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MEASUREMENT OF MATRIX FREQUENCY RESPONSE FUNCTIONS AND MULTIPLE COHERENCE FUNCTIONS

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FOREWORD

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ABSTRACT

The report describes fundamental concepts involved in the statistical analysis of multiple-input single-output time-invariant linear systems. The definitions of a matrix frequency response function and a multiple coherence function are presented. Also discussed are marginal and conditional (partial) coherence functions with emphasis on their interpretation.

Formulas for computing simultaneous confidence bands for all elements of the matrix frequency response function are presented. Obtaining these confidence bands require the use of the standard "F" distribution. Expressions for these confidence bands are given both as a function of the various types of coherences and of the elements of the spectral density matrix. The effect of the various quantities on the width of the confidence bands is discussed in detail. Confidence bands for the gains and phases of the frequency response functions are also developed.

The interpretation of linear system computational results in terms of a time invariant nonlinear system model is described. It is shown how the linear system results provide what may be thought of as a "best" linear fit to the nonlinear model. The multiple coherence function then gives a quantitative measure of goodness of this fit. In this sense the coherence function may be used to provide a test for system linearity.

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LIST OF SYMBOLS

B	small frequency bandwidth of spectral analysis
$e(t)$	extraneous noise variable
f	frequency variable
$H_{jk}(f)$	frequency response function between $x_j(t)$ and $x_k(t)$
$H(f)$	$1 \times q$ matrix frequency response function $[H_1(f), \ldots, H_q(f)]$
$H_{kR}(f)$	real part of $H_k(f)$.
$H_{kI}(f)$	imaginary part of $H_k(f)$
i	$\sqrt{-1}$, imaginary unit
K_k	nonlinear time invariant operator relating input $x_k(t)$ with output $y(t)$
L_{pk}	linear time invariant operator relating record $x_k(t)$ with $x_p(\cdot)$
n	number of effective degree-of-freedom in coherence estimate
N	$N = BT$, the number of degrees-of-freedom in the spectral analysis where B = analysis bandwidth and T = record length. Many other reports use the convention $N = 2BT$, double the value defined here. Use care when comparing results.
p	total number of records (time series) in analysis, i.e., the dimension of the multivariate random process
q	the number of input variables
$S_{jk}(f)$	cross-spectral density function of $x_j(t)$ and $x_k(t)$. Power spectral density function when $j = k$.
$v_{p_1+k}(t)$	that part of $x_{p_1+k}(t)$ which is a linear functional of $x_1(t), \ldots, x_{p_1}(t)$, $k = 1, \ldots, p - p_1$
$w_{p_1+k}(t)$	kth conditioned process, conditioned on $x_1(t), \ldots, x_{p_1}(t)$, $k=1, \ldots, p-p_1$
$x_k(t)$	kth random process
$\hat{x}$	random variable which is estimate of parameter x
x'	transpose of x when x denotes a matrix

$\|S_{jk}(f)\|$	matrix with elements $S_{jk}(f)$
$\|S^{jk}(f)\|$	inverse of matrix $\|S_{jk}\|$ with elements $S^{jk}(f)$
$\gamma^2_{j\cdot 1,2,\ldots,j-1,j+1,\ldots p}(f)$	multiple coherence function between variable $x_j(t)$ and variables $x_1(t),\ldots,x_{j-1}(t),x_{j+1}(t),\ldots,x_p(t)$
$\gamma^2_{p\cdot p-1,p-2,\ldots,p_1+1\|1,2,\ldots,p_1}$	multiple conditional coherence between $x_p(t)$ and $x_{p-1}(t), x_{p-2}(t),\ldots,x_{p_1+1}(t)$ when conditioned on $x_1(t)$, $x_2(t),\ldots,x_{p_1}(t)$.
$\phi(f)$	phase of $H(f)$
$\Sigma(f)$	$p \times p$ spectral density matrix. Elements are the spectral density functions $S_{jk}(f)$.
$\Sigma_{jk}(f)$	submatrix of $\Sigma(f)$ when $\Sigma(f)$ is partitioned
$\Sigma_{p_1+1,\ldots,p\|1,2,\ldots,p_1}(f)$	the conditional spectral density matrix of the variables $x_{p_1+1}(t),\ldots,x_p(t)$ conditioned on the variables $x_1(t),\ldots,x_{p_1}(t)$. That is, the spectral density matrix of the time series $x_{p_1+k}(t)$, $k=1,\ldots,p-p_1$
$\Sigma_{xx}(f)$	$q \times q$ spectral density matrix of input variables $x_1(t),\ldots,x_q(t)$
$\Sigma_{xy}(f)$ $[\Sigma_{yx}(f)]$	$q \times 1$ $[1 \times q]$ spectral density vector of inputs $x_1(t),\ldots,x_2(t)$ with output $y(t)$
$\Sigma_{yy}(t)$	1×1 spectral density matrix of output $y(t)$ [identically equal to $S_{yy}(f)$]
$\Sigma_{y\|x}(f)$	the conditional spectral density matrix of an output variable $y(t)$ conditioned on the input variables $x_1(t),\ldots,x_q(t)$
$\Sigma^{-1}(f)$	inverse matrix of matrix $\Sigma(f)$
$\Sigma_{x_kx_k}(f)$	the kth diagonal element of the matrix $\Sigma_{xx}(f)$
$\Sigma^{kk}_{xx}(f)$	the kth diagonal element of the matrix $\Sigma^{-1}_{xx}(f)$

1. THE MEASUREMENT OF THE VARIOUS TYPES OF COHERENCE

In measuring frequency response functions and in many other applications, for example, measuring the kind and degree of relation between simultaneously recorded vibration records, one is led to the problem of measuring coherences of multiple stationary random functions (time series). There are various types of coherences. Four types are discussed here. They are (a) multiple coherence, (b) marginal multiple coherence, (c) conditional (or partial) multiple coherence, and (d) marginal conditional multiple coherence. The various types of coherence mentioned above are all particular functions of the elements of a spectral density matrix of a multiple stationary time series. (Formulas for the various types of coherence will be stated subsequently.) A spectral density matrix of a multiple stationary time series is a function of frequency f, and coherences are then also functions of frequency f. In speaking of a spectral density matrix or a coherence, one is really speaking of a spectral density matrix or a coherence at a particular frequency f_0.

From finite length records (e. g., simultaneously measured vibration records) that are regarded to be a finite length sample of a multiple stationary time series, one computes in an appropriate manner sample spectral density matrices corresponding to a collection of frequencies. To be more precise, each sample spectral density matrix corresponding to a particular frequency f_0 in reality pertains to a (usually) small frequency band of bandwidth B centered at frequency f_0. It is convenient, however, to speak of the sample spectral density matrix at frequency f_0.

The sample counterparts or estimators for the various types of coherences mentioned above are obtained in the following manner. At a particular frequency f_0 , each sample coherence is the same function of the elements of the sample spectral density matrix at frequency f_0 as the corresponding true coherence is of the elements of the true spectral density matrix. Subject to certain hypotheses, the joint distribution of the elements of a sample spectral density matrix has been derived in closed form (Reference 1). Furthermore, it is demonstrated in Reference 1 that if the frequencies corresponding to the collection of sample spectral density matrices are spaced a suitable distance apart, the sample spectral density matrices are essentially independently distributed. (This necessary spacing is the analysis bandwidth B where B is defined in a reasonable manner.) Since sample coherences are functions of the elements of a sample spectral density matrix corresponding to a particular frequency, sample coherences corresponding to different frequencies are also essentially independently distributed if the frequency spacing mentioned above prevails. With such a frequency spacing, the statistical uncertainty of sample coherences may then be described separately at each frequency f_0.

