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Radar Determination of Velocity Data from an Observatory Satellite

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RADAR DETERMINATION OF VELOCITY DATA

FROM AN OBSERVATORY SATELLITE

by

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SUMMARY

We obtain for several geometric situations the expression for the probability of insufficiently accurate data on the velocity of a surface target as a function of the parameters scan rate, nearest approach distance of an observatory satellite, the detection radii of the radar and the probability of detection per scan. The actual number of observations taken per pass of the satellite is random and in one instance we give explicitly the expectation of the probability of accurate data. This expectation can be calculated in other cases.

This paper allows the determination of the capability of the radar necessary to produce an acceptable probability of obtaining accurate velocity information.

1. INTRODUCTION

Consider an observatory satellite moving in a straight line across the plane which at any given time observes via radar a sector of an annulus on the surface. This annulus, with satellite position at the center, we call the surveillance ring. This ring, we consider, encounters a target which is also moving linearly in the plane at a velocity much less than that of the satellite.

It is easily seen that if we regard the satellite as being fixed, the target still travels linearly across the surveillance ring. We assume for this argument that the satellite is fixed. Note that from this assumption it would follow that the surface was moving which would have to be taken into account, if we were interested in estimating the satellite's position on the surface. This particular problem already has been the subject of two previous documents, [2] and [3], and we are now interested in a different one namely, the probability of obtaining good data from the radar.

Before we proceed further, let us introduce the following notation:

R_1 is the minimum detection radius of the radar,

R_2 is the maximum detection radius,

D is the nearest approach distance of the target to the radar,

p is the probability of detection per scan of the target in the surveillance ring,

h is the distance the target moves relative to the satellite between scans.

T is the distance between observations resulting in detection that must be obtained to procure the velocity data needed,

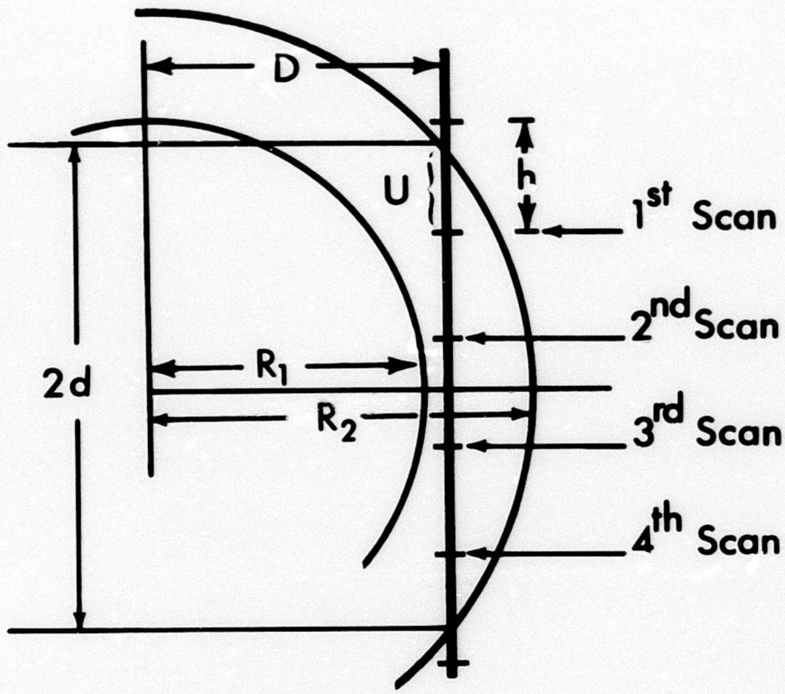
U is the distance the target moves into the annulus before being scanned.

At each sweep of the radar across the target, when the target is at least R_1 but less than a distance R_2 away, the event's detection or non-detection can occur with probabilities p and $q = 1 - p$, respectively. Moreover, successive sweeps of the radar are stochastically independent and so we have a sequence of Bernoulli trials as the target moves across the surveillance ring. We assume the probability of detection is zero when the target is outside the surveillance ring.

2. ONE-STAGE ACCEPTANCE WHEN THE DISTANCE OF NEAREST APPROACH EXCEEDS THE MINIMUM DETECTION RADIUS

We now assume that unsatisfactory data will be obtained if less than two detections are obtained or two or more detections are obtained but the first and last are less than a distance T apart.

