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1. Introduction and Summary Most analyses of reliability problems assume that the form of the underlying failure distribution(s) is known; the parameter may be assumed known or unknown. Families of failure distributions commonly used are the exponential, gamma, normal, and lognormal.

The weakness of such analyses is that the conclusions reached may be grossly in error if the assumption as to underlying failure distributions is incorrect. In fact, the error in the original assumption may be greatly compounded in arriving at the final conclusion, especially if the conclusion concerns a tail probability or if the system analyzed is complex.

In this paper we present an expository survey of recent research in reliability estimation based on assumptions made not simply for mathematical convenience but because they correspond to the physical situation. The class of statistical problems considered thus lies somewhere between parametric problems (in which the underlying failure distributions are assumed known up to a finite number of parameters) and nonparametric problems (in which no information is assumed available concerning the underlying failure distributions).

In Section 2 we summarize research concerning the errors resulting from using standard exponential life test and estimation procedures when in fact the distribution has an increasing failure rate or a decreasing failure rate. This raises the natural question: Given a sample from an unknown distribution, does the distribution have an increasing failure rate (or alternately decreasing failure rate)? In Section 3 we summarize research on tests for increasing (decreasing) failure rate. Assuming then that the assumption of an increasing (decreasing) failure rate is plausible, the next problem, discussed in Section 4, is the estimation of the failure rate function, the distribution, and the density. A different class of problems is discussed in Section 5, namely the estimation of an evolving reliability, based on observations obtained at successive stages of evolution. Unlike the usual treatment in the literature, no assumption is made concerning the mathematical form of the reliability growth; rather, only monotonicity of reliability in succeeding stages is assumed. Maximum likelihood estimators and conservative confidence bounds are obtained for parameters or functions arising in such reliability growth problems.

In order to present precisely the results of the succeeding sections we introduce the following definitions and notation. Let F be a right continuous distribution such that $F(0^-) = 0$. If F has a density f then $r(t) = \frac{f(t)}{\bar{F}(t)}$ is known as the <u>failure rate,</u> where $\bar{F}(t) = 1 - F(t)$ represents the survival probability or <u>reliability.</u> Physically, $r(t)dt$ is the conditional probability of failure in $(t, t+dt)$ given survival until time t . We may verify by differentiation that under the assumption of increasing (decreasing) failure rate $\log \bar{F}(t)$ is concave (convex) on $[0,\infty)$. More generally, whether a density exists or not, we shall say a distribution has an increasing failure rate (IFR) if $\log \overline{F}(t)$ is concave on $[0,\infty)$, and a decreasing failure rate (DFR) if $\log \overline{F}(t)$ is convex on $(0,\infty)$. Properties of IFR and DFR distributions are discussed in Barlow, Marshall, and Proschan (1963). Applications of such distributions to reliability problems are presented in Barlow and Proschan (1965).

2. Errors in Using Exponential Procedures When the Distribution Has Monotone Failure Rate

This section is based on Barlow and Proschan (1964). We give proofs initially to convey the type of mathematical argument involved; after that we refer the reader to the source paper for detailed proofs.

2.1 Censored Sampling Without Replacement Assume n items are put on life test simultaneously at time 0. Let $X_1 \leq \ldots \leq X_n$ denote the ordered observations corresponding to successive failures. If the failure distribution has exponential density

$$(2.1) \qquad g(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \quad , \quad x \geq 0 \quad , \theta > 0 \; ,$$

then

$$(2.2) \qquad \hat{\theta}_{rn} = \frac{1}{r}\left\{ \sum_{i=1}^{r} X_i + (n-r)X_r \right\} = \frac{1}{r} \sum_{i=1}^{r} (n-i+1)(X_i - X_{i-1}) \; , \; 1 \leq r \leq n \; ,$$

is the maximum likelihood (ML), minimum variance unbiased (MVU) estimator for θ based on the first r order statistics. (Epstein and Sobel, 1953) We shall show in Theorem 2.2 below that this estimator is biased high (low) when the underlying failure distribution is IFR (DFR). We will need

Lemma 2.1 Let $X_1 \leq \ldots \leq X_n$ be order statistics from F, IFR (DFR). Then the normalized spacings $(n-i+1)(X_i - X_{i-1})$ are stochastically decreasing (increasing) in i.