1.1 DEFINITION OF MULTIPLE COHERENCE

Formulas for and interpretations of the types of coherence mentioned above will now be stated.

Let $x_1(t), x_2(t), \ldots, x_{p_1}(t), x_{p_1+1}(t), \ldots, x_{p_1+p_2}(t)$ denote a $p_1 + p_2 = p$th order multiple stationary time series possessing the p x p spectral density matrix (at frequency f).

$$
\Sigma(f) = \left[\begin{array}{ccc|ccc}
S_{11}(f) & \cdots & S_{1p_1}(f) & S_{1,p_1+1}(f) & \cdots & S_{1p}(f) \\
\vdots & & \vdots & \vdots & & \vdots \\
S_{p_1,1}(f) & \cdots & S_{p_1p_1}(f) & S_{p_1,p_1+1}(f) & \cdots & S_{p_1p}(f) \\
\hline
S_{p_1+1,1}(f) & \cdots & S_{p_1+1,p_1}(f) & S_{p_1+1,p_1+1}(f) & \cdots & S_{p_1+1,p}(f) \\
\vdots & & \vdots & \vdots & & \vdots \\
S_{p1}(f) & \cdots & S_{pp_1}(f) & S_{p,p_1+1}(f) & \cdots & S_{pp}(f)
\end{array}\right] \qquad (1)
$$

In Eq. (1) the element $S_{jk}(f)$ of the matrix $\Sigma(f)$ denotes the cross-spectral density (at frequency f) between $x_j(t)$ and $x_k(t)$, $(j, k = 1, \ldots, p)$. A spectral density matrix $\Sigma(f)$ is always Hermitian non-negative definite. It will be presumed (for the present discussion) that the matrix $\Sigma(f)$ is positive definite, and hence non-singular. Let

$$
\Sigma(f) = \| S_{jk}(f) \| = \left[\begin{array}{c|c}
\Sigma_{11}(f) & \Sigma_{12}(f) \\
\hline
\Sigma_{21}(f) & \Sigma_{22}(f)
\end{array}\right] \qquad (2)
$$

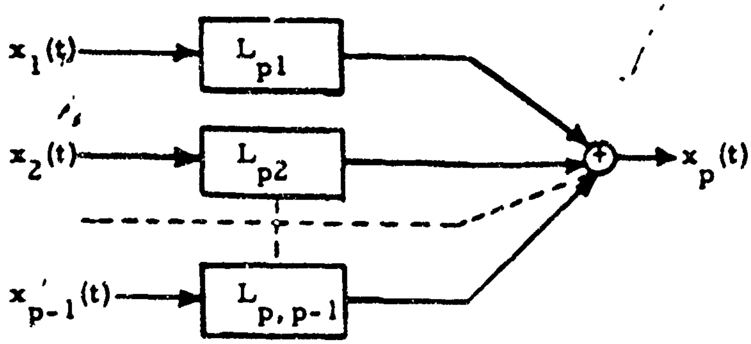
where the matrices $\Sigma_{11}(f)$, $\Sigma_{12}(f)$, $\Sigma_{21}(f)$, $\Sigma_{22}(f)$ in Eq. (2) are the submatrices of $\Sigma(f)$ indicated by the partitioning in Eq. (1). Let

$$\Sigma^{-1}(f) = \| S^{jk}(f) \| \tag{3}$$

The multiple coherence at frequency f between $x_p(t)$ and $[x_1(t), x_2(t), \ldots, x_{p-1}(t)]$ is given by the formula

$$\longrightarrow \quad \gamma^2_{p \cdot 1,2,\ldots,p-1}(f) = 1 - \frac{1}{S_{pp}(f)\, S^{pp}(f)} \tag{4}$$

The multiple coherence $\gamma^2_{p \cdot 1,2,\ldots,p-1}(f)$ ranges between zero and unity and measures or describes the degree to which (at frequency f) $x_p(t)$ is related to $[x_1(t), x_2(t), \ldots, x_{p-1}(t)]$ by means of linear time invariant operators L_{pk}, $k=1,\ldots,p-1$ acting on $x_k(t)$, $k=1,\ldots,p-1$ respectively. Stated another way, $\gamma^2_{p \cdot 1,2,\ldots,p-1}(f)$ measures or describes the degree to which (at frequency f) the system diagram indicated below prevails.



(5)

In Eq. (5), $L_{p1}, L_{p2}, \ldots, L_{p,p-1}$ denote linear time invariant operators. A multiple coherence of zero indicates no such relation prevails; the larger the multiple coherence is, the more nearly Eq. (5) represents the true relation with such a relation prevailing perfectly as multiple coherence becomes unity.

Since the subscripts on the $x(t)$ may be regarded as arbitrary labels, it is clear how Eq. (4) is a formula for other multiple coherences (at frequency f). For example, the multiple coherence at frequency f between $x_j(t)$ and the remaining (p-1) components of $\left[x_1(t), \ldots, x_p(t)\right]$ is given by

$$\gamma^2_{j \cdot 1, 2, \ldots, j-1, j+1, \ldots, p}(f) = 1 - \left(S_{jj}(f)\, S^{jj}(f)\right)^{-1}$$

Since any submatrix of $\Sigma(f)$ which is symmetric with respect to the main diagonal is a spectral density matrix (at frequency f) of selected components of $x_1(t), x_2(t), \ldots, x_p(t)$, one may employ such submatrices to compute other multiple coherences. Such multiple coherences are termed marginal multiple coherences or simply multiple coherences, when proper subscript notation indicates which components are involved. For example, if one considers $\left[x_1(t), \ldots, x_{p_1}(t)\right]$ then $\gamma^2_{p_1 \cdot 1, 2, \ldots, p_1 - 1}(f)$ is the multiple coherence at frequency f between $x_{p_1}(t)$ and $\left[x_1(t), \ldots, x_{p_1 - 1}(t)\right]$. To compute $\gamma^2_{p_1 \cdot 1, 2, \ldots, p_1 - 1}(f)$ one starts with the submatrix $\Sigma_{11}(f)$ of $\Sigma(f)$ and suitably applies the formula given by Eq. (4). It is clear that the interpretation of marginal multiple coherences is the same as that for multiple coherences.

1.2 CONDITIONAL (PARTIAL) COHERENCE

With respect to Eq. (1) there is a matrix computation that may be performed on $\Sigma(f)$ to yield a spectral density matrix of smaller dimensions. Such a smaller spectral density matrix is called a conditional (or partial) spectral density matrix. The formula for computing such a matrix and its interpretation will be explained with the partitioning and submatrices appearing in Eq. (1) and Eq. (2).

