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BUCKLING COEFFICIENTS FOR SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE

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ABSTRACT

This Note contains curves of buckling coefficients and formulas for the calculation of the critical external pressure of finite length, circular cylindrical shells with sandwich walls. The facings of the sandwich are isotropic, of equal or unequal thickness, of the same or different material, and their individual stiffness is not taken into account. The sandwich core is isotropic or orthotropic having natural axes in the axial, tangential, and radial directions of the cylinder. If the cores are very rigid, the method yields results that are substantially those of von Mises.

BUCKLING COEFFICIENTS FOR SANDWICH CYLINDERS OF FINITE LENGTH UNDER UNIFORM EXTERNAL LATERAL PRESSURE[1,2]

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Introduction

Design curves for the critical external radial pressure of circular cylindrical shells with sandwich walls, calculated according to the formulas developed at the Forest Products Laboratory (9),[4] are presented in this report. The sandwich cylinder walls have isotropic facings of equal or unequal thickness, and of the same or different materials, and orthotropic or isotropic cores. It is assumed that Poisson's ratio is the same for both facings. The natural axes of the orthotropic cores are axial, tangential, and radial. These formulas reduce substantially to those developed by von Mises (7, 10, 14) when the core is very rigid.

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[1] This Note is the latest revision of "Design Curves for the Buckling of Sandwich Cylinders of Finite Length under Uniform External Lateral Pressure," by Charles B. Norris and John J. Zahn. It was originally issued as Forest Products Lab. Rpt. 1869 in 1959, and revised as U.S. Forest Service Research Note FPL-07 in 1963.

[2] This progress report is one of a series (ANC-23, Item 57-3) prepared and distributed by the Forest Products Laboratory under U.S. Navy, Bureau of Naval Weapons Order No. 19-65-8005 WEPS and U.S. Air Force Contract No. AO 33(615)65-5002. Results here reported are preliminary and may be revised as additional data become available.

[3] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[4] Underlined numbers in parentheses refer to Literature Cited at the end of this report.

Much investigative work has been done on isotropic cylindrical shells subjected to external pressure since Forest Products Laboratory Report 1844-B, giving theoretical analysis of sandwich cylinders, was published. It was found that experiment sometimes yields critical loads that are less than those predicted by von Mises' theory (13). This has been attributed to two causes. First, the experimental cylinders contained imperfections that lowered the critical load (1, 2, 3, 8, 12). Second, energy levels associated with postbuckling configurations of the cylinder are lower than those just at buckling. The energy levels associated with postbuckling may be reached without snap-through buckling. The energy necessary for snap-through buckling may be supplied by vibration or shocks (4, 5, 6, 8). The curves in this report do not consider snap-through buckling or cylinders with imperfections. Sandwich cylinder walls, however, are much more perfect than their solid counterparts because they are thicker and the effect of an imperfection is in proportion to the ratio of its amplitude to the thickness of the cylindrical shell wall. Also, the curves neglect the stiffnesses of the individual facings. These stiffnesses add to the critical loads when the cylinders are short, and it is for short cylinders that snap-through is likely to occur (6).

Development of Formula for Design Curves

The formulas presented in this report for calculating the critical external pressure and the buckling coefficient are based on the work presented in Forest Products Laboratory Reports 1844-A and 1844-B (9) with the following notation:

E_1, E_2 Modulus of elasticity of the outer facing (1) and inner facing (2), respectively.

μ Poisson's ratio.

θ $\frac{G_{rz}}{G_{r\theta}}$.

G_{rz} Modulus of rigidity of core in the radial and axial directions.

$G_{r\theta}$ Modulus of rigidity of core in the radial and tangential directions.

d Thickness of the sandwich.

h Distance between facing centroids.

t_1, t_2 Thickness of outer and inner faces, respectively.

r Mean radius of the sandwich cylinder.

L Length of the cylinder.

β $\quad \frac{\pi^2 r^2}{L^2}$.

R $\quad \frac{E_1 t_1}{E_2 t_2}$.

N $\quad$ qr.

n $\quad$ Number of half-waves in the circumferences of the cylinder.

V $\quad \frac{2E_1t_1E_2t_2}{3rG_{r\theta}(1 - \mu^2)(E_1t_1 + E_2t_2)}$.

k $\quad \frac{qr(1 - \mu^2)}{E_1t_1 + E_2t_2}$.

q $\quad$ The external critical pressure on the cylinder.

A few parameters of Report 1844-B were generalized to dissimilar facings as follows:

Report 1844-B	Revised Parameter
$k = \frac{1}{1 + \frac{b}{a} - \frac{Et \log b/a}{E_c a}}$	$\psi = \frac{1}{1 + \frac{bE_1t_1}{aE_2t_2} - \frac{E_1t_1 \log b/a}{E_c a}}$
$\beta = \frac{E_c a(1 - \mu^2)}{Et}$	$W_i = \frac{E_c a(1 - \mu^2)}{E_i t_i} \quad i = 1, 2$
$\alpha = \frac{qa(1 - \mu^2)}{Et}$	$\Omega = \frac{qa(1 - \mu^2)}{E_i t_i} \quad i = 1, 2$

These generalized parameters were combined with the notation and derivation of equations given in Report 1844-B. The critical external pressure was then found by solving the revised form of equation 51 of Report 1844-B for the buckling coefficient. The determinant was simplified by taking modulus of elasticity of the core (E_c) to be infinite. For most core materials, except possibly for low-density foams, E_c is sufficiently large so that this assumption yields only slightly high values of the critical pressure. Before E_c is allowed to approach infinity, the first, third, and fourth columns of the determinant are multiplied by $\frac{G_{z\theta}}{E_c}$. Then when E_c approaches infinity, the expressions in rows 3, 4, 5, and 6 of column 3 approach zero. Further simplification of the 6 by 6 determinant was made by taking μ to be 1/3, multiplying row 1 by

$$\frac{G_{z\theta}a(1 - \mu^2)}{E_2 t_2}$$

and row 2 by

$$\frac{G_{z\theta}a(1 - \mu^2)}{E_1 t_1} \frac{a^2}{b^2}$$

Further reduction is accomplished by adding multiples of one row to another as indicated by the following sequence of substitutions, where subscripts indicate row number counting down from the top:

$$(R_1 + R_2) \frac{E_1 t_1 E_2 t_2}{G_{r\theta}a(1 - \mu^2)(E_1 t_1 + E_2 t_2)} \longrightarrow R_2$$

$$R_3 + 2R_2 \longrightarrow R_3$$

$$R_4 - R_3 \longrightarrow R_4$$

$$R_5 + (n^2 + 3\lambda^2) R_2 \longrightarrow R_5$$

$$R_6 + \left(n^2 + 3\lambda^2 \frac{a^2}{b^2}\right) R_2 \longrightarrow R_6$$

These substitutions cause the expressions in rows 2, 3, 4, 5, and 6 of column 3 and those in rows 3, 4, 5, and 6 of column 6 to become zero, and the 6 by 6 determinant was readily reduced by minors to a 4 by 4 determinant. The radii a and b were eliminated by the following equations obtained from the geometry of the cylinder:

$$a = r + \frac{h}{2}$$

$$b = r - \frac{h}{2}$$

and the following substitutions were made:

$$\Phi = \frac{4r}{d}$$

$$\phi = \frac{2h}{d}$$

After setting this determinant equal to zero and solving for k, the critical pressure can be found by

$$q = \frac{E_1 t_1 + E_2 t_2}{r(1 - \mu^2)} k$$

This represents a theoretical solution for the critical pressure with the assumption that the sandwich core modulus of elasticity is infinite.

The determinant was further simplified by a few approximations without any significant loss of accuracy. Examination of the parameters Φ and ϕ showed that Φ is large in comparison to ϕ. Therefore, any terms of $(\Phi \pm \phi)$ were taken as equal to Φ.

Making this simplification, the determinant reduces to the following expressions:

Row 1, column 1

$$- \frac{2}{(R + 1)n^2} + \frac{3}{2} V\left(1 - \frac{\beta}{3n^2}\right)$$

Row 2, column 1

$$\frac{2}{n^2} + 6V\frac{\phi}{\Phi}$$

Row 3, column 1

$$\frac{8R}{(R + 1)^2} \frac{(n^2 - 1)}{n^2} (n + 3\beta)\frac{\phi}{\Phi} + 3V\beta\left(\frac{1}{3} - \frac{\beta}{n^2} - k\right)$$

Row 4, column 1

$$\frac{8R}{(R + 1)^2} \frac{(n^2 - 1)}{n^2} (n^2 + 3\beta)\frac{\phi}{\Phi} + 3V\beta\left(\frac{1}{3} - \frac{\beta}{n^2} - k\right)$$

Row 1, column 2

$$+\frac{\beta}{3} + (n^2 - 1)(2k - 1)$$

Row 2, column 2

$$-\frac{2\phi}{\Phi}\left(n^2 - 1 + \frac{\beta}{3}\right)$$

Row 3, column 2

$$2\beta\left(-\frac{n^2 - 1}{3} + \beta + (n^2 - 1)k\right) + 2k(n^2 - 1)(n^2 + 3\beta)$$

Row 4, column 2

$$2\beta\left(-\frac{(n^2 - 1)}{3} + \beta + (n^2 - 1)k\right) + 2k(n^2 - 1)(n^2 + 3\beta)$$

Row 1, column 3

$$\frac{2\phi}{\Phi}$$

Row 2, column 3

$$0$$

Row 3, column 3

$$-\frac{R + 1}{R} + (n^2 + 3\beta)\frac{2\phi}{\Phi}$$

Row 4, column 3

$$R + 1 + (n^2 + 3\beta)\frac{2\phi}{\Phi}$$

Row 1, column 4

$$n^2 - \frac{\beta}{3}$$

Row 2, column 4

$$-\frac{4}{3}\frac{\phi}{\Phi}\beta$$

Row 3, column 4

$$4\beta n^2\left(\frac{1}{3}-\frac{k}{2}\right)-(n^2+3\beta)\frac{2}{3}\beta$$

Row 4, column 4

$$4\beta n^2\left(\frac{1}{3}-\frac{k}{2}\right)-(n^2+3\beta)\frac{2}{3}\beta$$

By setting this determinant equal to zero, the approximate solution for k is:

$$k=\frac{\frac{8}{9}+\frac{4R}{(R+1)^2}\alpha^2(n^2-1)\left(3+\frac{n^2}{\beta}\right)\left[\left(\frac{n^2}{\beta}-\frac{1}{3}\right)(n^2-1+\beta)-\frac{2}{3}\right]+\frac{8}{3}V\alpha\left(n^2+\frac{\beta}{3}\right)}{\left[\left(\frac{n^2}{\beta}+1\right)^2(n^2-1)+\frac{1}{3}\right]\left[1+3V\alpha\left(n^2+\frac{\beta}{3}\right)\right]}$$

where $\alpha=\frac{\phi}{\Phi}=\frac{h}{2r}$

To obtain this value of k, terms containing k^2 and $k\alpha^2$ were neglected. It was also assumed that terms $(1 \pm m\alpha) = 1$ where m is a small whole number.