Proof Assume F is IFR. Then

$$P[nX_1 \geq x] = \{\overline{F}(\tfrac{x}{n})\}^n \geq \{\overline{F}(\tfrac{x}{n-1})\}^{n-1}$$

since $\{\overline{F}(t)\}^{1/t}$ is decreasing in t by the log concavity of $\overline{F}$.

Let $F_u(x) = \frac{F(u+x)-F(u)}{\bar{F}(u)}$, the conditional distribution of an item of age u . Then since F is IFR, $F_u(x) \geq F(x)$. Given that $X_1 = u$ is observed, X_2-X_1 is distributed as the first order statistic from a sample of size n-1 each with distribution $F_u(x)$. Hence

$$P[(n-1)(X_2-X_1) \geq x \mid X_1 = u] = \{\bar{F}_u(\tfrac{x}{n-1})\}^{n-1} .$$

Conditioning on X_1 , we have

$$P[nX_1 \geq x] = \{\bar{F}(\tfrac{x}{n})\}^n \geq \{\bar{F}_u(\tfrac{x}{n-1})\}^{n-1} = P[(n-1)(X_2-X_1) \geq x \mid X_1 = u] , \ u \geq 0 .$$

Unconditioning,

$$P[nX_1 \geq x] \geq \int_0^\infty \{\bar{F}_u(\tfrac{x}{n-1})\}^{n-1} dG(u) = P[(n-1)(X_2-X_1) \geq x] ,$$

where $G(u) = 1- \{\bar{F}(u)\}^n$ is the distribution of X_1 .

Thus we have shown that nX_1 is stochastically larger than $(n-1)(X_2-X_1)$. In a similar manner we can show that $(n-i+1)(X_i-X_{i-1})$ is stochastically larger than $(n-i)(X_{i+1}-X_i)$ for $i = 2,3,\ldots,n$.

All inequalities are reversed for DFR distributions. ||

Now we can show

<u>Theorem 2.2</u> a) If F is IFR with mean θ , then $\theta \leq E\,\hat{\theta}_{rn} \leq \frac{n\theta}{r}$ for $r=1,2,\ldots n$. b) If F is DFR with mean θ , then $0 \leq E\hat{\theta}_{rn} \leq \theta$ for $1 \leq r \leq n$. All inequalities are sharp.

<u>Proof</u> a) From Barlow and Proschan (1965), p. 33, we know that $E\hat{\theta}_{1n} \geq \theta$. Also

$$h(r) = \sum_{i=1}^{r} \left\{E(n-i+1)(X_i-X_{i-1}) - \theta\right\}$$

exhibits at most one sign change as a function of r since $E(n-i+1)(X_i-X_{i-1})$

is decreasing in i by Lemma 2.1. But $h(1) \geq 0$ and $h(n) = 0$, so that $h(r) \geq 0$ for $r = 1,2,\ldots,n$. Hence $E\hat{\theta}_{rn} \geq \theta$. Clearly the bound is attained by the exponential distribution so that it is sharp.

To show the upper bound, note that $\sum_{i=1}^{r} X_i + (n-r)X_r \leq \sum_{i=1}^{n} X_i$ for every sample realization. Hence $Er\hat{\theta}_{rn} \leq n\theta$, so that $E\hat{\theta}_{rn} \leq \frac{n\theta}{r}$. Since equality is attained with distributions degenerate at θ (which is the limit of IFR distributions) the bound is sharp.

For the proof of b), see Barlow and Proschan (1964), Corollary 2.3. ‖

2.2 <u>Censored Sampling with Replacement</u> Suppose now that failed items are replaced at failure. In this case the bias of the usual exponential estimate for θ is even greater than in the non-replacement case.

Let X_i^* denote the time of the i^{th} failure when failed items are replaced. Then under the exponential assumption, the ML, MVU estimate of θ based on the first r failure times is $\hat{\theta}^*_{rn} = \frac{nX^*_r}{r}$. Then Barlow and Proschan (1964) prove

<u>Theorem 2.3</u> If the underlying failure distribution F is IFR with mean θ, then

$$\theta \leq E\hat{\theta}_{rn} \leq E\hat{\theta}^*_{rn} \leq \frac{n}{r} EX_r , \quad 1 \leq r \leq n .$$

2.3 <u>Inverse Binomial Sampling</u> Nadler (1960) has considered the following type of sampling. An item having life distribution F with mean θ is put on test until it fails or reaches age t, whichever occurs first. At this time the item is replaced by a fresh item. This procedure is repeated sequentially until r actual failures are observed. The number N of items that have to be tested until the r actual failures are obtained is a random variable. Nadler (1960) showed that when $F(x) = 1-e^{-x/\theta}$, an unbiased estimate of θ is

$$\hat{\theta}_r(t) = \frac{1}{r}\left\{\sum_{i=1}^{r} X_i + (N-r)t\right\}$$

where the $X_1,\ldots,X_r$ are the r life lengths not exceeding t.