Consider the $p_2 \times p_2$ Hermitian matrix defined by:

$$\Sigma_{p_1+1,\ldots,p|1,2,\ldots,p_1}(f) = \Sigma_{22}(f) - \Sigma_{21}(f)\,\Sigma_{11}^{-1}(f)\,\Sigma_{12}(f) \qquad (6)$$

Recall that $\Sigma_{22}(f)$ is a $p_2 \times p_2$ matrix (where $p_2 = p - p_1$), $\Sigma_{21}(f)$ and $\Sigma_{12}(f)$ are $p_2 \times p_1$ and $p_1 \times p_2$ respectively, and $\Sigma_{11}(f)$ is $p_1 \times p_1$. Since $\Sigma(f)$ is positive definite, it follows that $\Sigma_{p_1+1,\ldots,p|1,2,\ldots,p_1}(f)$ is positive definite (and hence also non-singular). Since $\Sigma_{p_1+1,\ldots,p|1,2,\ldots p_1}(f)$ is a $p_2 \times p_2$ Hermitian positive definite matrix it could be the spectral density matrix (at frequency f) of a p_2th order multiple stationary time series $\left[w_{p_1+1}(t),\ldots,w_p(t)\right]$. With respect to the discussion on multiple coherence of the previous section, it is possible to represent the p_2th order multiple stationary time series $\left[x_{p_1+1}(t),\ldots,x_{p_1+p_2}(t)\right]$ in the following form:

$$x_{p_1+1}(t) = L_{11}\,x_1(t) + L_{12}\,x_2(t) + \ldots + L_{1p_1}\,x_{p_1}(t) + w_{p_1+1}(t)$$

$$x_{p_1+2}(t) = L_{21}\,x_1(t) + L_{22}\,x_2(t) + \ldots + L_{2p_1}\,x_{p_1}(t) + w_{p_1+2}(t) \qquad (7)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{p_1+p_2}(t) = L_{p_21}\,x_1(t) + L_{p_22}\,x_2(t) + \ldots + L_{p_2p_1}\,x_{p_1}(t) + w_{p_1+p_2}(t)$$

The interpretation of Eq. (7) is as follows. In Eq. (7), $\left[x_1(t),\ldots,x_{p_1}(t),\ x_{p_1+1}(t),\ldots x_{p_1+p_2}(t)\right]$ is the original $p_1+p_2 = p$th order multiple stationary time series of the previous section. The L_{jk} of Eq. (7) represent linear time invariant operators. One furthermore has $\left[w_{p_1+1}(t),\ldots,w_{p_1+p_2}(t)\right]$, a p_2th order multiple stationary time series with each of its components $w_{p_1+k}(t)$, $(k=1,\ldots,p_2)$, possessing at all frequencies f a multiple coherence of zero with $[x_1(t), x_2(t), \ldots, x_{p_1}(t)]$. The preceding property of the $w_{p_1+k}(t)$ may be explained roughly as follows. The components $w_{p_1+k}(t)$, $k=1,\ldots,p_2$, are the original components $x_{p_1+k}(t)$, $k=1,\ldots,p_2$ with the linear "effect" of the remaining components $x_j(t)$, $j=1,\ldots,p_1$, subtracted out. Since the linear "effect" between $x_{p_1+k}(t)$, $k=1,\ldots,p_2$, and $x_j(t)$, $j=1,\ldots,p_1$, has been subtracted out to obtain each $w_{p_1+k}(t)$, the multiple coherence between each $w_{p_1+k}(t)$ and $[x_1(t),\ldots,x_{p_1}(t)]$ is zero. The $p_2 \times p_2$ spectral density matrix of $[w_{p_1+1}(t),\ldots,w_{p_1+p_2}(t)]$ in Eq. (7) is given by Eq. (6).

The representation Eq. (7) is unique, and one may write formulas expressing the frequency response functions $H_{jk}(f)$ corresponding to the L_{jk} in terms of the elements of the spectral density matrix $\Sigma(f)$ of Eq. (1). The formulas for the frequency response functions $H_{jk}(f)$ are not stated here since they are not needed for the present discussion. However, let

$$
\begin{aligned}
v_{p_1+1} &\equiv L_{11} x_1(t) + L_{12} x_2(t) + \ldots + L_{1p_1} x_{p_1}(t) \\
v_{p_1+2} &\equiv L_{21} x_1(t) + L_{22} x_2(t) + \ldots + L_{2p_1} x_{p_1}(t) \qquad (8) \\
&\cdots\cdots\cdots\cdots\cdots \\
v_{p_1+p_2} &\equiv L_{p_21} x_1(t) + L_{p_22} x_2(t) + \ldots + L_{p_2p_1} x_{p_1}(t) .
\end{aligned}
$$

From the discussion on multiple coherence of the previous section, it is clear that $[v_{p_1+1}(t),\ldots,v_{p_1+p_2}(t)]$ is a p_2th order multiple stationary time series with each of its components $v_{p_1+k}(t)$, $(k=1,\ldots,p_2)$, possessing at all frequencies f a multiple coherence of unity with $\left[x_1(t), x_2(t), \ldots, x_{p_1}(t)\right]$. That is, by construction, there is a perfect linear relation between $v_{p_1+k}(t)$, $k=1,\ldots,p_2$ and $x_j(t), j=1,\ldots,p_1$. Now, from Eq. (8), one may write Eq. (7) in the form (uniquely)

$$\begin{bmatrix} x_{p_1+1}(t) \\ x_{p_1+2}(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{p_1+p_2}(t) \end{bmatrix} = \begin{bmatrix} v_{p_1+1}(t) \\ v_{p_1+2}(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_{p_1+p_2}(t) \end{bmatrix} + \begin{bmatrix} w_{p_1+1}(t) \\ w_{p_1+2}(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_{p_1+p_2}(t) \end{bmatrix} \tag{9}$$

where the V component is perfectly coherent at all frequencies f and the W component is perfectly incoherent at all frequencies f, with $\left[x_1(t), \ldots, x_{p_1}(t)\right]'$. The conditional (or partial) spectral density matrix $\Sigma_{p_1+1,\ldots,p|1,2,\ldots,p_1}(f)$ therefore is the $p_2 \times p_2$ spectral density matrix of $\left[x_{p_1+1}(t), \ldots, x_{p_1+p_2}(t)\right]$ after subtracting from $\left[x_{p_1+1}(t), \ldots, x_{p_1+p_2}(t)\right]$ that part which is attributable to the linear time invariant operators acting on $\left[x_1(t), \ldots, x_{p_1}(t)\right]$.

A conditional (or partial) multiple coherence is a multiple coherence computed from a conditional (or partial) spectral density matrix. The formula for a conditional spectral density matrix is given by Eq. (6); the formula for a multiple coherence by Eq. (4). Appropriate identification of submatrices and use of Eqs. (6) and (4) enable conditional multiple coherences to be determined from $\Sigma(f)$ of Eq. (1).

The following example illustrates the notation for conditional multiple coherence. The multiple coherence (at frequency f) between $x_p(t)$ and $[x_{p-1}(t), \ldots, x_{p_1+1}(t)]$ after conditioning on $[x_1(t), x_2(t), \ldots, x_{p_1}(t)]$ is denoted by $\gamma^2_{p \cdot p-1, p-2, \ldots, p_1+1 | 1, 2, \ldots, p_1}(f)$. To be specific, if a two input $x_1(t)$ and $x_2(t)$, single output $x_3(t)$, linear system is being analyzed then the multiple conditional coherence between $x_1(t)$ and $x_3(t)$ conditioned on $x_2(t)$ is denoted by $\gamma^2_{1 \cdot 3 | 2}(f)$. This special case reduces to the ordinary (2-dimensional) coherence between $x_1(t)$ and $x_3(t)$ after conditioning on $x_2(t)$ and is discussed in detail in Reference 5 with a slight change in subscript notation. A conditional multiple coherence measures or describes the degree to which (at frequency f) a component $x_p(t)$ is related by linear time invariant operators to other components $x_{p-1}(t), \ldots, x_{p_1+1}(t)$ after "effects" due to linear time invariant relations with other components $x_1(t), x_2(t), \ldots, x_{p_1}(t)$ have been subtracted from $x_p(t), x_{p-1}(t), \ldots, x_{p_1+1}(t)$.

