{\delta}{\delta^{**}}\right)_{sp} \xi \varphi_1(\xi), \tag{5.6}$$

where $f = \frac{\delta}{w_0} \cdot \frac{dw_0}{dx}$ is the form parameter, not related to the quantity c_f.

Putting the value of τ_{cr} in (5.3), we find that

$$\omega = \omega_{1\,кр} + \left(-f\frac{\delta}{\delta^{**}}\right)^{1/2}_{кр}\int_{\xi_{1кр}}^{\xi}\left(\frac{\delta}{l}\right)_{кр}\sqrt{\xi\varphi_1(\xi)}\,d\xi. \qquad (5.7)$$

Assuming in (5.7) $\omega = 1$ and, correspondingly, the upper limit in the right integral to equal 1, we find that

$$\left(-f\frac{\delta}{\delta^{**}}\right)_{кр} = \left[\frac{1-\omega_{1\,кр}}{\int_{\xi_{1кр}}^{1}\left(\frac{\delta}{l}\right)_{кр}\sqrt{\xi\varphi_1(\xi)}\,d\xi}\right]^2. \qquad (5.8)$$

Thus, if the integrand and lower limits of integration in equations (5.7) and (5.8) are known, then the breakaway velocity profile in the turbulent nucleus of a boundary layer and the critical values of the parameters f, $\frac{\delta^{**}}{\delta}$, and H, are calculated, i.e., all the quantities, characterizing the phenomenon of breakaway.

Equations (5.7) and (5.8) can be solved, if we accept the assumption of the conservation of the dependence of the length of the path of mixing l on the form parameter f and we approximate tangential stresses profile by a cubic parabola.

Both of these assumptions always can be considered as a first approximation of the real distributive laws of the quantities l and τ in the critical cro s section, and in any case they do not emerge within the framework of assumptions, accepted in the contemporary semiempirical theories of a turbulent boundary layer.

Any further more precise definitions of these dependencies can be easily considered by the corresponding change of integrands in equations (5.7) and (5.8).

Regarding the lower limits of integration, i.e., $\omega_{1\,cr}$ and $\xi_{1\,cr}$, then for their determination in the general case knowledge of the law of stability of the viscous underlayer is necessary.

At $Re \to \infty$ $\xi_1 \to 0$, $\omega_1 \to 0$, and the critical parameters of the breakaway of the isothermal turbulent boundary layer at an impenetrable wall are determined by equations:

$$\left(-f\frac{\delta}{\delta^{**}}\right)_{sp} = \left[\int_0^1 \left(\frac{\delta}{l}\right)_{sp} \sqrt{\xi\varphi_1(\xi)}\, d\xi\right]^{-2},$$

$$\omega = \frac{\int_0^{\omega} \left(\frac{\delta}{l}\right)_{sp} \sqrt{\xi\varphi_1(\xi)}\, d\xi}{\int_0^1 \left(\frac{\delta}{l}\right)_{sp} \sqrt{\xi\varphi_1(\xi)}\, d\xi}. \qquad (5.9)$$

Taking as primary experimental fact logarithmic velocity profile at $\frac{dp}{dx} = 0$, we find that

$$\frac{l}{\delta} = \varkappa\xi\sqrt{\varphi_0}. \qquad (5.10)$$

According to formula (3.1) $\varkappa = 0.4$.

According to the other experimental data the value of this constant lies in the range 0.38 to 0.41, i.e., very close to the quantity, defined by Nikuradze.

Putting relations (5.5) and (5.10) in equation (5.9), we obtain the limiting velocity profile in the region of the breakaway of the turbulent layer:

$$\omega = \frac{\ln\left(2\sqrt{4\xi^2 + 2\xi} + 4\xi + 1\right)}{\ln\left(2\sqrt{6} + 5\right)}. \qquad (5.11)$$

Accordingly,

$$\left(-f\frac{\delta}{\delta^{**}}\right)_{sp} = 2\left[\frac{1}{\varkappa}\ln\left(2\sqrt{6} + 5\right)\right]^{-2}. \qquad (5.12)$$

From formula (5.11) it follows that for the accepted assumptions the limiting velocity profile in the cross section of the breakaway of the boundary layer does not depend on the empirical constants of turbulence. The limiting critical value of the form parameter f depends on the constant $\varkappa$.

At $\varkappa = 0.4$, for the considered consitions (T = const, $j_1 = 0$, $Re \to \infty$), we have[1]:

$$\left\{\begin{array}{l}\left(-f\frac{\delta}{\delta^{**}}\right)_{sp} = 0{,}062;\ \left(\frac{\delta^{**}}{\delta}\right)_{sp} = 0{,}30;\\ \left(\frac{\delta^{**}}{\delta}\right)_{sp} = 0{,}16;\ H_{sp} = 1{,}87;\ f_{sp} = 0{,}010.\end{array}\right\} \tag{5.13}$$

The velocity profile (5.11) is approximated by the exponential relation

$$\omega = \xi^{0.43}. \tag{5.14}$$

This value of the exponent for the cross section, in which breakaway of the turbulent layer from the impenetrable wall occurs, is very close to the value $n_{cr} = 0.5$, found earlier from other considerations in the works of Moss, Stratford, and Townsend. However, it seems to us that in these investigations less general assumptions are accepted and not all of the complex of parameters, characterizing the region of breakaway are obtained. In particular, the critical values of the form parameter f were not calculated.

From the given calculations it is clear that just the form parameter f preserves a finite critical value, even at $Re \to \infty$, whereas the critical value of the Buri-Loytsyanskiy form parameter

$$\Gamma_{sp} = \frac{2f}{c_{fo}} \tag{5.15}$$

approaches infinity (since at $Re \to \infty$, $c_{fo} \to 0$). In the region of not very large values of Re number the quantity Γ_{cr} has fully finite values.

Qualitatively, this conclusion was earlier substantiated in the works of L. E. Kalikhman and G. M. Bam-Zelikovich.

In Fig. 26 is given a comparison of the velocity profile, calculated by formula (5.14), with results of the measurements of

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[1]Values of δ^* and δ^{**} are found by formulas (1.24) and (1.25) upon substitution in them of ω from (5.11) and $\rho = \rho_0$.

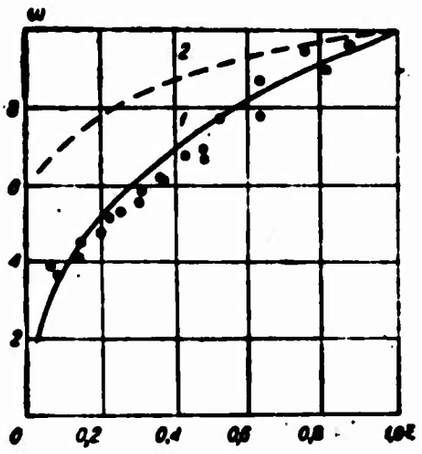


Fig. 26. Comparison of calculated velocity profile at the point of breakaway of the turbulent boundary layer with experimental data: 1) calculation by formula (5.11); 2) calculation according to the 1/7 power law; O — experiments of I. Nikuradze; ● — experiments of A. I. Leont'ev, A. N. Oblivin, P. N. Romanenko.

I. Nikuradze and A. I. Leont'ev, A. N. Oblivin, and P. N. Romanenko.

It is clear that the theoretical limiting velocity distributions in the prebreakaway region of flow is satisfactorily confirmed by the experimental data for finite **Re** numbers.

The critical value of the form parameter H from the experiments of I. Nikuradze are equal to 1.8, from the experiments of E. Grushwitz — 1.9, i.e., very close to the found theoretical value.

According to some other experiments the quantity H_{cr} attains 2-2.5. It is possible that this increase of the value H_{cr} is connected with the roughness of the streamlined surface or some other sort of factors, not considered by the accepted approximations of the functions $l(y)$ and $\tau(y)$.

5.2. Condition of Stability of the Viscous Underlayer

During streamlining of an impenetrable plate by an isothermal flow with $\frac{dp}{dx} = 0$, the condition of stability of the viscous underlayer can be written in the form

$$Re_1 = \frac{w_1 y_1}{\nu} = \eta_{10}^2 = \text{const}. \qquad (5.16)$$

In the two-layered diagram of a turbulent boundary layer $\eta_{10} = 1.6$ and $Re_1 = 134$ [- ? - original illegible]. During gradient flow the condition of stability is changed due to a change of the configuration of the velocity profile in the viscous underlayer. The character of

this change can be clarified by proceeding from the following considerations.

Let us assume that formula (5.14) approximately describes the velocity profile in the cross section of breakaway and at finite **Re** numbers. Such an assumption is more valid the larger the **Re** number. Furthermore, it is also confirmed by the experimental data (see Fig. 26).

In the viscous underlayer the velocity distribution is determined by formula (2.21), which at $c_f = 0$ takes the form

$$\omega = -\frac{1}{2} f_{кр} \left(\frac{\delta}{\delta^{**}}\right)^2_{кр} \cdot Re^{**} \xi^2. \qquad (5.17)$$

Intersection of profiles (5.14) and (5.17) at the values of the critical parameters from (5.13) will give the values of $\xi_{1\ cr}$ and $\omega_{1\ cr}$ to a first approximation.

We have

$$\xi_{1\ кр} \approx \frac{2.84}{Re^{**0.04}};$$

$$\omega_{1\ кр} \approx \frac{1.57}{Re^{**0.27}}. \qquad (5.18)$$

Hence,

$$Re_{кр} = \omega_{1\ кр} \xi_{1\ кр} \left(\frac{\delta}{\delta^{**}}\right)_{кр} Re^{**} \approx \frac{1.57 \cdot 2.84}{0.16} \approx 28.$$

In addition the sum of the exponents for **Re**** in formulas (5.18) is rounded off to 1 with accuracy up to 0.1, which is fully permissible, taking into account the approximate character of these dependencies.

Thus, the Reynolds number for the viscous underlayer essentially decreases with an increase of divergence.

Obviously, such a decrease of stability of the viscous underlayer is connected with a corresponding distortion of the velocity profile with an increase of the pressure gradient. In some measure this circumstance is taken into account in the Reynolds number, constructed from the velocity derivative on the external boundary of the viscous underlayer:

$$\dot{Re}_1 = \left(\frac{\partial w_x}{\partial y} \cdot \frac{y^2}{\nu}\right)_{y=y_1}. \qquad (5.19)$$

At (dp/dx) = 0 $\dot{\mathbf{Re}}_1$ is identically equal to $\mathbf{Re}_1$ or η_{10}^2. Assuming, as in all the preceding calculations, the quantity $\eta_{10} = 11.6$, we find that at f = 0 $\mathbf{Re}_1 = 134$.

At the point of breakaway

$$\dot{Re}_1 = -f_{sp}\left(\frac{\delta}{\delta^{**}}\right)^3_{sp} Re^{**3}_{1\,sp}. \qquad (5.20)$$

Introducing here the value of the critical parameters from (5.13) and (5.18) and rounding off to two the exponent for $\mathbf{Re}^{**}$ number in the cube of the quantity $\xi_{1\,cr}$, we find that

$$\dot{Re}_1 \approx \frac{0{,}01 \cdot 2{,}34}{0{,}16^3} = 57.$$

Thus, $\dot{\mathbf{Re}}_1$ actually is stabler than $\mathbf{Re}_1$.

The condition

$$\dot{Re}_1 = \text{const} \qquad (5.21)$$

is identical to the Stseblevskiy condition introduced earlier

$$\left(\frac{y}{\nu}\sqrt{\frac{\tau}{\rho}}\right)_{y=y_1} = \text{const} = \eta_{10}. \qquad (5.22)$$

5.3. Approximation of the Relation Between the Form Parameters H and f

Above was shown the method of calculation of the values of the form parameters at the point of breakaway. The value of the form parameters at (dp/dx) = 0 is calculated from the experimental universal velocity profile $\varphi(\eta)$.

According to the degree of the population of the velocity profile that occurs with an increase of the form parameter f, the magnitude of the exponent n decreases. Assuming that at $f \to \infty$ $n \to \infty$ from (3.13), we find that in this case $H \to 1$.

Thus, we have the conditions:

$$\left.\begin{array}{ll} f = f_{sp}; & H = H_{sp}; \\ f = 0; & H = H_0; \\ f = \infty: & H = 1. \end{array}\right\} \qquad (5.23)$$

In addition $2 > H > 1$, i.e., changes minutely in the segment from f_{cr} to $f = \infty$.

All these conditions are well satisfied by the simple interpolation formula:

$$\frac{H-1}{H_{\infty}-1} = \left(\frac{1_{exp}-1}{H_0-1}\right)\tilde{f}. \qquad (5.24)$$

where $\tilde{f} = (f/f_{cr})$.

To the logarithmic velocity profile corresponds the value

$$H_{\infty} = 1 + \frac{\sqrt{2c_{f_0}}}{\varkappa - \sqrt{2c_{f_0}}}. \qquad (5.25)$$

Consequently,

$$H = H(f; Re^{**}). \qquad (5.26)$$

In Fig. 27 is given a comparison of calculations by formula (5.24) with the experimental data of I. Nikuradze, plotted in the coordinates $(\tilde{H}; f)$, where $\tilde{H} = (H/H_{cr})$.

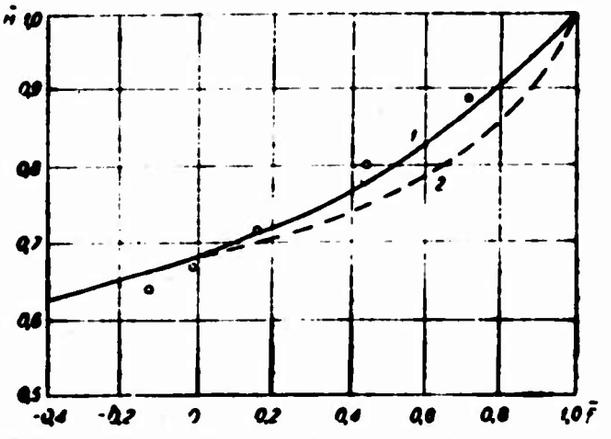


Fig. 27. Dependence of the parameter H on the parameter f: 1) by formula (5.24); 2) calculation for a laminar boundary layer; points from the experiments of Nikuradze.

We can see the satisfactory agreement of calculation with experiment.

5.4. Law of Resistance for an Isothermal Boundary Layer on an Impenetrable Surface at $(dp/dx) \neq 0$

From equations (2.6) and (5.4) it follows that the velocity profile in the turbulent nucleus of an isothermal boundary layer on an impenetrable curved surface is determined by the equation:

$$\omega = \omega_1 + \int_{\xi_1}^{\xi} \frac{\delta}{l} \sqrt{\Psi \frac{c_{f0}}{2} \varphi_0(\xi) - \frac{\delta}{\delta^{**}} \Gamma \xi \varphi_1(\xi)}\, d\xi. \qquad (5.27)$$

where, as before, $\Psi = (c_f/c_{f0})_{Re^{**}}$.

Distribution of velocities in the viscous underlayer is described by equation (2.21).

The condition of stability we will write by means of the quantity Re_1.

Taking into account equation (2.19) for the case T = const and $j_1 = 0$, we have

$$\Psi \frac{c_{f0}}{2} \xi_1^2 - \frac{\delta}{\delta^{**}} f \xi_1^3 = \frac{\dot{Re}_1}{\left(Re^{**} \frac{\delta}{\delta^{**}}\right)^2} \qquad (5.28)$$

At the point of breakaway $\Psi = 0$ and from (5.28), (5.19) follows.

The system of equations (5.27), (2.21), and (5.28) determines the quantity Ψ, if the function $\frac{l}{\delta}(\xi)$; $\varphi_0(\xi)$; $\varphi_1(\xi)$; $\dot{Re}_1(\tilde{f})$ and $c_{f0}(Re^{**})$ are known.

Actually, by assuming in (5.27) $\omega = 1$ and correspondingly the upper limit of the right integral is also equal to 1, we have the system of equations:

$$1 - \omega_1 = \int_{\xi_1}^{1} \frac{\delta}{l} \sqrt{\Psi \frac{c_{f0}}{2} \varphi_0(\xi) - \frac{\delta}{\delta^{**}} \Gamma \xi \varphi_1(\xi)}\, d\xi;$$

$$\omega_1 = \frac{\delta}{\delta^{**}} Re^{**} \left(\Psi \frac{c_{f0}}{2} \xi_1 - \frac{1}{2} \frac{\delta}{\delta^{**}} f \xi_1^2\right); \qquad (5.29)$$

$$\left(\Psi \frac{c_{f0}}{2} \xi_1^2 - \frac{\delta}{\delta^{**}} f \xi_1^3\right)\left(\frac{\delta}{\delta^{**}} Re^{**}\right)^2 = \dot{Re}_1.$$

When the indicated functions are known, then the system of equations (5.29) is solved in a first approximation, if we introduce in

value δ^{**}/δ, defined via the quantity H from the formulas (3.13), and the quantity H, in turn, is determined by approximation (5.24).

Further, by introducing in equation (5.27) the thus found values of Ψ; ξ_1; and ω_1, we will calculate the quantity δ^{**}/δ in a second approximation and again solve the system of equations (5.29). As a result in the second approximation are calculated not only Ψ; ξ_1; and ω_1, but also δ^{**}/δ and H.

The authors together with N. N. Kirillova and G. P. Zykin solved the system of equations (5.28) in a first approximation. In addition the quantity l/δ was determined by formula (5.10) at $\varkappa = 0.4$; the functions $\varphi_0(\xi)$ and $\varphi_1(\xi)$ – by formulas (5.5); the quantity c_{f0} – by formula (3.7); the quantity H – by formula (5.24); the quantity H_{cr} was taken equal to 1.87.

Thus, in the semiempirical theroy of a turbulent boundary layer developed here, as a primary experimental fact is taken a definite universal velocity profile $\varphi(\eta)$, and not some special hypothesis on the length of the path of mixing.[1]

The full system of equations of the examined first approximation takes the form:

$$\omega_1 = 1 - 2{,}5\int_{\xi_1}^{\xi}\frac{1}{\xi}\sqrt{\Psi\frac{c_{f0}}{2}+\frac{\delta}{\delta^{**}}|f_{xp}|f\frac{\xi}{2\xi+1}}\,d\xi;$$
$$\omega_1 = \frac{\delta}{\delta^{**}}Re^{**}\left(\Psi\frac{c_{f0}}{2}\xi_1+\frac{1}{2}\frac{\delta}{\delta^{**}}|f_{xp}|f\,\xi_1^2\right); \qquad (5.30)$$

[1]In the majority of works $l = \varkappa y$ (Prandtl formula) or $l = \varkappa[(dw_x/dy)/(d^2w_x/dy^2)]$ (Karman formula) is assumed. "Constants of turbulence" $\varkappa$ and η_{10} are determined from the same experimental velocity profile. Experiment shows that these dependences are sufficiently accurate only in a narrow prewall region. Usually $\tau = \tau_{wall} = \text{const}$ is assumed, i.e., a condition valid also in direct proximity to the wall.

The hypothesis on the conservation of the function $l(y)$, naturally, is also necessary in these theories.

$$\Psi \frac{c_{f0}}{2}\xi_1^3 + \frac{\delta}{\delta^{**}}|f_{sp}|\tilde{f}\xi_1^3 = \frac{Re_1}{\left(\frac{\delta}{\delta^{**}}Re^{**}\right)^3};$$

$$\sqrt{\frac{2}{c_{f0}}} = 5{,}5 + 2{,}5\ln\left(\frac{\delta}{\delta^{**}}Re^{**}\sqrt{\frac{c_{f0}}{2}}\right);$$

$$\frac{\delta}{\delta^{**}} = \frac{H(H+1)}{H-1};$$

$$H - 1 = (H_0 - 1)\left(\frac{0{,}87}{H_0 - 1}\right)^{\tilde{f}}; \qquad (5.30)$$

$$H_0 - 1 = \frac{\sqrt{\frac{c_{f0}}{2}}}{0{,}2 - \sqrt{\frac{c_{f0}}{2}}}.$$

The critical parameters are determined preliminarily from this system of equations at $\Psi = 0$ and $\tilde{f} = 1$.