Following the proof in Barlow and Proschan (1964), we show

Theorem 2.4 If F is IFR (DFR) with mean θ, then $E\hat{\theta}_r(t) \geq (\leq)\ \theta$.

Proof Let F be IFR, Z_i denote test time elapsed between the $i\text{-}1^{st}$ failure time and the i^{th} failure time, $i=1,2,\ldots,r$, where the 0^{th} failure time is defined to be 0. Then

$$\hat{\theta}_r(t) = \frac{1}{r}\sum_{i=1}^{r} Z_i .$$

Next consider an alternative testing procedure differing in that replacement occurs only upon failure. Let Z_i' = test time elapsed between the $i\text{-}1^{st}$ failure and the i^{th} failure under the alternate testing procedure. Now since F is IFR, Z_i is stochastically larger than Z_i'. It follows that

$$E\hat{\theta}_r(t) = \frac{1}{r}\sum_{i=1}^{r} EZ_i \geq \frac{1}{r}\sum_{i=1}^{r} EZ_i' = \theta .$$

The inequality is reversed when F is DFR. ||

Additional results are obtained in Barlow and Proschan (1964) for other sampling procedures which illustrate the general thesis that the use of exponential life test and estimation procedures when the underlying distribution is IFR (DFR) yields estimates of mean life which are biased high (low).

3. Tests for Increasing (Decreasing) Failure Rate

3.1 The Test It is apparent from Section 2, that we should be cautious about the use of exponential life test procedures if we suspect that the underlying failure distribution may be IFR (DFR). Thus it is appropriate now to consider tests for constant vs. increasing (decreasing) failure rate. In this section we summarize results obtained in Proschan and Pyke (1965).

Let $X_1 \leq X_2 \leq \cdots \leq X_n$ be an ordered sample from the distribution F, with density f, where $f(t) = 0$ for $t < 0$, and failure rate $r(t)$. We wish to choose between the following:

Null Hypothesis, H_0 : $r(t) = \lambda$, λ an unknown positive constant.

Alternative Hypothesis, H_1: $r(t)$ is strictly increasing.

The (nonparametric) test statistic is computed as follows. Let $D_1^* = X_1$, $D_2^* = X_2 - X_1, \ldots, D_n^* = X_n - X_{n-1}$, the spacings, and $D_1 = nD_1^*$, $D_2 = (n-1)D_2^*, \ldots, D_n = D_n^*$, the normalized spacings. Let

$$V_{ij} = \begin{cases} 1 & \text{if } D_i \geq D_j \quad \text{for } i, j=1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

The test statistic is

$$V_n = \sum_{\substack{i,j=1 \\ i<j}}^{n} V_{ij} . \tag{3.1}$$

We reject the null hypothesis at the α level of significance if $V_n > v_{n\alpha}$, where $v_{n\alpha}$ is determined so that $P[V_n \geq v_{n\alpha} \mid H_0] = \alpha$.

Heuristically we may justify the test as follows. Under the null hypothesis, $D_1, D_2, \ldots, D_n$ are independently distributed, each with density $\lambda e^{-\lambda t}$, as shown in Epstein and Sobel (1953), so that the D_i are stochastically identical and therefore the V_{ij} are equally likely to be 0 or 1. However, under the alternative hypothesis, each V_{ij}, and consequently V_n, tends to be larger, so that rejection of the null hypothesis occurs for large values of V_n. Since under the null hypothesis, the distribution of V_n is known, we have available $v_{n\alpha}$.

3.2 Distribution Under the Null Hypothesis Under H_0 all orderings of $D_1,\ldots,D_n$ are equally likely. Thus we may use the results obtained by Kendall (1938) and Mann (1945) for the number of orderings of $D_1,\ldots,D_n$ with exactly k inversions of indices; an inversion of indices $i < j$ occurs when $D_i \geq D_j$. Both Mann and Kendall provide tables for $P[V_n \leq k]$, $n \leq 10$. For $n > 10$, we may approximate $P[V_n \leq k]$ closely using the fact that V_n is asymptotically normal with mean $\mu_n = \frac{n(n-1)}{2}$ and variance $\sigma_n^2 = \frac{(2n+5)(n-1)n}{72}$. Thus $v_{n\alpha}$ is available for given α.