A marginal conditional multiple coherence is a marginal multiple coherence computed from a conditional spectral density matrix. The previous discussions on marginal multiple coherence and conditional multiple coherence indicate how marginal conditional multiple coherence is to be interpreted. The following example illustrates the notation for marginal conditional multiple coherence. The marginal multiple coherence (at frequency f) between $x_p(t)$ and the components $[x_{p-1}(t), x_{p-2}(t)]$ of $[x_{p-1}(t), x_{p-2}(t), x_{p-3}(t), \ldots, x_{p_1+1}(t)]$ after conditioning on $[x_1(t), x_2(t), \ldots, x_{p_1}(t)]$ is denoted by $\gamma^2_{p \cdot p-1, p-2 | 1, 2, \ldots, p_1}(f)$. More specifically, if a three input $[x_i(t), i=1,2,3]$ single output $x_4(t)$ linear system is under consideration, then $\gamma^2_{1 \cdot 4 | 2}(f)$ is the marginal multiple coherence between the input $x_1(t)$ and the output $x_4(t)$. In this case the conditioning is on $x_2(t)$ and the third input $x_3(t)$ is effectively ignored.

1.3 THE DISTRIBUTION OF SAMPLE COHERENCE

From the above discussion on coherence and sample coherence it is clear that (at a particular frequency f_0) the various sample coherences corresponding to the various types of coherence are, in general, different functions of the elements of the sample spectral density matrix $\hat{\Sigma}(f_0)$ corresponding to the spectral density matrix $\Sigma(f_0)$ of Eq. (1). (The hat "^" notation will denote a sample [estimate] of the indicated quantity.) The statistical distribution corresponding to each type of sample coherence defined above has been derived, and the results obtained in closed form. (See References 1 and 2.) The statistical distributions of the various types of sample coherence are, in general, different. That is clearly to be expected. However, the probability density function of the distribution of the four types of sample coherence defined above may be expressed by the following general formula:

Let

$$
\begin{aligned}
n &= \text{effective number of degrees-of-freedom,} \\
p &= \text{effective number of records} \\
\gamma^2 &= \text{true coherence} \qquad (10) \\
\hat{\gamma}^2 = y &= \text{sample coherence}
\end{aligned}
$$

The probability density function of any type of sample coherence defined above is then given by

$$
C(y|n, p, \gamma^2) = \frac{\Gamma(n)}{\Gamma(p-1)\,\Gamma(n-p+1)} (1-\gamma^2)^n \, y^{p-2} (1-y)^{n-p} F(n, n; p-1; \gamma^2 y), \; (0 \leq y \leq 1) \qquad (11)
$$

In Eq. (11), $F(n, n; p-1; \gamma^2 y)$ is the hypergeometric function with the indicated parameters and variables. The method of determining the parameters n, p, γ^2 of Eq. (10) for the various types of sample coherences defined previously is now described.

With respect to Eq. (11) let N = BT denote the effective number of degrees-of-freedom of the spectral density estimator $\hat{\Sigma}(f_0)$ of $\Sigma(f_0)$. For a sample multiple coherence or a sample marginal multiple coherence n = N. For any sample conditional coherence $n = N - p_1$ where p_1 denotes the number of components that have been conditioned. For any type of sample multiple coherence, the parameter p is given by the total number of components involved in the coherence relation (not in general the total number of components of the multiple stationary time series). The parameter γ^2 is always the true value of coherence whatever the type. Examples: With reference to Eq.(11) and the previous discussion

a. For $\hat{\gamma}^2_{p\cdot 1,2,\ldots,p-1}(f_0)$ one has n = N, p = p, and $\gamma^2 = \gamma^2_{p\cdot 1,2,\ldots,p-1}(f_0)$.

b. For $\hat{\gamma}^2_{p_1\cdot 1,2,\ldots,p_1-1}(f_0)$ one has $n=N, p=p_1$, and $\gamma^2=\gamma^2_{p_1\cdot 1,2,\ldots,p_1-1}(f_0)$.

c. For $\hat{\gamma}^2_{p\cdot p-1,p-2,\ldots,p_1+1|1,2,\ldots,p_1}(f_0)$ one has $n = N-p_1$, $p = p_2$,

and $\gamma^2 = \gamma^2_{p\cdot p-1,p-2,\ldots,p_1+1|1,2,\ldots,p_1}(f_0)$.

d. For $\hat{\gamma}^2_{p\cdot p-1,p-2|1,2,\ldots,p_1}(f_0)$ one has $n = N-p_1, p = 3$,

and $\gamma^2 = \gamma^2_{p\cdot p-1,p-2|1,2,\ldots,p_1}(f_0)$.

For more concrete examples, assume a three input $[x_i(t), i=1,2,3]$ single output, $x_4(t)$, linear system with N degrees-of-freedom in the measurements.

a. The sample multiple coherence between the output $x_4(t)$ and the inputs is $\hat{\gamma}^2_{4\cdot 1,2,3}(f)$ and one has n=N, p=4 and $\gamma^2 = \gamma^2_{4\cdot 1,2,3}(f)$.

b. The sample marginal multiple coherence between the input $x_3(t)$ and the inputs $x_1(t)$ and $x_2(t)$ while ignoring the output $x_4(t)$ is $\hat{\gamma}^2_{3\cdot 1,2}(f)$ and one has n=N, p=3 and $\gamma^2 = \gamma^2_{3\cdot 1,2}(f)$.

c. The sample conditional multiple coherence between the output $x_4(t)$ and the two inputs $x_2(t)$ and $x_3(t)$ conditioned on $x_1(t)$ is $\hat{\gamma}^2_{4\cdot 3,2|1}(f)$ and one has $n=N-1$, $p=3$, and $\gamma^2 = \gamma^2_{4\cdot 3,2|1}(f)$.

d. The sample marginal conditional multiple coherence between the output $x_4(t)$ and the input $x_3(t)$ while conditioning on $x_1(t)$ and ignoring $x_2(t)$ is $\hat{\gamma}^2_{4\cdot 3|1}(f)$ and one has $n=N-1$, $p=2$, and $\gamma^2 = \gamma^2_{4\cdot 3|1}(f)$.

Tables of the cumulative distribution function corresponding to the probability density function, Eq. (11), have been calculated for the parameter p ranging from two through ten, and for n such that $p \leq n \leq 20$ (Reference 3). The rules for n and p described above may be applied in order to properly make use of these tables.

