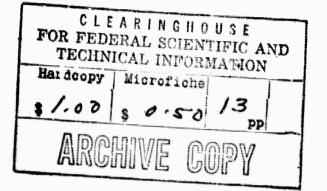GENERAL ATOMIC DIVISION
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REMARKS ON SIMILARITY SOLUTIONS

FOR HYPERVELOCITY IMPACT

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SUMMARY

The following report discusses the use of similarity flows in certain hydrodynamic impact problems, as a basis for approximate analytical solutions. After a qualitative discussion of various ways in which such solutions may be obtained, a _single_ derivation is obtained of a class of one-parameter similarity flows for problems in 1, 2 and 3 dimensions, which includes substantially all those used previously, and allows the superposition of constant uniform motions. The perturbation equations which relate the asymptotic form of the actual solution to such similarity flows (under some conditions) are explicitly formulated and the equations of the characteristic surfaces which determine their behavior are derived.

I. INTRODUCTION

Because of their non-linear form, the equations of hydrodynamics (and the problems they represent) are not amenable to general analytical solutions. On the other hand, they are often so complex that direct numerical solutions, if at all possible, become uneconomic and/or unconvincing without some kind of analytical guidance. This is particularly true in the case of problems involving shocks, where the singular nature of the flow may lead to serious numerical difficulties if its implications for the solution are not properly understood.

It therefore becomes necessary to utilize those analytical methods that are available to provide at least a qualitative understanding, if possible, of the solution of the problem posed, and even if not, a guide to the numerical approach. There are a variety of such techniques available in the various branches of hydrodynamics, and though they are by no means always adequate, they often provide the first step of an approximate analytical approach which meets the need described above. These methods involve one or more of the following:

1.   the use of special solutions, e.g., conformal
     mapping in stationary incompressible flows,
     Legendre's transformation in more general problems,
     and "similarity" (homological) solutions in problems
     involving explosions and/or boundary layers;

2.   perturbation techniques around such solutions (in
     particular singular perturbation methods for
     boundary layer problems); and

3.   integral methods based on the conservative properties
     of the flow. (These seem particularly applicable to
     problems involving shocks.)

In the case of hypervelocity impacts, one is confronted with problems which lack complete symmetry and involve severe discontinuities (shocks), as well as regions of smooth flow, and so we may expect a need for all the methods cited above in order to provide a satisfactory analytical foundation for understanding both the experimental and numerical results. Despite their general difficulty, impact problems have certain characteristics which suggest that the above cited methods are particularly applicable. The initial and boundary conditions are usually sufficiently localized in space and time as to effectively approximate an explosion, and a correspondingly similar "similarity" solution. The deviation of these conditions from those necessary for a similarity solution, as well as the strong deviation near the shock, is confined to a relatively small region of space and time, making possible effective perturbation methods (albeit of a singular nature in the second case).

There are thus several approaches available for providing an approximate analytical framework for understanding and further developing quantitative (i.e. generally numerical) solutions of hypervelocity impact problems. The first step in each of these is the establishment of some suitable "approximating" similarity solution. This may then be used in various ways:

1. as the first stage in an approximation or perturbation procedure in some parameter of the problem (e.g. the shock strength), (Sakurai);[5]

2. as a general form (with variable parameters) in an integral method, in which the parameters are determined phenomenologically from the conservation laws (Chernyi);[2] (see also Teichmann[3] and Stuart[4])

3. as an asymptotic solution, in which case it must be shown that the initial and boundary variations decay suitably as the process develops.

(1) and (2) above have received a good deal of attention in the cases of point explosions and hypersonic flow, in most cases with a certain amount of success. In the case of hypervelocity impact the problems are more difficult (largely because of equation-of-state limitations). Nevertheless some progress has been made: method (1) has been applied by Rae,[6] using results of Sakurai[5] and Oshima in the explosive case, and a version of method (2) has been used by Raizer.[1] The third method indicated above does not seem to have been used. It involves rather more mathematical profundity even in its formulation, but if successfully carried out provides a more satisfactory conceptual framework for both of the other methods cited, as well as for new ones which may be developed. It is not proposed to carry such a rather difficult program through here. What will be done below is to give a urified derivation of the various one dimensional similarity solutions used by Raizer and others, and briefly to formulate the mathematical problems of the asymptotic behavior of actual solutions.