3.3 Unbiasedness of Test Following Proschan and Pyke (1965) it is easy to show that the test statistic V_n is unbiased, i.e., $P[V_n \geq v_{n\alpha} \mid H_1] \geq \alpha$ for $0 < \alpha < 1$, $n = 2,3,\ldots$.

To prove unbiasedness, make the transformation $X_i' = -\log \bar{F}(X_i)$. If $Z_i'(Z_i)$, $i = 1,\ldots,n$, are the unordered $X_i'(X_i)$, $i = 1,\ldots,n$, then $P[Z_i' > u] = P[\log \bar{F}(Z_i) < -u] = P[\bar{F}(Z_i) < e^{-u}] = e^{-u}$. Thus each Z_i' is distributed according to the exponential distribution with unit mean. Moreover since the $Z_1,\ldots,Z_n$ are independent, so are the $Z_1',\ldots,Z_n'$.

Next let $D_i' = (n-i+1)(X_i' - X_{i-1}')$ $i = 1,\ldots,n$, where $X_0' = 0$. Now note that X_i' is a convex increasing function of X_i which is 0 for $X_i = 0$. It follows that $D_i' \geq D_j'$ implies $D_i \geq D_j$ for $i < j,i,$

$j=1,\ldots,n$. Thus $V_{ij} \geq V'_{ij}$, $i < j, i, j=1,\ldots,n$, where $V'_{ij} = 1$ if $D'_i \geq D'_j$. Hence $V_n \geq V'_n$, where $V'_n = \sum_{i<j} V'_{ij}$, so that $P[V_n \geq v_{n\alpha} \mid H_1] \geq \alpha$ for $0 < \alpha \leq 1$, $n = 2,3,\ldots$, by Lehmann (1959,p.73).

3.4 Asymptotic Distribution Under Alternative Hypothesis Proschan and Pyke (1965) prove that the statistic V_n when suitably normalized is asymptotically normally distributed for a wide class of alternatives.

Theorem 3.1 Let the distribution F be an absolutely continuous IFR with failure rate function r . Let ρ exist and be continuous a.e. on $(0,1)$, where $\rho(u) = 1/r(F^{-1}(u))$, the reciprocal of the failure rate after it is transformed by means of F^{-1} onto the unit interval. Then $n^{-3/2}(V_n - \mu)$ converges in law as $n \to \infty$ to a normal random variable with mean 0 and variance σ^2 , where

$$(3.2) \qquad \mu = \int_0^1\int_u^1 \rho(u)[\rho(u) + \rho(v)]^{-1}\, dv\, du ,$$

$$(3.3) \qquad \sigma^2 = \sigma_S^2 + 2\sigma_{SR} + \sigma_R^2 ,$$

$$\sigma_S^2 = \left\{\int_0^1\int_0^v\int_0^v + \int_0^1\int_v^1\int_v^1 - 2\int_0^1\int_v^1\int_0^v\right\}\rho(u)\rho(w)\rho^2(v)[\{\rho(u)+\rho(v)\}\{\rho(v)+\rho(w)\}$$
$$\{\rho(u)\rho(w) + \rho(u)\rho(v) + \rho(v)\rho(w)\}]^{-1} du\, dw\, dv ,$$

$$\sigma_{SR} = \int_0^1\int_u^1 \rho(u)\rho(v)[\rho(u) + \rho(v)]^{-2}[\beta(u) - B(v)]dv\, du ,$$

$$\sigma_R^2 = \int_0^1 \beta^2(u)du , \text{ and}$$

$$\beta(w) = (1-w)^{-1}\left\{\int_w^1 \rho'(u)(1-u)\int_u^1 \rho(v)[\rho(u)+\rho(v)]^{-2}dv\, du - \int_w^1 \rho'(v)(1-v)\int_0^v \rho(u)[\rho(u)+\rho(v)]^{-2}\, du\, dv\right\}$$