Calculations were carried out on the electronic computer of the Calculating Center of the Siberian Section of the Academy of Sciences of the USSR. The results are given in Table 5.1 and in Figs. 28, 29, and 30.

Table 5.1. Values of the Parameters of an Isothermal Boundary Layer at the Point of Breakaway from a First Approximation

Re^{**}	$2\cdot10^3$	$1\cdot10^4$	$5\cdot10^4$	$1\cdot10^5$	$5\cdot10^5$	$1\cdot10^6$
ω_{1or}	0,133	0,077	0,0416	0,0354	0,0294	0,0183
$\bar{v}_{1or}$	0,0183	0,00625	0,00215	0,00135	0,00046	0,00029
$\left(\frac{\delta^{**}}{\delta}\right)_{or}$	0,16	0,16	0,16	0,16	0,16	0,16
$-f_{or}$	0,010	0,010	0,010	0,0100	0,0010	0,0010
$-\Gamma_{or}$	4,7	6,7	9,0	10,0	12,8	14,1
H_{or}	1,87	1,87	1,87	1,87	1,87	1,87

As can be seen, the thickness of the viscous underlayer increases in the region of diffusion flow, and the velocity on the external boundary of the viscous underlayer decreases. Besides the larger the **Re** number, then the relatively stronger are these changes.

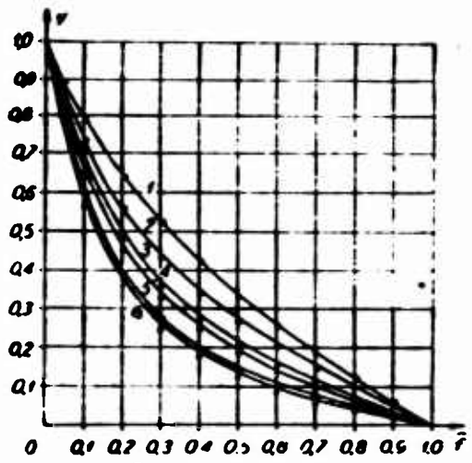


Fig. 28. Dependence of the relative value of the coefficient of friction Ψ on the form parameter $\tilde{f}$ and $\mathbf{Re}^{**}$ number. Solid line — calculation by the system of equations (5.30) at $\dot{\mathbf{R}}\mathbf{e}_1^{1/2} = 11.6-3.9\,\tilde{f}$; points — calculation by the system of equations (5.30) at $\mathbf{Re}_1^{1/2} = 11.6$. 1) $\mathbf{Re}^{**} = 2\cdot10^3$; 2) $\mathbf{Re}^{**} = 1\cdot10^4$; 3) $\mathbf{Re}^{**} = 5\cdot10^4$; 4) $\mathbf{Re}^{**} = 1\cdot10^5$; 5) $\mathbf{Re}^{**} = 5\cdot10^5$; 6) $\mathbf{Re}^{**} = 1\cdot10^6$.

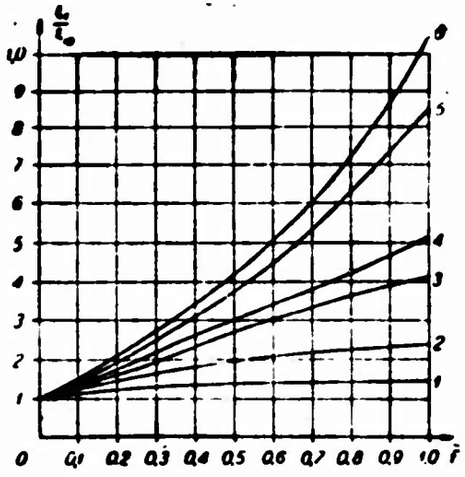


Fig. 29. Change of relative thickness of the viscous underlayer with the value of the form parameter $\tilde{f}$. $\mathbf{Re}^{**}$ number the same, as in Fig. 28.

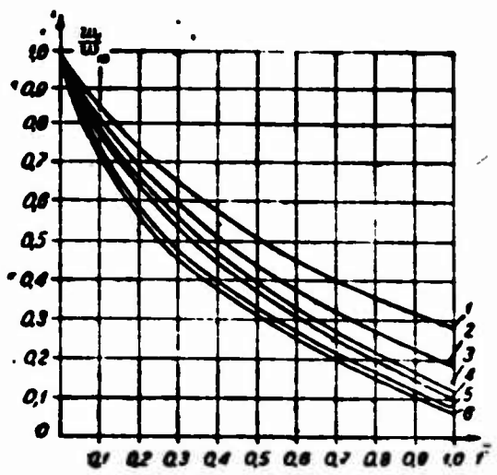


Fig. 30. Changes of relative velocity on the boundary of the viscous underlayer with the value of the form parameter $\tilde{f}$. $\mathbf{Re}^{**}$ number the same, as in Fig. 28.

The relative change of the coefficient of friction in the diffusion region is very significant and depends both on the form parameter and Reynolds number, i.e.,

$$\Psi = \Psi(\tilde{f};\ Re^{**}). \tag{5.3}$$

In addition, the greater the value of Re^{**}, the more rapidly the coefficient of friction decreases with the growth of the parameter $\tilde{f}$.

Calculations were performed for two conditions of stability: $\sqrt{Re}_1 = \eta_{10} = 11.6$ and $\sqrt{Re}_1 = 11.6 - 4.0f$. The latter is a linear interpolation between the value of the quantity $\left(\frac{y}{\nu}\sqrt{\frac{\tau}{\rho}}\right)_{y=y_1}$, determined experimentally for the case $\tilde{f} = 0$, and the value of this quantity at $\tilde{f} = 1$, calculated from the limiting velocity profile.

Calculations by both conditions of stability practically coincide, i.e., possible inaccuracy in approximation of the function $Re_1(\tilde{f})$ is immaterial for calculation of the parameters of a turbulent boundary layer.

5.5. Solution of Equation of Momentum for an Isothermal Boundary Layer on an Impenetrable Surface

It is convenient to write the momentum equation in the form:

$$\frac{dRe^{**}}{dX} + f_{кр}\,Re_L\,(1 + H_{кр}\,\tilde{H})\,\tilde{f} = Re_L\,\Psi_1 - \frac{f_{cr}}{2}. \tag{5.32}$$

Here $Re^{**} = (w_0\delta^{**}/\nu)$ is the flow value of the Reynolds number, constructed according to the thickness of the momentum loss;

$Re_L = (w_0 L/\nu)$ is the Reynolds number, constructed according to the local value of the velocity w_0 and the characteristic linear dimension of a streamlined surface L;

$X = (x/L)$ is the relative distance from the inlet edge;

f_{cr}; H_{cr} are the values of the form parameters f and H at the point of breakaway of the boundary layer;

$\tilde{f} = (f/f_{cr})$; $\tilde{H} = (H/H_{cr})$ are relative values of the form parameters;

Ψ_f is relative change of coefficient of friction under the influence of a pressure gradient.

$\tilde{H}$ and Ψ_f are complex functions of $\tilde{f}$ and Re^{**}. Therefore, an exact solution of the momentum equation can be obtained only by numerical methods for a given velocity distribution outside the boundary layer.

The solution is radically simplified by linearization of equation (5.31).

We will represent this equation in the form:

$$\frac{2}{c_{f0} Re_L} \cdot \frac{dRe^{**}}{dX} = F(\tilde{f}). \tag{5.33}$$

where

$$F(\tilde{f}) = \Psi_f - (1 + H_{sp}\tilde{H})\tilde{f}\Gamma_{sp}. \tag{5.34}$$

At $\tilde{f} = 0$ we have:

$$\Psi_f = 1;\ F(\tilde{f}) = 1;$$

at $\tilde{f} = 1$ we have:

$$\Psi_f = 0;\ \tilde{H} = 1;\ F(\tilde{f}) = 1 - (1 + H_{sp})\Gamma_{sp}.$$

Linear interpolation gives the formula:

$$F(\tilde{f}) \approx 1 - (1 + H_{sp})\tilde{f}\Gamma_{sp}. \tag{5.35}$$

Calculations by the approximate formula (5.35) coincide very closely with calculations by the exact formula (5.34) upon substitution of the values of $\tilde{H}$ and $\tilde{f}$, which are determined by formula (5.23) and the graph in Fig. 28, at $|f_{cr}| \approx 0.01$.

Introducing in (5.33) the value of $F(\tilde{f})$ from formula (5.35) and c_{f0} from formula (3.14), we obtain the equation:

$$\frac{dRe^{**}}{dX} + (1 + H_{sp})\frac{Re^{**}}{w_0} \cdot \frac{dw_0}{dx} = \frac{B}{2} \cdot \frac{Re_L}{Re^{**m}}. \tag{5.36}$$

Integrating this equation, we find that:

$$Re^{**} = \tilde{w}_0^{-\frac{1+m}{1+m}} \left[\frac{1+m}{2} B Re_0 \int_{X_1}^{X} \tilde{w}_0^{\frac{1+m}{1+m}} dX + (Re^{**}\tilde{w}_0^{\frac{1+m}{1+m}})_{X_1}^{\frac{1}{1+m}}\right]^{\frac{1}{1+m}}, \tag{5.37}$$

where

$$x_1 = 1 + H_{sp}; \quad x_2 = 1 + (1 + m)(1 + H_{sp}), \quad \bar{w}_0 = \frac{w_0}{w_{01}}; \quad Re_0 = \frac{w_{01} L}{\nu};$$

w_{01} is the velocity of the current incident on the leading edge of the streamlined body; X_1 is a dimensionless coordinate of the beginning of the turbulent boundary layer.

By equation (5.37) the $\mathbf{Re}^{**}$ is distributed along the contour of the body, covered by the turbulent boundary layer. In addition, the flow velocity outside the boundary layer $w_0(X)$ should be given.

The local value of the form parameter is determined by formula

$$f = \frac{Re^{**}}{Re_0 \bar{w}_0} \cdot \frac{d\bar{w}_0}{dX}. \tag{5.38}$$

For given values of f and $\mathbf{Re}^{**}$ the value of Ψ_f is given by the graph in Fig. 28, and by formula

$$c_f = \Psi c_{f0} \tag{5.39}$$

the local value of coefficient of friction is calculated.

The values of the coefficients B and m during solution of the momentum equation are selected for a given interval of values of $\mathbf{Re}^{**}$ numbers in accordance with Table 3.1.

The quantity c_{f0} in formula (5.39) can be calculated by formula (3.8).

For not very large values of the form parameter $\tilde{f}$, it is possible to put H = 1.3 and Ψ = 1 in the momentum equation. In this case

$$Re^{**} = \tilde{\bar{w}}_0^{-2.3}\left[\frac{1+m}{2} B Re_0 \int_{X_1}^{X} \tilde{\bar{w}}_0^{3.3 - 2.3m}\, dX + (Re^{**} \tilde{\bar{w}}_0^{2.3})^{1-mX_1}\right]^{\frac{1}{1-m}}. \tag{5.40}$$

For $\mathbf{Re}^{**}$ numbers $< 10^4$ it is possible to assume m = 0.25 and B = 0.0258. Then

$$Re^{**} = \tilde{\bar{w}}_0^{-2.3}\left[0{,}0161\, Re \int_{X_1}^{X} \tilde{\bar{w}}^{3.3}\, dX + (Re^{**} \tilde{\bar{w}}_0^{2.3})^{1.25X_1}\right]^{0.8}. \tag{5.41}$$

In the works of Gruschwitz, Dengot, and Tetervin it is shown that the form parameter H may be a good criterion for determination of the point of breakaway of the turbulent boundary layer. Besides, as was noted above, the critical value of the form parameter H, found experimentally by Gruschwitz, practically coincides with those calculated by us. In the opinion of the above researchers, distribution of the form parameter H along the contour of a streamlined body depends not only on the local value of the $\mathbf{Re}^{**}$ number, but also on the distributive law of pressure, i.e., on function $w_0[H]$. In this connection empirical relations of the type,

$$\frac{\delta^{**}}{L} \cdot \frac{dH}{dX} = \dot{H}(\tilde{f}; Re^{**}),$$

are introduced.

The degree of reliability of these relations is vague, inasmuch as the corresponding experimental data are not published in sufficient volume.

The presence of some sort of influence of the character of the function $w_0(H)$ on the local parameters of the boundary layer does not contradict the general considerations about similarity and, within the framework of our theory, should be taken into consideration by introduction of the corresponding correction in the approximation of the tangential stress profile.

Regarding the quantities f_{cr}, H_{cr}, and $\Psi(f; \mathbf{Re}^{**})$, the influence on them of the indicated factor has not yet been observed.

5.6. Law of Heat Transfer in the Diffusion Region of a Quasiisothermal Boundary Layer on an Impenetrable Wall

The presence of a pressure gradient essentially disturbs the similarity between the processes of friction and heat transfer in the boundary layer. In addition the process of heat transfer possesses

a significant degree of conservation which is revealed already upon comparison of the distributive laws of the tangential stresses and heat flux along the cross section of the boundary layer.

By formulas (2.35) and (2.74) for an impenetrable surface (b = 0):

$$\left.\begin{aligned}\tilde{\tau} &= 1 + \Lambda\xi - (2\Lambda + 3)\xi^2 + (\Lambda + 2)\xi^3;\\ q &= 1 - 3\xi^2 + 2\xi^3.\end{aligned}\right\} \qquad (5.42)$$

Within the limits of the given approximation the distribution of the heat flux, in general, does not depend on the pressure gradient, whereas the distribution of tangential pressures changes substantially with a change of the quantity (dp/dx).

In Fig. 31 is shown the distribution of tangential stresses and heat flux along the cross section of the boundary layer at the point of breakaway, calculated by formulas (5.42). In the wall region the path of the curves $\tilde{q}(\xi)$ and $\left(2\tau/\rho_0 w_0^2\right)(\xi)$ is quite different.

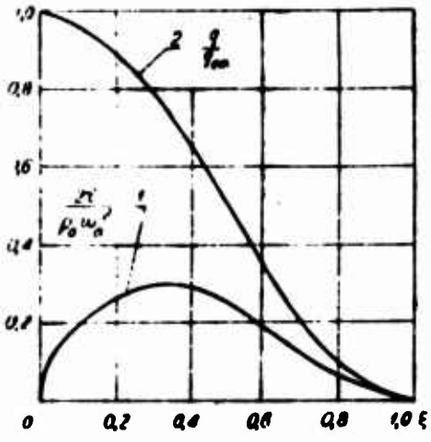


Fig. 31. Distribution of tangential stresses 1) and heat flux 2) at the point of breakaway of the boundary layer by formulas (5.42).

Let us carry out a calculation of the intensity of heat transfer at the point of breakaway at **Pr** ≈ 1 and $\delta_t < \delta$. In this case $l_t \approx l$ and equation (2.64) for quasiisothermal conditions will take the form

$$\tilde{q}\,St \approx \left(\frac{l}{\delta}\right)^2 \frac{\partial w}{\partial \xi} \cdot \frac{\partial \theta}{\partial \xi} \qquad (5.43)$$

or, taking into account (5.10),

$$St \approx 0{,}16\,\xi^2 \frac{\partial w}{\partial \xi} \cdot \frac{\partial \theta}{\partial \xi}. \qquad (5.44)$$

Taking the velocity distribution from formula (5.14), we have

$$St \approx 0{,}0685 \frac{\int_{\xi_1}^{1} \frac{\partial \theta}{\partial \xi}\, d\xi}{\int_{\xi_1}^{\xi} \frac{d\xi}{\xi^{1.43}}} = 0.0295\, \xi_1^{0.43} \frac{1 - \theta_1}{1 - \left(\frac{\delta}{\delta_t}\xi_1\right)^{0.43}}. \qquad (5.45)$$

Here

$$\theta_1 = \frac{\alpha y_1}{\lambda} = St\,Pr\,Re^{**}\frac{\delta}{\delta^{**}}\xi_1.$$

Disregarding the quantity $[(\delta/\delta_t)\xi_1]^{0.43}$ as compared to 1 and assuming **Pr** = 1, we obtain the value:

$$St_{sp} \approx \frac{0{,}0295\,\xi_1^{0.43}}{1 + 0{,}0295\left(\frac{\delta}{\delta^{**}}\right)_{sp}\xi_1^{1.43}Re^{**}}. \qquad (5.46)$$

Putting in (5.46) the value of $(\delta^{**}/\delta) = 0.16$ and ξ_1 from formula (5.18), we have

$$St_{sp} \approx \frac{0{,}046}{Re^{**0.27}(1 + 0{,}71\,Re^{**})^{0.1}}. \qquad (5.47)$$

Assuming **St** = 0.0129 $\mathbf{Re}^{**-0.25}$, we find that in the considered conditions

$$\frac{St_{sp}}{St_0} \approx \frac{3{,}5}{Re^{**0.12} + 0{,}7\,Re^{**0.12}}. \qquad (5.48)$$

By formula (5.48), in the region of $\mathbf{Re}^{**}$ numbers $\approx (3\text{-}10)\cdot 10^3$ the ratio $\mathbf{St}_{cr}/\mathbf{St}_0$ is equal, on the average, to 1.

With growth of the **Re** number the critical value of the **St** number becomes less than **St**.

Thus, a theoretical appraisal shows that for the practically most frequently met values of $\mathbf{Re}^{**}$, the law of heat transfer almost does not change with a change of the pressure gradient. This important conclusion is well confirmed by the experimental data shown in Figs. 32, 33, and 34. It is distinctly clear that upon a significant decrease of the coefficient of friction and an abrupt deformation of the velocity profile with an increase of the diffusion nature of the flow, the values of the criterion **St** and the temperature profile almost do not change.

Nonetheless, the theory shows that at $\mathbf{Re} \to \infty$ the value of **St**, although slowly, approaches zero. This tendency also is revealed in experiments.

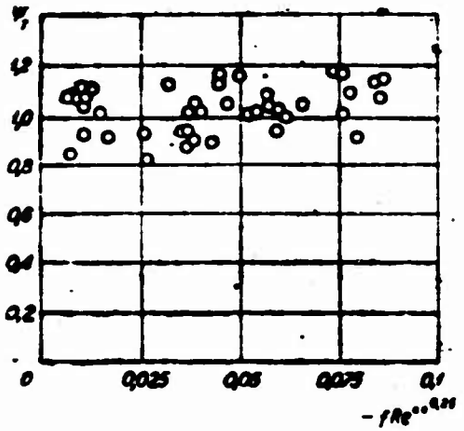


Fig. 32. Influence of the pressure gradient on heat transfer according to the experiments of A. I. Leont'ev, A. N. Oblivin, and P. N. Romanenko.

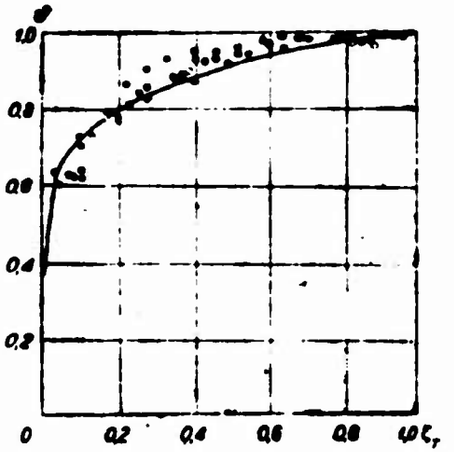


Fig. 33. Distribution of temperatures in the boundary layer for different values of the form parameter: ○ — $f\,Re^{**0.25} = -0.0740$; ⊕ — $f\,Re^{**0.25} = -0.0550$; ◐ — $f\,Re^{**0.25} = -0.0282$; ◑ — $f\,Re^{**0.25} = 0.0049$; ◑ — $f\,Re^{**0.25} = 0.0121$.