A lower and upper limit exist for the buckling coefficient k. The lower limit is associated with the infinitely long shell for which $\beta = 0$ and this limit from the approximate formula for k is given by

$$\beta = 0; \qquad k=\frac{4R(n^2-1)\alpha^2}{(R+1)^2(1+3V\alpha n^2)}$$

and for $V = 0$ this formula is minimum for $n = 2$ and is

$$\beta = 0; \qquad V = 0; \qquad k=\frac{12R\alpha^2}{(R+1)^2}$$

The upper limit for k is associated with the usual shear instability type of buckling of the sandwich wall which occurs when $n = \infty$ and the formula for k then is given by

$$n = \infty; \qquad k = \frac{4R\alpha}{3(R + 1)^2 V}$$

Substitution of this value for k into the formula for q results in a critical hoop compression per unit length of cylinder of $N = qr = hG$.

Description of Design Curves

Using the approximate equation, values for k were determined for various values of V, n, $\frac{L}{r}$, and α^2. These values of k are plotted (figs. 1 to 16) for values of $\frac{L}{r}$ ranging from 1 to 100. Curves are given for values of V equal to 0, 0.5, 1.0, and 1.5. For each of these values of V, curves are given for seven values of α^2 ranging from 10^{-6} to 10^{-4}. These curves apply to sandwich cylinders with isotropic facings and isotropic cores or orthotropic cores with their natural axes parallel to the axial, tangential, and radial directions of the cylinders. The critical pressure is given by

$$q = \frac{E_1 \ldots}{r \ldots} + \frac{E_2 t_2}{\ldots \mu^2)} k$$

From a solution of the exact determinant, it was found that the modulus of rigidity of the core associated with the radial and axial directions G_{Rz} has very little influence on the critical pressure. It does not enter the formulas for the limits of critical pressure. The ratio $\theta = \frac{G_{Rz}}{G_{R\theta}}$ does affect the critical pressure for small $\frac{L}{r}$ values, but for reasonable values of θ the critical pressure is affected only slightly. The ratio θ does not appear in the approximate solution for k.

The accuracy of the approximate solution was checked by making a comparison of the approximate and exact solutions for values of $V = 0$ and $\alpha = 0.01$. This comparison is shown on the top curve of figure 1 where the small circles represent the exact solution. This shows that, except for small $\frac{L}{r}$ values, the error encountered with the approximate solution is less than 1 percent. For larger values of V and smaller values of α, the error diminishes. For small values of $\frac{L}{r}$, it is doubtful that either solution gives good results.

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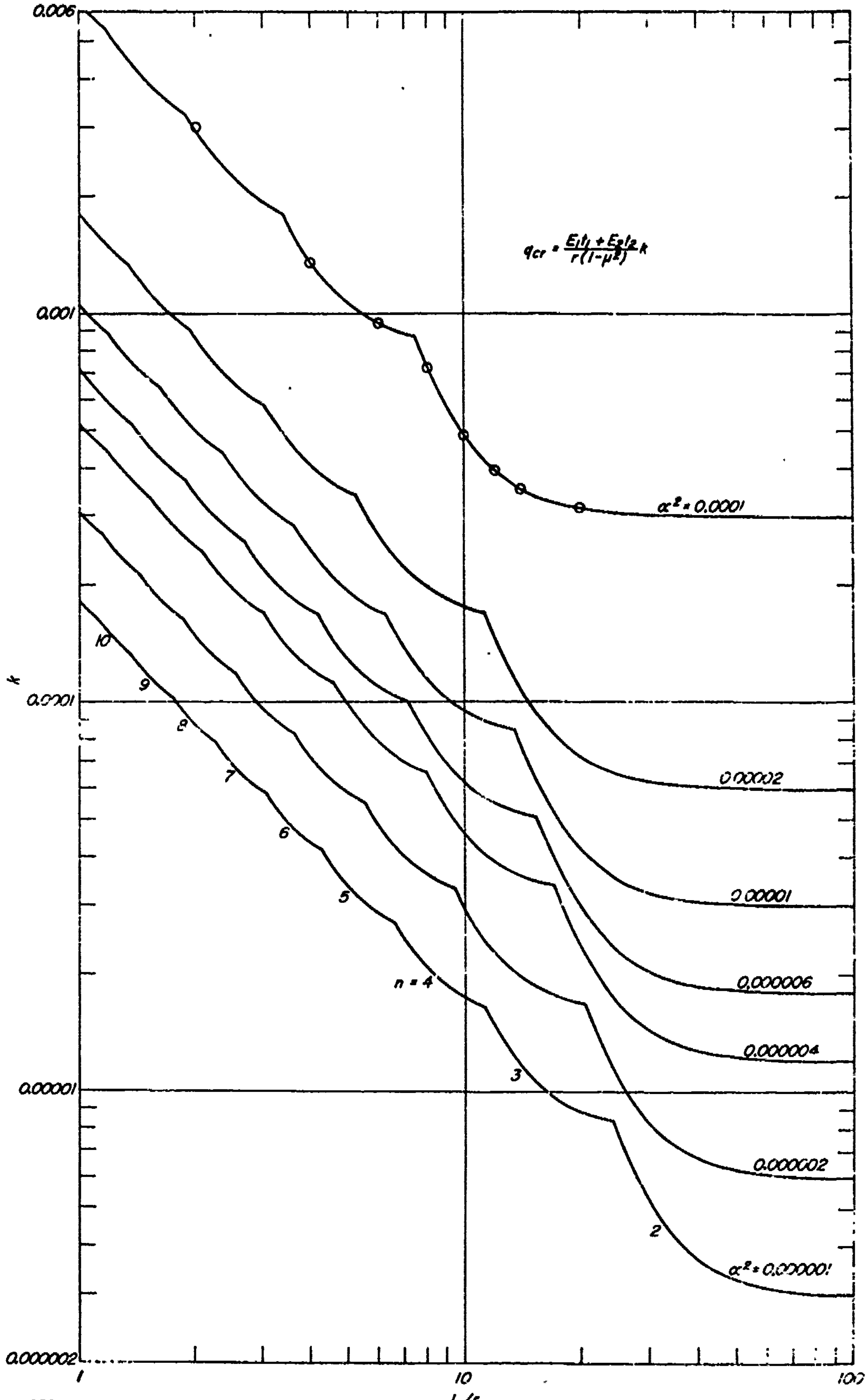
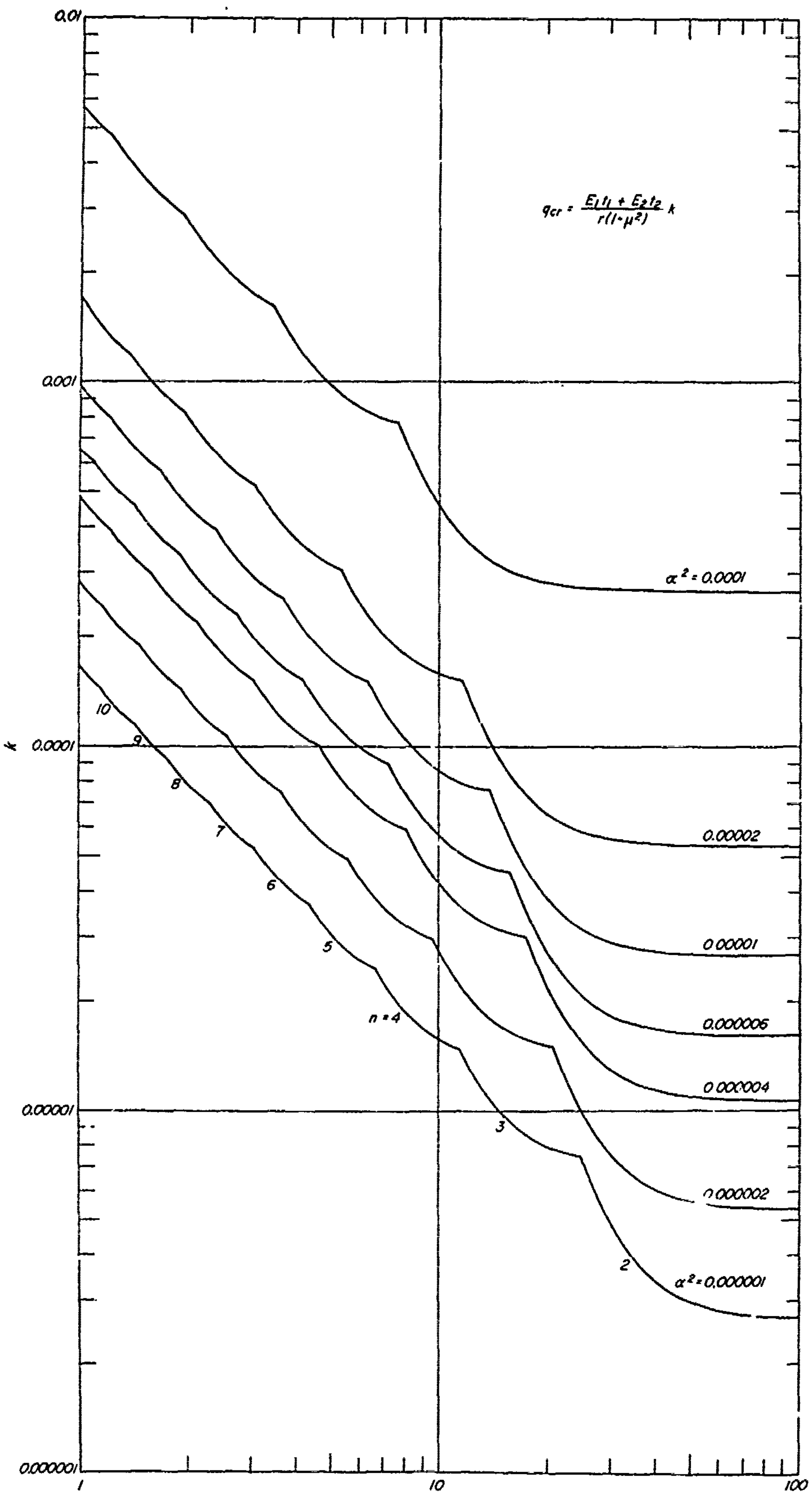