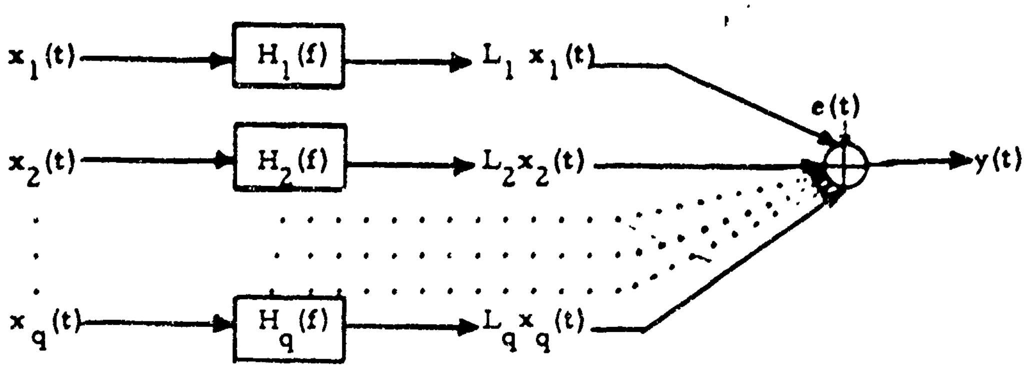
2. THE MEASUREMENT OF MATRIX FREQUENCY RESPONSE FUNCTIONS

Consider q time functions $x_1(t), \ldots, x_q(t)$ and a time function $y(t)$ related by the equation:

$$y(t) = L_1\, x_1(t) + L_2\, x_2(t) + \ldots + L_q\, x_q(t) + e(t) \quad . \tag{12}$$

In Eq. (12) the L_k, $k=1,\ldots,q$ denote linear time invariant operators possessing corresponding frequency response functions $H_k(f)$, $k=1,\ldots,q$. The function $e(t)$ is presumed to be a zero mean stationary Gaussian random function statistically independent of the functions $x_k(t)$, $k=1,\ldots,q$. The spectral density $S_e(f)$ of $e(t)$ is presumed unknown. The frequency response functions $H_k(f)$, $k=1,\ldots,q$ are also presumed to be unknown.

Equation (12) may be viewed as expressing the multiple-input single-output (with extraneous noise) block diagram illustrated below.



(13)

The 1 x q (complex valued) matrix

$$H(f) \equiv \left[H_1(f), \ldots, H\ (f)\right] \equiv \left[H_{1R}(f) + iH_{1I}(f), \ldots, H_{qR}(f) + iH_{qI}(f)\right] \tag{14}$$

is called the matrix frequency function of Eq. (12) or equivalently of the system described by the block diagram.

2.1 FREQUENCY RESPONSE FUNCTION ESTIMATES

Suppose a single finite realization $0 \leq t \leq T$ of the functions $x_1(t), \ldots, x_q(t)$, $y(t)$ of Eq. (12) is observed (recorded). From the finite length observed records of $[x_1(t), \ldots, x_q(t),\ y(t)]$ simultaneous confidence bands for the elements of the matrix frequency response function $H(f_0)$ at a particular frequency f_0 are to be determined. A discussion analogous to that of Section 1 establishes that estimators for the matrix frequency response function at a collection of frequencies are essentially statistically independent if the frequencies are spaced suitably apart. Thus, with such a frequency spacing, simultaneous confidence bands for the elements of the matrix frequency response function $H(f_0)$ may be independently determined at each particular frequency f_0.

An estimator $\hat{H}(f_0)$ for the matrix frequency response function $H(f_0)$ is obtained in the following manner. The finite length $(0 \leq t \leq T)$ of records $[x_1(t), \ldots, x_q(t),\ y(t)]$ are treated as if they were a finite realization of a (q+1)th order multiple stationary time series. Proceeding by the method of spectral estimation (Reference 1) a (q+1)x(q+1) sample spectral density matrix at frequency f_0

$$\hat{\Sigma}(f_0) \equiv \left[\begin{array}{c|c} \hat{\Sigma}_{xx}(f_0) & \hat{\Sigma}_{xy}(f_0) \\ \hline \hat{\Sigma}_{yx}(f_0) & \hat{\Sigma}_{yy}(f_0) \end{array}\right] \tag{15}$$

is then computed. It is presumed that the degrees-of-freedom parameter n associated with $\hat{\Sigma}(f_0)$ satisfies $n \geq q + 1$. It is also assumed that the $q \times q$ matrix $\hat{\Sigma}_{xx}(f_0)$ is non-singular. The estimator $\hat{H}(f_0)$ for $H(f_0)$ is then

$$\hat{H}'(f_0) \equiv \hat{\Sigma}_{xx}^{-1}(f_0)\ \hat{\Sigma}_{xy}(f_0) \equiv \left[\hat{H}_1(f_0), \ldots, \hat{H}_q(f_0)\right] \equiv \left[\hat{H}_{1R}(f_0) + i\hat{H}_{1I}(f_0), \ldots, \hat{H}_{qR}(f_0) + i\hat{H}_{qI}(f_0)\right] \tag{16}$$

2.2 CONFIDENCE BANDS FOR MATRIX FREQUENCY RESPONSE FUNCTIONS

At frequency f_0 the sample conditional spectral density of $y(t)$ conditioned on $[x_1(t), \ldots, x_q(t)]$ is

$$\hat{\Sigma}_{y|x}(f_0) \equiv \hat{\Sigma}_{yy}(f_0) - \hat{\Sigma}_{yx}(f_0)\,\hat{\Sigma}_{xx}^{-1}(f_0)\,\hat{\Sigma}_{xy}(f_0). \tag{17}$$

Define the two quantities

$$\begin{aligned} \hat{A}(f_0) &\equiv \hat{\Sigma}_{y|x}(f_0) \\ \hat{B}(f_0) &\equiv \overline{\left(\hat{H}(f_0) - H(f_0)\right)}\,\hat{\Sigma}_{xx}(f_0)\left(\hat{H}(f_0) - H(f_0)\right)' \end{aligned} \tag{18}$$

As a special case of general results summarized in Reference 4, it follows that under appropriate hypotheses the quantity $\left(\frac{n-q}{q}\right) \cdot \frac{\hat{B}(f_0)}{\hat{A}(f_0)}$ possesses the standard F distribution with 2q and 2(n-q) degrees-of-freedom. The cumulative distribution of

$$\frac{\hat{A}(f_0)}{\hat{A}(f_0) + \hat{B}(f_0)} \equiv \left[1 + \frac{\hat{B}(f_0)}{\hat{A}(f_0)}\right]^{-1} \tag{19}$$

is therefore directly and easily determined from the cumulative F distribution. Without going into hypotheses details, it is nevertheless important to mention here that the hypotheses do not involve statistical distribution conditions on the input functions $x_1(t), \ldots, x_q(t)$. For example, the input functions $x_1(t), \ldots, x_q(t)$ are permitted to be nonstationary or nonrandom, etc.