The effective utilization of a similarity flow (or, indeed, any other flow) as a first step in an approximation procedure depends strongly on its simplicity and perspicuity. In all cases of interest, both for explosions and impacts, attention has been restricted to one-dimensional (or more strictly speaking, one-parameter) solutions, though the physical problem itself may of course be multi-dimensional. This restriction implies a substantial degree of geometric symmetry, more specifically, limitation to plane, spherically symmetric, and axial or transverse cylindrically symmetric problems. The question of any deviations from such symmetry will not even be touched on here.

II. SIMPLE SIMILARITY FLOWS

One thus considers the basic hydrodynamic equations (for a perfect gas) in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \ (\vec{v} \cdot \nabla \vec{v}) = - \nabla p$$

$$\frac{\partial}{\partial t} \ (p\rho^{-\gamma}) + \vec{v} \cdot \nabla(p\rho^{-\gamma}) = 0$$

Following the general method of Michal[7] as outlined by Morgan[8] and Manohar,[9] one searches for a one parameter set of transformations

$$t \rightarrow bt$$

$$\vec{r} \rightarrow b^{\alpha} \ \vec{r}$$

$$\vec{v} \rightarrow b^{\beta} \ \vec{v}$$

$$\rho \rightarrow \rho$$

$$p \rightarrow b^{\delta} p$$

which leave the equations invariant. (Note the requirement $\rho \rightarrow \rho$ imposed by the presence of an exterior medium.) One easily finds

$$\beta = \alpha - 1$$

$$\delta = 2(\alpha - 1) \ .$$

Introducing

$$\vec{s} = \vec{r}/t^{\alpha}$$

$$\vec{V} = \vec{v}/t^{\alpha-1}$$

$$P = p/t^{2(a-1)}$$

the equations become

$$\rho\nabla \cdot \vec{V} + (\vec{V} - \alpha\vec{s}) \cdot \nabla\rho = 0$$

$$\rho[(\alpha-1)\vec{V} + (\vec{V}-\alpha\vec{s}) \cdot \nabla\vec{V}] = -\nabla P$$

$$2(\alpha-1)[P\rho^{-\alpha} + (\vec{V}-\alpha\vec{s}) \cdot \nabla(P\rho^{-\gamma})] = 0$$

All the quantities are now functions of $\vec{s}$ alone, and ∇ now applies to $\vec{s}$.

For simplicity (tractability!) one now searches for a solution of the form

$$\vec{V} = \vec{A} + \Gamma \vec{s}$$

where $\vec{A}$ is a constant vector and Γ a constant dyadic (matrix). Then

$$\nabla \cdot \vec{V} = \text{trace } \Gamma = a, \text{ say}$$

Let

$$\Xi = \Gamma - \alpha 1$$

and

$$\vec{U} = \vec{V} - \alpha \vec{s}$$

$$= \vec{A} + \Xi \vec{s}$$

the equations then have the form

$$(\vec{A} + \Xi\vec{s}) \cdot \nabla\rho + a\rho = 0$$

$$(\Gamma+(\alpha-1)1)\vec{A} + \Gamma(\Gamma-1)\vec{s} + \frac{1}{\rho} \nabla P = 0$$

$$(A + \Xi s) \cdot \nabla(P\rho^{-\gamma}) + 2(\alpha-1)P \rho^{-\gamma} = 0$$

The first and last equations may be solved by the method of characteristics. Placing

$$\psi = \log \rho/\rho_0$$

one has

$$\frac{d\vec{s}}{d\psi} = -\frac{1}{a} (A + \Xi \vec{s})$$

$$\frac{d\vec{U}}{d\psi} = -\frac{1}{a}\,\Xi\,\vec{U}$$

with

$$\vec{s} = \vec{s}_0\,, \qquad \rho = \rho_0\,, \qquad \vec{V} = \vec{V}_0\,, \qquad \vec{A} = \vec{V}_0 - \Gamma\vec{s}_0\,.$$

for the initial conditions. Hence

$$\vec{U} = \vec{V}_0 - \Gamma\,\vec{s}_0 + \Xi\,\vec{s} = e^{-\frac{\psi}{a}\,\Xi}\,(\vec{V}_0 - \alpha\vec{s}_0)\,.$$

Similarly, putting $\chi = \log\frac{P}{P_0}$ the third equation gives

$$\vec{U} = e^{-\frac{(\chi-\gamma\psi)}{2(\alpha-1)}\,\Xi}\,(\vec{V}_0 - \alpha\vec{s}_0)$$

whence

$$\chi = (\gamma + \frac{2(\alpha-1)}{a})\,\psi$$

$$\frac{P}{P_0} = (\frac{\rho}{\rho_0})^{\gamma + \frac{2(\alpha-1)}{a}}$$

$$\nabla P = (\gamma + \frac{2(\alpha-1)}{a})P\,\frac{\nabla\rho}{\rho}\,.$$

Inserting these quantities into the second equation one finds (eventually!)