Whenever $R_1 < D < R_2$, we have the following situation:



MOTION OF TARGET RELATIVE TO SATELLITE

The length of the path of the target when it is within the surveillance annulus is $2d$ and

$$d = \sqrt{R_2^2 - D^2} \quad . \tag{2.0.8}$$

The number of scans of the radar is N where

$$N = \left[\frac{2d - U}{h} \right] + 1 \tag{2.0.9}$$

with $[x]$ denoting the largest integer less than x for x any real number. Because we have a sequence of N Bernoulli trials each with probability of success p, we introduce the notation for the cumulative binomial probability

$$B(k:n,p) = \sum_{j=0}^{k} \binom{n}{j} p^{j}(1-p)^{n-j},$$

this being standard since the classic text of Feller [1]. The probability of obtaining less than two detections, that is either zero or one detection, is

$$B(1:N,p) = q^{N} + Npq^{N-1}. \tag{2.1}$$

Now the probability that we have at least two detections when the first detection occurs at the r^{th} scan and the last at the $r+s^{th}$ scan is seen to be

$$\prod_{i=1}^{n-1} q_i p_r p_{r+s} \prod_{i=r+s+1}^{N} q_i = p^2 q^{N-s-1},$$

where p_i is the probability of detection at the i^{th} scan. Hence, the probability of at least two detections with the first and the last less than or equal k units apart, where

$$k = [T/h], \tag{2.1.1}$$

is

$$Q_1 = \sum_{s=1}^{k}\sum_{r=1}^{N-s} p^2 q^{N-s-1} = \sum_{s=1}^{k}(N - s)p^2 q^{N-s-1} .$$

Letting the derivative with respect to the variable q be denoted by D_q,

$$Q_1 = p^2\sum_{s=1}^{k} D_q(q^{N-s}) = p^2 D_q\left(\sum_{s=1}^{k} q^{N-s}\right) = p^2 D_q\left(\frac{q^{N-k} - q^N}{1 - q}\right)$$

by using the fact that for any positive integers N,k

$$\sum_{s=1}^{k} q^{N-s} = \frac{q^{N-k} - q^N}{1 - q} . \qquad (2.1.5)$$

Now

$$D_q\left(\frac{q^{N-k} - q^N}{1 - q}\right) = \frac{q^{N-k} + (N-k)pq^{N-k-1} - (q^N + Nq^{N-1})}{p^2}$$

$$= \frac{B(1:N-k,p) - B(1:N,p)}{p^2} . \qquad (2.1.6)$$

Hence

$$Q_1 = B(1:N-k,p) - B(1:N,p). \qquad (2.2)$$

Thus the probability of bad or insufficiently accurate data for the first case is obtained by adding Equations (2.1) and (2.2) and is

$$\boxed{Q_2 = B(1:N-k,p) = (N-k)pq^{N-k-1} + q^{N-k}} \qquad (2.3)$$

where N is given in (2.0.9), k is given in (2.1.1).

Now the distance the target penetrates into the surveillance ring before it is scanned by the radar is a random variable which we called U. Thus, any function of U is also random e.g. N and expressions Q_1 and Q_2 above.

If we assume U is uniform on $(0,h)$ and we let

$$[d/h] = n \qquad (d/h)-n = f$$

be the integral and fractional part of d/h, one can see

$$N = \begin{cases} n+1 & \text{with probability} \quad f \\ n & \text{"} \qquad \text{"} \qquad 1-f. \end{cases} \tag{2.3.5}$$

Thus, we have by taking the expectation, the expected probability of obtaining unsatisfactory data

$$EQ_2 = fB(1{:}n-k+1,p) + (1-f)B(1{:}n-k,p). \tag{2.3.6}$$

Now this form lends itself to easy use with tables of the cumulative binomial distribution e.g. the Harvard tables [4].