3.5 Asymptotic Relative Efficiency To compare the V_n test with other possible tests we use the criterion of asymptotic relative efficiency (ARE). For some specified set of alternatives indexed by θ , the ARE of one sequence of tests based on a sequence of asymptotically normal test statistics $\{T_n\}$ against a second sequence of tests based on the asymptotically normal test statistics $\{\tau_n\}$ is defined as

$$(3.4) \qquad \{[\mu_T'(\theta_0)]^2/\sigma_T^2(\theta_0)\}\{[\mu_\tau'(\theta_0)]^2/\sigma_\tau^2(\theta_0)\}^{-1}$$

whenever it exists. In (3.2), $\mu_T(\theta)$ and $\sigma_T^2(\theta)$ denote the limiting mean and variance respectively of $\{T_n\}$, μ_T' denotes the derivative of μ_T with respect to θ and θ_0 denotes the null hypothesis. A similar interpretation is understood relative to τ .

(a) Likelihood-ratio test for Weibull alternatives Suppose that H_0 is as before, but H_1 is specialized to the case in which the underlying distribution is the Weibull with increasing failure rate:

$$F(x) = 1-e^{-\lambda x^\theta} \quad , \quad \lambda > 0 \ , \ \theta > 1 \ , \ x \geq 0 \ ,$$

where λ assumed known. We may readily verify that the likelihood ratio test is to reject H_0 if $T_n^W > c_\alpha$, with $T_n^W = \sum_{i=1}^{n} (1-\lambda X_{ni})\log X_{ni}$. Also, we may calculate for the derivative of the limiting mean and for the limiting variance of T_n^W , $\mu'_{T^W}(1)=(\ell n\ \lambda + \gamma - 1)^2 + \frac{\pi^2}{6} = \sigma^2_{T^W}(1)$, where $\gamma = .577215\ldots$, so that $\{\mu'_{T^W}(1)\}^2/\sigma^2_{T^W}(1) = (\ell n\ \lambda + \gamma - 1)^2 + \frac{\pi^2}{6}$.

For V_n , using (3.2) and (3.3), after some calculation we find

$$\{\mu_V'(1)\}^2/\sigma_V^2(1) = \frac{9}{4}(\log 2)^2 \ .$$

Thus the ARE of V_n with respect to T_n^W reduces to $ARE_W = \dfrac{1.0809}{(\ell n\ \lambda - .4228)^2 + 1.6449)}$. As $\lambda \to 0$ or ∞ , $ARE_W \to 0$. For all $\lambda > 0$, $ARE_W \leq .6571$; equality is attained for $\ln \lambda = .4228$.

(b) Likelihood-ratio test for Gamma alternatives Next assume that H_1 is specialized to the case in which the underlying distribution is the Gamma with increasing failure rate. The density is

$$f(x) = \frac{\lambda^\theta x^{\theta-1} e^{-\lambda x}}{\Gamma(\theta)} , \quad \theta \geq 1 .$$

The likelihood ratio test is to reject H_0 if

$$T_n^G = \sum_{i=1}^{n} \log X_{ni} > c_\alpha .$$

By calculation similar to that in (a), we obtain for the ARE of V_n relative to T_n^G:

$$ARE_G = .2040 ,$$

independent of λ.

In a similar fashion, Proschan and Pyke (1965) compute the ARE of V_n relative to the likelihood ratio test against the Gamma when the true distribution is the Weibull, and the ARE of V_n relative to the likelihood ratio test against the Weibull when the true distribution is the Gamma. For certain values of the parameters in each case, the ARE is > 1, implying that the V_n test is better.

Other nonparametric tests for IFR (DFR) are possible. For example, the coefficient of variation under H_0 is 1, and ≤ 1 under H_1. Moreover, as shown in a forthcoming paper by Marshall, Olkin, and Proschan, the sample coefficient of variation is stochastically smaller under H_1 than under H_0. Thus an unbiased test of H_0 vs. H_1 may be constructed using as test statistic the sample coefficient of variation.

4. Maximum Likelihood Estimation for Distributions with Monotone Failure Rate

Suppose now we have established either from physical considerations or from tests on observed data that the underlying failure distribution is IFR or DFR. How do we estimate the density f, distribution F, or failure rate function r ?

Grenander (1956) derives the maximum likelihood estimator (MLE) assuming F is IFR. Marshall and Proschan (1965) present an alternate derivation of the MLE in the IFR case and also obtain a MLE for the DFR case. They show that these estimators are consistent.