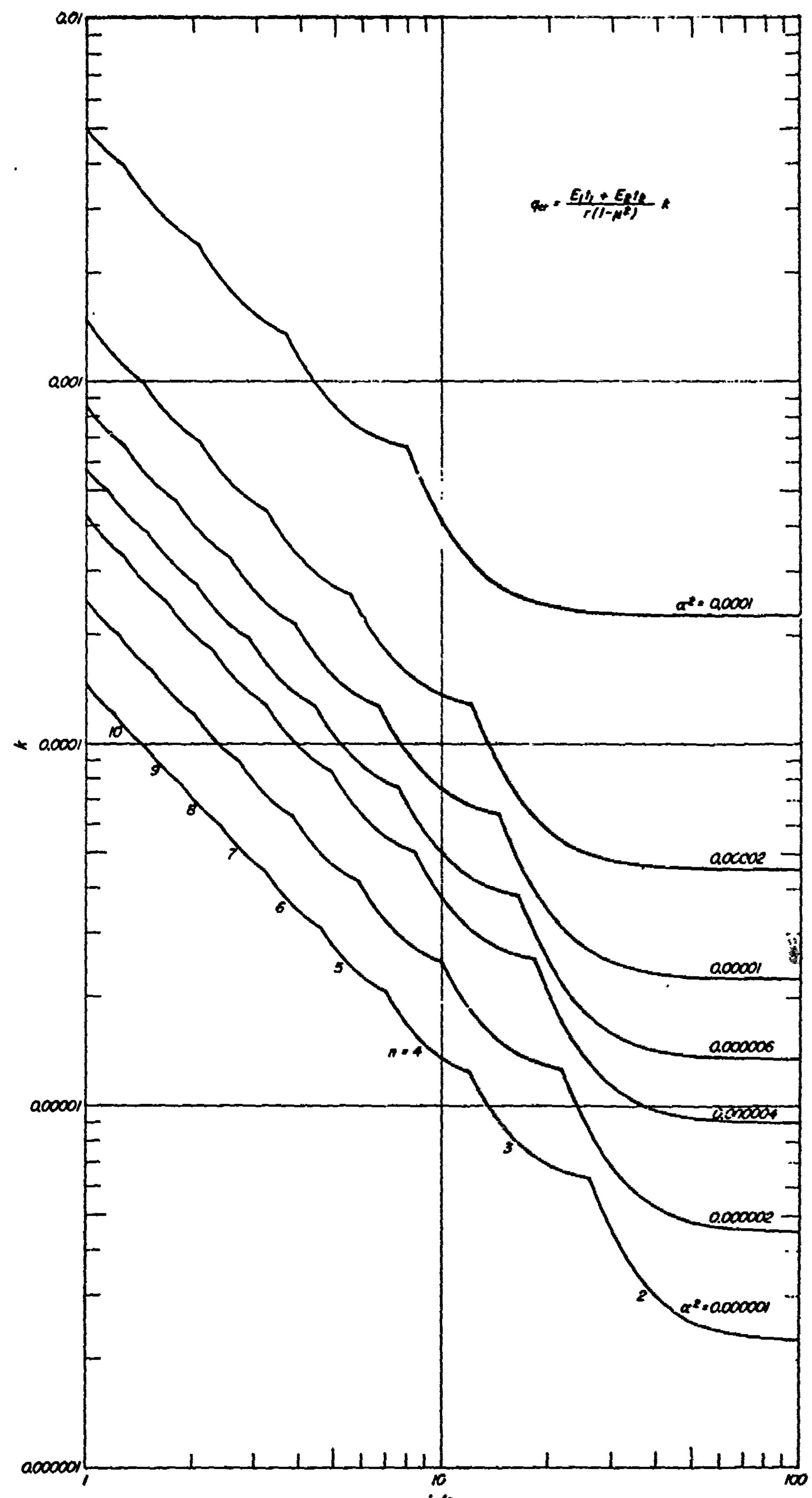
M 123 661

Figure 1.--Values of k for V = 0, and for $\frac{E_1t_1}{E_2t_2} = 1$.



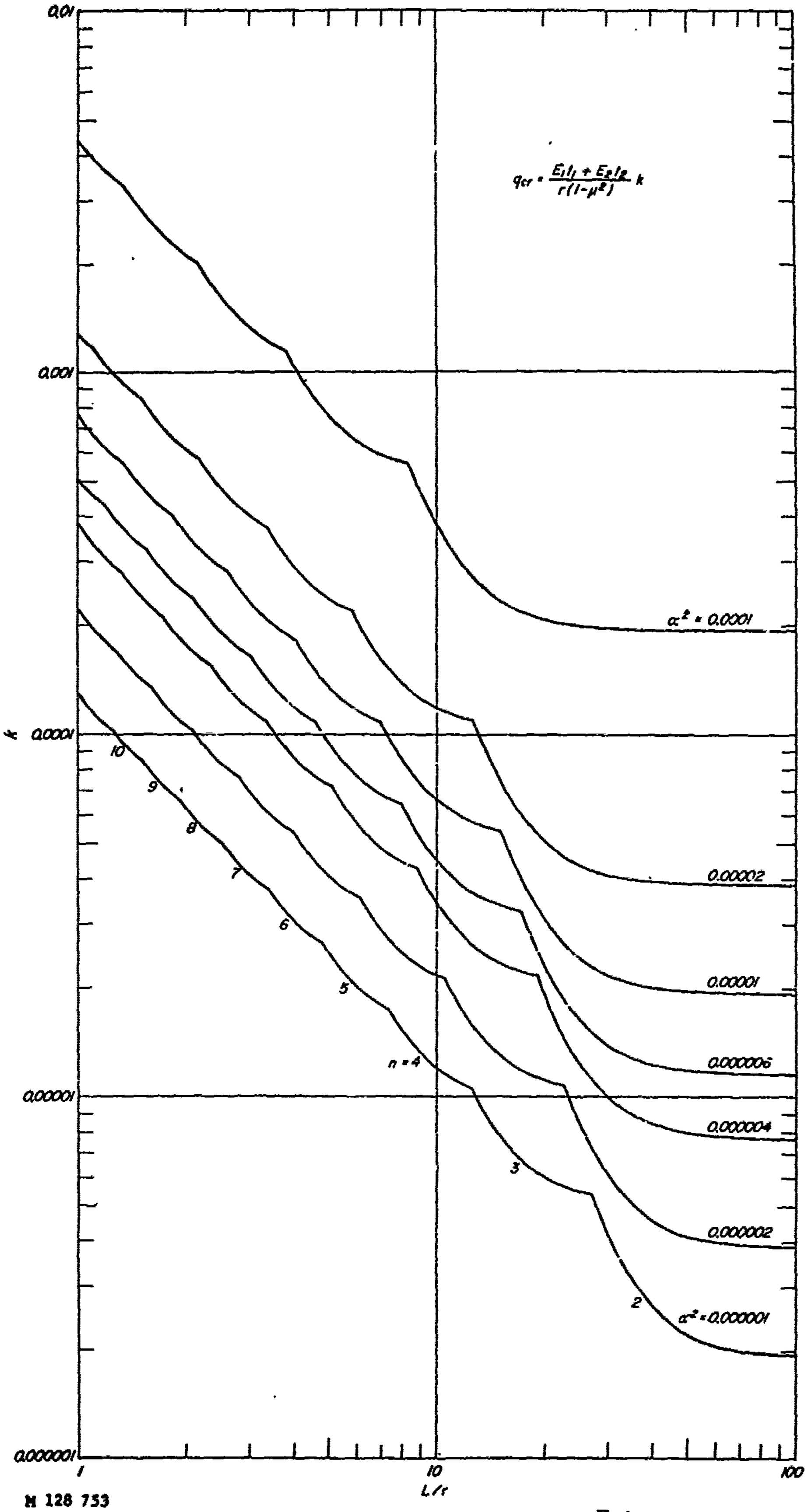
M 128.750

Figure 2.--Values of k for V = 0, and for $\frac{E_1 t_1}{E_2 t_2} = 2$.