One can obtain alternate forms by manipulation e.g.

$$EQ_2 = fp(n-k)q^{n-k} + (1-f)(n-k)pq^{n-k-1} + q^{n-k},$$

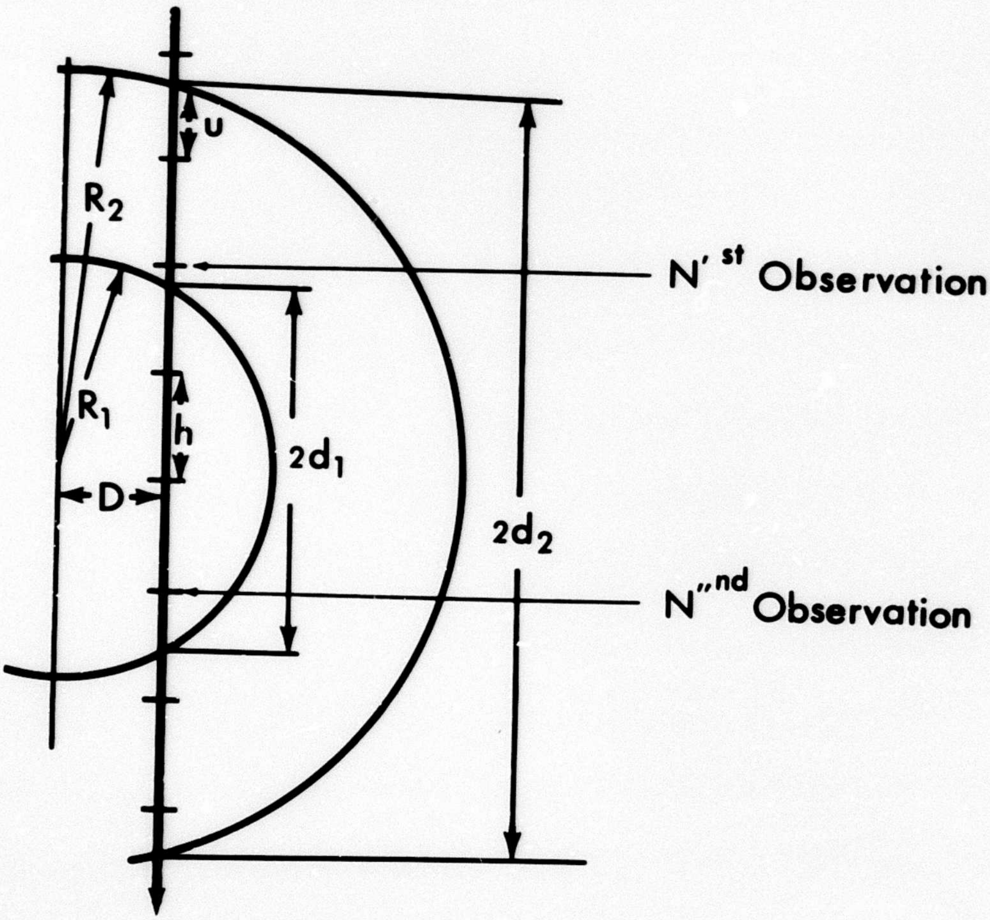
and again

$$EQ_2 = \Big((n-k)p(fq + 1 - f) + q\Big)q^{n-k-1} \tag{2.4}$$

as a final form for the probability sought for Case I.

3. ONE-STAGE ACCEPTANCE WHEN THE NEAREST APPROACH DISTANCE IS LESS THAN THE MINIMUM DETECTION RADIUS

We now consider the case when $0 < D < R_1$. At this distance, the relative motion of the target to the satellite carries it inside the blind range of the radar. The situation is as follows:



MOTION OF TARGET RELATIVE TO SATELLITE

We now number the scans in the order of their occurrence inside the surveillance ring from the top down. And we write

p_j = probability of detection at the j^{th} scan $j=1,\ldots,N$

where

$$N = \left[\frac{2d_2 - U}{h}\right] + 1 \tag{3.1}$$

$$d_2 = \sqrt{R_2^2 - D^2}\,, \qquad d_1 = \sqrt{R_1^2 - D^2}\,.$$

Also let N' be the number of observations obtained in passing through the first segment of the ring.

$$N' = [(d_2 - d_1 - U)/h] + 1 \tag{3.2}$$

and $N'' + 1$ be the number of the first observation obtained in the passage through the second segment of the ring,

$$N'' = [(d_1 + d_2 - U)/h] + 1. \tag{3.3}$$

Since in this case we have

$$p_j = \begin{cases} 0 & j = N'+1, \ldots, N'' \\ p & \text{otherwise} \end{cases}$$