4.1 Derivation of MLE   Let $X_1 \le X_2 \le \cdots \le X_n$ be a sample of n ordered observations from F, IFR. Using the fact that

$$\bar{F}(x) = e^{-\int_{-\infty}^{x} r(z)dz} , \tag{4.1}$$

we may express the log likelihood L as

$$L = \sum_{i=1}^{n} \log f(X_i) = \sum_{i=1}^{n} \log r(X_i) - \sum_{i=1}^{n} \int_{-\infty}^{X_i} r(z)dz . \tag{4.2}$$

Without further constraint, L can be made arbitrarily large by taking $f(X_n)$ correspondingly large. Consequently, we first consider the class $\mathcal{F}^M$ of IFR distributions with failure rate bounded by M. We shall first find the unique distribution $\hat{F}_n^M$ in $\mathcal{F}^M$ for which (4.2) is maximized. Then letting $M \to \infty$, we shall find $\hat{F}_n^M$ converges in distribution to an estimator $\hat{F}_n$ which we call the MLE for F in $\mathcal{F}$, the class of IFR distributions.

As shown by Grenander (1956), L is maximized over $\mathcal{F}^M$ by a distribution with failure rate constant between observations. (Heuristically, this is apparent, since on the one hand the failure rate cannot decrease between observations, and on the other hand, any increase in failure rate <u>between</u> a pair of observations can only result in diminishing the possible values of the density at the remaining observations.) Thus we may replace L by the function

$$(4.3) \qquad \sum_{i=1}^{n} \log r(X_i) - \sum_{i=1}^{n-1} (n-i)(X_{i+1}-X_i)r(X_i) .$$

The maximization of (4.3) subject to $r(X_1) \leq \ldots \leq r(X_n) = M$ is performed by Grenander (1956); it can also be performed as a direct application of Brunk (1958), Corollary 2.1 and the discussion following (see also van Eeden (1956, 1957)). This yields for r (corresponding to F in $\mathcal{F}^M$) the estimator

$$(4.4) \qquad \hat{r}_n^M(X_i) = \min(\min_{v \geq i+1} \max_{u \leq i} \{ \frac{1}{v-u} [r_u^{-1} + \ldots + r_{v-1}^{-1}]\}^{-1}, M) \quad ,$$

where $r_n = M$ and

$$(4.5) \qquad r_j = [(n-j)(X_{j+1}-X_j)]^{-1} \quad \text{for } j=1,2,\ldots,n-1 .$$

The estimator $\hat{r}_n^M$ given in (4.4) differs in form from the one given in Grenander (1956) but is equivalent to it.

The maximization which yields (4.4) may be described as follows. First, find the maximum of (4.4) obtaining (4.5). If there is a reversal, say $r_i > r_{i+1}$, then set $r(X_i) = r(X_{i+1})$ in (4.3) and repeat the procedure. After at most n steps of this kind, a monotone estimator is obtained. The maximum derived with $r(X_i) = r(X_{i+1})$ can be directly obtained by replacing r_i and r_{i+1} by their harmonic mean, $[\frac{1}{2}(r_i^{-1} + r_{i+1}^{-1})]^{-1}$. Succeeding steps amount to further such averaging which is extended just to the point necessary to eliminate all reversals. It can be seen that this is exactly what is called for in (4.4) taking into account $r(x) \leq M$. (In this connection see also Ayer, Brunk, Ewing, Reid, and Silverman (1955) and Brunk (1955).) The resulting estimator $\hat{r}_n^M$ is of the form

$$\hat{r}_n^M = \begin{cases} 0, & x < X_1 \\ \min(r_{n_i+1,n_{i+1}}, M), & X_{n_i+1} \leq x < X_{n_{i+1}+1} \\ M, & x \geq X_n , \end{cases}$$

Where $r_{1n_1} \leq r_{n_1+1,n_2} \leq \cdots \leq r_{n_k+1,n-1}$, $0 = n_0 < n_1 < \ldots < n_k < n-1$, and $r_{n_i+1,n_{i+1}}$ is the harmonic mean of $r_{n_i+1}, r_{n_i+2}, \ldots, r_{n_{i+1}}$. The n_i are determined by the rule which determines the extent of the averaging.