Given a probability (confidence level) p_0 $(0 < p_0 < 1)$ one may then, using the distribution result stated above, determine the corresponding unique constant $a_0 (0 < a_0 < 1)$ so that

$$\text{Prob}\left[a_0 < \frac{\hat{A}(f_0)}{\hat{A}(f_0) + \hat{B}(f_0)}\right] = p_0 \tag{20}$$

From Eq. (18) and Eq. (20), using various algebraic results and inequalities (Reference 4), the simultaneous confidence bands for all the elements of the matrix frequency response function $H(f_0)$ stated below are obtained.

$$\text{Prob}\left[\begin{array}{c}\left|H_k(f_0) - \hat{H}_k(f_0)\right|^2 \leq \left(\frac{1}{a_0} - 1\right)\hat{\Sigma}_{y|x}(f_0)\,\hat{\Sigma}_{xx}^{kk}(f_0)\\ k = 1,\ldots,q\end{array}\right] \geq p_0 \qquad (21)$$

In Eq. (21), $\hat{\Sigma}_{y|x}(f_0)$ is given by Eq. (17) and $\hat{\Sigma}_{xx}^{kk}(f_0)$, $k = 1,\ldots,q$ denote the indicated diagonal elements of the q x q matrix $\hat{\Sigma}_{xx}^{-1}(f_0)$ where the matrix $\hat{\Sigma}_{xx}(f_0)$ is defined in Eq. (15).

2.3 CONFIDENCE AS FUNCTION OF COHERENCE

For the case where the inputs $x_1(t),\ldots,x_q(t)$ are stationary random functions, it is desirable for interpreting and describing results related to the measurement of frequency response functions to rephrase Eq. (21). From Eq. (15), $\hat{\Sigma}_{xx}(f_0)$ is the q x q sample spectral density matrix at frequency f_0 of the multiple inputs $x_1(t),\ldots,x_q(t)$. Also from Eq. (15) one may regard $\hat{\Sigma}_{xx}(f_0)$ to be the q x q sample marginal spectral density matrix of the (q+1)x(q+1) sample spectral density matrix $\hat{\Sigma}(f_0)$ marginal on $x_1(t),\ldots,x_q(t)$.

From Reference 1, the sample multiple coherence at frequency f_0 between the input $x_k(t)$ and the other (q-1) inputs $x_1(t),\ldots,x_{k-1}(t),x_{k+1}(t),\ldots,x_q(t)$ is given by:

$$\hat{\gamma}^2_{x_k\cdot x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_q}(f_0) = 1 - \left(\hat{\Sigma}_{x_k x_k}(f_0)\,\hat{\Sigma}_{xx}^{kk}(f_0)\right)^{-1}, \quad (k=1,\ldots,q) \qquad (22)$$

In Eq. (22), $\hat{\Sigma}_{x_k x_k}(f_0)$ denotes the sample spectral density at frequency f_0 of $x_k(t)$ and $\hat{\Sigma}_{xx}^{kk}(f_0)$ is defined at the conclusion of Section 2.2. From Eq. (22) one obtains

$$\hat{\Sigma}^{kk}_{xx}(f_0) = \frac{1}{\left(1 - \hat{\gamma}^2_{x_k \cdot x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q}\right) \hat{\Sigma}_{x_k x_k}(f_0)} \tag{23}$$

Let $\hat{\gamma}^2_{y \cdot x_1, x_2, \ldots, x_q}(f_0)$ denote the sample multiple coherence at frequency f_0 between the output $y(t)$ and the q inputs $x_1(t), x_2(t), \ldots, x_q(t)$. On applying the equation for sample multiple coherence (Reference 1) to the matrix $\hat{\Sigma}(f_0)$ given by Eq. (15), performing the necessary matrix calculations and using Eq. (17), one obtains

$$\hat{\Sigma}_{y|x}(f_0) = \left[1 - \hat{\gamma}^2_{y \cdot x_1, x_2, \ldots, x_q}(f_0)\right] \hat{\Sigma}_{yy}(f_0) \tag{24}$$

From Eqs. (23) and (24) one may write Eq. (21) in the form

$$\text{Prob}\left[\begin{array}{c} |H_k(f_0) - \hat{H}_k(f_0)|^2 \leq \left(\frac{1}{a_0} - 1\right) \dfrac{\left[1 - \hat{\gamma}^2_{y \cdot x_1, x_2, \ldots, x_q}(f_0)\right] \hat{\Sigma}_{yy}(f_0)}{\left[1 - \hat{\gamma}^2_{x_k \cdot x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q}\right] \hat{\Sigma}_{x_k x_k}(f_0)} \\ (k = 1, \ldots, q) \end{array}\right] \geq P_0 \tag{25}$$

Equation (25) expresses the simultaneous confidence bands for all the elements of the matrix frequency response function $H(f_0)$ in terms of the sample multiple coherence between the output and all the inputs, the sample multiple coherence between each input and all the other inputs, and the ratio between the sample output spectral density to the sample input spectral densities.

From Eq. (25) one may discern how the various sample coherences, sample spectral densities, and parameters q and n govern the accuracy with which frequency response functions are measured. One notes, for example, that the accuracy with which frequency response functions are measured;

a. improves as the sample multiple coherence between the output and inputs $\hat{\gamma}^2_{y\cdot x_1, x_2, \ldots, x_q}(f_0)$ increases toward unity,

b. diminishes as sample multiple coherences between inputs increase,

c. improves as sample input spectral densities increase,

d. improves as the degrees-of-freedom parameter n increases (since for fixed p_0 and q the "constant" $(a_0^{-1} - 1)$ diminishes as n increases).

The distributions of the sample coherences appearing in Eq. (25) are given by Eq. (11). For $\hat{\gamma}^2_{y\cdot x_1, x_2, \ldots, x_q}(f_0)$ the parameter p in Eq. (11) is $p = q + 1$. For the sample coherences $\hat{\gamma}^2_{x_k \cdot x_1, \ldots, x_{k-1}\, x_{k+1}, \ldots, x_q}$ $(k=1, \ldots, q)$ the parameter p in Eq. (11) is $p = q$. In all cases γ^2 in Eq. (11) denotes the true value of the respective coherences.

The simultaneous confidence bands on all the elements $H_k(f_0)$ $(k=1, \ldots, q)$ of the matrix frequency response function $H(f_0)$ given by Eq. (25) are determined by the sample frequency response functions, the sample coherences, and the sample spectral densities appearing in Eq. (25). It is important to notice that one requires no a priori knowledge of the frequency response functions, the coherences, or the spectral densities to determine the confidence bands on the elements of the matrix frequency response function $H(f_0)$ in using Eq. (25).

One may also use Eq. (25) as a guide in planning measurement programs or experiments to determine the elements of a matrix frequency response function H(f). When that is done, a priori estimates or knowledge of sample coherences and sample spectral densities expected to be obtained are substituted in Eq. (25) and the results yield apriori estimates of accuracies with which the elements of a matrix frequency response function H(f) will be determined. If one is limited to only measuring the inputs and output one is then only able to estimate the degrees-of-freedom parameter n (or alternatively the lengths of record and frequency resolutions) needed to approximately achieve desired accuracy in measuring frequency response functions. If one, say in some experiment, is able to control or select the inputs to some degree more may be achieved. In that regard, the previous remarks (on how coherences and spectral densities govern the accuracy with which frequency response functions are measured) become especially helpful. For example, one would, if possible, select inputs that are incoherent with each other.