The probability of zero detections is

$$\prod_{i=1}^{N} q_i = q^{N'+N-N''}\,.$$

The probability of one detection is

$$\sum_{j=1}^{N} p_j \prod_{i \neq j} q_i = \sum_{j=1}^{N'} pq^{N'+N-N''-1} + \sum_{j=N''+1}^{N} pq^{N'+N-N''-1}$$

$$= (N' + N - N'')pq^{N'+N-N''-1}\,. \tag{3.6}$$

Hence, as before we see the probability of less than two detections is

$$B(1{:}N' + N - N'', p). \tag{3.6.5}$$

The probability of two or more detections with the first detection at the r^{th} scan and the last detection at the $r + s^{th}$ scan is

$$\prod_{i=1}^{r-1} q_i \quad p_r p_{r+s} \prod_{i=r+s+1}^{N} q_i = \begin{cases} p^2 q^{N-s-N''+N'-1} & \text{for } r+s \le N' \\ & \text{or } r \ge N''+1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the probability of at least two detections with the first and last detections less than or equal k units apart, with $k = [T/h]$, is

$$Q_2' = \sum_{s=1}^{k} \sum_{r=1}^{N-s} \left(\prod_{i=1}^{r-1} q_i \right) p_r p_{r+s} \left(\prod_{i=r+s+1}^{N} q_i \right).$$

$$Q_2' = \sum_{s=1}^{k} \sum_{r=1}^{N-s} p^2 q^{N-s-N''+N-1} \left(\{r \ge N''+1\} + \{r + s \le N'\} \right)$$

where $\{x \le t\}$ is the indicator function of the relation taking the value one for all values of the variable for which it is true and the value zero otherwise. Now we have

$$Q_2' = \sum_{s=1}^{k} p^2 q^{N-s-N''+N'-1} \left((N-s-N'')^{+} + (N'-s)^{+} \right)$$

where again we have introduced the notation

$$(x)^{+} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

$$Q_2' = p^2 q^{N'} \sum_{s=1}^{k} (N-s-N'')^{\dagger} q^{N-s-N''-1} + p^2 q^{N-N''} \sum_{s=1}^{k} (N'-s)^{\dagger} q^{N'-s-1}$$

$$= p^2 q^{N'} D_q \left(\sum_{s=1}^{\{k,N-N''\}} q^{N-N''-s} \right) + p^2 q^{N-N''} D_q \left(\sum_{s=1}^{\{k,N'\}} q^{N'-s} \right)$$

where in this paper we make the convention that

$$\{a,b\} = \min(a,b). \tag{3.7}$$

Now using the result (2.1.6) we have

$$Q_2' = q^{N'} \Big(B(1:N-N''-\{k,N-N''\},p) - B(1:N-N'',p) \Big)$$
$$+ q^{N-N''} \Big(B(1:N-\{k,N'\},p) - B(1:N',p) \Big)$$

and by adding (3.6.5) and Q_2 above we have the probability of bad data for this case, namely,

$$Q_3 = B(1:N+N'-N'',p) + q^{N'} (B(1:N-N''-\{k,N-N''\},p) - B(1:N-N'',p))$$
$$+ q^{N-N''} (B(1:N'-\{k,N'\},p) - B(1:N'-p)).$$

But we notice that

$$B(1:N+N'-N'',p) - q^{N'} B(1:N-N'',p) - q^{N-N''} B(1:N',p)$$
$$= -q^{N-N''+N'} \tag{3.7.1}$$

hence, the answer is

$$\boxed{Q_3 = q^{N'}B(1:N-N''-\{k,N-N''\},p) + q^{N-N''}B(1:N'-\{k,N'\},p) - q^{N'-N''+N}} \quad (3.8)$$

To find the expected value of this expression, we are reduced to a tedious evaluation of special cases. Take as the integral and fractional parts of the following:

$$d_i/h = n_i + f_i \qquad i=1,2,$$

with $V = U/h$ being a random variable uniform on $(0,1)$,

$$N = [2n_2+2f_2-V] + 1 = \begin{cases} 2n_2 & \text{if} \quad 0 > 2f_2 - V > -1 \\ 2n_2 + 1 & \text{if} \quad 1 > 2f_2 - V > 0 \\ 2n_2 + 2 & \text{if} \quad 2 > 2f_2 - V > 1 \end{cases}$$

$$N' = [n_2-n_1+f_2-f_1-V] + 1 = \begin{cases} n_2 - n_1 + 1 & \text{if} \quad 1 > f_2 - f_1 - V > 0 \\ n_2 - n_1 & \text{if} \quad 0 > f_2 - f_1 - V > -1 \\ n_2 - n_1 - 1 & \text{if} \quad -1 > f_2 - f_1 - V > -2 \end{cases}$$

$$N'' = [n_1+n_2+f_1+f_2-V] + 1 = \begin{cases} n_1 + n_2 + 2 & \text{if} \quad 2 > f_1 + f_2 - V > 1 \\ n_1 + n_2 + 1 & \text{if} \quad 1 > f_1 + f_2 - V > 0 \\ n_1 + n_2 & \text{if} \quad 0 > f_1 + f_2 - V > -1 \end{cases}$$