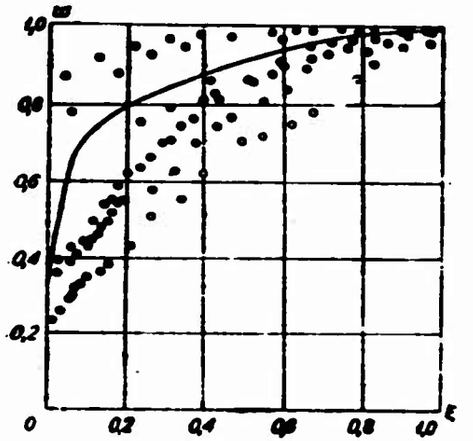


Fig. 34. Distribution of velocities in the boundary layer with the same values of the form parameter as in Fig. 33. Solid lines on both figures correspond to distributions at $(dp/dx) = 0$: ◑ — $f\,Re^{**0.25} = -0.00906$; ⊗ — $f\,Re^{**0.25} = -0.0550$; ◓ — $f\,Re^{**0.25} = 0.0049$; ◒ — $f\,Re^{**0.25} = -0.0282$; ○ — $f\,Re^{**0.25} = -0.0740$; ◑ — $f\,Re^{**0.25} = 0.0112$.

5.7. Solution of the Energy Equation for a Quasiisothermal Boundary Layer on an Impenetrable Surface

The equation of the energy of a boundary layer (1.26) during constancy of the physical properties of the flow and an impenetrable wall can be written in the form

$$\frac{d\mathrm{Re}_t^{**}}{dX} + \frac{\mathrm{Re}_t^{**}}{\Delta T}\cdot\frac{d\Delta T}{dX} = St\cdot \mathrm{Re}_L = \frac{Nu}{X\cdot Pr}. \qquad (5.49)$$

Here $X = (x/L)$ is the relative length;

$\mathrm{Re}_L = (w_0 L/\nu)$ is the Reynolds number, calculated from the local value of the velocity w_0 and from the characteristic linear dimension of the body — L;

$St = (\alpha/c\gamma w_0)$ is the local value of the Stanton number;

$Nu = (\alpha x/\lambda)$ is the local value of the Nusselt number.

Due to the conservation character of the law of heat transfer relative to the pressure gradient, in the majority of practical calculations (at $\mathrm{Re}^{**} < 10^4$), it is possible to assume the value of the St number equal to its magnitude at $(dp/dx) = 0$. Consequently, for quasiisothermal flow formula (3.35) can be used. Putting this value of St in (5.49), we obtain a linear equation relative to the quantity Re_1^{**1+m}. Carrying out the calculation, we find that

$$\mathrm{Re}_t^{**} = \frac{1}{\Delta T}\left[\frac{1+m}{2Pr^{0.75}} B\mathrm{Re}_0 \int_{X_1}^{X} \bar{w}_0 \Delta T^{1+m} dX + (\mathrm{Re}_t^{**}\Delta T)_{X_1}^{1+m}\right]^{\frac{1}{1+m}}, \qquad (5.50)$$

where $\mathrm{Re}_0 = (w_{01}L/\nu)$ is the Reynolds number, calculated on the velocity of the approaching stream w_{01} and the characteristic length of the body L;

$w_0 = (w_0/w_{01})$ is the relative velocity outside the boundary layer.

Putting the value of Re_t^{**}, determined from formula (5.50), in formula (3.35), we find the local value of the Stanton number and correspondingly the heat transfer coefficient. In addition, the

functions $w_0(x)$ and $\Delta T(x)$ should be given beforehand.

5.8. Influence of Nonisothermalness on the Parameters of Breakaway of the Boundary Layer from an Impenetrable Surface

The velocity distribution in the region of the breakaway of a nonisothermal layer on an impenetrable wall is determined by equation

$$\int_{\omega_1}^{1} \sqrt{\frac{\rho}{\rho_0}(1-\omega)}\,d\omega = 2{,}5\sqrt{\left(-\frac{\delta}{\delta^{**}}f'\right)_{sp}}\int_{\xi_1}^{\xi}\frac{d\xi}{\sqrt{2\xi+\xi}}. \qquad (5.51)$$

The solution of the right integral is expressed by formula (5.8).

From here we obtain, that in the limiting case, when

$$Re \to \infty,\ \xi_1 \to 0,\ \omega_1 \to 0,\ \beta \to 0,$$

$$\left(-\frac{\delta}{\delta^{**}}f'\right)_{sp} \doteq \left(\frac{\int_0^1 \sqrt{\frac{\rho}{\rho_0}}\,d\omega}{2{,}5\int_0^1 \frac{d\xi}{\sqrt{2\xi+\xi}}}\right)^2. \qquad (5.52)$$

At $\rho = \rho_0$ we obtain formula (5.9) for the limiting case ($\omega_1 = 0$).

Thus, the relationship of the limiting critical values of the form parameter for nonisothermal and isothermal flows is determined by integral

$$\frac{\left(\frac{\delta}{\delta^{**}}f'\right)_{sp}}{\left(\frac{\delta}{\delta^{**}}f'\right)_{sp_0}} = \left(\int_0^1 \sqrt{\frac{\rho}{\rho_0}}\,d\omega\right)^2. \qquad (5.53)$$

The value of this integral for a constant coefficient of nonsimilarity of the temperature and velocity fields was calculated in Chapter III. For gradient flow in the general case $\varepsilon = \varepsilon(\xi)$. Thus, during isothermal flow at the point of breakaway the velocity profile is determined approximately by formula (5.14). At the same time in the region of Re^{**} numbers, at least up to 10^4, the law of heat transfer almost does not change with a change of the form parameter, i.e., $n_t \approx n_{t0} \approx (1/7)$. For these conditions from formula (2.57):

$$\varepsilon = \varepsilon_0 \xi^{0.29}, \tag{5.54}$$

where ε_0 is the value of the coefficient of nonsimilarity of the temperature and velocity fields at $(dp/dx) = 0$.

But at $\mathbf{Re} \to \infty$ $\mathbf{St}_{cr} \to 0$, i.e., the processes of friction and heat transfer in the diffusion region become similar. Therefore, inasmuch as integral (5.53) expresses the relationship of the limiting critical values of the form parameter, its solution coincides with formula (3.48).

This result will also apply for finite **Re** numbers. Actually, even for moderate values of $\mathbf{Re}^{**}$, from formula (5.54) the quantity $\varepsilon > 0.5\varepsilon_0$ already at $\xi = 0.1$; at $\xi = 0.5$ $\varepsilon = 0.82\varepsilon_0$; at $\xi = 1$ $\varepsilon = \varepsilon_0$. At the same time the same quantity ε, as was shown in Chapter III, weakly affects the considered integral.

From the above it follows that:

a) during moderate flow velocities of a gas

$$\frac{\left(\frac{\delta}{\delta^{**}} f\right)_{sp}}{\left(\frac{\delta}{\delta^{**}} f\right)_{sp0}} \approx \left(\frac{2}{1 + \overline{\psi} + 1}\right)^2; \tag{5.55}$$

b) during significant flow velocities of a gas[1]

$$\frac{\left(\frac{\delta}{\delta^{**}} f\right)_{sp}}{\left(\frac{\delta}{\delta^{**}} f\right)_{sp0}} \approx \frac{1}{\psi^* - 1}\left[\arcsin \frac{2(\psi^* - 1) + \Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}} - \arcsin \frac{\Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}}\right]^2. \tag{5.56}$$

Taking into account the value of n_{cr}, obtained in (5.14), we find that the limiting velocity field at the point of breakaway is determined by equation

$$\int_0^{\infty} \sqrt{\frac{T_0}{T}}\, d\omega \approx \xi^{0.43} \int_0^1 \sqrt{\frac{T_0}{T}}\, d\omega. \tag{5.57}$$

[1]The formulas (5.55) and (5.56) were obtained previously by L. E. Kalikhman, but not as limiting.

For subsonic flows, when $\psi^* \approx 1$,

$$\omega \approx \frac{\psi - \left[\psi - \left(\psi - 1\right)\xi^{0.43}\right]^2}{\psi - 1}. \qquad (5.58)$$

For flows with high velocities, when ψ^* is noticeably larger than unity,

$$\omega \approx \frac{E}{2(\psi^* - 2)} \sin\left\{\left[\arcsin\frac{2(\psi^* - 1) + \Delta\psi}{E} - \arcsin\frac{\Delta\psi}{E}\right]\xi^{0.43} + \arcsin\frac{\Delta\psi}{E}\right\} - \frac{\Delta\psi}{2(\psi^* - 1)}, \qquad (5.59)$$

where

$$E = \sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}.$$

In Fig. 35 is shown the influence of the temperature factor on the limiting velocity profile at the point of breakaway of a boundary layer of gas during subsonic flow. In the same place is given a graph for the adiabatic supersonic flow of gas at $\psi^* = 6$ which corresponds to $M = 5$ at $r = 1$ and $M = 5.3$ at $r = 0.9$.

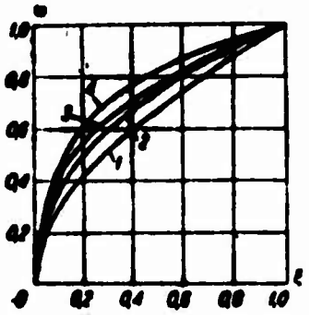


Fig. 35. Influence of heat transfer on the limiting velocity profile at the point of breakaway:
1) $\psi^* = 1$, $\psi = 0.25$;
2) $\psi^* = 1$, $\psi = 1.0$;
3) $\psi^* = 1$, $\psi = 2.0$;
4) $\psi^* = 6$, $\psi = 6$.

The temperature factor comparatively weakly deforms the velocity profile at the point of breakaway of a boundary layer of gas from an impenetrable wall.

In Figs. 36, 37, and 38 are given the values of the limiting critical parameters f_{cr}; H_{cr}; $(\delta/\delta^{**})_{cr}$ in dependence on the temperature factor ψ at subsonic flows ($\psi^* \approx 1$).

The quantity H_{cr} almost linearly changes with the growth of the temperature factor ψ. Cooling the surface ($\psi < 1$) increases the stability of diffusion flow.

During heating of a streamlined surface ($\psi > 1$) the stability of diffusion flow decreases. In Fig. 39 is shown the change of the quantity of the critical value of the form parameter f_{cr} during supersonic streamlining of a body.

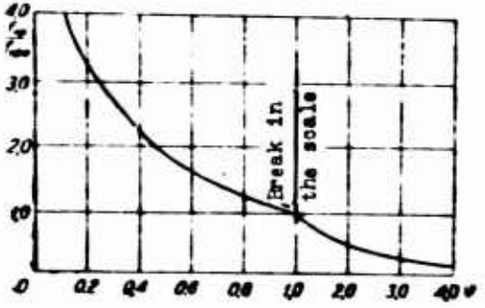


Fig. 36. Dependence of f_{cr} on ψ during subsonic flow of a gas.

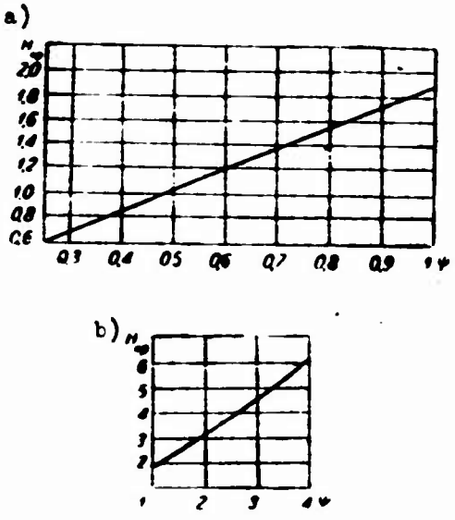


Fig. 37. Dependence of H_{cr} on ψ during subsonic flow of a gas: a) cooling of the wall, b) heating of the wall.

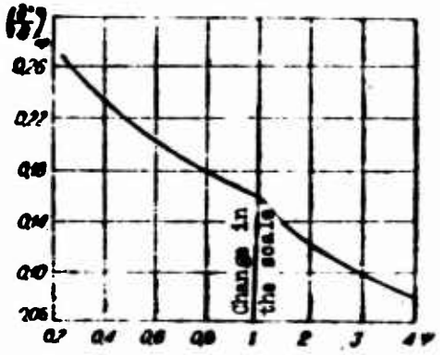


Fig. 38. Dependence of $(\delta^*/\delta)_{cr}$ on ψ during subsonic flow of a gas.

As can be seen, during high flow velocities the region of existence of a stable turbulent boundary layer at $(dp/dx) > 0$ is extremely limited.

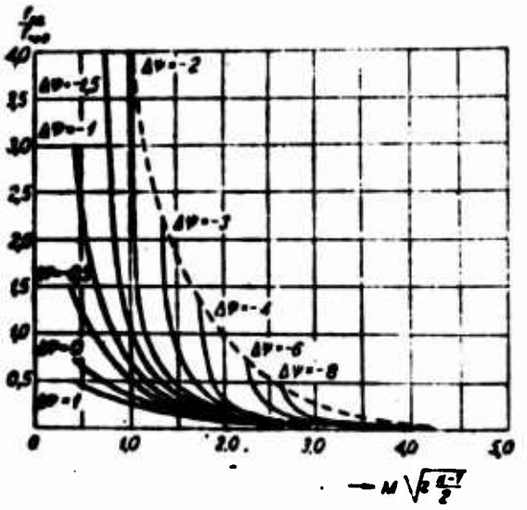


Fig. 39. Dependence of f_{cr} on ψ^{**} and $\Delta\psi$.

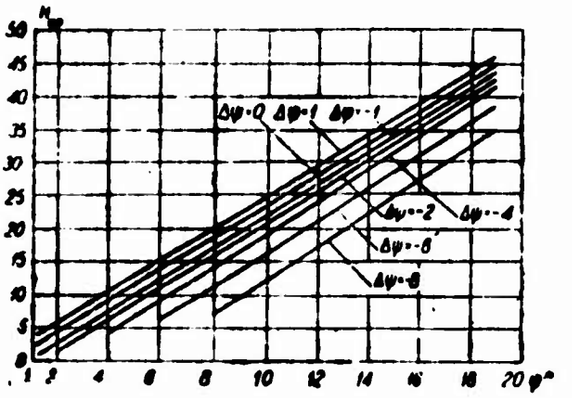


Fig. 40. Dependence of H_{cr} on ψ^{*} and $\Delta\psi$.

In Fig. 40 is given a graph for the value of H_{cr} during supersonic flow of a gas.[1]

5.9. Solution of Equations of Energy and Momentum for a Nonisothermal Boundary Layer on an Impenetrable Surface During Subsonic Flow of a Gas

The integral energy equation for the region of finite values of Re_t^{**} numbers, in which one may assume that the **St** number little depends on the form parameter f, will take the form

$$\frac{d\,Re_T^{**}}{dX} + \frac{Re^{**}}{\Delta T}\cdot\frac{d\,\Delta T}{dX} = Re_L\,\psi_T\,St_\infty \qquad (5.60)$$

where

$$\psi_T = \left[\frac{2}{\sqrt{\psi - 0{,}2(\psi-1)\sqrt{c_{f_0}} + 1}}\right]^2. \qquad (5.61)$$

is the correction for nonisothermalness taking into account the finiteness of the **Re** number.

[1] A more conservative parameter $(f\cdot H)_{cr}$ which is explained by mutual compensation in the change of f_{cr} and H_{cr}.

Formula (5.61) is obtained from formula (3.58) at $\psi^* \approx \varepsilon \approx 1$.

The limiting value of the function Ψ_t during subsonic flow of a gas is determined by formula (3.55), which coincides with the corresponding limiting formula (3.47) for the ratio of the coefficients of friction.

In practice in calculations it is possible to be limited by the limiting formulas for Ψ_t or to introduce this function, referred to the mean value of c_{f0} for a given streamlining. Then $\Psi_t \approx \Psi(\psi)$, i.e., it can be considered independent of $\mathbf{Re}^{**}$. The solution of equation (5.60) under these conditions has the same form as equation (5.50), but under the integral sign stands the product $\Psi_t \tilde{w}_0 \Delta T^{1+m}$. The value of the local Stanton number is determined for the obtained value of the $\mathbf{Re}_t^{**}$ number by formula

$$St = \Psi_\tau St_0. \tag{5.62}$$

where $\mathbf{St}_0$ is determined by formula (3.35).

During linear interpolation of the function $F(\tilde{f})$ its value during nonisothermal flow will be determined by formula:

$$F(\tilde{f}) \approx \Psi_\tau - (1 + H_{ap}) \frac{2J}{c_{f0}}. \tag{5.63}$$

For the coefficient of friction we will obtain the expression

$$c_f = \Psi_\tau \Psi_f c_{f0}, \tag{5.64}$$

where Ψ_f is the correction for the influence of the pressure gradient during isothermal flow.

As was explained in Chapter III, the function Ψ_t for the coefficient of friction differs from the analogous function for the coefficient of heat transfer by the form of calculation of the coefficient of nonsimilarity of the velocity and temperature fields ε. It was also shown there, that for the most frequently met ratios, $0.5 < (\delta^{**}/\delta_t^{**}) < 2$, in practical calculations the influence of the coefficient ε on Ψ_t can be ignored and $\varepsilon = 1$ can be assumed.

Solution of the momentum equation takes the form

$$Re^{**} = \exp\left(-\frac{J}{1+m}\right)\left[\frac{B}{2} Re_0 \int \psi_T \tilde{w}_0 \times \right. \\ \left. \times \exp(J)\, dX + c_1\right]^{\frac{1}{1-m}}, \tag{5.65}$$

where

$$J = \int (1 + H_{kp}) \frac{d\tilde{w}_0}{\tilde{w}_0}. \tag{5.66}$$

Here H_{cr} is a function of the temperature factor ψ.

During a constant wall temperature, the integral (5.65) is simplified and can be written in the form

$$Re^{**} = \tilde{w}_0^{-n}\left[\frac{1+m}{2} B Re_0 \int_{X_1}^{X} \tilde{w}_0^{1-(1+m)n} \cdot \psi_T\, dX + \right. \\ \left. + (Re^{**}\tilde{w}_0^{n})_{X_1}^{1+m}\right]^{\frac{1}{1-m}}, \tag{5.67}$$

where

$$n = 1 + H_{kp}.$$

In that case, when it is necessary to determine the distribution of the wall temperature at a given law of feed (discharge) of heat $q_{wall}(x)$, the energy equation is written in the form

$$\frac{d(\Delta T Re^{**})}{dx} = \frac{q_{cr}(x)}{R c_p \mu_0}. \tag{5.68}$$

Hence

$$Re_T^{**} = \frac{\int_{x_1}^{x} q_{cr}\, dx}{R c_p \mu_0 \Delta T} + Re_{t, X_1}^{**}. \tag{5.69}$$

Taking into account formulas (5.62) and (3.35), we find that at $Re_{t,X_1}^{**} = 0$ the local difference of temperatures of the wall and the flow

$$\Delta T = \left[\frac{2 q_{cr} L}{B \lambda_0 \psi_T Pr^{0.33+m} Re_0 \tilde{w}_0}\left(\int_0^x \frac{q_{cr}}{\lambda_0}\, dx\right)^m\right]^{\frac{1}{1+m}}. \tag{5.70}$$

Putting in (5.70) B = 0.0258, m = 0.25, and Ψ_t from the limiting formula (3.55), we have

$$\Delta T = \left[19{,}4 \frac{(\sqrt{\psi} + 1)^2 q_{cr} L}{\lambda_{\infty} Pr^{0.5} Re_{\infty} \tilde{w}_0} \left(\int_0^x \frac{q_{cr}}{\lambda_0} dx \right)^{0.25} \right]^{0.8}. \qquad (5.71)$$

Here, as before,

$$\psi = \frac{T_{cr}}{T_0}, \quad \tilde{w}_0 = \frac{w_0}{w_{max}}, \quad Re_0 = \frac{w_{max} L}{\nu_0}.$$

5.10. Resolution of the Equations of Energy and Momentum During Supersonic Flow of Gas

At $\psi^* > 1$ the energy equation preserves the form of equation (5.60), if we assume:

$$\Delta T = T_{cr}^* - T_{cr};$$
$$Re_{\tau}^{**} = \frac{\rho_{\infty} w_0 \delta^{**}}{\mu_{00}}; \qquad (5.72)$$
$$Re_L^{**} = \frac{\rho_0 w_{max} L}{\mu_{00}},$$

where μ_{00} is the viscosity of the gas at the temperature of stagnation outside the boundary layer.