$$\vec{s} = \Xi^{-1}(1 - e^{-\frac{\psi}{a}\,\Xi})\,\vec{V}_0 + \vec{s}_0 + \alpha\Xi^{-1}(1 - e^{-\frac{\psi}{a}\,\Xi})\,\vec{s}_0$$

$$\vec{U}\cdot(\Gamma + (\alpha-1)1)\,(\vec{V}_0 - \Gamma\,\vec{s}_0) + \vec{U}\cdot(\Gamma - 1)\,\Gamma\,\vec{s} - (a\gamma + 2(\alpha-1))\frac{P}{\rho} = 0$$

$$(\vec{A} + \Xi\,\vec{s})\cdot(\Gamma + (\alpha-1)1)\,\vec{A} + (\vec{A} + \Xi\,\vec{s})(\Gamma - 1)\,\Gamma\,\vec{s}$$

$$= \frac{P_0}{\rho_0}(a\gamma + 2(\alpha-1))(\frac{\rho}{\rho_0})^{\gamma + \frac{2(\alpha-1)}{a} - 1}$$

$$= \vec{A}\cdot(\Gamma + (\alpha-1)1)\,\vec{A} + \vec{A}\,[\widetilde{\Xi}\,(\Gamma + (\alpha-1)1)\,\Xi + (\Gamma - 1)\,\Gamma]\,\vec{s}$$

$$+ \vec{s}\,[\Xi\,(\Gamma - 1)\,\Gamma]\,\vec{s}$$

Since Ξ is a 3 dimensional matrix (dyadic) all functions of Ξ may be written as the sum of 3 terms $\ldots 1 + \ldots \Xi + \ldots \Xi^2$; the coefficients above will involve the proper values c_i of Ξ, and the functions $e^{-(\psi/a)c_i}$, and possibly their derivatives up to second order. Since $\rho = \rho_o e^{\psi}$, the coefficients of terms involving derivatives must vanish. Since only one term appears on the right of the equation, the bilinear term $\vec{s}$ must vanish on the left. Hence one must have

$$\tilde{\Xi}\,(\Gamma - 1)\,\Gamma = 0$$

i.e., $\qquad\qquad (\Gamma - \alpha 1)\,(\Gamma - 1)\,\Gamma = 0$

To avoid inessential complications for the present let $\tilde{\Gamma} = \Gamma$, i.e. Γ is supposed symmetric, implying $\nabla \times \vec{V} = 0$ a reasonable assumption for many applications. Thus

$$(\Gamma - \alpha 1)\,(\Gamma - 1)\,\Gamma = 0$$

Hence α and/or 1 and/or 0 must be proper values of Γ. The case $\Gamma = 0$ corresponds to uniform motion, and is uninteresting. The case

$$\Gamma = \left\|\begin{matrix} 1 & & \\ & 0 & \\ & & 0 \end{matrix}\right\|$$

corresponds to a plane situation, the case

$$\Gamma = \left\|\begin{matrix} 1 & & \\ & 1 & \\ & & 0 \end{matrix}\right\|$$

to a cylindrical (line source) configuration, and the case $\Gamma = 1$ to a spherically symmetric or a point cylindrically symmetric situation. The presence of the vector $\vec{A}$ (which is generally independent of Γ) allows a superposition of uniform and expanding (contracting) motions. The parameter α must be determined by the exact shock conditions, by the properties of the perturbation solution, or by the integral relations.

The specific solutions for these cases are given by Raizer,[1] and their combination with conservation laws discussed there.

III. ASYMPTOTIC BEHAVIOR; PERTURBATION EQUATIONS

If φ is a vector representing the dependent variables, the hydro-dynamic equations have the form

$$\frac{\partial \varphi}{\partial t} = X(\varphi)$$

where X is a differential operator linear in the derivatives. A similarity transformation (of the type considered above) implies the existence of a transformation operator S such that

$$\Phi = S^{-1} \varphi$$

is independent of t. If

$$\Psi = \varphi - S \Phi \quad ,$$

then

$$\frac{\partial \Psi}{\partial t} = X(\varphi) - X(S\Phi)$$

$$= X(S\Phi - \Psi) - X(S\Phi)$$

$$\simeq X'(S\Phi) \Psi$$

plus higher order terms. These first order perturbation equations have the formal solution

$$\Psi(t) = e^{\int_0^t X'(S\Phi) \, dt} \Psi(o) \quad .$$

Thus, if the set of equations admits such a similarity transformation S, it is conceivable that a similarity solution Φ can be found such that $\Psi(t) \to 0$ as $t \to \infty$, even though $\Psi(0) \neq 0$ may be large.