Now the evaluation of the expectation of (3.8) with respect to the random variables N,N',N'' would be so tedious to write out and would convey such little information additional to (3.8) that we refrain from doing so.

4. THE TWO-STAGE ACCEPTANCE WITH NEAREST APPROACH EXCEEDING THE MINIMUM DETECTION RADIUS

We adopt the same probabilistic model as before only now the criterion for insufficient data we take as being less than two detections or two detections separated by at most a distance T_1 or three or more detections separated by at most T_2.

As before, we have in the first case the probability of less than two detections being

$$B(1:N,p) = q^N + Npq^{N-1}. \tag{4.1}$$

The probability of exactly two detections with the first and last less than or equal $k_1 = [T_1/h]$ units apart is

$$\sum_{s=1}^{k_1}\sum_{r=1}^{N-s} p^2q^{N-2} = \sum_{s=1}^{k_1}(N-s)p^2q^{N-2} = \frac{k_1(2N-k_1-1)}{2}\,p^2q^{N-2}. \tag{4.2}$$

The probability of three or more detections with the first and last detections less than or equal $k_2 = [T_2/h]$ units apart is easily seen to be by comparison with the argument of Section 2

$$\sum_{s=2}^{k_2}\sum_{r=1}^{N-s} p^2(1-q^{s-1})q^{N-s-1} = p^2\sum_{s=2}^{k_2}(N-s)(q^{N-s-1}-q^{N-2})$$

$$= p^2 D_q \left(\sum_{s=2}^{k_2} q^{N-s} \right) - \frac{(k_2-1)(2N-k_2-2)}{2} p^2 q^{N-2}$$

$$= B(1:N-k_2,p) - B(1:N-1,p) - p^2 q^{N-2} \frac{(k_2-1)(2N-k_2-2)}{2} . \quad (4.3)$$

Thus, the probability of insufficient data is the sum of (4.1), (4.2) and (4.3) which is

$$Q_4 = B(1:N,p) + \frac{k_1(2N-k_1-1)}{2} p^2 q^{N-2} + B(1:N-k_2,p) - B(1:N-1,p)$$
$$- p^2 q^{N-2} \frac{(k_2-1)(2N-k_2-2)}{2}$$

since $B(1:N,p) - B(1:N-1,p) = -(N-1)p^2 q^{N-2}$

$$\boxed{Q_4 = p^2 q^{N-2} \left(\frac{k_1(2N-k_1-1)}{2} - \frac{(k_2-1)(2N-k_2-2)}{2} - N + 1 \right) + B(1:N-k_2,p).} \quad (4.4)$$

By letting $k_1 = k_2$, we see this reduces to the preceding result of Equation (2.3). We do not bother with the expectation of Q_4.

5. THE TWO-STAGE ACCEPTANCE WITH NEAREST APPROACH LESS THAN THE MINIMUM DETECTION RADIUS

By direct analogy with Section 3, we have the probability of zero detections or one detection is

$$B(1:N'+N-N'',p). \tag{5.1}$$

The probability of exactly two detections with the first at the r^{th} scan and the last at the $r+s^{th}$ scan is

$$\prod_{i=1}^{r-1} q_i \, p_r \prod_{i=r+1}^{r+s-1} q_i \, p_{r+s} \prod_{i=r+s+1}^{N} q_i = \begin{cases} p^2 q^{N'+N-N''-2} & r+s \le N' \text{ or } r \ge N''+1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the probability of two detections with the first and last less than or equal $k_1 = [T_1/h]$ units apart is

$$Q_4' = \sum_{s=1}^{k_1} \sum_{r=1}^{N-s} p^2 q^{N'+N-N''-2} (\{r+s \le N'\} + \{r \ge N''+1\}).$$