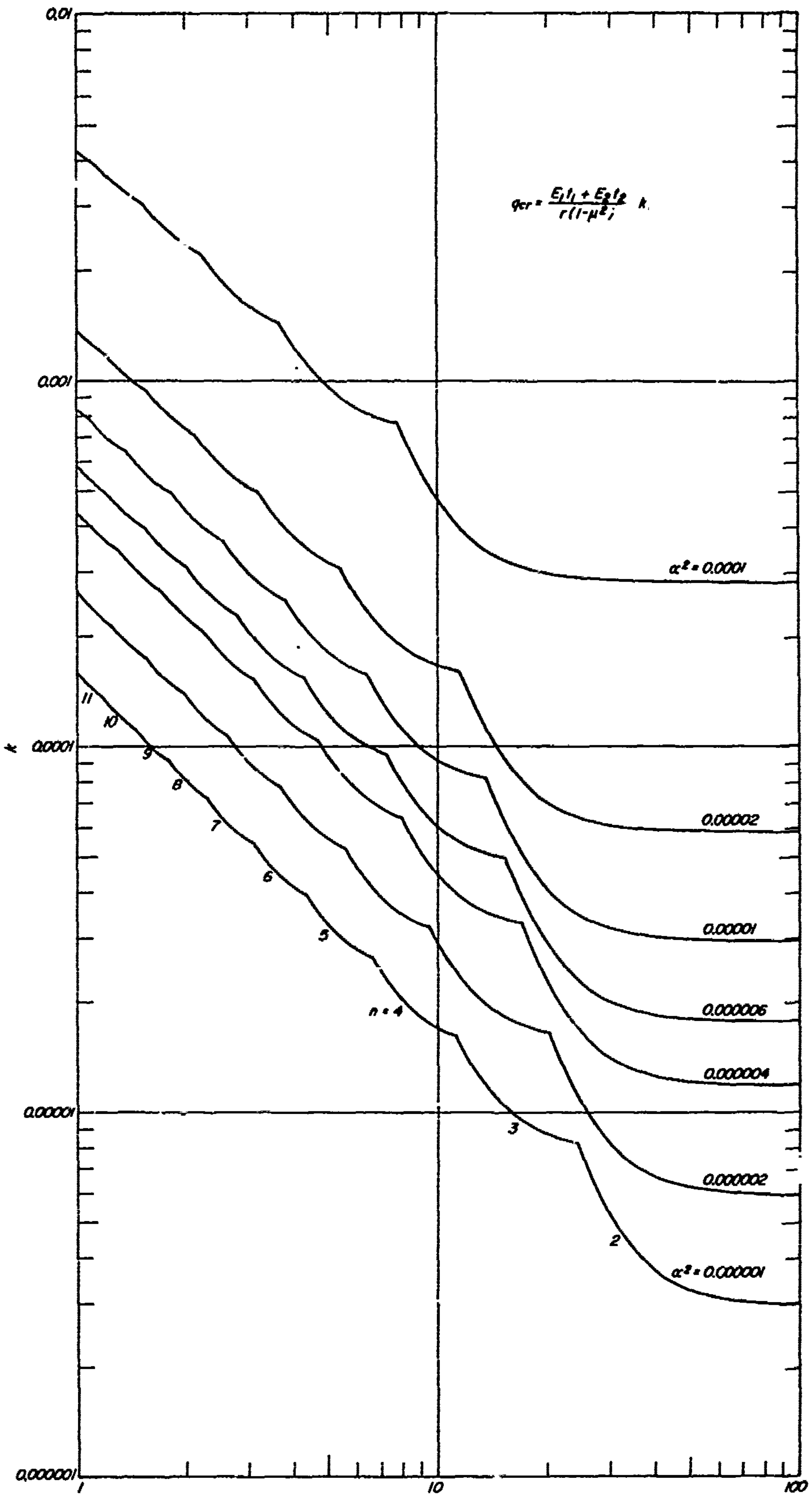
M 128 751

Figure 3.--Values of k for V = 0, and for $\frac{E_1 t_1}{E_2 t_2} = 3$.



M 128 753

Figure 4.--Values of k for V = 0, and for $\frac{E_1t_1}{E_2t_2} = 4$.



M 128 748

Figure 5.--Values of k for V = 0.5, and for $\frac{E_1 t_1}{E_2 t_2} = 1$.

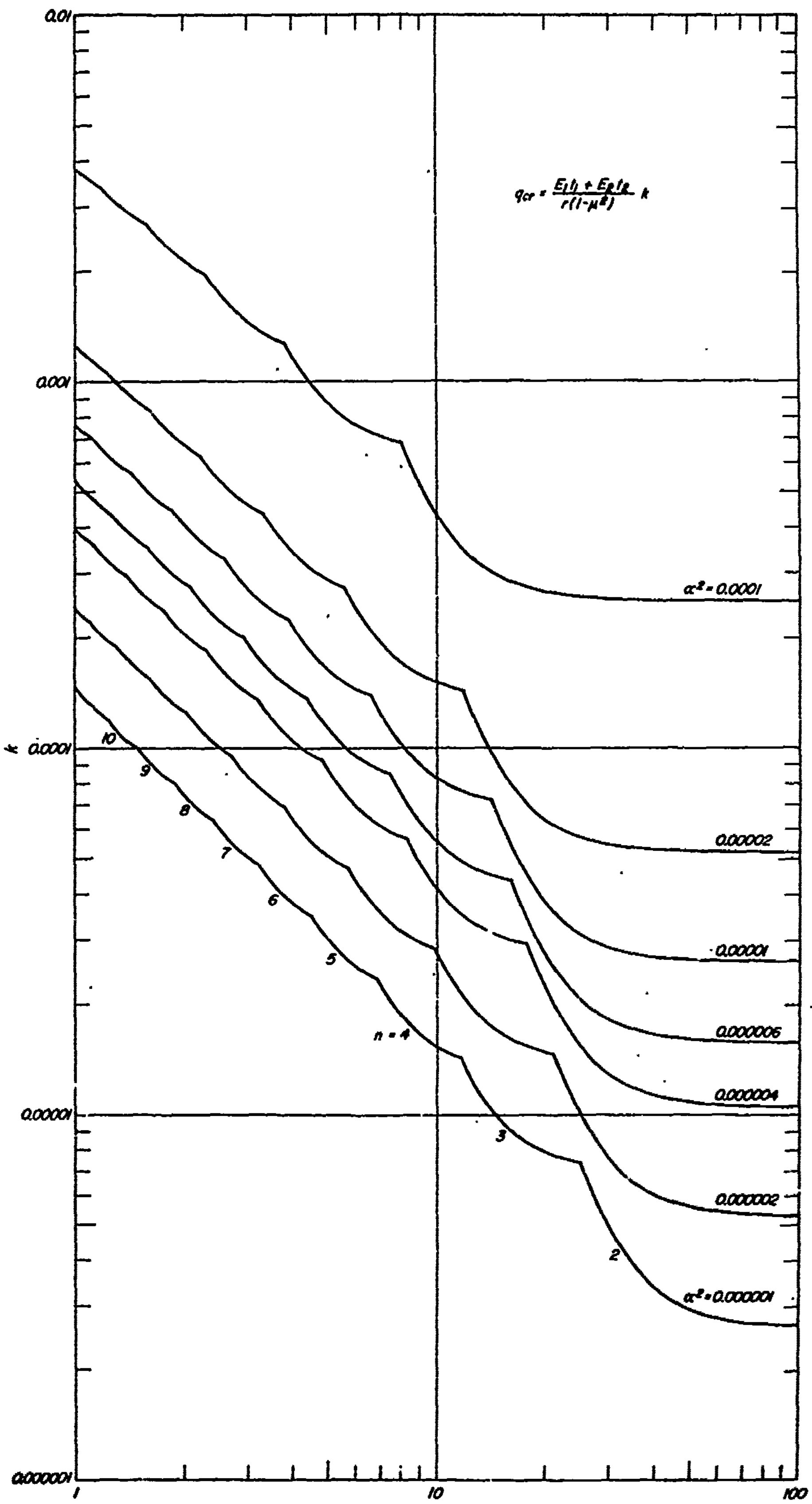


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Figure 6.--Values of k for V = 0.5, and for $\frac{E_1t_1}{E_2t_2} = 2$.