Even if the equations do not admit an exact similarity solution (e.g., due to equation of state difficulties), it is possible that approximations to the equations do so, and that the remaining terms may be treated as an inhomogeneous perturbation, which under certain conditions $\to 0$ as $t \to \infty$ in analogy to the right hand side of $\Psi(t)$ above.

To conclude, it is of interest to write down the perturbation equations for the type of similarity transformation given above. Using the notation above, one writes

$$\rho(t,\vec{r}) \rightarrow \rho(t,\vec{s}) = R(\vec{s}) + G(t,\vec{s})$$

$$\vec{v}(t,\vec{r}) \rightarrow \vec{v}(t,\vec{s}) = t^{\alpha-1} \vec{V}(\vec{s}) + \vec{w}(t,\vec{s})$$

$$p(t,\vec{r}) \rightarrow p(t,\vec{s}) - t^{2(\alpha-1)} \underline{P}(\vec{s}) + q(t,\vec{s})$$

where $R(\vec{s})$ has now been written in place of $\rho(\vec{s})$. The perturbation equations have the form

$$\left(\frac{\partial}{\partial t} + \sum_1^3 D^i \frac{\partial}{\partial s_i} + C \right) \vec{h} = 0$$

where $\vec{h} = (\sigma, \vec{w}, q)$.

$$D^i = \begin{pmatrix} \frac{1}{t} U_i, & t^{-\alpha} R, & t^{-\alpha} R, & t^{-\alpha} R, & 0 \\ 0, & \frac{1}{t} U_i, & 0, & 0, & t^{-\alpha}/R \\ 0, & 0, & \frac{1}{t} U_i, & 0, & t^{-\alpha}/R \\ 0, & 0, & 0, & \frac{1}{t} U_i, & t^{-\alpha}/R \\ 0, & \gamma t^{\alpha-2} P, & \gamma t^{\alpha-2} P, & \gamma t^{\alpha-2} P, & \frac{1}{t} U_i \end{pmatrix}$$

with $\vec{U} = \vec{V} - \alpha \vec{s}$, and

$$
\begin{pmatrix}
\frac{1}{t} \nabla_s \cdot \vec{V}, & t^{-\alpha} (\nabla_s R)_1 & t^{-\alpha} (\nabla_s R)_2, & t^{-\alpha} (\nabla_s R)_3, & 0 \\[2ex]
-\frac{t^{\alpha-2}}{R^2} (\nabla_s P)_1, & \frac{1}{t} \nabla_{s1} V_1, & \frac{1}{t} \nabla_{s2} V_1, & \frac{1}{t} \nabla_{s3} V_1, & 0 \\[2ex]
-\frac{t^{\alpha-2}}{R^2} (\nabla_s P)_2, & \frac{1}{t} \nabla_{s1} V_2, & \frac{1}{t} \nabla_{s2} V_2, & \frac{1}{t} \nabla_{s3} V_2, & 0 \\[2ex]
-\frac{t^{\alpha-2}}{R^2} (\nabla_s P)_3, & \frac{1}{t} \nabla_{s1} V_3, & \frac{1}{t} \nabla_{s2} V_3, & \frac{1}{t} \nabla_{s3} V_3, & 0 \\[2ex]
0, & t^{\alpha-2} \nabla_{s1} P, & t^{\alpha-2} \nabla_{s2} P, & t^{\alpha-2} \nabla_{s3} P, & \frac{\gamma}{2} \nabla_s \vec{V}
\end{pmatrix}
$$

These perturbation equations have the characteristic surfaces determined by

$$
\varphi_t + \frac{1}{t} \vec{U} \cdot \nabla_s \varphi = 0
$$

$$
\varphi_t + \frac{1}{t} \vec{U} \cdot \nabla_s \varphi = \pm \sqrt{\frac{\gamma P}{R t^2}} \; (\Sigma \varphi_i)
$$

The solutions of the equations have not yet been derived for the cases considered here.

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