Now for $i=1,2,$ let

$$C_i = \sum_{s=1}^{k_i} \sum_{r=1}^{N-s} \{r+s \le N'\} + \{r \ge N''+1\} = \sum_{s=1}^{k_i} \left((N'-s)^{+} + (N-N''-s)^{+}\right)$$

$$= \sum_{s=1}^{\{k_i,N'\}} (N'-s) + \sum_{s=1}^{\{k_i,N-N''\}} (N-N''-s)$$

$$C_i = \frac{\{k_i,N'\}(2N'-\{k_i,N'\}-1) + \{k_i,N-N''\}(2N-2N''-\{k_i,N-N''\}-1)}{2} .$$

$$\therefore \quad Q_4' = C_1 p^2 q^{N'+N-N''-2}$$

The probability of three or more detections with the first at the r^{th} scan and the last at $r + s^{th}$ scan by analogy with previous work is

$$= \begin{cases} p^2(1-q^{s-1})q^{N'+N-N''-s-1} & \text{if} \quad r + s \leq N' \\ & \text{or} \\ & \quad r \geq N'' + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the probability of three or more detections with the first and last less than or equal k_2 units apart is

$$\sum_{s=1}^{k_2}\sum_{r=1}^{N-s} p^2(1-q^{s-1})q^{N'+N-N''-s-1}\left(\{r + s \leq N'\} + \{r \geq N'' + 1\}\right)$$

$$= p^2\sum_{s=1}^{k_2} (1-q^{s-1})q^{N'+N-N''-s-1}\left((N'-s)^+ + (N-s-N'')^+\right)$$

$$= p^2\sum_{s=1}^{k_2} \Big((N'-s)^+ q^{N'+N-N''-s-1} - (N'-s)^+ q^{N'+N-N''-2} + (N-N''-s)^+ q^{N'+N-N''-s-1}$$

$$- (N-N''-s)^+ q^{N'+N-N''-2}\Big)$$

$$= p^2q^{N-N''}D_q\left(\sum_{s=1}^{\{k_2,N'\}} q^{N'-s}\right) + p^2q^{N'}D_q\left(\sum_{s=1}^{\{k_2,N-N''\}} q^{N-N''-s}\right) - C_2p^2q^{N'+N-N''-2}$$

$$= q^{N-N''}\Big(B(1{:}N'-\{k_2,N'\},p)-B(1{:}N',p)\Big)-C_2p^2q^{N'+N-N''-2}$$
$$+ q^{N'}\Big(B(1{:}N-N''-\{k_2,N-N''\},p)-B(1{:}N-N'',p)\Big).$$

Hence, the probability that we have insufficiently accurate data for this case is

$$Q_5 = B(1{:}N'+N-N'',p) + p^2q^{N'+N-N''-2}(C_1-C_2)$$
$$+ q^{N-N''}\Big(B(1{:}N'-\{k_2,N'\},p)-B(1{:}N',p)\Big)$$
$$+ q^{N'}\Big(B(1{:}N-N''-\{k_2,N-N''\},p)-B(1{:}N-N'',p)\Big)$$

which by the simplification of (3.7.1) is

$$\boxed{\begin{aligned} Q_5 = {} & p^2q^{N'+N-N''-2}(C_1-C_2) - q^{N-N''+N'} \\ & + q^{N-N''}B(1{:}N'-\{k_2,N'\},p) + q^{N'}B(1{:}N-N''-\{k_2,N-N''\},p) \end{aligned}}$$

which of course agrees with (3.8) if $k_1 = k_2$. Again we refrain from writing out the expectation of Q_5.

BIBLIOGRAPHY

[1] Feller, William: An Introduction to Probability Theory and Its Applications, Vol. I, Second Edition, John Wiley & Sons, 1957.

[2] Saunders, S. C. and Johnson, D. L.: On the Estimation of Location Coordinates from Data Taken from an Orbiting Satellite, Boeing document D1-82-0370, Boeing Scientific Research Laboratories, Seattle, Washington, 1964.

[3] Saunders, S. C.: Further Statistical Problems of Tracking a Target from an Observatory Satellite, Boeing document D1-82-0390, Boeing Scientific Research Laboratories, Seattle, Washington, 1964.

[4] Tables of the Cumulative Binomial Probability Distribution, Harvard University Press, 1955.