The integral of this equation has the form:

$$Re_{\tau}^{**} = \frac{1}{\Delta T} \left[\frac{1+m}{2 Pr^{0.75}} B Re_{\infty} \int_{X_1}^{X} \Psi_{\tau} \left(\frac{\mu_{00}}{\mu_0} \right)^m \times \right.$$
$$\left. \times U (1 - U^2)^{\frac{1}{k-1}} \Delta T^{1+m} dX + (Re_{\tau}^{**} \Delta T^{1+m})_{X_1} \right]^{\frac{1}{1+m}}. \qquad (5.73)$$

Here

$$Re_{\infty} = \frac{w_{max} L}{\nu_{\infty}}, \quad U = \frac{w_0}{w_{max}}, \quad w_{max} = \sqrt{\frac{2}{A} g c_p T_0^*}.$$

The function Ψ_t is calculated from the respective formulas of Chapter III for supersonic flow. The limiting value of Ψ at $\varepsilon \approx 1$ is determined by formula (3.58).

The Stanton number will be determined by the formula

$$St = \frac{B \Psi_{\tau}}{2 Pr^{0.75}} \left(\frac{\mu_0}{\mu_{00} Re_{\tau}^{**}} \right)^m.$$

where the value of $\mathbf{Re}_t^{**}$ is found from equation (5.73).

Taking into account the compressibility, the integral of the momentum equation will take the form

$$Re^{**} = \exp\left(-\frac{f}{1+m}\right)\left[\frac{B}{2} Re_{\infty} \int_0^X \Psi_T \left(\frac{\mu_\infty}{\mu_0}\right)^m \times \right.$$
$$\left. \times U (1-U)^{\frac{1}{k-1}} \exp(fJ)\, dX + c\right]^{\frac{1}{1+m}}, \qquad (5.74)$$

where

$$Re^{**} = \frac{\rho_\infty w_\infty \delta^{**}}{\mu_\infty}.$$

At T_{wall} = const

$$Re^{**} = U^{x}\left[\frac{1+m}{2} B\, Re_{\infty} \int_{X_1}^{X} \Psi_T \left(\frac{\mu_\infty}{\mu_0}\right)^m \times \right.$$
$$\left. \times (1-U^{\frac{1}{k-1}})\, U^{1+(1+m)x}\, dX + (Re^{**} U^{x})_{X_1}^{m}\right]^{\frac{1}{1+m}}, \qquad (5.75)$$

where

$$x = 1 + H_{sp}.$$

The value of the form parameter is determined by formula

$$f = \frac{Re^{**}}{Re_{\infty} U^{x} (1-U^{\frac{1}{k-1}})} \cdot \frac{dU}{dX}. \qquad (5.76)$$

The critical value of the form parameter is determined from the graphs of Fig. 39.

The drag coefficient is calculated by formula

$$c_f = \Psi_f \Psi_T \left(\frac{\mu_\infty}{\mu_0}\right)^m c_{f0}, \qquad (5.77)$$

where c_{f0} is referred to the value of $\mathbf{Re}^{**}$, determined by equation (5.74) or (5.75).

In practice, equation (5.75) can be used and in the case of not very strong changes of the wall temperature along the length of the contour. Connected with this is the fact that errors in the determination of the value of $\mathbf{Re}^{**}$ weakly affect the value of c_{f0} due to the small value of the exponent m in the law of resistance for a

turbulent boundary layer.

5.11. Solution of the Equations of Energy and Momentum for Axisymmetric Flow of a Gas

For an axisymmetric boundary layer the equation of energy and momentum can be written in the form:

$$\frac{dRe_T^{**}}{dX} + Re_T^{**}\left(\frac{1}{\Delta T}\frac{d\Delta T}{dX} - \frac{1}{R}\cdot\frac{dR}{dX}\right) = Re_L St; \qquad (5.78)$$

$$\frac{dRe^{**}}{dX} + \frac{Re^{**}}{R}\cdot\frac{dR}{dX} = Re_L F(\bar{f}), \qquad (5.79)$$

where R is the current radius.

The law of heat transfer remains practically the same, as for a flat boundary layer.

The integral of equation (5.78) differs from the integral of the energy equation of a flat boundary layer only by the fact that the product $R\Delta T$ enters into it instead of the value of ΔT.

The integral of equation (5.79) at T_{wall} = const has the same form as formula (5.75), but instead of the value of U^n in it one should put the product RU^n. During subsonic flow in formula (5.67) the quantity $\tilde{w}_0^n$ is replaced by Rw_0^n.

During flow of a gas in a cylindrical nozzle (internal problem) the integral of the energy equation is conveniently written in the form

$$Re_T^{**} = \left[\frac{(1+m)\,B\,Re_{01}}{2\,Pr^{0.75}\,\Delta T^{1+m}}\left(\frac{D_{cp}}{D}\right)^{1+m}\left(\frac{k-1}{k+1}\right)^{0.5}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\times\right.$$
$$\left.\times\int_0^X \psi_T\,\Delta T^{1+m}\left(\frac{D_{cp}}{D}\right)^{1-m}dX\right]^{\frac{1}{1+m}}. \qquad (5.80)$$

Besides, it is assumed that the turbulent layer starts from the inlet section of the nozzle.

Equation (5.80) is obtained from the basic solution of the energy equation, if we bear in mind that during flow of a gas in a nozzle, with a good degree of accuracy, it is possible to assume

$$U(1-U^2)^{\frac{1}{k+1}} = \left(\frac{k-1}{k+1}\right)^{0.5}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\frac{\Omega_{cr}}{\Omega}, \qquad (5.81)$$

:eby in this case $(\Omega_{cr}/\Omega) = (D_{cr}/D)^2$,

:e Ω is the cross-sectional area. In these formulas D_{cr} is the ıeter of the critical cross section of the nozzle.

The value of the local Nusselt number will be determined by ıula

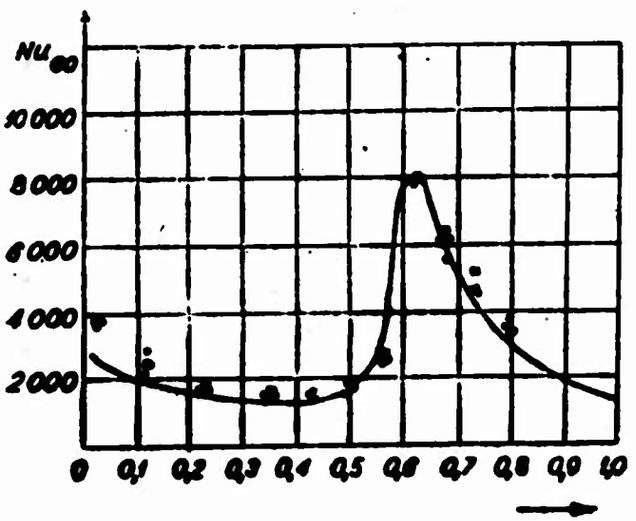
$$Nu_{x0} = \frac{B}{2} Pr^{0.25} Re_{\infty} \psi^* Re_T^{n-m}\left(\frac{k-1}{k+1}\right)^{0.5}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\left(\frac{D_x}{D}\right)^2, \qquad (5.82)$$

:e

$$Nu_{x0} = \frac{\alpha L}{\lambda_{\infty}}.$$

characteristic linear dimension L should be one and the same in Nu and Re numbers.

The quantity ψ^* during flow of a gas in a nozzle is a single- ıed function of the dimensionless cross-sectional area of the noz- that essentially simplifies calculation of the function $\Psi_t(X)$ and :ulation of the thermal boundary layer.



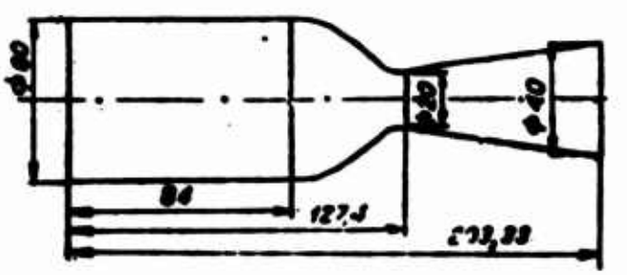


Fig. 41. Comparison of calculations of heat transfer in a super- ıonic nozzle by formula (5.82) with the experimental data of A. I. Leont'ev.

In Fig. 41 is given a comparison of the calculations of heat ınsfer in a supersonic nozzle by the proposed method with the results

of an experimental investigation. It is possible to ascertain a very good agreement of theory and experiment.

5.12. Solution of the Equations of Energy and Momentum During Subsonic Flow of a Gas on a Permeable Slightly Curved Surface

The laws obtained in Chapter IV of friction and heat exchange for a permeable plate can be used for solution of the equations of momentum and energy during streamlining by a subsonic, nonisothermal, turbulent boundary layer of a permeable slightly curved surface.

The energy equation for the considered case is written in the form

$$\frac{d\,Re_T^{**}}{dX} + \frac{Re_T^{**}}{\Delta T}\cdot\frac{d\Delta T}{dX} = Re_L\,St_0(\Psi + b_T), \tag{5.83}$$

where the function Ψ is determined by formula (4.30), i.e.,

$$\Psi = \Psi_T\left(1 - \frac{b_T}{b_{T,cr}}\right)^2 \tag{5.84}$$

where Ψ_t is chosen from formula (3.47).

In the general case, the quantities Ψ_t, b_t, and $b_{t,cr}$ are functions of the coordinate X. Then the integral of equation (5.83) has the form

$$Re_T^{**} = \frac{1}{\Delta T}\left\{\frac{1+m}{2Pr^{0.75}}\,B\,Re_0\int_{X_1}^{X}\bar{w}_0\left[\Psi_T\left(1-\frac{b_T}{b_{T,cr}}\right)^2 + b_T\right]\times\right.$$
$$\left.\times\Delta T^{1+m}\,dX + (Re_T^{**}\Delta T)_{X_1}^{1+m}\right\}^{\frac{1}{1+m}}. \tag{5.85}$$

For the given functions $w_0(x)$, $\Delta T(x)$, and $b_t(x)$ the value of Re_t^{**} is calculated from equation (5.85) as a function of the coordinate X.

Distribution of the feed of the cooling gas along the length of the circuit is calculated by formula

$$\frac{j_1}{\rho_0 w_0} = \frac{c_{p0}}{c_{p1}}\,St_0\,b_T. \tag{5.86}$$

The local Stanton number is determined by formula

$$St = \Psi_T\left(1 - \frac{b_T}{b_{T,cr}}\right)^2 St_0. \tag{5.87}$$

The local heat flux

$$q_{cr} = g c_{p0} \rho_0 w_0 \Delta T St. \tag{5.88}$$

From the known consumption of the cooling gas its initial temperature is determined:

$$T_1 = T_{cr} - \frac{q_{cr}}{g c_{p1} j_1}. \tag{5.89}$$

During evaporation and sublimation the surface temperature usually is given. The quantity of gas (vapor) discharged from this surface is subject to determination.

In the considered case

$$q_{cr} = r g j_1, \tag{5.90}$$

where r is the latent heat of vaporization.

Hence

$$\Psi = \frac{St}{St_0} = K b_t \tag{5.91}$$

where $K = (r/c_{p1}\Delta T)$ is the criterion of phase transition.

On the other hand, the value of Ψ is determined by expression (5.84). Combining formulas (5.84) and (5.91), we find that

$$b_t = b_{t.\,ap} - \frac{K b_{t.\,ap}^2}{2\Psi_t}\left(\sqrt{\frac{4\Psi_t}{K b_{t.\,ap}^2} + 1} - 1\right). \tag{5.92}$$

Introducing the value of Ψ from (5.91) and b_t from (5.92) in equation (5.83), we find

$$Re_t^{**} = \frac{1}{\Delta T}\left\{\frac{1+m}{2\rho^{0.75}} B Re_0 \int_{X_1}^{X}\left[b_{t.\,ap} - \frac{K b_{t.\,ap}^2}{2\Psi_t} \times \left(\sqrt{\frac{4\Psi_t}{K b_{t.\,ap}^2} + 1} - 1\right)\right] \bar{w}_0 \Delta T^{1+m} dX + (Re_t^{**}\Delta T)_{X_1}^{1+m}\right\}^{\frac{1}{1+m}}. \tag{5.93}$$

From the found value of Re_t^{**} the value of St_0 is determined; the value of b_t from formula (5.92); and from formula (5.86) the flow rate of the cooling gas.

For relatively small intensities of evaporation or sublimation (which is usually the case) it is possible to be limited during calculation of $\mathbf{Re}_t^{**}$ to the first approximation, i.e., to conduct a calculation of this quantity by formula (5.40) taking into account the nonisothermalness of the flow from Section 5.6.

We will consider the case, when the parameters are given of the external flow, the wall temperature, and the initial temperature of the cooling gas, passing into the boundary layer through a porous surface. It is necessary to determine the needed distribution of the cooling gas along the circuit of the warmed surface. In this case

$$\overline{\rho} = b_t \frac{\Delta T_1}{\Delta T}. \tag{5.94}$$

where $\Delta T_1 = T_{wall} - T_1$ and T_1 is the temperature of the gas at the entrance to a porous surface.

The value of b_t will be determined by formula (5.92) upon replacement in it of the value of the criterion $\mathbf{K}$ by the ratio $(\Delta T_1/\Delta T)$. The value of $\mathbf{Re}_t^{**}$ will be determined by formula (5.93) upon the same replacement of $\mathbf{K}$ by $(\Delta T_1/\Delta T)$. Further calculations are also conducted just as in the preceding problem.

For an axisymmetric boundary layer, solution of the energy equation has the same form as in the case of a flat boundary layer, but instead of ΔT the product $R\Delta T$ enters, where R is the radius of the streamlined body. The whole remaining course of calculation does not change.

The momentum equation for a flat boundary layer on a permeable wall of small curvature has the form

$$\frac{d\mathbf{Re}^{**}}{dX} - \frac{\mathbf{Re}^{**}}{w_e}\cdot\frac{dw_e}{dX}(1+H) = \mathbf{Re}_L(\overline{\rho}+b_t\frac{B}{2}\mathbf{Re}^{**}). \tag{5.95}$$

where

$$\Psi = \Psi_\tau \left(1 - \frac{b}{b_{cr}}\right)^3. \tag{5.96}$$

We have the dependences $\Psi_t(x)$, $b(x)$, $b_{cr}(x)$. Assuming $H = H_0 = 1.3$, we obtain

$$Re^{**} = \tilde{w}_0^{-2.3} \left(\frac{1+m}{2} B Re_{x_1} \int_{x_1}^{x} \tilde{w}_0^{2.3} \cdot^{2.3m} \left[\Psi_\tau \left(1 - \frac{b}{b_{cr}}\right)^3 + b\right] \times \right.$$
$$\left. \times dX + (Re^{**} \tilde{w}_0^{2.3})_{x_1}^{1+m}\right)^{\frac{1}{1+m}}. \tag{5.97}$$

For an axisymmetric boundary layer the solution remains the same, but instead of the quantity $\tilde{w}_0^{2.3}$ the product $R\tilde{w}_0^{2.3}$ appears.

The local value of the coefficient of friction will be determined by formula

$$c_f = \Psi_\tau \left(1 - \frac{b}{b_{cr}}\right)^3 c_{f_0}. \tag{5.98}$$

When the distribution of the flow rate of injected gas $j_1(x)$ along the contour of a streamlined body is known, the problem is solved by the method of successive approximations. The first, and fully sufficient approximation for the determination of Re_t^{**} is calculated by the formulas for an impenetrable wall of small curvature.

The methods presented allow calculation of heat transfer and friction on a porous surface of the frontal part of bodies (flame regulators, the frontal parts of a sphere and a transversely streamlined cylinder, the head part of axisymmetric bodies).

C H A P T E R VI

FLOW IN A PIPE

Definitions of Cyrillic Items in Order of Appearance

ст = wall = wall

т = t = Thermal, Turbulent

6.1. Distribution of Velocities, Friction, and Heat Transfer During Quasiisothermal, Stabilized Flow

During flow in a pipe a flow stabilization section and a section with stabilized flow are distinguished. Stabilized flow approaches after merging of the boundary layers, which arise in the inlet section of the pipe, and for isothermal conditions is characterized by constancy of all the flow parameters. In the inlet section (stabilization section) the flow parameters are changed in connection with the build-up of the boundary layer on the wall of the pipe. A diagram of the flow in the inlet section of the pipe is depicted in Fig. 42.

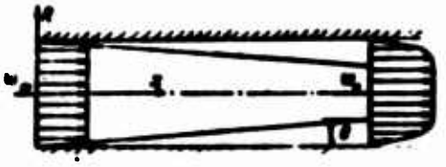


Fig. 42. Diagram of the build-up of the boundary layer in the inlet section of a cylindrical pipe.

During stabilized isothermal, turbulent flow the velocity distribution in a smooth pipe is well described by formula (3.1). This is explained by the fact that flow in a pipe is nozzle flow with comparatively small values of the form parameter. The pressure drop in a pipe is determined by formula

$$\frac{-dp}{dx} = \zeta \frac{\rho_0 \bar{w}^2}{2D}, \qquad (6.1)$$

where ζ is the drag coefficient;

D is the internal diameter of the pipe;

$\bar{w}$ is the average consumption flow velocity;

ρ_0 is the density of the flowing medium, referred to the average consumption temperature of the stream.

During isothermal flow $\rho = \rho_0 = \text{const}$.

From the condition of equilibrium (Fig. 43) it follows that

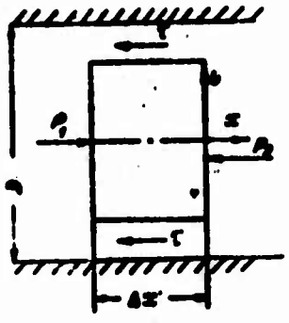


Fig. 43. Diagram of the action of forces on an elementary cylinder of fluid in a pipe.

$$\frac{-dp}{dx} \pi R^2 = \tau 2 \pi R, \qquad (6.2)$$

i.e.,

$$\left.\begin{aligned} \tilde{\tau} &= \frac{R}{R_0} = 1 - \tilde{y}; \\ 4c_f &= \zeta. \end{aligned}\right\} \qquad (6.3)$$

The average consumption velocity

$$\bar{w} = \frac{2}{\pi R_0^2} \int_0^{R_0} \rho w R \, dR. \qquad (6.4)$$

Putting in (6.4) the velocity profile (3.1) and assuming $\rho = \text{const}$, we find that

$$\bar{w} = U_* \left(1.75 + 2.5 \ln \frac{U_* R_0}{\nu}\right). \qquad (6.5)$$

where

$$U_* = \bar{w} \sqrt{\frac{\zeta}{8}}.$$

From here the law of resistance follows:

$$\frac{1}{\sqrt{\zeta}} = 0.88 \ln \left(Re_D \sqrt{\zeta}\right) - 0.9. \qquad (6.6)$$

In the region $5 \cdot 10^5 < R_D < 1 \cdot 10^5$ the Blasius formula gives good results

$$\zeta = \frac{0.316}{Re_D^{0.25}}, \qquad (6.7)$$

where

$$Re_D = \frac{\bar{w} D}{\nu}.$$

The velocity distribution to the n = 1/7 power corresponds to formula (6.7).

The form parameter f can be calculated by formula (2.36),

$$f = -\frac{\delta}{\tau_{wall}} \cdot \frac{dp}{dx} \cdot \frac{\delta^{**}}{\delta} \cdot \frac{c_f}{2}. \qquad (6.8)$$

In the region of stabilized flow, when the boundary layers merged, $\delta = R_0$. According to formula (6.2) $-\frac{dp}{dx} = \frac{2\tau_{wall}}{R_0}$.

Hence during isothermal stabilized flow in a pipe

$$f = \frac{\delta^{**}}{\delta} c_f. \qquad (6.9)$$

At n = 1/7

$$f = \frac{0{,}0308}{Re_D^{0.25}}, \qquad (6.10)$$

i.e., at $Re_D > 10^4$

$$f < 0{,}003; \qquad (6.11)$$

At $Re_D > 10^5$

$$f < 0{,}002. \qquad (6.12)$$

The law of heat transfer in the region of stabilized flow of a gas in a pipe is described by formula,

$$Nu_D = 0{,}023\, Pr^{0.4}\, Re_D^{0.8}. \qquad (6.13)$$

where

$$Nu_D = \frac{\alpha D}{\lambda}.$$

Correspondingly

$$St = \frac{0{,}023}{Pr^{0.6} Re_D^{0.2}}. \qquad (6.14)$$

6.2. Nonisothermal, Stabilized Flow of a Gas

Stabilized flow in a pipe of constant cross section can exist only at $M < 1$.