2.4 CONFIDENCE BANDS FOR GAIN AND PHASE

In most applied work one customarily expresses frequency response functions in terms of gains and phases. From Eq. (25) simultaneous confidence bands on all the real parts, the imaginary parts, the gains, and the phases of the frequency response functions $H_k(f_0)$, k=1,...,q, may be obtained. With reference to Eq. (25), let

$$\hat{r}_k^2(f_0) = (a_0^{-1}-1)\frac{\left(1-\hat{\gamma}^2_{y\cdot x_1,x_2,\ldots,x_q}(f_0)\right)\hat{\Sigma}_{yy}(f_0)}{(1-\hat{\gamma}^2_{x_k\cdot x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_q})\hat{\Sigma}_{x_kx_k}(f_0)}, \quad (k=1,\ldots,q) \tag{26}$$

Furthermore, let

$$H_k(f_0) = H_{kR}(f_0) + iH_{kI}(f_0) \equiv |H_k(f_0)|e^{i\phi(f_0)}, \quad (k=1,\ldots,q) \tag{27}$$

and

$$\hat{H}_k(f_0) \equiv \hat{H}_{kR}(f_0) + i\hat{H}_{kI}(f_0) \equiv |\hat{H}_k(f_0)|e^{i\hat{\phi}(f_0)}, \quad (k=1,\ldots,q) \tag{28}$$

Equation (27) defines the real parts $H_{kR}(f_0)$, the imaginary parts $H_{kI}(f_0)$, the gains $|H_k(f_0)|$, and the phases $\phi_k(f_0)$ of the frequency response functions $H_k(f_0)$, k=1,...,q. Equation (28) defines the sample real parts $\hat{H}_{kR}(f_0)$, the sample gains $|\hat{H}_k(f_0)|$, the sample imaginary parts $\hat{H}_{kI}(f_0)$, and the sample phases $\hat{\phi}_k(f_0)$ of the frequency response functions $H_k(f_0)$, k=1,...,q. Consider the diagram sketched in Figure 1 below.

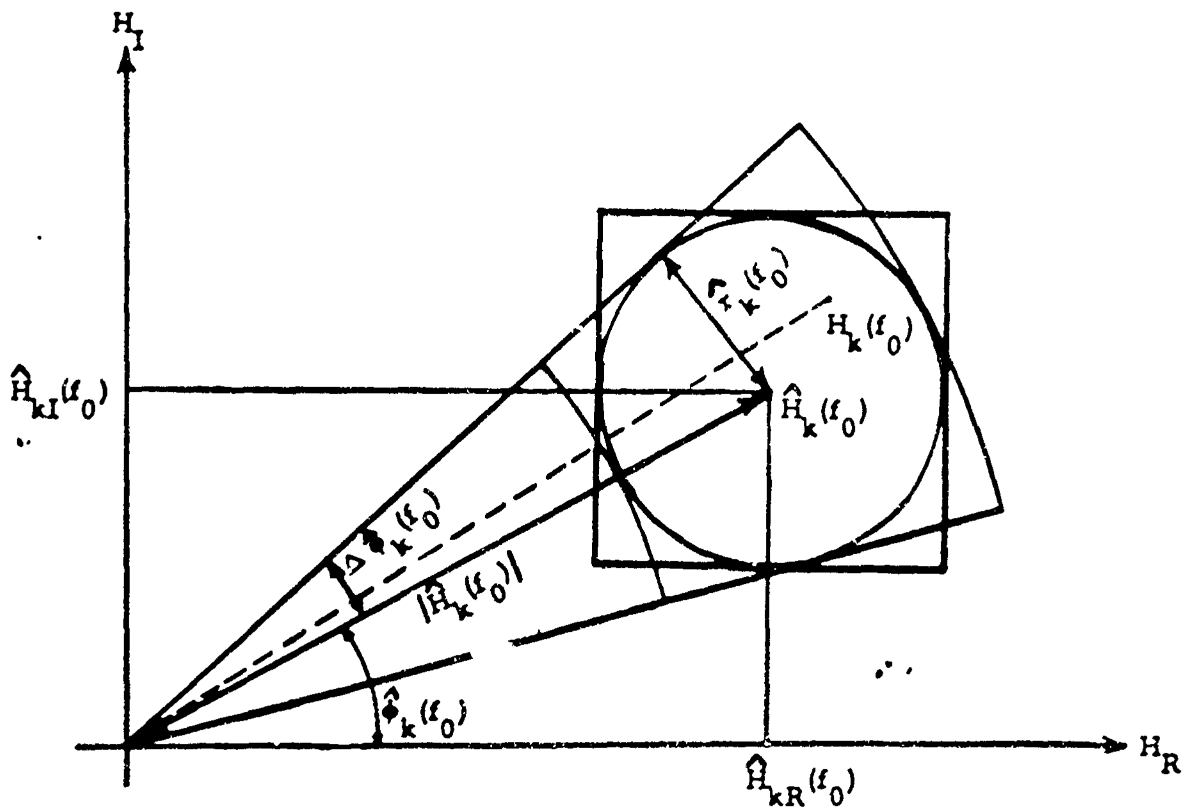


Figure 1. Confidence Band Diagram

In drawing the diagram of Figure 1, it is presumed that $\hat{r}_k(f_0) < |\hat{H}_k(f_0)|$. One has

$$\Delta\hat{\phi}_k(f_0) \equiv \text{Arc sin} \frac{\hat{r}_k(f_0)}{|\hat{H}_k(f_0)|} \quad , \ (k=1,\ldots,q) \tag{29}$$

From Eq. (25), Figure 1, and the various defining equations above, one obtains the simultaneous confidence band statement:

$$
\text{Prob}\left[\begin{array}{c}
\hat{H}_{kR}(f_0) - \hat{r}_k(f_0) \le H_{kR}(f_0) \le \hat{H}_{kR}(f_0) + \hat{r}_k(f_0) \\
\hat{H}_{kI}(f_0) - \hat{r}_k(f_0) \le H_{kI}(f_0) \le \hat{H}_{kI}(f_0) + \hat{r}_k(f_0) \\
|\hat{H}_k(f_0)| - \hat{r}_k(f_0) \le |H_k(f_0)| \le |\hat{H}_k(f_0)| + \hat{r}_k(f_0) \\
\hat{\phi}_k(f_0) - \Delta\hat{\phi}_k(f_0) \le \phi_k(f_0) < \hat{\phi}_k(f_0) + \Delta\hat{\phi}_k(f_0) \\
(k=1,\ldots,q)
\end{array}\right] \ge P_0 \qquad (30)
$$

8. THE APPLICATION OF COHERENCE FUNCTIONS TO NONLINEAR MULTIPLE-INPUT SINGLE-OUTPUT TIME INVARIANT SYSTEMS

Consider q time functions $x_1(t), \ldots, x_q(t)$ and a time function $y(t)$ now related by the equation:

$$y(t) = K_1 x_1(t) + K_2 x_2(t) + \ldots + K_q x_q(t) \tag{31}$$

In Eq. (31) the K_k, $k=1,\ldots,q$ denote time invariant operators here not (necessarily)linear. The operators K_k, $k=1,\ldots,q$ are presumed to be unknown. Equation (31) describes a multiple-input single-output (possibly) nonlinear time invariant system.