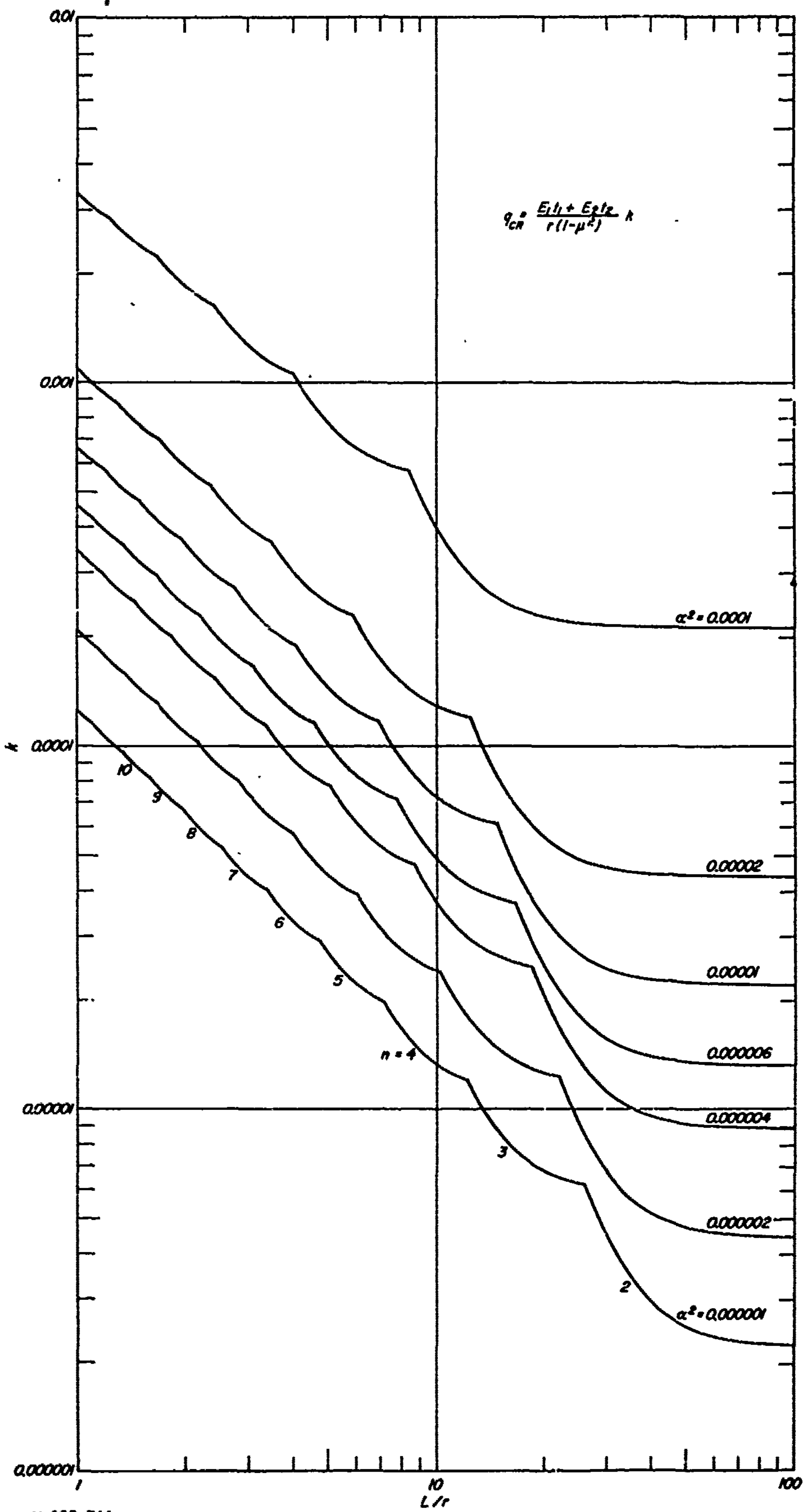

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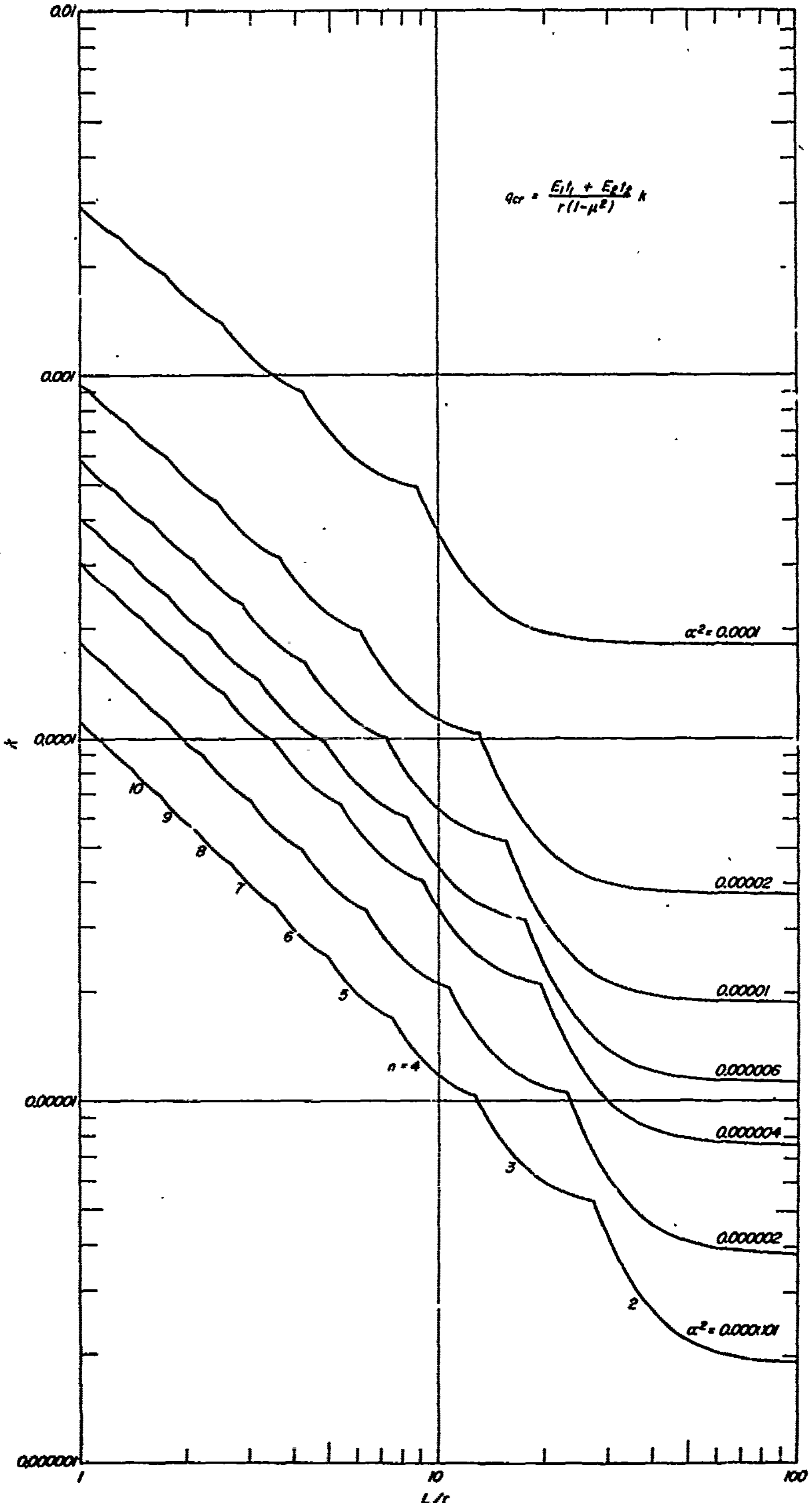
Figure 7.--Values of k for V = 0.5, and for $\frac{E_1t_1}{E_2t_2} = 3$.



M 128 741

Figure 8.--Values of k for V = 0.5, and for $\frac{E_1t_1}{E_2t_2} = 4$.

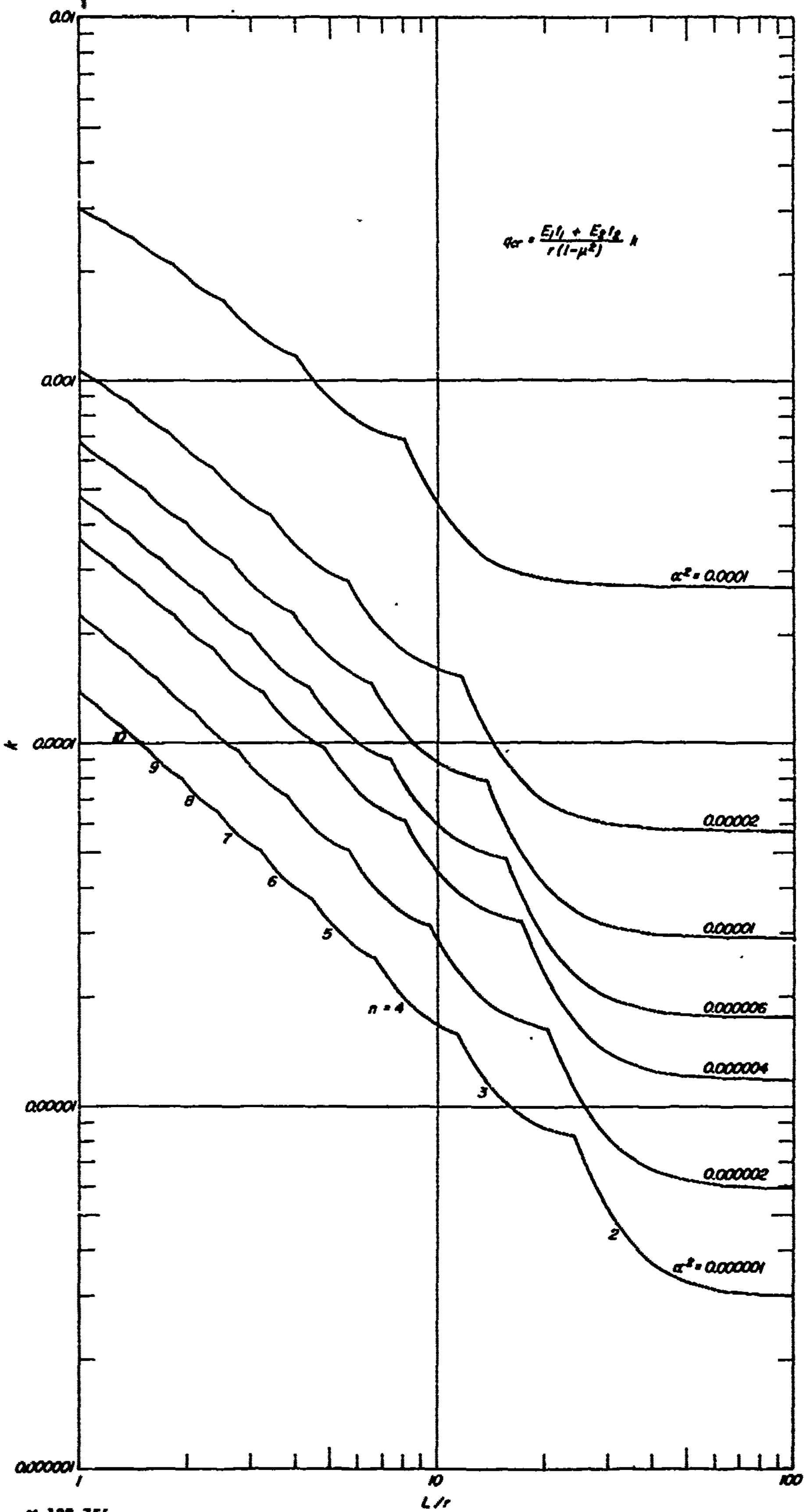


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Figure 9.--Values of κ for V = 1, and for $\frac{E_1 t_1}{E_2 t_2} = 1$.