Inasmuch as in this case the boundary layers are merged, then condition Re = idem is equivalent to the condition Re** = idem. Therefore, the limiting formula (3.47) for given case takes the form

$$\varepsilon = \left(\frac{c}{c_*}\right)_{Re_D} \left(\frac{2}{1+\psi+1}\right)^2 \qquad (6.15)$$

In Fig. 44 is given a comparison of calculations by formula (6.15) with experimental data. The limiting formula in this case also correctly depicts the real character of the investigated relation.

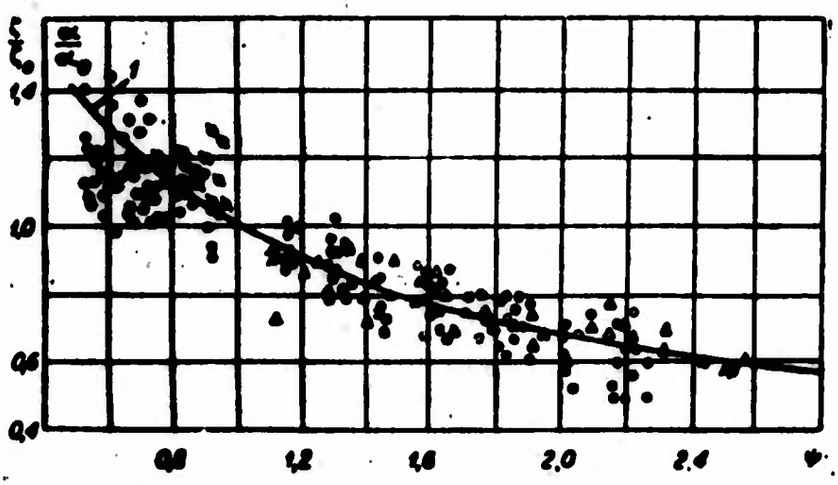


Fig. 44. Influence of temperature factor on friction and heat transfer during stabilized flow of gas in a pipe: 1 — calculation by formula (6.15), ◑ — Jordan, ○ — Greber, ● — Il'in, Δ — Ivashchenko, ▱ — NASA (averaged by series). Experiments on the resistance of friction: ○ — Il'in.

If we consider the finiteness of Re the numbers, then we should use formula (3.58), which at $\psi^* \approx 1$ takes the form

$$\varepsilon = \left[\frac{2}{\sqrt{\psi - 0{,}2(\psi-1)}\sqrt{c_{fm}+1}}\right]^2. \qquad (6.16)$$

For stabilized flow in a pipe this formula should be written with consideration of the ratio $\frac{w_m}{w}$, inasmuch as in the theory of a

boundary layer the coefficient of friction is referred to the maximum velocity: during flow in pipes this coefficient pertains to the average consumption velocity. Proceeding from these considerations we can write that for stabilized conditions

$$\psi \approx \left[\frac{2}{\sqrt{\psi - 4{,}1(\psi - 1)h^2\sqrt{\zeta_0}} + 1}\right]^2, \qquad (6.17)$$

where

$$h = \frac{w}{w_m}.$$

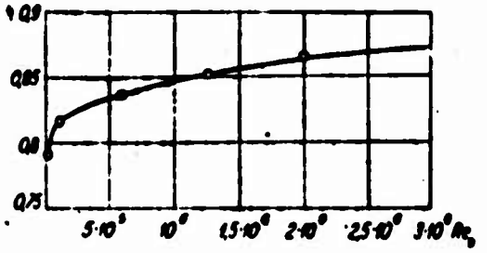


Fig. 45. Dependence of $h = \frac{w}{w_m}$ on Re_D during isothermal flow of a gas in a pipe.

At values of the temperature factor $\psi > 1$ the velocity profiles become less populated and the value of h less than during isothermal flow. At $\psi < 1$ the velocity profiles are more populated and the value of h is larger than during isothermal flow.

In Fig. 45 is shown the dependence of h on Re_D during isothermal flow in smooth pipes.

In practice at $\psi > 1$ it is possible to use the limiting formula (6.15), and at $\psi < 1$ by formula (6.17), assuming $h \approx 0.9$.

6.3. Calculation of Heat Transfer and Friction in the Inlet Section of a Cylindrical Pipe

In the inlet section of the pipe a boundary layer is developed just as during external streamlining of a body as long as the opposite points of the external boundary of the layer do not touch.

The over-all flow rate of a gas along the whole pipe is constant, and the flow velocity in the undisturbed nucleus of flow changes due to the build-up of the boundary layer. Taking into account this circumstance all the equations of a turbulent boundary layer derived above can be used.

We will assume that upon entry into the pipe, the velocity distribution is uniform, and a turbulent boundary layer immediately arises. The latter condition is always fulfilled because of the turbulence causing action of the finite thickness of the leading edge.

We will be limited by the case, when the pipe is warmed, starting from the leading edge. The build-up of dynamic and thermal boundary layers occurs simultaneously along the whole length of the inlet section of the pipe. Correspondingly, the parameters of the gas in the undisturbed nucleus of the flow also change.

The equation of continuity of the flow will be written in the form

$$\rho_{01} w_{01} R_0^2 = 2\int_0^{R_0} \rho w \, dR = \text{const}, \tag{6.18}$$

where ρ_{01} and w_{01} are the density of the gas and flow velocity in the inlet section of the pipe.

For a cylindrical pipe according to formula (1.39)

$$\delta^* = \int_0^{\delta} \left(1 - \frac{\rho w}{\rho_0 w_0}\right)\left(1 - \frac{y}{R_0}\right) dy. \tag{6.19}$$

Taking into account this expression of the thickness of displacement, it is possible to bring the equation of continuity (6.18) to the form

$$\rho_{01} w_{01} = \rho_0 w_0 \left(1 - 2\frac{\delta^*}{R_0}\right), \tag{6.20}$$

where ρ_0 and w_0 are the density and flow velocity outside the boundary layer in the cross section x.

The density of the gas ρ_0 is realted to the density ρ_{00} for the parameters of stagnation (p_{00}; T_0^*) and the dimensionless velocity U by known the expression

$$\frac{\rho_0}{\rho_{00}} = (1 - U^2)^{\frac{1}{k-1}}, \tag{6.21}$$

which has already been used in Chapter V during derivation of formula (5.73).

Combining equations (6.20) and (6.21), we find that

$$2\cdot\frac{\delta^{**}}{R_0}=1-\frac{U_1}{U}\left(\frac{1-U_1^2}{1-U^2}\right)^{\frac{1}{k-1}}, \tag{6.22}$$

where

$$U=w_0\sqrt{\frac{A}{2gc_pT_0^*}}\quad\text{and}\quad U_1=w_{01}\sqrt{\frac{A}{2gc_pT_0^*}}. \tag{6.23}$$

Introducing the parameter H, we obtain the value

$$Re^{**}=\frac{Re_{\infty}}{4H}\left[U(1-U^2)^{\frac{1}{k-1}}-U_1(1-U_1^2)^{\frac{1}{k-1}}\right], \tag{6.24}$$

where, as before

$$Re^{**}=\frac{\rho_0 w_0\delta^{**}}{\mu_{\infty}}\quad\text{and}\quad Re_{\infty}=\frac{\rho_{\infty}w_{max}D}{\mu_{\infty}}.$$

The momentum equations will take the form

$$\frac{dRe^{**}}{dX}+\frac{Re^{**}}{w_0}\cdot\frac{dw_0}{dX}(1+H)=U(1-U^2)^{\frac{1}{k-1}}\times\Psi\frac{B}{2}Re_{\infty}\left(\frac{\mu_0}{\mu_{\infty}Re^{**}}\right)^m. \tag{6.25}$$

where

$$X=\frac{x}{D}.$$

We will be limited by flows for not very large values of the M number. Then supersonic flow of a gas in a cylindrical pipe, which is always diffusion flow, flows at values of the form parameter f, not close to the critical value. In this case the function Ψ can be equated to the function Ψ_t, i.e., will depend only on the parameters ψ^* and ψ. On these parameters also depends the quantity H.

As L. E. Kalikhman showed, the quantity

$$H'=\frac{1}{\delta^{**}}\int_0^{\delta}\left(1-\frac{w_x}{w_0}\right)\frac{\rho}{\rho_0}dy \tag{6.26}$$

depends on the parameters of nonisothermalness significantly weaker than the quantity H.

Between the quantities H and H' there exists the dependence

$$H = \frac{1}{1-U^2}\left(U^2 + H' - \frac{\Delta T}{T_0^*} H_t' \frac{Re_t^{**}}{Re^{**}}\right). \qquad (6.27)$$

where

$$H_t' = \frac{1}{\delta_t^{**}} \int_0^{\delta_t} (1-u)\frac{\rho}{\rho_\infty}\, dy. \qquad (6.28)$$

Upon solution of equation (6.25) it is possible in practice to assume $\frac{Re_t^{**}}{Re^{**}} H_t' = H_t = H_0$ during isothermal flow, i.e., 1.3.

Then

$$H = \frac{n+1}{1-U^2}\left[U^2 + 1{,}3\left(1 - \frac{T_{ст}^* - T_{ст}}{T_0^*}\right)\right]; \qquad (6.29)$$

$$Re^{**} = \frac{Re_\infty\left[(1-U^2)^{\frac{1}{k-1}} U - (1-U_1^2)^{\frac{1}{k-1}} U_1\right](1-U^2)}{4\left[U^2 + 1{,}3\left(1 - \frac{T_{ст}^* - T_{ст}}{T_0^*}\right)\right]}. \qquad (6.30)$$

The momentum equation will be rewritten thus:

$$\frac{dRe^{**}}{dX} + \frac{Re^{**}}{U(1-U^2)}\left[1 + 1{,}3\left(1 - \frac{(T_{ст}^* - T_{ст})}{T_0^*}\right)\right]\frac{dU}{dX} =$$

$$= \frac{B}{2}\eta^* \left(\frac{\mu_*}{\mu_\infty Re^{**}}\right)^m U(1-U^2)^{\frac{1}{k-1}} Re_\infty. \qquad (6.31)$$

Putting in this equation Re^{**} from equation (6.30), we obtain

$$\left[\frac{U(1-U^2)^{\frac{1}{k-1}} - U_1(1-U_1^2)^{\frac{1}{k-1}}}{1{,}69\left(1 - \frac{T_{ст}^* - T_{ст}}{T_0^*}\right)}\right]^m \left\{\left[1{,}3\left(1 - \frac{T_{ст}^* - T_{ст}}{T_0^*}\right) + \right.\right.$$

$$\left. - U^2\right] \cdot \left[(1-U^2)^{\frac{k}{k-1}} - \frac{2k}{k-1} U^2 (1-U^2)^{\frac{1}{k-1}} + \right.$$

$$+ 2UU_1(1-U_1^2)^{\frac{1}{k-1}}\Big] - \Big[(1-U^2)^{\frac{k}{k-1}} U -$$

$$\left. - (1-U^2) U_1 (1-U_1^2)^{\frac{1}{k-1}}\right] 2U\Big\} +$$

$$+\frac{\left[U(1-U^2)^{\frac{1}{k-1}}-U_1(1-U_1^2)^{\frac{1}{k-1}}\right]^{1+m}(1-U^2)^m}{U\left[1{,}3\left(1-T_{cr}^*-T_{cr}\right)+U^2\right]^{1+m}}\times$$

$$\times\left[1-1{,}3\left(1-\frac{T_{cr}^*-T_{cr}}{T_0^*}\right)\right]dU=$$

$$=2B\varphi\left(\frac{4\mu_0}{\rho_0 w_{0m}}\right)^m U(1-U^2)^{\frac{1}{k-1}}dX. \qquad (6.32)$$

For a given law of change of the wall temperature and known velocities and stagnation parameters at the inlet in the channel, equation (6.32) allows us to determine the law of the velocity change in the undisturbed nucleus of the flow along the length of the pipe X.

From equation (6.30) the change of Re^{**} along the length of the pipe is calculated and by formula (5.77) the local values of the coefficients of friction are determined.

The values of the Nusselt number are determined by formula

$$Nu=\frac{c_f}{2}Pr^{0.4}Re. \qquad (6.33)$$

It is necessary to note that the presented method of calculation can be applied only if impactless entrance of a gas into a cylindrical pipe takes place. For the region of subsonic gas velocities it is possible to consider $U^2 \ll 1$. In this case we have T_{wall} = const

$$\left\{\left[1+1{,}3\left(1-\frac{T_{cr}^*-T_{cr}}{T_0^*}\right)\right](1+m)+1\right\}\int_1^{\tilde{w}_0}\frac{(\tilde{w}_0-1)^m}{\tilde{w}_0}d\tilde{w}_0-$$

$$-\left[1+1{,}3\left(1-\frac{T_{cr}^*-T_{cr}}{T_0^*}\right)\right]\frac{(\tilde{w}_0-1)^{1+m}}{\tilde{w}_0}=$$

$$=2B\varphi\left(\frac{4}{U_1 Re_{1w}}\right)^m\left[1+1{,}3\left(1-\frac{T_{cr}^*-T_{cr}}{T_0^*}\right)\right]^{1+m}X, \qquad (6.34)$$

where

$$\tilde{w}_0=\frac{w_0}{w_{0a}}.$$

Assuming m = 0.25, B = 0.0258, and taking the limiting value of the function Ψ_t for subsonic flow, we obtain

$$\left[(1+1.3\psi)\frac{\delta}{4}+1\right]\left[4(\tilde{w}_0-1)^{0.25}-\frac{1}{\sqrt{2}}\ln\times\right.$$
$$\times\frac{(\tilde{w}_0-1)^{0.5}+\sqrt{2}(\tilde{w}_0-1)^{0.25}+1}{(\tilde{w}_0-1)^{0.5}-\sqrt{2}(\tilde{w}_0-1)^{0.25}+1}-\sqrt{2}\operatorname{arctg}\times$$
$$\left.\times\frac{\sqrt{2}(\tilde{w}_0-1)^{0.25}}{1-(\tilde{w}_0-1)^{0.50}}\right]-(1+1.3\psi)\frac{(\tilde{w}_0-1)^{1.25}}{\tilde{w}_0}=$$
$$=\frac{0.4\psi^{1.25}X}{(\sqrt{\psi}+1)^3 Re_1^{0.25}}. \qquad (6.35)$$

where

$$Re_1=\frac{w_{in}D}{\nu_0}.$$

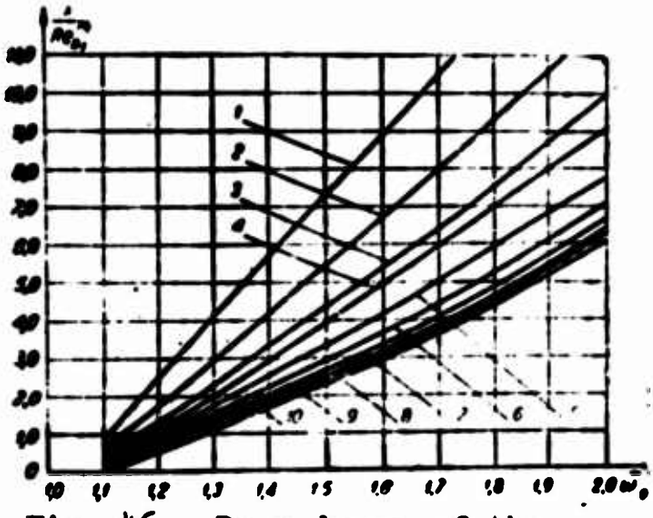


Fig. 46. Dependence of the parameter $X\cdot Re_{D_1}^{-0.25}$ on $\tilde{w}_0$ and ψ by equation (6.35):
1) $\psi = 0.4$; 2) $\psi = 0.6$;
3) $\psi = 0.8$; 4) $\psi = 1.0$;
5) $\psi = 1.4$; 6) $\psi = 1.8$;
7) $\psi = 2.2$; 8) $\psi = 2.6$;
9) $\psi = 3.0$; 10) $\psi = 4.0$.

In Fig. 46 is shown the dependence of the parameter $X\cdot Re_{D_1}^{-0.25}$ on $\tilde{w}_0$ and ψ, calculated by equation (6.35).

Using this graph, we can determine the velocity change in the nucleus of flow along the length of the pipe for given parameters of the gas at the pipe inlet and for a given wall temperature.

For subsonic velocities from equation (6.30) we have

$$Re^{*\psi}=\frac{Re_1(\tilde{w}_0-1)}{5.2\psi}. \qquad (6.36)$$

The local coefficients of friction are determined form the law of resistance:

$$c_f=\frac{0.0516}{(\sqrt{\psi}+1)^2 Re^{*0.25}}. \qquad (6.37)$$

The local coefficients of heat transfer are calculated by formula (6.33). Using equation (6.35) from equation (5.40), generalized for nonisothermal flow, we obtain the value

$$\mathrm{Re}_t^{**} = \frac{\mathrm{Re}_1(\tilde{\omega}_0 - 1)}{5.2\psi}\left\{(2 + 1.3\psi) - \frac{1.25 + 1.62\psi}{(\tilde{\omega}_0 - 1)^{1.25}} \times \right.$$
$$\times\left[4(\tilde{\omega}_0 - 1)^{0.25} - \sqrt{2}\arctg\frac{\sqrt{2}(\tilde{\omega}_0 - 1)^{0.25}}{1 - (\tilde{\omega}_0 - 1)^{0.50}} - \right.$$
$$\left.\left. - \frac{1}{\sqrt{2}}\ln\frac{(\tilde{\omega}_0 - 1)^{0.50} + \sqrt{2}(\tilde{\omega}_0 - 1)^{0.25} + 1}{(\tilde{\omega}_0 - 1)^{0.50} - \sqrt{2}(\tilde{\omega}_0 - 1)^{0.25} + 1}\right]\right\}^{0.8}. \qquad (6.38)$$

Thus, during flow of a gas ($\mathrm{Pr} \approx 1$) in the initial section of the pipe we have

$$\frac{q_w^{**}}{\vartheta^{**}} = \left\{(2 + 1.3\psi) - \frac{1.25 + 1.62\psi}{(\tilde{\omega}_0 - 1)^{1.25}}\left[4(\tilde{\omega}_0 - 1)^{0.25} - \right.\right.$$
$$- \frac{1}{\sqrt{2}}\ln\frac{(\tilde{\omega}_0 - 1)^{0.5} + \sqrt{2}(\tilde{\omega}_0 - 1)^{0.25} + 1}{(\tilde{\omega}_0 - 1)^{0.5} - \sqrt{2}(\tilde{\omega}_0 - 1)^{0.25} + 1} -$$
$$\left.\left. - \sqrt{2}\arctg\frac{\sqrt{2}(\tilde{\omega}_0 - 1)^{0.25}}{1 - (\tilde{\omega}_0 - 1)^{0.5}}\right]\right\}^{0.8}. \qquad (6.39)$$

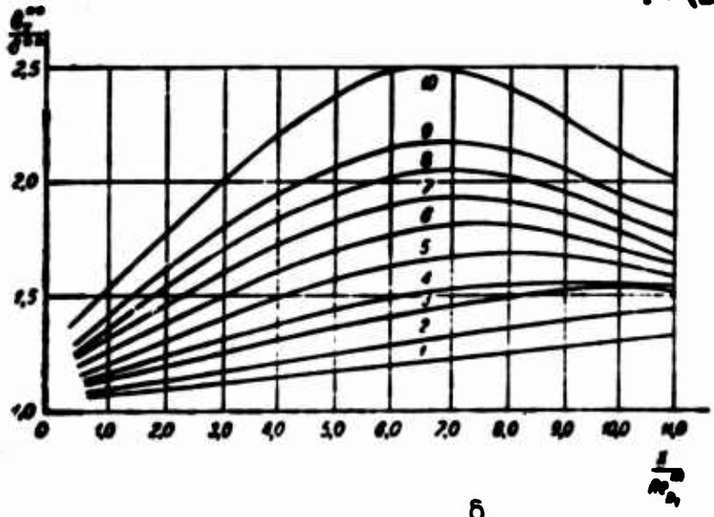


Fig. 47. Dependence of $\frac{\delta_t}{\delta^{**}}$ on $X \cdot \mathrm{Re}_{D_1}^{-0.25}$ and ψ by formula (6.39):
1) $\psi = 0.4$; 2) $\psi = 0.6$; 3) $\psi = 0.8$; 4) $\psi = 1.0$; 5) $\psi = 1.4$; 6) $\psi = 1.8$; 7) $\psi = 2.2$; 8) $\psi = 2.6$; 9) $\psi = 3.0$; 10) $\psi = 4.0$.