Suppose a single finite realization $0 \leq t \leq T$ of each of the functions $x_1(t), \ldots, x_q(t)$, $y(t)$ is observed (recorded). Suppose furthermore that the finite length $(0 \leq t \leq T)$ of records $\left[x_1(t), \ldots, x_q(t), \; y(t) \right]$ are treated as if they were a finite realization of a $(q+1)$th order multiple stationary time series. Proceeding by the method of spectral estimation of Reference 1, a $(q+1)x(q+1)$ sample spectral density matrix at frequency f_0

$$\hat{\Sigma}(f_0) \equiv \left[\begin{array}{c|c} \hat{\Sigma}_{xx}(f_0) & \hat{\Sigma}_{xy}(f_0) \\ \hline \hat{\Sigma}_{yx}(f_0) & \hat{\Sigma}_{yy}(f_0) \end{array} \right] \tag{32}$$

may then be computed. It is presumed that the degrees-of-freedom parameter n associated with $\hat{\Sigma}(f_0)$ satisfies $n \geq q+1$, and that the $q \times q$ matrix $\hat{\Sigma}_{xx}(f_0)$ is nonsingular.

The reader will note that the (possibly) nonlinear time invariant system described by Eq. (31), in general, may be different from the linear time invariant system described by Eq. (12). The reader will, however, also note that from the sample spectral density matrix $\hat{\Sigma}(f_0)$ of Eq. (32) one may formally compute all the sample entities described in Section 2. The topic that is now briefly discussed is the relevance, interpretation, and usefulness of such sample entities in relation to a multiple-input single-output (possibly) nonlinear time invariant system described by Eq. (31).

In the present discussion it is presumed that $\left[x_1(t), \ldots, x_q(t)\right]$ are multiple stationary random functions. Since the operators K_k, $k=1,\ldots,q$ are time invariant, it follows that $\left[x_1(t), \ldots, x_q(t), y(t)\right]$ are also multiple stationary random functions. Furthermore, it follows that $\hat{\Sigma}(f_0)$ of Eq. (32) is then an estimator of the $(q+1)x(q+1)$ spectral density matrix $\Sigma(f)$ of $\left[x_1(t), \ldots, x_q(t), y(t)\right]$ at frequency f_0. Even though $y(t)$ is determined by $x_1(t), \ldots, x_q(t)$ in the (possibly) nonlinear manner described by Eq. (31), there is an interpretation that enables the relation between $y(t)$ and $x_1(t), \ldots, x_q(t)$ to be described by the block diagram illustrated by Eq. (13). Stated another way, one may write Eq. (31) in the form of Eq. (12) provided one properly defines $L_1, \ldots, L_q$ and $e(t)$ of Eq. (12) in relation to Eq. (31). Since $\left[x_1(t), \ldots, x_q(t), y(t)\right]$ are multiple stationary random functions, there exists a unique decomposition of $y(t)$ where

$$y(t) = y_L(t) + y_e(t) \tag{33}$$

In Eq. (33), $y_L(t)$ is the part of $y(t)$ that is related to $x_1(t), \ldots, x_q(t)$ by the equation

$$y_L(t) = L_1 x_1(t) + L_2 x_2(t) + \ldots + L_q x_q(t) \tag{34}$$

where the L_k, $k=1,\ldots,q$ in Eq. (34) denote linear time invariant operators possessing corresponding frequency response functions $H_k(f)$, $k=1,\ldots,q$. In Eq. (33), $y_e(t)$ is the part of $y(t)$ that is multiply incoherent with $x_1(t), \ldots, x_q(t)$. If the $(q+1)x(q+1)$ spectral density matrix of $\left[x_1(t), \ldots, x_q(t), y(t)\right]$ of Eq. (31) is

$$\Sigma(f) \equiv \left[\begin{array}{c|c} \Sigma_{xx}(f) & \Sigma_{xy}(f) \\ \hline \Sigma_{yx}(f) & \Sigma_{yy}(f) \end{array}\right] \qquad (35)$$

then the $H_k(f)$ corresponding to the L_k , $k=1,\ldots,q$, of Eq. (34) are given by

$$H(f) \equiv \left[H_1(f),\ldots,H_q(f)\right]' \equiv \Sigma_{xx}^{-1}(f)\,\Sigma_{xy}(f) \qquad (36)$$

From the preceding discussion one then has the block diagram (13) holding for Eq. (31) where $H_k(f)$, $k=1,\ldots,q$ are given by Eq. (36) and e(t) of Eq. (13) is replaced by $y_e(t) = y(t) - y_L(t)$. One has $y_e(t)$ multiply incoherent with $x_1(t),\ldots,x_q(t)$. From Eq. (33) one may interpret $y_e(t)$ to be that part of y(t) of Eq. (31) that is unaccounted for by the linear time invariant operators L_k, $k=1,\ldots,q$ of Eq. (34) that "best" approximate y(t) by acting on the inputs $x_1(t),\ldots,x_q(t)$. In summary, one is able to write Eq. (31) in the form Eq. (12) provided e(t) of Eq. (12) is replaced by $y_e(t)$. In Section 2, e(t) was presumed to be a zero mean stationary Gaussian random function statistically independent of the input functions $x_k(t)$, $k=1,\ldots,q$. Here, $y_e(t)$ is a random function multiply incoherent of the input functions $x_k(t)$, $k=1,\ldots,q$. The applicability of the results of Section 2 to describing (or approximating) nonlinear time invariant systems by linear time invariant systems therefore depends on how the differing properties of $y_e(t)$ and e(t) affect the results of Section 2.

Generally speaking, the sample entities of Section 2 maintain their relevance, interpretation, and usefulness. For example, Eq. (16) for $\hat{H}'(f_0)$ in the (possibly) nonlinear context of this section is now interpreted to be an estimator at frequency f_0 for the matrix frequency response function $H'(f)$ of Eq. (36). The sample multiple coherence $\hat{\gamma}^2_{y\cdot x_1,\ldots,x_q}(f_0)$ at frequency f_0 between the output y(t) and the q inputs $x_1(t),\ldots,x_q(t)$ is now interpreted to be an estimator of the multiple coherence $\gamma^2_{y \cdot x_1, \ldots, x_q}(f_0)$, i.e., an estimator of the degree to which the output $y(t)$ at frequency f_0 is related by linear time invariant operators to the q inputs $x_1(t), \ldots, x_q(t)$.

The remaining question concerns the applicability of the sampling distribution and confidence band results of Sections 1 and 2. The output $y(t)$ of a time invariant nonlinear system is, in general, a non-Gaussian random function even if the inputs $x_1(t), \ldots, x_q(t)$ are multiple stationary Gaussian random functions. The sampling distribution and confidence band results of Sections 1 and 2 are based on Gaussian theory prevailing in the frequency domain. The computation of the sample entities of Section 2 inherently involves "narrow band frequency filtering." An important result often observed in practice is that many stationary non-Gaussian random functions become nearly Gaussian when so "filtered." Thus, one may expect the distribution and confidence band results of Sections 1 and 2 to be approximately valid for many multiple stationary non-Gaussian random functions. In that regard, one may expect the sampling distribution and confidence band results of Sections 1 and 2 to be in many cases approximately applicable to the method of studying nonlinear systems described above.

The reader will note that the methods of this section indicate how a (possibly) nonlinear time invariant system may be (a) approximated by a linear time invariant system, and (b) provide a measure (sample multiple coherence) of how accurate (at each frequency f_0) such a linear time invariant system approximation is. The methods of this section may therefore also be roughly used to test the hypothesis that a time invariant system is linear. The general idea is that a time invariant system capable of being suitably approximated by a linear time invariant system may for many practical purposes be regarded as a linear time invariant system.

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