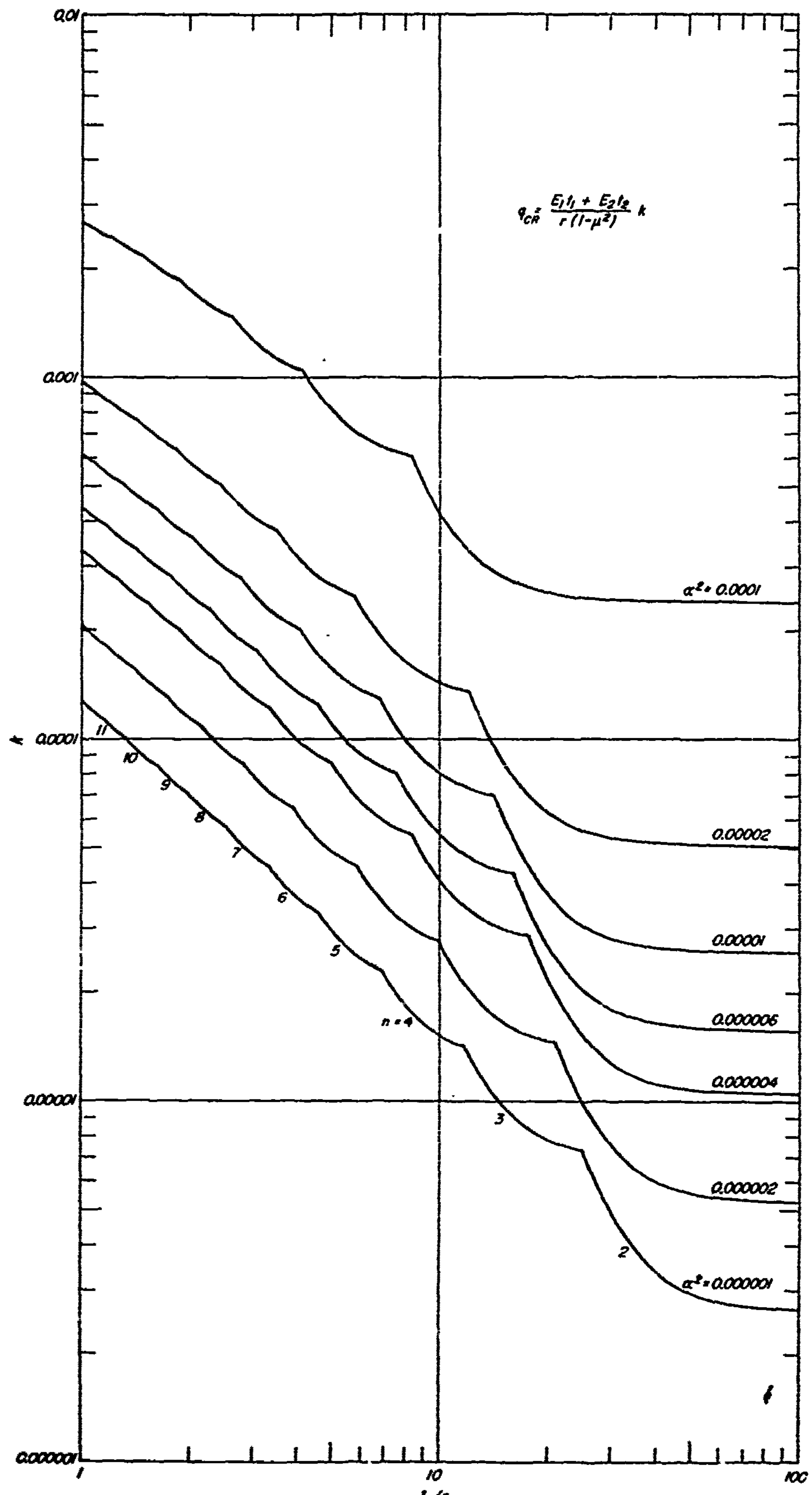

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Figure 10.--Values of k for V = 1, and for $\frac{E_1 t_1}{E_2 t_2} = 2$.

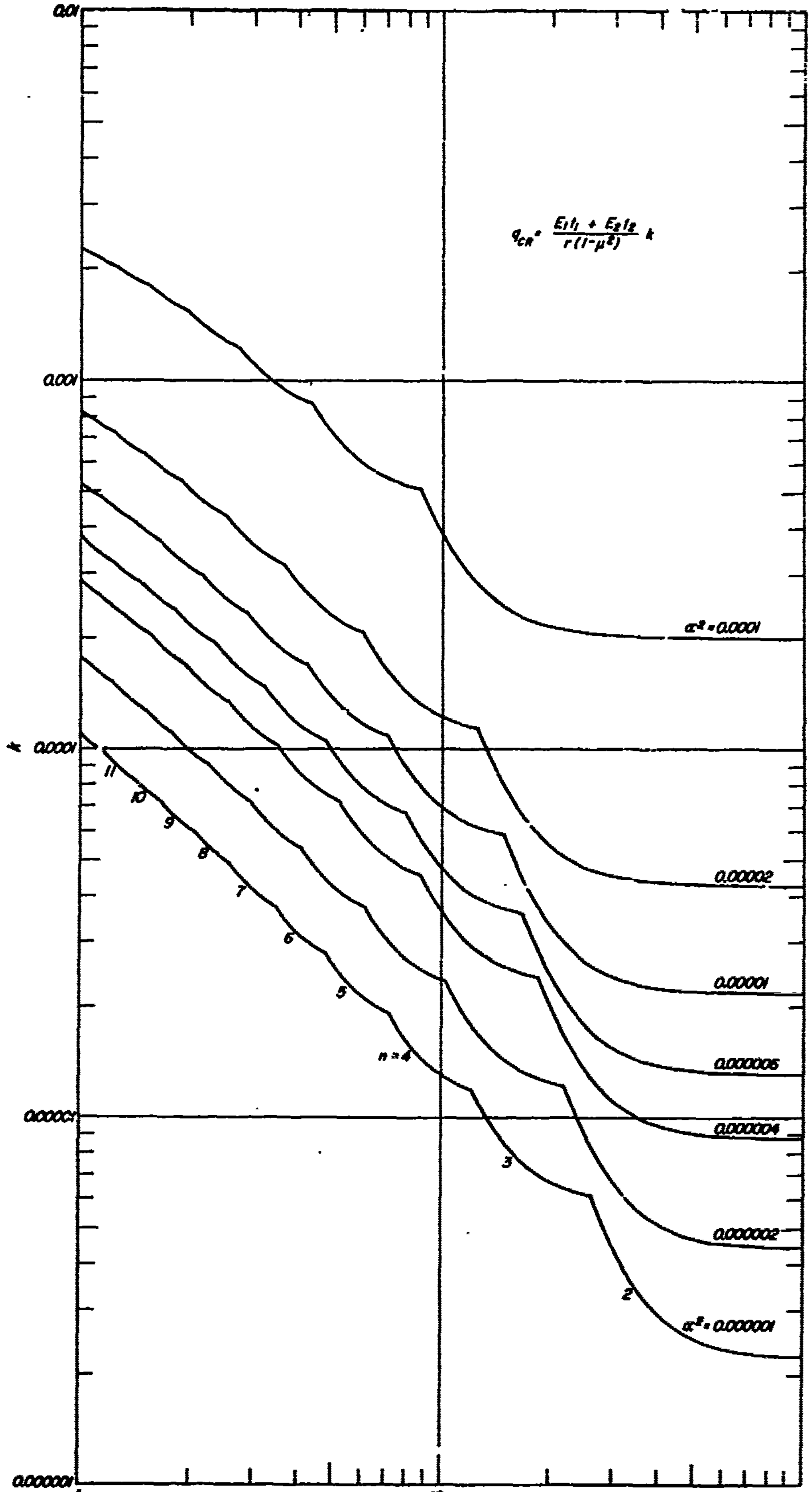

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Figure 11.--Values of k for V = 1, and for $\frac{E_1 t_1}{E_2 t_2} = 3$.