In Fig. 47, is shown the $\frac{\mathrm{Re}_t^{**}}{\mathrm{Re}^{**}}$ on $X \cdot \mathrm{Re}_{D_1}^{-0.25}$ and ψ. As can be seen from the graph, during subsonic gas velocities in the initial section of a cylindrical pipe, simultaneous build-up of thermal and dynamic layers $\psi < 1$ and $\mathrm{Re}_t^{**} \approx \mathrm{Re}^{**}$. Taking into account this circumstance it is possible to propose a convenient method of

generalization of the experimental data on heat transfer i.. the initial section of a cylindrical pipe.[1]

From equation (5.39) it follows that

$$Re_1^{**} = \frac{\int_0^x q_{c1}\,dx}{g c_{p\infty} \Delta T}. \tag{6.40}$$

Further we have the relation:[2]

$$St = \frac{q_{c1} D}{Re_{D_2} c_{p\infty} \Delta T}; \tag{6.41}$$

$$Re_D = U_{11} - U^{2,1^{\frac{1}{k-1}-\frac{3}{4}}} \cdot Re_{\infty}; \tag{6.42}$$

$$\frac{\rho}{\rho_\infty} = 1 - U^{2,1^{\frac{k-1}{k+1}}}. \tag{6.43}$$

Thus, by measuring the distribution of the static pressures, the wall temperatures, and the heat flux along the length of the pipe by these equations, it is possible to construct the experimental dependence of St on Re_t^{**} .

If in the experiments measurements of the static pressures are not carried out, then, by taking into account the condition $Re_t^{**} \approx Re^{**}$, it is possible to calculate the Stanton number for subsonic velocities by the formula

$$St \approx \frac{q_{c.n}}{(Re_1 + 5{,}2 \div Re_1^{**}) g c_{p\infty} \Delta T}. \tag{6.44}$$

In Fig. 48 are shown the results of treatment, by the proposed method, of the experiments of B. S. Petukhov, V. L. Lel'chuk, B. V. Dedyakin, V. N. Fedorov, A. I. Leoont'ev, I. A. Kozhinov, and S. I. Kosterin. All these data are reduced to the conditions $\psi = 1$ by the limiting formula (3.55). As can be seen, the average line, drawn through all the experimental points, is described by formula

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[1]Method was developed jointly with V. K. Fedorov.

[2]During derivation of formula (6.42) it is assumed that $\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{3/4}$

$$St_0 = \frac{0.014}{Pr^{0.75} Re_t^{0.25}}. \qquad (6.45)$$

which coincides with the relation

$$St_0 = \frac{c_{f0}}{2 Pr^{0.75}}. \qquad (6.46)$$

where c_{f0} is calculated by formula (3.8) upon substitution in it of Re_t^{**}.

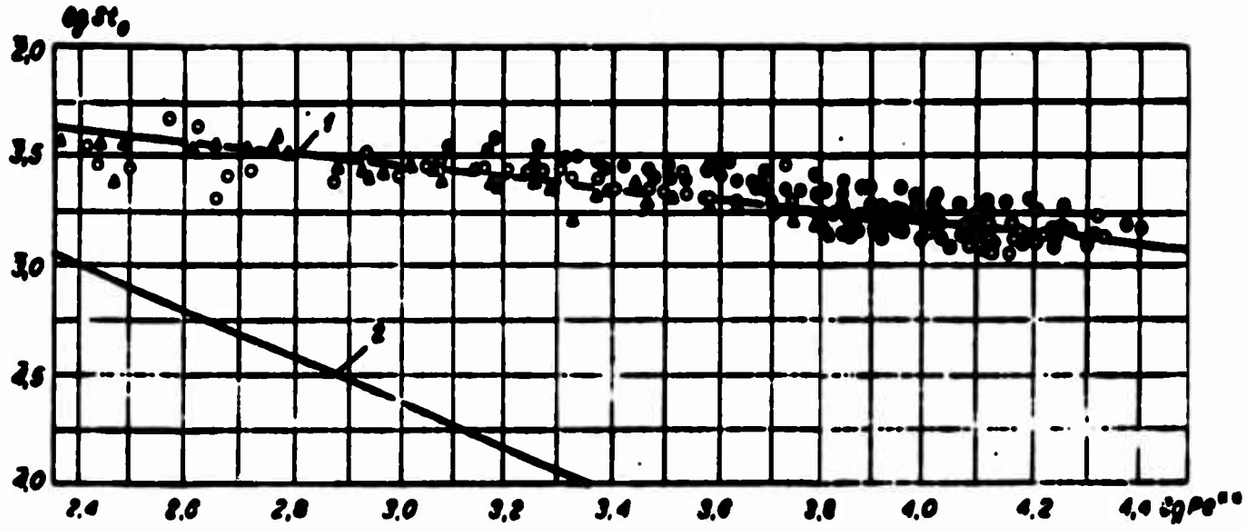

Fig. 48. Results of treatment of the experimental data on heat transfer in the initial section of a cylindrical pipe: 1 – turbulent conditions $St = \frac{0.0143}{Re^{0.25} Pr^{0.5}}$; 2 – laminar conditions $St = \frac{0.22}{Re\ Pr^{1/3}}$; ● – experiments of B. V. Dedyakin and V. L. Lel'chuk; ▲ – experiments of B. S. Petukhov; O – experiments of I. A. Kozhinov, S. I. Kosterin, Leont'ev, and V. N. Fedorov.

It is necessary to emphasize that these experimental data, presented by their authors in the usual dimensionless number treatment, diverge noticeably.

The obtained result distinctly confirms the generality of the laws of heat transfer and friction in a turbulent boundary layer for

the internal and exterior tasks of aerodynamics.

In Fig. 49 are shown the results of treatment, by the proposed method, of the experimental data on heat transfer for pipes, nozzles, plates, and nose cones. These experiments embrace a wide range of changes of the numbers M and $\Delta\psi$. In spite of a significiant variance of the points, all of them are grouped around a line, corresponding to formula (6.46).

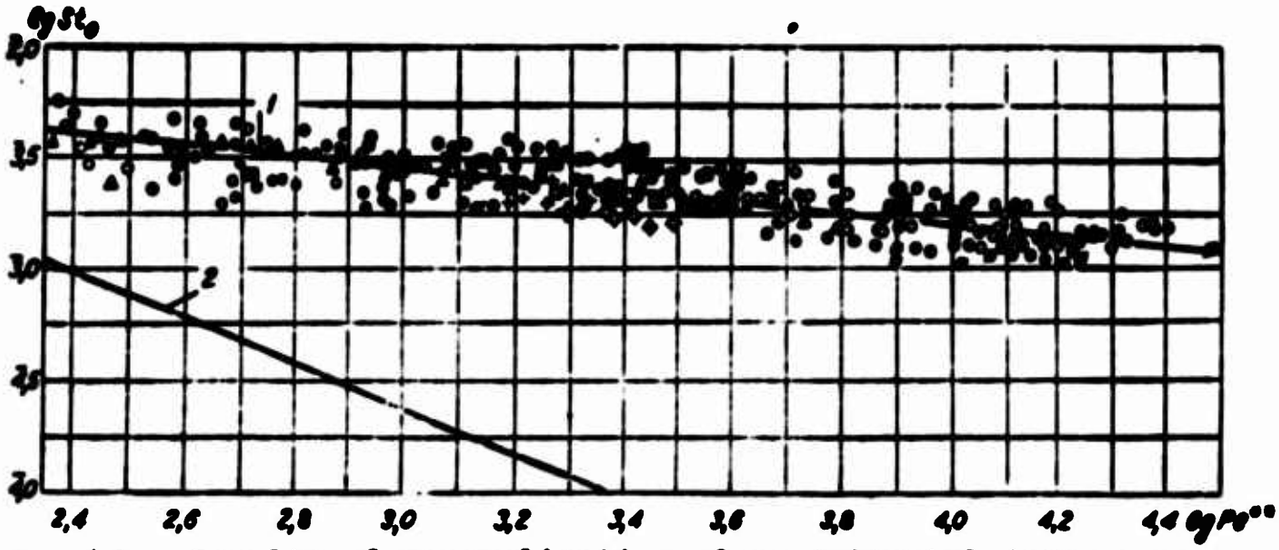

Fig. 49. Results of generalization of experimental data on convective heat transfer in a turbulent boundary layer of compressible gas; 1 — turbulent conditions; 2 — laminar conditions; ◘ ■ ◑ ⊖ — V. K. Fedorov, (pipe), $St = \frac{0.0143}{Pe^{0.25} \cdot Pr^{0.5}}$, ● — Pappas (plates); ⌀ — Fisher and Norris (V-2 rocket); ⌖ — Eber (cone-cylinder); ◐ — Petukhov (plate); ○ — Fedorov (pipe M < 1); $St = \frac{0.22}{Pe\ Pr^{1/2}}$, □ — Bradfield (cone); ◑ — Leont'ev (nozzle); ▲ — Petukhov (pipe); ⊗ — Sveshnikov (pipe); + — Fallis (plate); ● — Lel'chuk, Dedyakin (pipe).

The equation of a thermal boundary layer can be written in:

$$\frac{dPe^{**}}{dX} + Pe^{**} \cdot \frac{d}{dX}\left[\ln\left(1 - \frac{\psi}{2}\right)\right] = St \cdot Pe_D, \qquad (6.47)$$

where

$$Pe^{**} = \frac{w_0 \xi_T^{**}}{a_0}, \quad Pe_D = \frac{w_0 D}{a_0}.$$

or taking into account equations (6.20), (6.22)

$$\frac{d\,Pe^{**}}{dX} + Pe^{**} \cdot \frac{d}{dX}\left[\ln(1-\psi)\right] = \\ = St_0 \Psi_*\,(Pe_{D_1} + 5{,}2 \cdot \psi\, Pe^{**}).$$

where

$$Pe_{D_1} = \frac{w_0 \cdot D}{a_0}.$$

For the case ψ = const

$$\frac{d\,Pe^{**}}{dX} = \frac{B\,(Pe_{D_1} + 5{,}2\,\psi \cdot Pe^{**})}{Pe^{**m} \cdot Pr^{0.5}} \cdot \Psi_*. \qquad (6.48)$$

For subsonic velocities of a gas (and m = 0.25) after integration we obtain

$$X = \frac{Pr^{0.5}}{\left(\frac{2}{1+\sqrt{\psi}}+1\right)^2 B}\left\{\frac{4 \cdot Pe^{**0.25}}{5{,}2\psi} - \frac{Pe_{D_1}^{0.25}}{\sqrt{2}\,(5{,}2\psi)^{1.25}} \times \right. \\ \times \left[\ln \frac{Pe^{**0.5} + 1{,}\sqrt{2} \cdot \left(\frac{Pe_{D_1} \cdot Pe^{**}}{5{,}2\psi}\right)^{0.25} + \left(\frac{Pe_{D_1}}{5{,}2\psi}\right)^{0.5}}{Pe^{**0.5} - 1{,}\sqrt{2} \cdot \left(\frac{Pe_{D_1} \cdot Pe^{**}}{5{,}2\psi}\right)^{0.25} + \left(\frac{Pe_{D_1}}{5{,}2\psi}\right)^{0.5}} + \right. \\ \left. \left. + 2 \operatorname{arc\,tg} \frac{1{,}\sqrt{2} \cdot \left(\frac{Pe_{D_1} \cdot Pe^{**}}{5{,}2\psi}\right)^{0.25}}{\left(\frac{Pe_{D_1}}{5{,}2\psi}\right)^{0.5} - Pe^{**0.5}}\right]\right\}. \qquad (6.49)$$

From equation (6.49) we obtain a change of the parameter Pe^{**} along the length of a pipe for given parameters of a gas at the inlet of a pipe and for a given wall temperature.

Local values of the Nusselt number and heat flux are determined by equations,

$$Nu = \frac{B\left(\frac{2}{1+\sqrt{\psi}}+1\right)^2}{Pe^{**0.25} \cdot Pr^{0.5}}\,(Pe_{D_1} + 5{,}2\,\psi \cdot Pe^{**}) \qquad (6.50)$$

and

$$q_{cm} = \frac{B \cdot \left(\frac{2}{1+\psi+1}\right)^2 \cdot T_0' (1-\psi) \lambda_m}{Pe^{**0.25} \cdot Pr^{0.5} \cdot D} \times \quad (6.51)$$

$$\times (Pe_{D_1} + 5.2\psi \cdot Pe^{**}).$$

For the case of a given constant thermal load along the length of a pipe from equation (6.47) we obtain

$$Pe^{**} = \frac{Nu_1 X}{(1-\psi)}. \quad (6.52)$$

where

$$Nu_1 = \frac{q_{cr} \cdot l}{\lambda_m \cdot T_m}.$$

In the range of a change of ψ from 0 to 3.0 the function ψ_t can be expanded in a series and can be bounded by the first term, i.e., assume (3.49):

$$\psi_t' = \frac{2}{1+\psi}.$$

Then

$$St = \frac{2R}{Pe^{**0.25}(1+\psi)Pr^{0.5}} = \frac{Nu_1}{Pe_D(1+\psi)}. \quad (6.53)$$

From equations (6.20) and (6.22) it follows that

$$Pe_D = Pe_{D_1} + 5{,}2\psi \cdot Pe^{**}. \quad (6.54)$$

As a result we have a system of three equations (6.52), (6.53), and (6.54) with three unknowns ψ, Pe^{**} and Pe_D.

For determination of the change of Pe^{**} along the length of the pipe, from equations (6.52), (6.53), and (6.54) we have

$$X = 3{,}36\, Nu_1^{-1}\left[Nu_1 Pe^{**0.25} Pr^{0.5} - 0{,}0286\, Pe_{D_1} - 0{,}149\, Pe^{**} - \right.$$

$$\left. - \sqrt{(0{,}0286\, Pe_{D_1} + 0{,}149\, Pe^{**} - Nu_1 Pe^{**} Pr^{0.5})^2 - 1{,}19\, Nu_1 Pe^{**1.25} Pr^{0.5}}\right]. \quad (6.55)$$

Knowing local the values of Re^{**}, from equation (6.52) we determine the local values of the wall temperature:

$$\varphi = 1 + \frac{Nu_1}{Re^{**}} X.$$

The proposed method of calculation can be also be extended to the case of an arbitrary law of supply or removal of heat along the length of the channel. Only in this case instead of equation (6.52) we should use equation (6.47). Further considerations remain the same.

6.4. Heat Transfer to a Vapor During Critical Pressures

The theory of the limiting laws of friction and heat transfer can be applied also to flows of gases, not obeying the Clapeyron-Mendeleev equation of state. The problem is solved most simply for the bounded temperature integrals in that case, when there is possible a linear approximation of the dependence of the gas density on the temperature (or enthalpy).

In this case for a uniform medium

$$\frac{\rho_0}{\rho} = \varphi_1 - (\varphi_1 - 1)\theta. \qquad (6.56)$$

where, as in Chapter IV,

$$\varphi_1 = \frac{\rho_0}{\rho_{cr}}.$$

The limiting law of heat transfer at $c_p \approx$ const for subsonic flow is expressed by formula

$$\frac{St}{St_0} = \left(\frac{2}{\sqrt{\varphi_1 + 1}}\right)^2. \qquad (6.57)$$

For an ideal gas

$$\varphi_1 = \varphi = \frac{T_{cr}}{T_0}.$$

In more complicated cases, for instance during a significant change of the heat capacity, it is possible to obtain if not exact quantitative, then in any case correct qualitative results.

Let us consider in this plan the character of the changes of the coefficient of heat transfer during flow of a vapor in the region of near critical (in a thermodynamic sense) parameters.

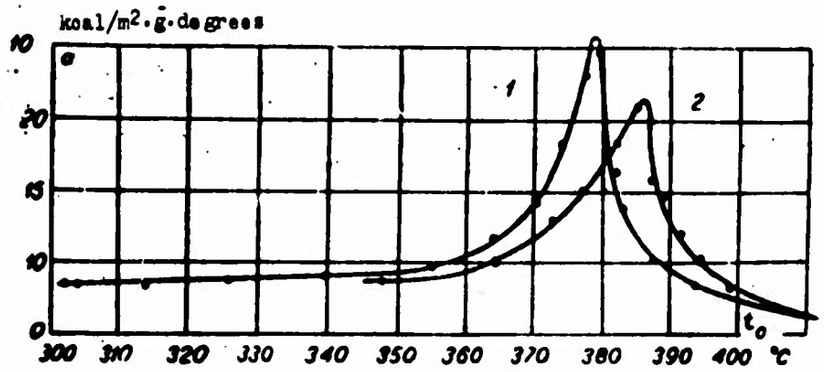

Fig. 50. Change of coefficient of heat transfer during flow of water and steam in the region of critical parameters from the experiments of A. A. Armand et al.: 1 — P = 240 atm; 2 — P = 260 atm; q = 1.7 · 10^5 kcal/m2 · hr; γw = 650 kg/m2 · sec.

In Fig. 50 are shown the results of the experiments of A. A. Armand on the determination of the coefficients of heat transfer during flow of steam having supercritical parameters in a cylindrical pipe. As can be seen, during passage through the critical temperature there is observed a sharply expressed maximum of the coefficient of heat transfer. Such a course of the curve $\alpha(T)$ is analogous to the course of the dependence $c_p(T)$ in the region of critical parameters.

In Fig. 51 is shown a comparison of the results of the experiments of Z. L. Miropolskiy, M. A. Styrikovich, and M. E. Shitsman with calculations by formula (6.13). No unambiguous relation between the formula and the experimental data on the graph is observed. An analogous result was also obtained in the above-mentioned work of

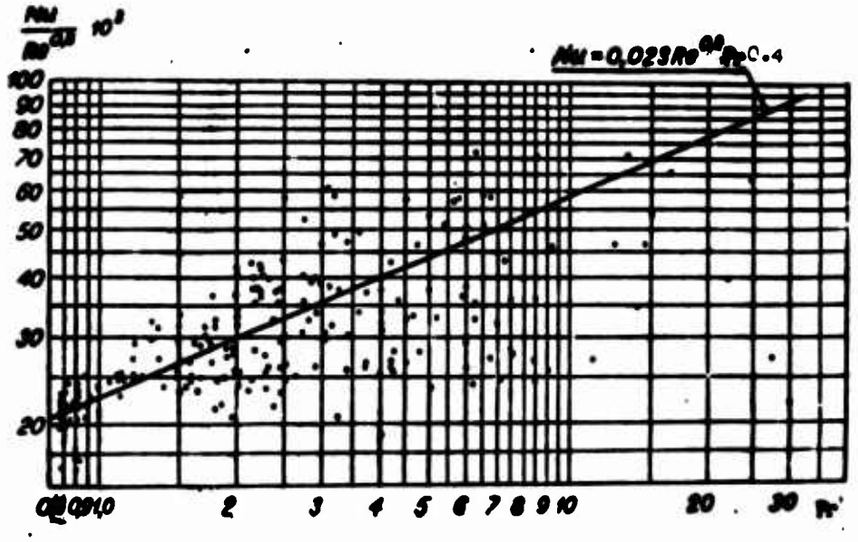

Fig. 51. Comparison of experiments of Z. L. Miropolskiy, M. A. Styrikovich, and M. E. Shitsman with Formula (6.13).

A. A. Armand. At the same time, according to the experiments of V. E. Doroshchuk and V. L. Lel'chuk, at $\psi \approx 1$ formula (6.13) is also valid in the region of supercritical parameters (Fig. 52). Consequently, deviation of the experiments with calculations by formula (6.19) are connected with the nonisothermalness of the stream.