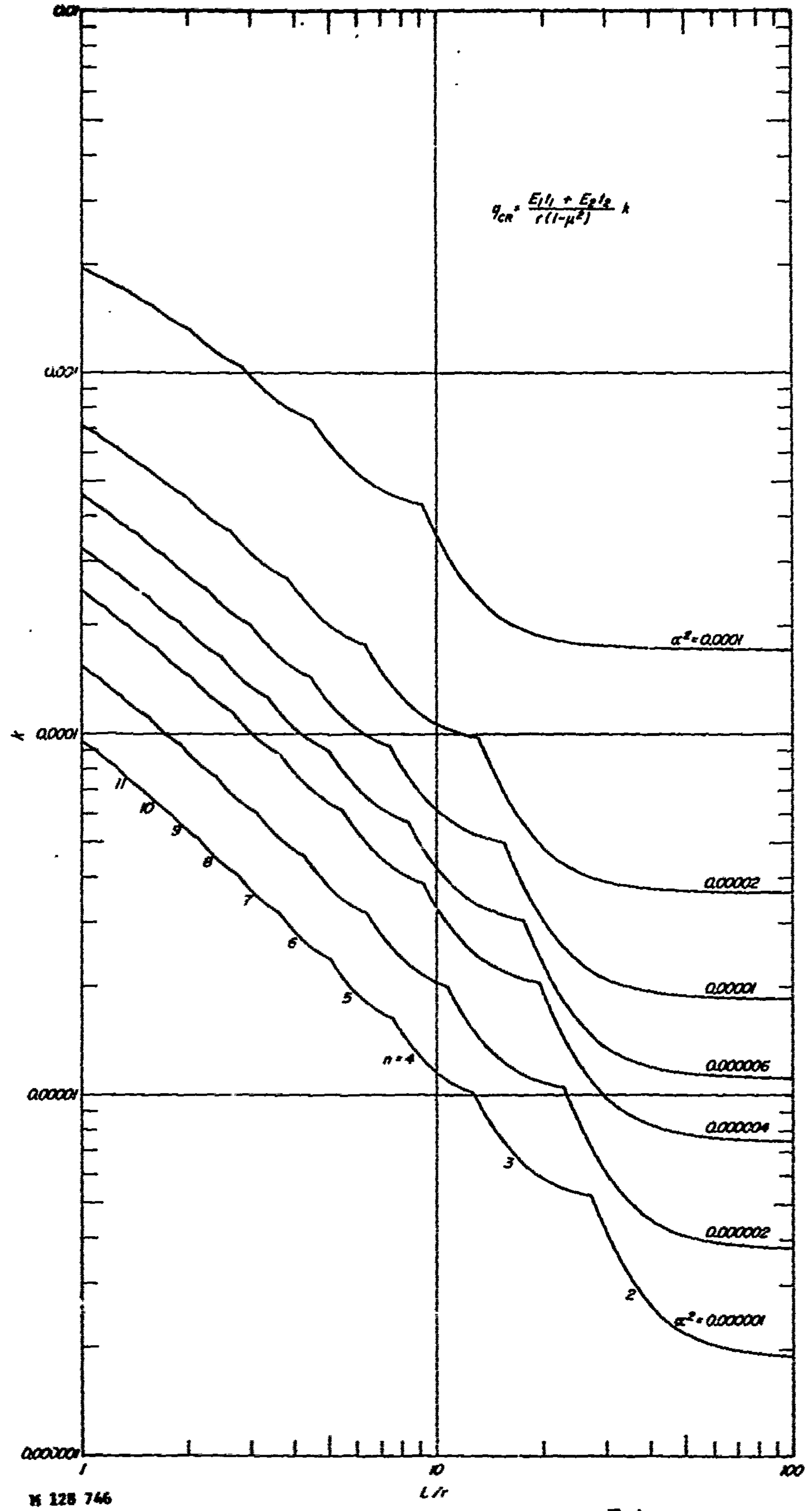

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Figure 12.--Values of k for V = 1, and for $\frac{E_1 t_1}{E_2 t_2} = 4$.

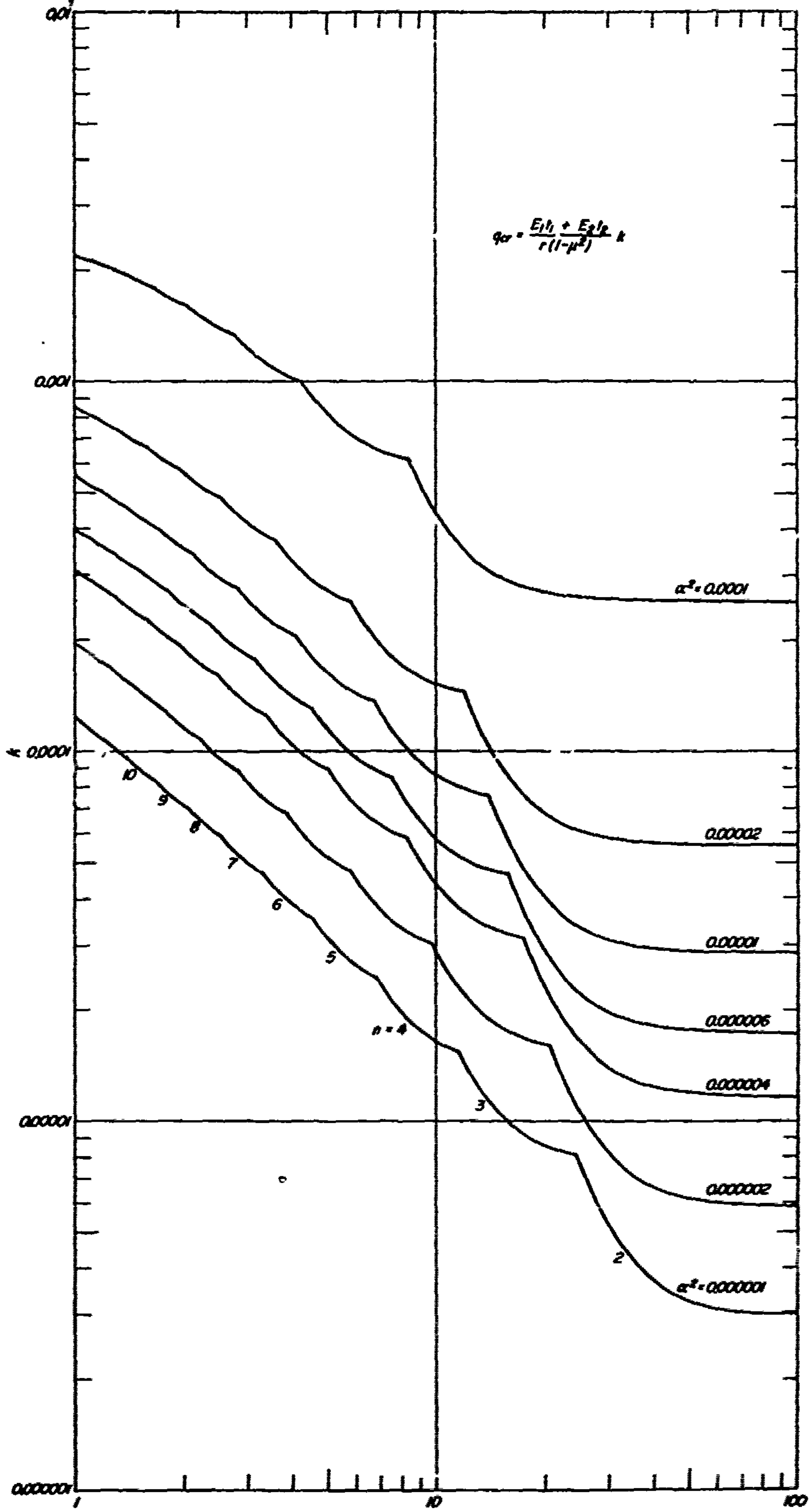

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Figure 13.--Values of k for V = 1.5, and for $\frac{E_1t_1}{E_2t_2} = 1$.