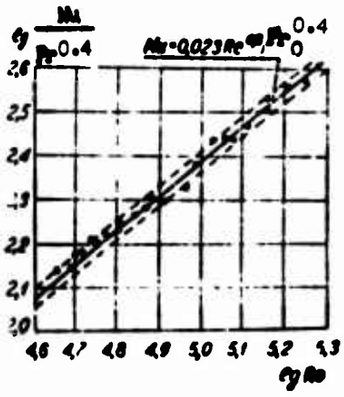

Fig. 52. Heat transfer for critical parameters and quasi-isothermal conditions from the experiments of V. E. Doroshchuk and V. L. Lel'chuk.

During simultaneous change of the quantities c_p and ρ the intensity of turbulent heat transfer will be determined by the relation

$$q = \rho l^2 \frac{dw}{dy} \cdot \frac{di}{dy}, \qquad (6.58)$$

where i is the specific enthalpy.

The limiting relative law of heat transfer is derived from equation (6.58) by the already known method:

$$\frac{St}{St_\infty} = \left(\int_0^1 \sqrt{\frac{\rho}{\rho_0}} d\theta \right) \qquad (6.59)$$

But in distinction from equation (2.69) in the given equation

$$\left.\begin{aligned} \vartheta &= \frac{t_{cr} - t}{t_{cr} - t_0}; \\ St &= \frac{q_{cr}}{\varepsilon h_0 w_0 (t_{cr} - t_0)}. \end{aligned}\right\} \qquad (6.60)$$

In the region of near critical parameters the specific volume of the vapor is almost a linear function of the enthalpy, i.e., for definite intervals of Δi formula (6.47) can be applied, if in it we introduce the value of ϑ from (6.60). Then the integral (6.59) is expressed by formula (6.57), if we compare the heat flux q_{wall}. We have

$$\frac{St}{St_0} = \frac{q_{cr}}{q_{cr0}} = \left(\frac{2}{\sqrt{\psi_1} + 1}\right)^2. \qquad (6.61)$$

Multiplying and dividing the left part of formula (6.61) by Δt, we find that it is applicable to the ratio of the coefficients of heat transfer.

In Figs. 53 and 54 are given the results of the treatment of the experimental data on heat transfer to vapors of water, carbon dioxide, and oxygen in the region of near critical parameters. The logarithmic straight line. drawn on this graph, is calculated by the formula

$$Nu_0 = 0{,}023\, Pr_0^{0.4} Re_0^{0.8} \left(\frac{2}{\sqrt{\psi_1} + 1}\right)^2. \qquad (6.62)$$

Physical properties, entering into the criterions Nu_0, Pr_0 and Re are referred to the average stream temperature by the best content.

It is possible to ascertain that the introduction of a correction by formula (6.61), to a significant degree, approaches the experimental

data with the formula for quasi-isothermal conditions. Thus, also in this case the influence of the nonisothermalness basically is expressed via a change of the vapor density along the cross section of the turbulent stream.

V. V. Krasnoshchekov and A. I. Protopopov proposed the empirical dependence

$$\frac{St}{St_0} = \left(\frac{\mu_0}{\mu_{ст}}\right)^{0,11}\left(\frac{\lambda_{ст}}{\lambda_0}\right)^{0,31}\left(\frac{c_{p,x}}{c_{p,0}}\right)^{0,35}, \qquad (6.63)$$

$$c_{p,x} = \frac{i_{ст}^{*} - i_0}{T_{ст} - T_0},$$

where

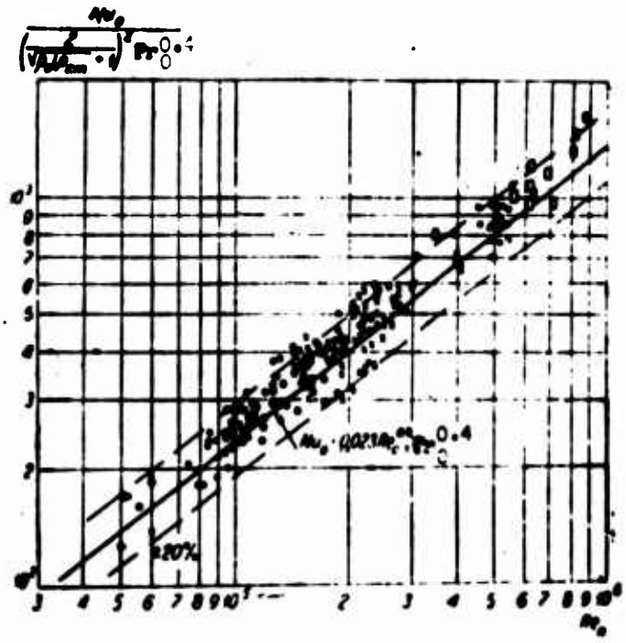

Fig. 53. Comparison of experimental data on convective heat transfer in the region of critical parameters with the limiting formula (6.62), ○ – experiments of Z. L. Miropolskyi and M. E. Shitsman (H_2O); ○ – experiments of A. A. Armand (H_2O); ● – experiments of Bringer (CO_2); □ – experiments of Dickinson (H_2O); × – experiments of V. S. Protopopov (CO_2).

In the region of the near critical state of the vapor the first of these factors is practically equal to 1.

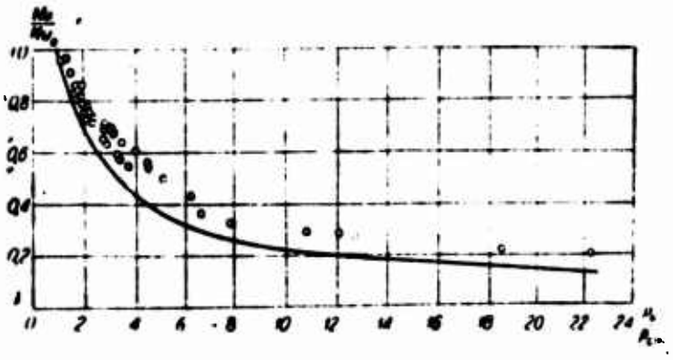

Fig. 54. Comparison of experimental data on convective heat transfer to oxygen in the region of critical parameters with the limiting formula (6.53). -- by formula (6.52); O — experiments of Powell.

Formula (6.63) gives qualitatively the same relation between the change of the coefficient of heat transfer and the temperature difference, as the limiting formula (6.62). It is possible to note in connection with this, that during criterial treatment it is expedient to introduce the ratio $\frac{\rho_{wall}}{\rho_0}$.

Naturally, the given analysis cannot pretend to a full reflection of the whole variety of factors, determining such a complicated process, as heat transfer for near critical parameters of the vapor. However, from data presented it is clear that at least one of the important sides of this phenomenon is explained by the theory of the limiting relative laws of heat transfer in a gaseous medium.

CONCLUSION

Definitions of Cyrillic Items in Order of Appearance

cT = wall = wall

T = t = Thermal, Turbulence

Kp = cr = critical

The proposed theory of the limiting relative laws of friction and heat transfer in a turbulent boundary layer of gas is based on two known theoretical conditions, which express the relation between turbulent friction and the averaged motion in a flat boundary layer in the form of equations (2.2) and (2.4).

The problem of the influence on the coefficient of friction of the nonisothermalness, compressibility, diffusion, and pressure gradient is reduced to the integral

$$Z = \int_{m_1}^{1} \frac{dm}{\sqrt{\psi \cdot \frac{\rho_1}{\rho} \cdot \frac{\tilde{\tau}}{\tilde{\tau}_0}}},$$

where

$$\psi = \left(\frac{c_f}{c_{f_0}}\right)_{Re^{**}}.$$

An analogous equation is also obtained for the relation

$$\psi_q = \left(\frac{St}{St_0}\right)_{Re_T^{**}}.$$

This integral at $Re \to \infty$ in a number of cases has a limiting expression of the form

$$\int_0^1 \frac{d\bar{m}}{\sqrt{\bar{\psi}\frac{\rho_0}{\rho}\cdot\frac{\bar{\zeta}}{\bar{\zeta}_0}}} = 1. \qquad (2.30)$$

Distribution of tangential stresses along the cross section of the boundary layer $\tau = \frac{\tau}{\tau_{wall}}$ is approximated by a polynomial, the coefficients of which are calculated from the boundary conditions, ensuing from the determination of the boundary layer.

The relation between the gas density and the velocity profile is established via the equation of state and coefficient of nonsimilarity of the stagnation temperature and velocity fields.

Thus, problem is reduced to an analytic investigation of the influence of the nonisothermalness, mass transfer, and pressure gradient on the relative changes of the coefficients of friction and heat transfer.

Values of the coefficients c_{f0} and St_0 can be determined on the basis of the available reliable experimental data for isothermal streamining of a plate.

For streamlining of an impenetrable plate by an unbounded stream of ideal gas an exact solution of the limiting equation (2.30), expressed by formula (3.47) for subsonic and by formula (3.48) for supersonic flows is obtained. This solution does not contain empirical coefficients (including "constants of turbulence") and is not connected with any sort of special type of semi-empirical theories of turbulence.

The weak dependency of the ratios $\frac{c_f}{c_{f0}}$ and $\frac{St}{St_0}$ on the Re^{**} and Re_t^{**} numbers allowed us to extend the limiting laws to flows with finite values of the Reynolds numbers.

For an impenetrable plate in practice a first approximation expressed by formula (3.58) appears to be sufficient.

In the case of a permeable surface the influence of the Re number on the relative laws of resistance and heat transfer is still less, due to perturbations, introduced in the near-wall region by a transverse stream of substance. Qualitatively correct and quantitatively satisfactory theoretical formulas are obtained in an approximation, expressed by the equation (for a plate)

$$\int_0^{\psi} \frac{d\psi}{\sqrt{(1+b\psi)\frac{b}{2}}} = 1,$$

where $b = \dfrac{2\rho_{wall}\, w_{wall}}{c_{f0}\, \rho_0\, w_0}$ is the wall permeability factor.

At $\psi = 0$ this integral gives the critical value of the factor b.

It is shown, that for subsonic flows of a gas the exact limiting solutions are approximated very well by formula

$$\psi = 4\left(\frac{1-\frac{b}{b_{cr}}}{\sqrt{\psi}+1}\right)^2. \qquad (4.30)$$

For values of b_{cr} solutions are given that take into account the influence of nonisothermalness and heterogeneity of the boundary layer.

For flows with significant positive pressure gradients it appeared to be possible theoretically to determine the critical parameters at the point of breakaway of the boundary layer from an impenetrable wall. It is shown that breaking away is characterized by a critical value of the form parameter f, where by the quantity f_{cr} is a weak function of the Re^{**} number. Dependencies of the critical parameters of breakaway on the temperature factor and the M number are obtained.

In connection with the fact that the critical value of the form parameter f remains practically constant, but not the form parameter

$\Gamma = \frac{2f}{c_{f0}}$, as was assumed previously, a new method of solution of the momentum equation is proposed. General solutions of the equations of energy and momentum for flat and axisymmetric boundary layers on a impenetrable wall are taking into account the nonisothermalness, compressibility, and pressure gradient.

For permeable walls solutions of the equations of energy and momentum for relatively small values of the form parameter f are given. The relative influence of nonisothermalness and mass transfer on friction and heat transfer in the turbulent flow of gas is connected with a change of the density in the nucleus of the boundary layer and minutely depends on the molar viscosity and thermal conductivity.

Thus, establishment of the fact of the existence of the limiting relative laws of friction and heat transfer for a turbulent gas stream allowed us to give a certain, logically consecutive method of research of the considered problem.

A P P E N D I C E S

A P P E N D I X I

SUMMARY OF BASIC CALCULATION FORMULAS

I.I. Streamlining of an Impenetrable Plate

Coefficient of friction during isothermal flow[1]

$$c_{f_0} = \frac{2}{(2{,}5 \ln \mathrm{Re}^{**} + 3{,}8)^2}. \qquad (I.1)$$

Coefficient of heat transfer during quasi-isothermal flow[1]

$$St_0 = \frac{2 Pr^{-0.75}}{\left(2{,}5 \ln \mathrm{Re}_\tau^{\infty} + 3{,}8\right)^2}. \qquad (I.2)$$

Limiting relative laws of resistance and heat transfer ($\varepsilon \approx 1$ is assumed):

a) during subsonic flow

$$\Psi = \frac{4}{\left(\sqrt{\psi} + 1\right)^2}; \qquad (I.3)$$

b) during supersonic flow

$$\Psi = \frac{1}{\psi^0 - 1}\left[\arcsin \frac{2(\psi^0 - 1) + \Delta\psi}{\sqrt{4(\psi^0 - 1)(\psi^0 + \Delta\psi) + (\Delta\psi)^2}} - \arcsin \frac{\Delta\psi}{\sqrt{4(\psi^0 - 1)(\psi^0 + \Delta\psi) + (\Delta\psi)^2}}\right]. \qquad (I.4)$$

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[1]During use of the exponential approximation,

$$c_{f_0} = B\,\mathrm{Re}^{**-m} \text{ and } St_0 = \frac{B}{2} Pr^{-0.75}\,\mathrm{Re}_\tau^{\infty - m},$$

where the coefficients B and m are taken from Table 3.1.

Relative laws of resistance and heat transfer taking into account the finiteness of the Re number ($\varepsilon \approx 1$):

a) during subsonic flow

$$\psi = \frac{1}{\left[\sqrt{\psi - 8{,}2(\psi - 1)\sqrt{c_{f_0}} + 1}\right]^2}; \tag{I.5}$$

b) during supersonic flow

$$\psi = \frac{1}{(\psi^* - 1)\left(1 - 8{,}2\sqrt{c_{f_0}}\right)^2} \times \left[\arcsin \frac{2(\psi^* - 1) + \Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}} - \right.$$

$$\left. - \arcsin \frac{16{,}4(\psi^* - 1)\sqrt{c_{f_0}} + \Delta\psi}{\sqrt{4(\psi^* - 1)(\psi^* + \Delta\psi) + (\Delta\psi)^2}}\right]^2. \tag{I.6}$$

The relation of the average coefficients of friction and heat transfer at T_{wall} = const and a turbulent layer, starting from the leading edge of the plate,

$$\left(\frac{c_F}{c_{F.0}}\right)_{Re_x} = \left(\frac{c_f}{c_{f_0}}\right)_{Re_x} = \psi^{\frac{1}{1+m}}. \tag{I.7}$$

Average coefficient of friction for an isothermal turbulent layer, developed from the leading edge of the plate

$$c_{F0} = 0{,}455\,(\lg Re_x)^{-2.58}. \tag{I.8}$$

Coefficient of heat transfer for a dynamic turbulent boundary layer, developed from the leading edge of the plate and the initial heat-insulated section with length x_0

$$St = \frac{0{,}0129}{Pr^{0{,}75}\,Re_x^{*0{,}25}}\left(\frac{x - x_0}{x}\right)^{0{,}086}. \tag{I.9}$$

I. II. Streamlining of a Permeable Plate during Subsonic Flow (limiting laws)

Relative law of resistance

$$\psi = 4\left(\frac{1 - \frac{b}{b_{sp}}}{\sqrt{\psi} + 1}\right)^2; \tag{II.1}$$

$$b = \frac{2j_1}{c_{p_0} \rho_0 w_0}. \qquad \text{(II.2)}$$

Critical value of the wall permeability parameter:

a) isothermal, uniform boundary layer

$$b_{cr} = 4; \qquad \text{(II.3)}$$

b) nonisothermal, uniform boundary layer at $\psi < 1$

$$b_{cr} = \frac{1}{1-\psi}\left(\ln \frac{1+\sqrt{1-\psi}}{1-\sqrt{1-\psi}}\right)^2; \qquad \text{(II.4)}$$

c) nonisothermal, uniform boundary layer at $\psi > 1$

$$b_{cr} = \frac{1}{\psi - 1}\left(\arccos \frac{2-\psi}{\psi}\right)^2. \qquad \text{(II.5)}$$

a graph of the function $b_{cr}(\psi)$ is given Fig. 17;

d) isothermal, nonuniform boundary layer at $R' > R_0$ ($\mu < \mu_0$), where μ is the molecular weight of the gas

$$b_{cr} \approx 1 + 3\frac{\mu'}{\mu_0}; \qquad \text{(II.6)}$$

e) isothermal, nonuniform boundary layer at $R' < R_0$ ($\mu' > \mu_0$)

$$b_{cr} \approx 1{,}47 + 2{,}53\frac{\mu'}{\mu_0}; \qquad \text{(II.7)}$$

f) nonisothermal, nonuniform boundary layer

$$b_{cr} \approx 0{,}25\, b_{cr1} + b_{cr2}, \qquad \text{(II.8)}$$

where b_{cr1} and b_{cr2} are calculated, respectively, by formulas (II.4), (II.5), and (II.6).

The relative law of heat transfer has the same form as (II.1), but instead of the parameter b the thermal parameter of wall

permeability is introduced:

$$b_t = \frac{c_{p1} f_1}{c_{p\infty} \rho_\infty w_\infty St_\infty}. \qquad \text{(II.9)}$$

Values of $b_{t,\,cr} = b_{cr}$.

I. III. Critical Parameters of Breakaway of the Boundary Layer from an Impenetrable Wall

Critical parameters at point of breakaway of isothermal boundary layer:

$$-f_{tp} \approx 0{,}010; \qquad \text{(III.1)}$$

$$H_{sp} \approx 1{,}87; \qquad \text{(III.2)}$$

$$\left(\frac{\delta^{**}}{l}\right)_{sp} = 0{,}16. \qquad \text{(III.3)}$$

See also Table 5.2.

Critical parameters during nonisothermal, subsonic flow, see Figs. 36, 37, and 38.

Critical value of parameters during supersonic flow, see Figs. 39 and 40.

Change of parameter $H = \frac{\delta^*}{\delta^{**}}$

$$H = 1 + 0{,}87\bar{f}\left(\frac{\sqrt{2x_{fp}}}{0{,}4 - \sqrt{2x_{fp}}}\right)^{1-\bar{f}}. \qquad \text{(III.4)}$$

where

$$\bar{f} = \frac{f}{f_{sp}}.$$

The relative change of the coefficient of friction under the influence of a pressure gradient is determined from Fig. 28.

The St number in the range of values $Re^{**} < 10^5$ can be considered practically independent of the pressure gradient.

I. IV. Integrals of the Energy Equation

Subsonic stream on an impenetrable surface

$$Re_\tau^{**} = \frac{1}{\Delta T}\left[\frac{1+m}{2Pr^{0,75}} B Re_0 \int_{X_1}^{X} \Psi_\tau \tilde{w} \Delta T^{1+m} \cdot dX + (Re_\tau^{**} \Delta T)_{X_1}^{1+m}\right]^{\frac{1}{1+m}}. \quad (IV.1)$$

The quantity ψ_t is calculated by formula (I.3).

Supersonic flow on an impenetrable plate

$$Re_\tau^{**} = \frac{1}{\Delta T}\left[\frac{1+m}{2Pr^{0,75}} B Re_\infty \int_{X_1}^{X} \Psi_\tau \left(\frac{\mu_\infty}{\mu_0}\right)^m U (1-U^2)^{\frac{1}{k-1}} \times \Delta T^{1+m} dX + (Re_\tau^{**} \Delta T)_{X_1}^{1+m}\right]^{\frac{1}{1+m}}. \quad (IV.2)$$

The quantity ψ_t is calculated by formula (I.4).

During supersonic flow

$$\Delta T = T_{c\tau}^{*} - T_{c\tau}. \quad (IV.3)$$

The local value of the Stanton number, determined from the calculated value of Re_t^{**},

$$St = \Psi_\tau \left(\frac{\mu_0}{\mu_\infty}\right)^m St_0. \quad (IV.4)$$

During subsonic flow $\mu_{00} = \mu_0$.

Subsonic flow on a permeable surface

$$Re_\tau^{**} = \frac{1}{\Delta T}\left\{\frac{1+m}{2Pr^{0,75}} B Re_0 \int_{X_1}^{X} \tilde{w}_0 \left[\Psi_\tau \left(1 - \frac{b_\tau}{b_{\tau,w}}\right)^s + b_\tau\right] \Delta T^{1+m} dX + (Re_\tau^{**} \Delta T)_{X_1}^{1+m}\right\}^{\frac{1}{1+m}}. \quad (IV.5)$$

Distribution of supply of cooling gas along the length of the contour (coordinate X)

$$\frac{j_1}{\rho_\infty w_\infty} = \frac{c_{p,w}}{c_{p,1}} b_\tau St_0. \quad (IV.6)$$

Local value of the Stanton number

$$St = \Psi_\tau \left(1 - \frac{b_\tau}{b_{\tau.\,кр}}\right)^2 St_0. \qquad \text{(IV.7)}$$

Local heat flux

$$q_{ст} = g\, c_{p_0} \rho_0 w_0 \Delta T St. \qquad \text{(IV.8)}$$

Initial temperature of the cooling (injected through the wall into the boundary layer) gas

$$T_1 = T_{ст} - \frac{q_{ст}}{g\, c_{p1} f_1}. \qquad \text{(IV.9)}$$

During evaporation and sublimation,

$$q_{ст} = r g j_1; \qquad \text{(IV.10)}$$

$$\frac{St}{St_0} = b_\tau K; \qquad \text{(IV.11)}$$

$$K = \frac{r}{c_{p1} \Delta T}; \qquad \text{(IV.12)}$$

$$b_\tau = b_{\tau,\,кр} - \frac{K b_{\tau.\,кр}^2}{2\Psi_\tau}\left(\sqrt{\frac{4\Psi_\tau}{K b_{\tau.\,кр}} + 1} - 1\right); \qquad \text{(IV.13)}$$

$$Re_\tau^{**} = \frac{1}{\Delta T}\left\{\frac{1+m}{2Pr^{0.75}} B Re_0 \int_{X_1}^{X} \left[b_{\tau,\,кр} - \frac{K b_{\tau.\,кр}^2}{2\Psi_\tau} \times \right.\right.$$

$$\left.\left. \times \left(\sqrt{\frac{4\Psi_1}{K b_{\tau.\,кр}} + 1} - 1\right)\right] \bar{w}_0 \Delta T^{1+m} \cdot dX + \right.$$

$$\left. + (Re_\tau^{**} \Delta T)_{X_1}^{1+m}\right\}^{\frac{1}{1+m}}. \qquad \text{(IV.14)}$$

The quantity Ψ_t is calculated by formula (I.3).

In the case of gas injection and given values of T_1, T_{wall}, and T_0 in formulas (IV.10) – (IV.13) instead of the criterion K is introduced the ration

$$\frac{T_{ст} - T_1}{T_{ст} - T_0}.$$

I.V. Integrals of the Momentum Equation

Subsonic flow on an impenetrable surface

$$Re^{**} = \tilde{w}_0^{-z}\left[\frac{1+m}{2} B Re_0 \int_{X_1}^{X} w^{1+(1+m)z} \Psi_T \, dX + (Re^{**}\tilde{w}^z)_{X_1}^{1+m}\right]^{\frac{1}{1+m}}; \quad (V.1)$$

where

$$z = 1 + H_{cp}.$$

The quantity Ψ_t is determined by formula (I.3). The quantity H_{cr} is determined from the graph of Fig. 37.

Local value of the coefficient of friction, determined from the calculated value of Re^{**},

$$c_f = \Psi_T \cdot \Psi_f \cdot c_{f0}, \quad (V.2)$$

where Ψ_f is determined from Fig. 28.

Value of the form parameter

$$f = \frac{Re^{**}}{Re_0 w_0} \cdot \frac{dw_0}{dx}. \quad (V.3)$$

Critical value form parameter is determined from the graph of Fig. 36.

Supersonic flow on an impenetrable plate

$$Re^{**} = U^{-z}\left[\frac{1+m}{2} B Re_\infty \int_{X_1}^{X} \Psi_T \left(\frac{\mu_w}{\mu_0}\right)^m (1-U^2)^{\frac{1}{k-1}} \times U^{1+(1-m)z} \cdot dX + (Re^{**}U^z)_{X_1}^{1+m}\right]^{\frac{1}{1+m}}. \quad (V.4)$$

The quantity Ψ_t is determined by formula (I.4). The quantity H_{cr} is determined from the graph of Fig. 40.

The value of the form parameter for the calculated value of Re^{**}

$$f = \frac{Re^{**}}{Re_\infty U^2 (1-U^2)^{\frac{1}{k-1}}} \cdot \frac{dU}{dX}. \quad (V.5)$$

The critical value of the form parameter is determined from the graph of Fig. 39.

The local value of the coefficient of friction for Re^{**}, determined by (V.4),

$$c_f = \Psi_\tau \Psi_f \left(\frac{\mu_0}{\mu_\infty}\right)^m c_{f0}. \tag{V.6}$$

Subsonic flow on a permeable surface at small $\tilde{f}$

$$Re^{**} = \tilde{w}_0^{-2.3}\left\{\frac{1+m}{2} B Re_0 \int_{X_1}^{X} \tilde{w}_0^{3.3+2.3m}\left[\Psi_\tau\left(1-\frac{b}{b_{кр}}\right)^8 + b\right] dX + (Re^{**}\tilde{w}_0^{2.3})_{X_1}^{1+m}\right\}^{\frac{1}{1+m}}. \tag{V.7}$$

Local value of the coefficient of friction for Re^{**} by (V.7)

$$c_f = \Psi_\tau \left(1-\frac{b}{b_{кр}}\right)^8 c_{f0}. \tag{V.8}$$

I.VI. Integrals of the Equations of Energy and Momentum for Axisymmetric Flow

During axisymmetric flow the integrals of the energy equation have the same form as in the case of flat flow (see Section I.IV), but instead of the quantity ΔT the product $R\Delta T$ enters, where R is the radius of the surface of the body.

Correspondingly, in the integrals of the momentum equation during subsonic flow instead of the quantiy $\tilde{w}_0^{\chi}$, the quantity $R\,\tilde{w}_0^{\chi}$ is introduced.

Local coefficient of heat transfer in an axisymmetric nozzle

$$Nu_\infty = \frac{B}{2} Pr^{0.25} Re_{0\infty} \Psi_\tau \left(\frac{k-1}{k+1}\right)^{0.5}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \times \left(\frac{D_{кр}}{D}\right)^2 Re_1^{\omega-m}, \tag{VI.1}$$

where

$$Re_\tau^{\infty} = \left[\frac{(1+m)\,B\,Re_{\infty}}{2Pr^{0,75}\,\Delta T^{1+m}}\left(\frac{D_{\tau p}}{D}\right)^{1-m}\left(\frac{k-1}{k+1}\right)^{0.5}\times\right.$$

$$\left.\times\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\cdot\int_{X_1}^{X}\Psi_\tau\,\Delta T^{1+m}\left(\frac{D_{\tau p}}{D}\right)^{1-m}dX\right]^{\frac{1}{1+m}}. \qquad \text{(VI.2)}$$

A P P E N D I X II

EXAMPLES OF CALCULATIONS

A. Heat Transfer in a Supersonic Nozzle

There exists a supersonic nozzle, the geometric dimensions of the flow-through part of which are shown in Fig. 41 and in the first two lines of Table II.1. The parameters of drag of air in the nozzle: $p_{00} = 4.49$; $T_0^* = 5.99.6^\circ K$.

Temperature of the wall $T_{wall} = 347^\circ K$ is constant along the length of the nozzle.

Distribution of the local coefficients of heat transfer along the length of the nozzle and the corresponding distribution of heat fluxes on the wall is found:

$$q = \alpha (T_{cr}^* - T_{cr}).$$

Parameters of air $T = T_0^*$: $Pr_{00} = 0.725$;

$$\mu_{00} = 3{,}12 \cdot 10^{-6} \text{ kg·sec/m}^2; \quad c_p = 0{,}251 \text{ kcal/kg·deg};$$

$$\rho_{00} = \frac{p_{00}}{gRT_0^*} = \frac{4{,}49 \cdot 10^4}{9{,}81 \cdot 29{,}27 \cdot 599{,}6} = 2{,}60 \text{ kg·sec/m}^4;$$

$$\lambda_{00} = 0{,}0381 \text{ kcal/m·hr·deg}.$$

Hence

$$Re_{in} = \frac{0{,}203}{1{,}2 \cdot 10^{-6}} \sqrt{427 \cdot 2 \cdot 9{,}81 \cdot 0{,}251 \cdot 599{,}6} = 1{,}9 \cdot 10^7.$$

Take m = 0.25 and B = 0.0129. Results of further calculations

are given in Table II.1.

B. Cooling of a Porous Plate

A plate of length 1 m is streamlined by a flow of air with a temperature $t_0 = 2000^\circ C$, and is cooled by cold air, passed through the pores of the plate, flow velocity of the hot air $w_0 = 50$ m/sec. It is necessary to determine the specific flow rates of the coolant air, needed to maintain the temperature of the plate constant, equal to $t_{wall} = 500^\circ C$.

The initial temperature of the coolant air is $t_1 = 30^\circ C$ (diagram of problem is given in Fig. II.1).

By formula (II.4) we determine the critical parameter of injection

$$b_{cr} = \frac{1}{1-0.34}\left[\ln\frac{1+\sqrt{0,66}}{1-\sqrt{0,66}}\right]^2 = 7,793;$$

$$\psi = \frac{773}{2273} = 0,34; \qquad \Psi_1 = \left(\frac{2}{\sqrt{0,34}+1}\right)^2 = 1,60.$$

By formula (IV.13) we determine the quantity b_t:

$$b_t = 7,793 - \frac{800-30}{2000-800}\cdot\frac{7,793^2}{1,60}\times$$

$$\times\left(\sqrt{\frac{4\cdot 1,60(2000-500)}{7,793(800-30)}}+1-1\right) = 2,44.$$

By formula (5.94) we determine the parameter Ψ:

$$\Psi = 2,44\cdot\frac{800-30}{2000-500} = 0,764.$$

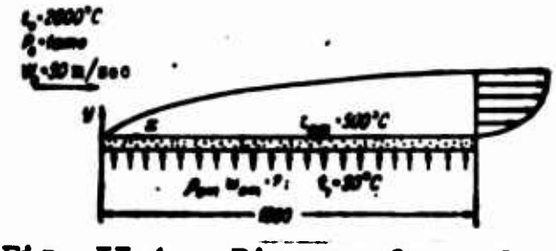

Fig. II.1. Diagram for calculation of the cooling of a porous plate.

Table II.1. Combined Table of Calculation of the Thermal Boundary Layer of a Nozzle

$X = \frac{x}{L}$	0	0,483	0,486	0,492	0,477	0,508	0,536	0,551	0,575	0,600	0,637	0,676	0,725	0,774	0,823	0,876	0,922	0,971	1,00
$\frac{F_{cr}}{F}$	0,1112	0,1112	0,1112	0,1198	0,1285	0,1472	0,1810	0,235	0,445	0,866	1,0	0,782	0,626	0,514	0,430	0,363	0,311	0,271	0,250
φ^*	1,00078	1,00078	1,00078	1,00087	1,00101	1,0013	1,00184	1,00403	1,013	1,0685	1,18	1,478	1,6705	1,835	2,0067	2,1670	2,324	2,472	2,557
φ	0,5804	0,5804	0,58043	0,5805	0,5806	0,5807	0,5811	0,5829	0,5875	0,6197	0,6844	0,8872	0,9888	1,0843	1,1638	1,2572	1,3479	1,4337	1,483
φ	1,2888	1,2888	1,2882	1,2881	1,2880	1,2879	1,2872	1,2860	1,2780	1,2389	1,1778	1,0285	0,9465	0,9028	0,8588	0,7979	0,7746	0,7314	0,709
$\left(\frac{D}{D_{cr}}\right)^{1,25}$	3,949	3,949	3,888	3,788	3,686	3,886	2,909	2,348	1,66	1,104	1,0	1,167	1,342	1,526	1,696	1,381	2,072	2,261	3,379
$\varphi \frac{F_{cr}}{F}\left(\frac{D}{D_{cr}}\right)^{1,25}$	0,5804	0,5804	0,5711	0,5814	0,5886	0,6295	0,6798	0,7699	0,9447	1,1717	1,1778	0,9310	0,7953	0,7049	0,6102	0,5447	0,4994	0,4451	0,422
$\int_0^X \varphi\left(\frac{D}{D_{cr}}\right)^{0,75} dX$	0	0,2235	0,2486	0,2884	0,2717	0,2883	0,3046	0,3270	0,355	0,3808	0,4202	0,4718	0,5158	0,5633	0,5910	0,6220	0,6520	0,6830	0,691
Re_t^{**}	0	1825	1888	1889	1257	1410	1636	2157	2886	4343	4972	4770	4686	4435	4280	4090	3980	3818	3713
δ^*	-	0,000000	0,000000	0,000000	0,000000	0,00342	0,00329	0,00307	0,00281	0,002562	0,00256	0,001994	0,001885	0,001891	0,001893	0,001826	0,001586	0,001515	0,00148
Re_{δ^*}	-	1445	1489	1520	1882	1783	2116	2773	4472	7982	8884	5521	4133	3278	2672	2086	1768	1453	1310

By formula (V.7) we determine Re^{**}:

$$Re_L = \frac{80 \cdot 1}{465 \cdot 10^{-6}} = 1{,}1 \cdot 10^6; \quad \nu_0 = 455 \cdot 10^{-6} \frac{m^2}{sec};$$

$$\rho_0 = 0{,}0153 \frac{kg \cdot sec^2}{m^4};$$

$$Re^{**} = (1{,}1 \cdot 10^6 \cdot 0{,}0128 \cdot 3{,}204 \cdot 1{,}25)^{0.8} X^{0.8} = 1000\, X^{0.8}.$$

By formula (1.1) we determine the local values of c_{f0}. After substitution of the results of the preceding calculations we have

$$\frac{c_{f0}}{2} = \frac{0{,}00226}{X^{0.2}}.$$

We determine the flow rate of coolant air:

$$\gamma_{cr} w_{cr} = \gamma_0 w_0 \frac{c_{f0}}{2} b = \frac{7{,}5 \cdot 2{,}44 \cdot 0{,}00226}{X^{0.2}}.$$

We calculate the local values of the coefficients of friction,

$$c_f = c_{f0} \Psi = \frac{0{,}00342}{X^{0.2}}.$$

Table II.2. Results of Calculation of Cooling of a Porous Plate

X	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
$\gamma_{wall} w_{wall}$ $\left[\frac{kg}{m^2 sec}\right]$	0,0555	0,0568	0,0525	0,0493	0,0472	0,0458	0,0443	0,0432	0,0421	0,0413
$c_f \cdot 10^3$	5,45	4,75	4,38	4,13	3,96	3,82	3,7	3,62	3,52	3,44
$q_{wall} \cdot 10^{-4}$ $\left[\frac{kcal}{m^2 hr}\right]$	2,66	2,3	2,13	2,0	1,92	1,85	1,8	1,75	1,71	1,675

We determine the specific heat flux:

$$q_{cr} = c_p \cdot \gamma_{cr} w_{cr} (t_{cr} - t_1) =$$

$$= \frac{0{,}24 \cdot 7{,}5 \cdot 2{,}44 \cdot 0{,}00226 (500 - 30)}{X^{0.2}}.$$

Results of the calculations are reduced in Table II.2 and are presented in Fig. II.2.

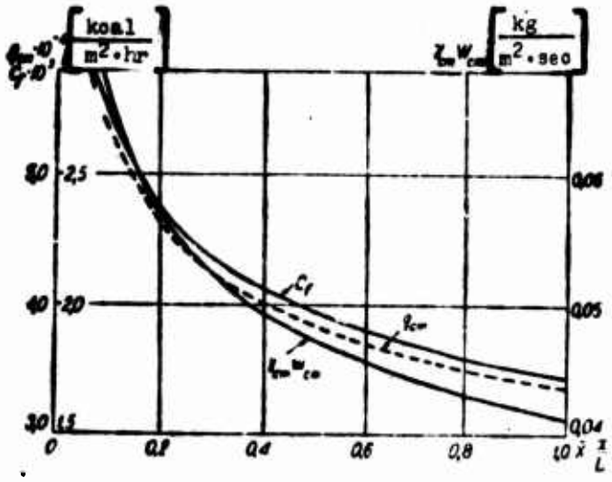

Fig. II.2. Results of calculation of cooling of a porous plate.

C. Heat Transfer in the Inlet Section of the Pipe

In a cylindrical pipe of D = 0.011 m air enters, with parameters at the inlet, $T_0 = 297.5^\circ K$; G = 22.7 kg/hour, $\mu = 1.83 \cdot 10^{-6}$. The temperature of the wall of the pipe is constant and equal to $T_{wall} = 372.5^\circ K$. It is required to determine the distribution of the specific heat flux along the length of the pipe.

We find that

$$Re_{D_1} = \frac{4 \cdot 22.7}{3600 \cdot 9,81 \cdot 1,83 \cdot 10^{-6} \cdot 3,14 \cdot 0,11} = 4,06 \cdot 10^4;$$

$$Pr_0 = 0,725; \quad Pe_{D_1} = Re_{D_1} Pr = 2,95 \cdot 10^4;$$

$$\psi = 1,25.$$

Equation (6.49), taking into account the given parameters, is reduced to the form:

$$X = 60\left\{0,616 \cdot Pe^{**0,25} - 0,89\left[2,3 \lg \times \right.\right.$$
$$\left.\left. \times \frac{Pe^{**} + 11,68\, Pe^{**0,25} + 67,4}{Pe^{**0,5} - 11,68\, Pe^{**0,25} + 67,4} + 2 \operatorname{arc\,tg} \frac{11,68 \cdot Pe^{**0,25}}{67,4 - Pe^{**0,5}}\right]\right\}.$$

Assigning the values Pe^{**} = 200, 400, 1000, 2000, 4000, by this equation we determine the corresponding values of X.

By formula (6.50) we determine the specific heat flux. Taking into account the given parameters, equation (6.50) takes the form:

$$q_{cr} = 19{,}8 \frac{(2{,}95 \cdot 10^4 + 6{,}5\, Pe^{**})}{Pe^{**0.8}}.$$

Results of the calculations are reduced in Table II.3. In Fig. II.3 are presented the results of calculations and experimental data of B. S. Petukhov, obtained for the indicated parameters. As can be seen from the graph, the agreement of the analytic calculation with the experiment on heat flows as well as by values of Pe^{**} are fully satisfactory. On the same graph are presented the results of calculations for streamlining of a flat plate by the formula of B. S. Petukhov. From the graph it is clear that the best agreement with experiment is given by the proposed method of calculation.

Table II.3. Results of calculation of Heat Transfer in the Inlet Section of the Pipe

Pe^{**}	340	440	1140	3400	4000
X	1,49	3,74	10,45	22,4	47,8
$q_{wall} 10^{-4} \frac{kcal}{m^2 \cdot hr}$	16,2	14,1	12,6	12,52	13,7

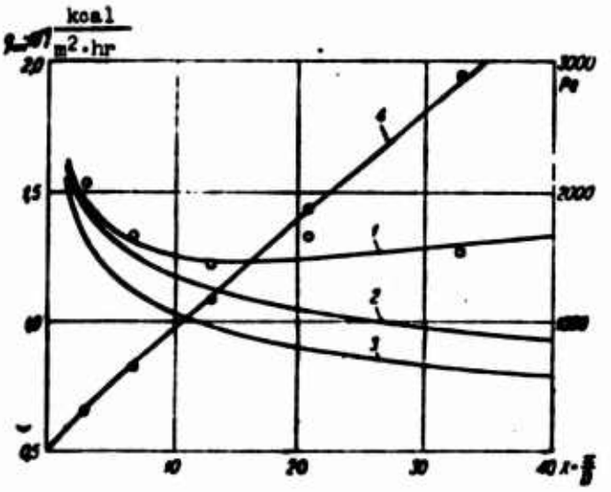

Fig. II.3. Results of calculation of heat transfer in the inlet section of the pipe: 1 – q_{wall} according to the method of the authors 2 – q_{wall} according to the method of B. S. Petukhov, 3 – q_{wall} by the empirical criterial formula for stabilized heat transfer in a pipe, 4 – Pe according to the method of the authors, O – experimental data of B. S. Petukhov for q_{wall}; ⊗ – experimental data of B. S. Petukhov for Pe^{**}.

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