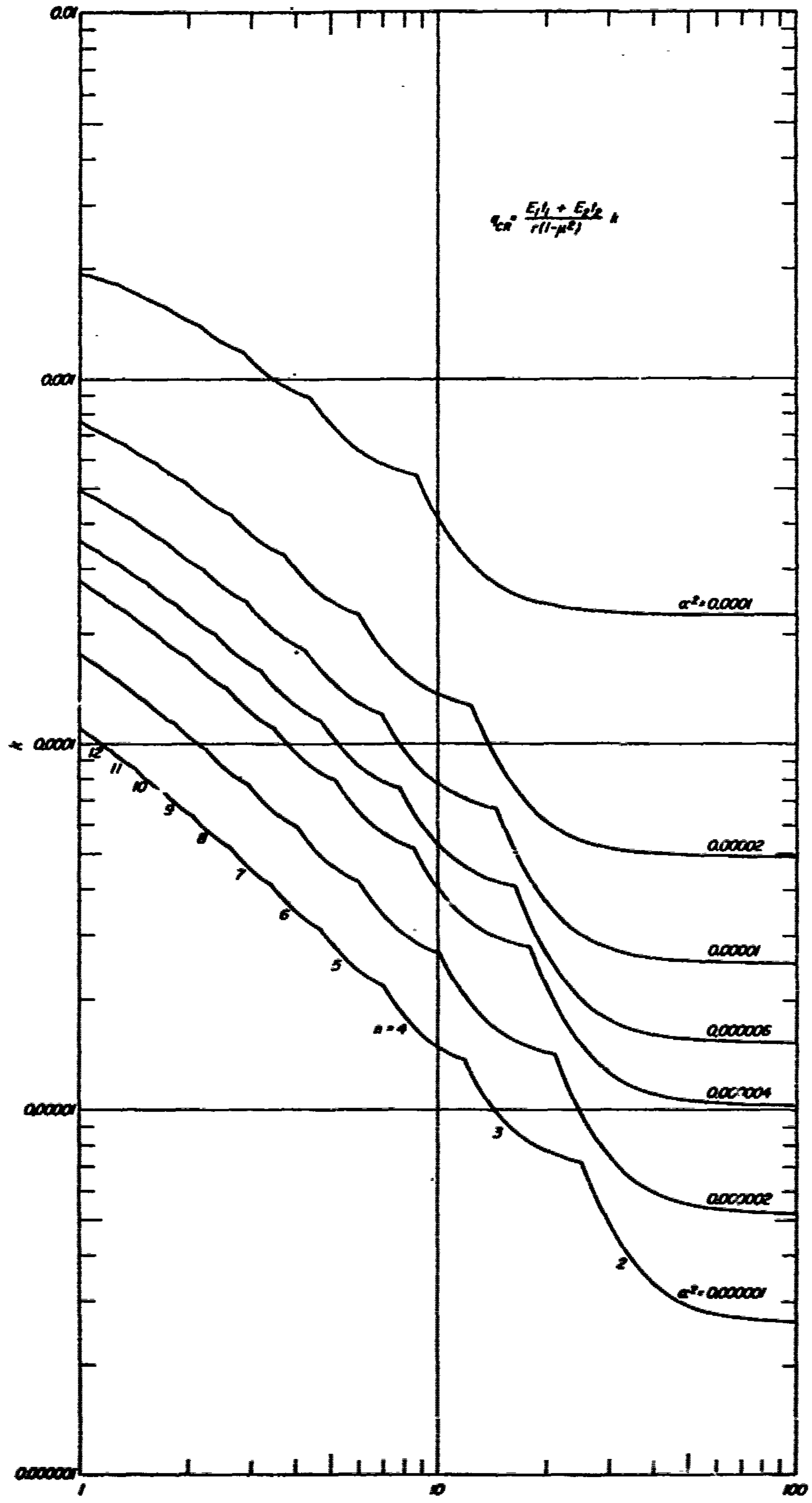

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Figure 14.--Values of k for V = 1.5, and for $\frac{E_1 t_1}{E_2 t_2} = 2$.

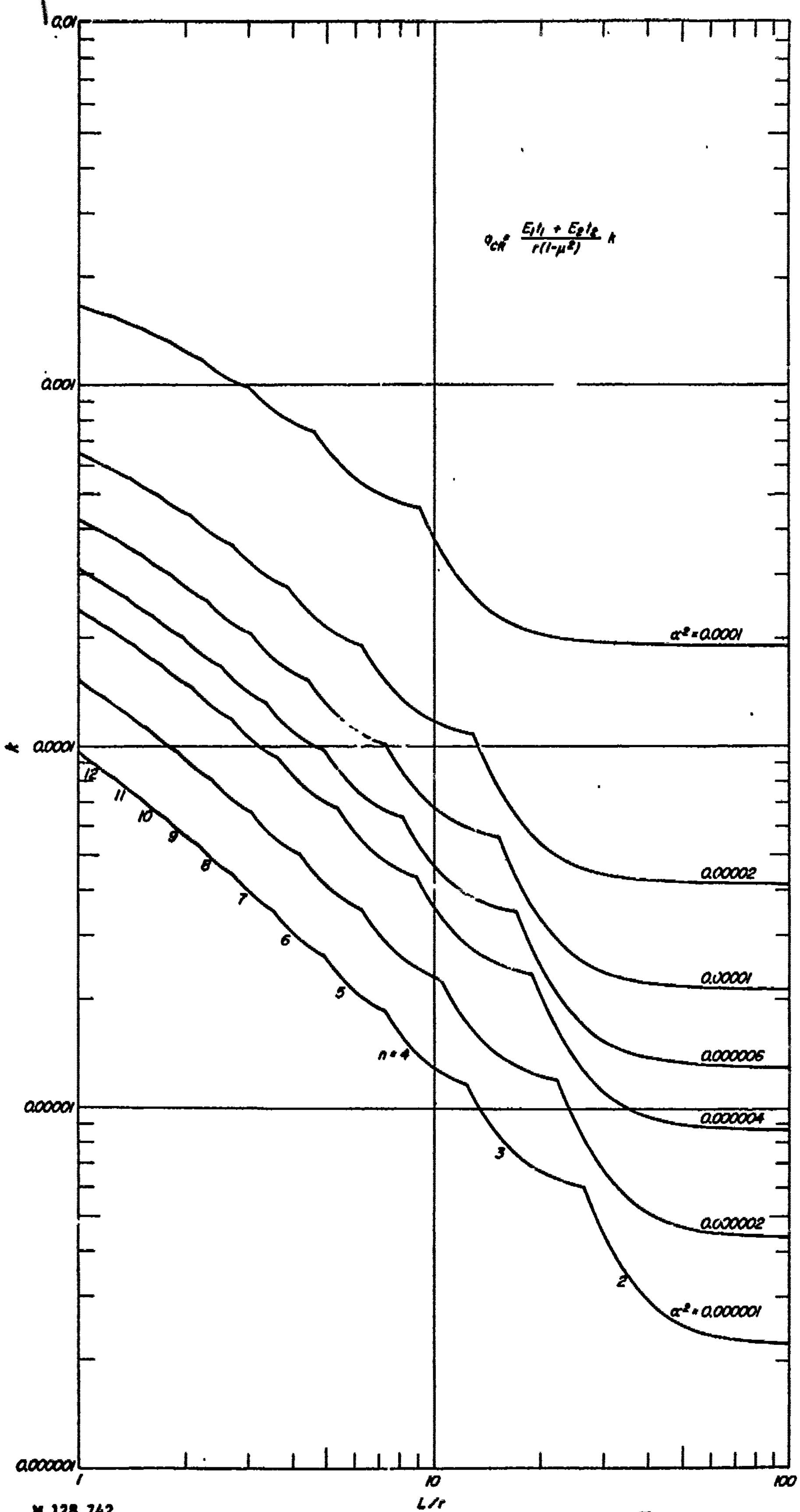

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Figure 15.--Values of k for V = 1.5, and for $\frac{E_1 t_1}{E_2 t_2} = 3$.

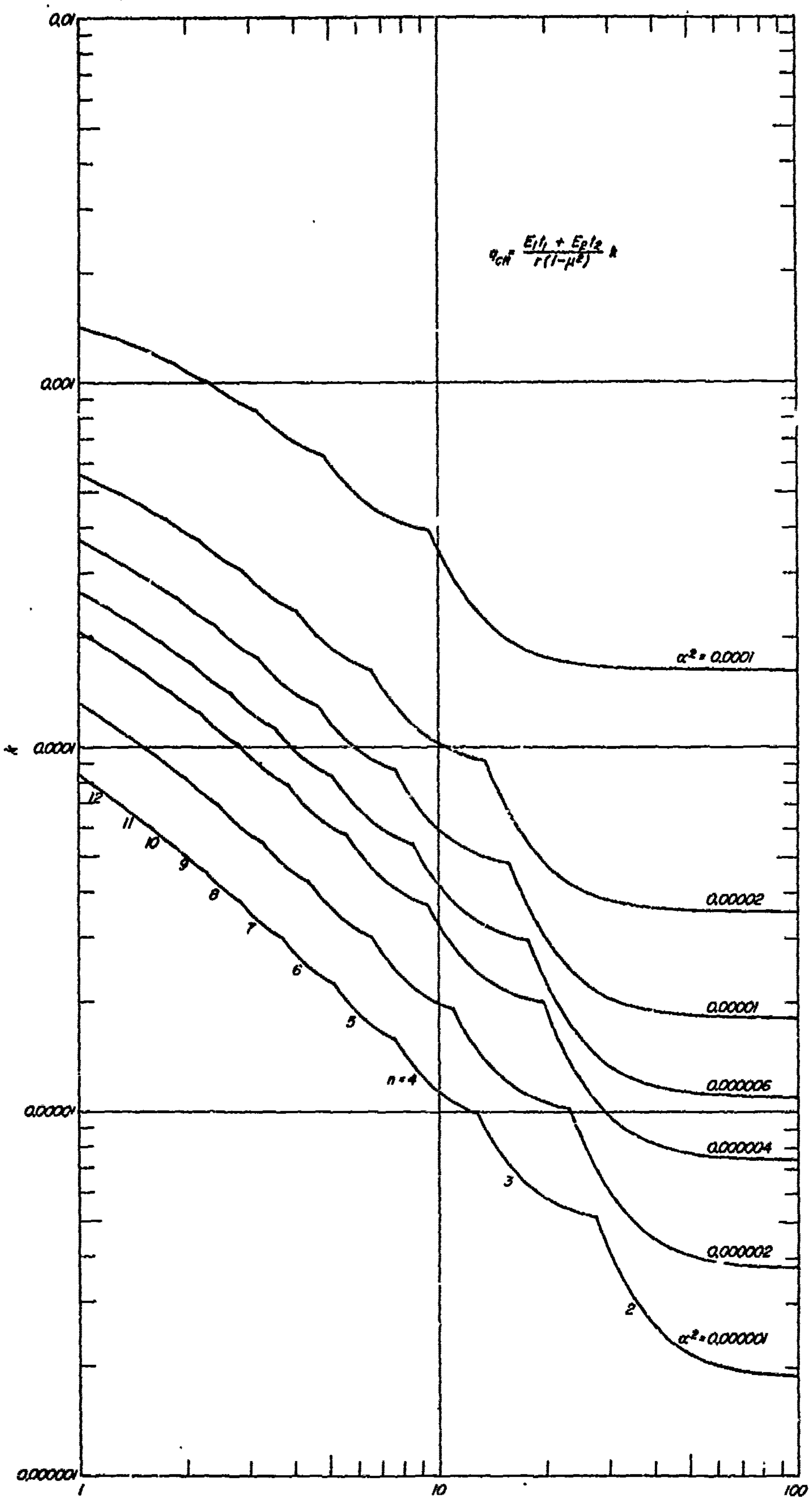

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Figure 16.--Values of k for V = 1.5, and for $\frac{E_1t_1}{E_2t_2} = 4$.

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Buckling coefficients for sandwich cylinders of finite length under uniform external lateral pressure, by E. W. Kuenzi, B. Bohannan, and G. H. Stevens. Madison, Wis., F.P.L., 1965.
26 pp., illus. (U.S. FS res. note FPL-0104)

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