The estimator for r corresponding to F in $\mathcal{F}$ is obtained by letting $M \to \infty$ in (4.4), and is given by

$$\hat{r}_n(X_i) = \min_{v \geq i+1} \max_{u \leq i} [v-u][(n-u)(X_{u+1}-X_u) + \ldots + (n-v+1)(X_v-X_{v-1})]^{-1}, \tag{4.6}$$

$i=1,2,\ldots,n-1$ and $\hat{r}_n(X_n) = \infty$. For the remaining values of x, $\hat{r}_n(x)$ is 0 for $0 \leq x < X_1$, ∞ for $x > X_n$, and constant between observations. The corresponding estimators $\hat{F}_n$ and $\hat{f}_n$ for F and f are obtained from $\hat{r}_n$ using (4.1) and the relation $\hat{f}_n(x) = \hat{r}_n(x)\hat{\bar{F}}_n(x)$.

4.2 Consistency In Marshall and Proschan (1965), the consistency of the MLE of F, f, and r are proven. The results may be summarized as follows:

Theorem 4.1 If r is increasing, then for every t_0,

$$(4.7) \qquad r(t_0^-) \le \liminf \hat{r}_n(t_0) \le \limsup \hat{r}_n(t_0) \le r(t_0^+)$$

with probability one.

Corollary 4.2 If r is increasing, then for all t, $\lim_{n\to\infty} \hat{F}_n(t) = F(t)$ with probability one.

Corollary 4.3 If r is increasing and continuous on $[a,b]$, then

(i) $\lim_{n\to\infty} \sup_{a\le t\le b} |\hat{r}_n(t) - r(t)| = 0$,

(ii) $\lim_{n\to\infty} \sup_{-\infty<t<\infty} |\hat{F}_n(t) - F(t)| = 0$,

(iii) $\lim_{n\to\infty} \sup_{a\le t\le b} |\hat{f}_n(t) - f(t)| = 0$,

each with probability one.

4.3 Additional Results With respect to a certain metric $\hat{r}_n(t)$ is closer to $r(t)$ than is $r_n(t)$, where

$$r_n(t) = \begin{cases} 0 & \text{for } 0 \le t < X_1 \\ \{(n-j)(X_{j+1}-X_j)\}^{-1} & \text{for } X_j \le t < X_{j+1},\ j=1,2,\dots,n-1 \\ \infty & \text{for } X_n \le t < \infty . \end{cases}$$

Note that $r_n(t)$ represents the "unaveraged" estimate of the failure rate, i.e., the estimate that does not take into account the requirement that $r(t)$ be increasing. Let $F_n(t)$ be the usual empirical distribution. Then the following inequality, a special case of the results of Brunk (1961), is obtained directly in Marshall and Proschan (1965).

__Theorem 4.4__ With probability one,

$$\int_{-\infty}^{\beta^-}\{r_n(t) - r(t)\}^2 dF_n(t) \geq \int_{-\infty}^{\beta^-}\{\hat{r}_n(t)-r(t)\}^2 dF_n(t)+\int_{-\infty}^{\beta^-}\{r_n(t)-\hat{r}_n(t)\}^2 dF_n(t) .$$

In addition, Barlow and Proschan (1965) obtain the MLE's of r, f, and F for F DFR and show the consistency of these estimators. Finally, they discuss maximum likelihood estimation in the case of discrete IFR and DFR distributions.

5. Estimation of Reliability Growth

An important group of reliability problems consists of those in which the reliability of an evolving system is to be estimated at successive stages of its evolution. Most studies in the literature have assumed a priori knowledge of the form of the function governing reliability growth. (See for example, Lloyd and Lipow (1962), Chapter 11, Wolman (1963), and Corcoran, Weingarten, and Zehna (1964).) Unfortunately, in many cases the only a priori knowledge actually available is that the reliability at successive stages of evolution is monotonically increasing. In the present section we shall show how to obtain a maximum likelihood estimator and establish conservative confidence bounds for reliability assuming monotonicity in reliability at successive stages. A more detailed treatment will appear in forthcoming papers by Barlow, Proschan, and Scheuer.

5.1 Maximum Likelihood Estimation of Reliability Growth A system is being improved at successive stages of development, corresponding, say, to basic design changes. At stage i the failure distribution of system life length is F_i ; no assumption is made as to the form of $F_1,\ldots,F_k$ or the relationship among them except that $F_1(t) \geq F_2(t) \geq \ldots \geq F_k(t)$ for all $t \geq 0$. Independent life length observations $X_{i1},\ldots,X_{in_i}$ are obtained at stage i , $i=1,2,\ldots,k$. From these we wish to estimate $F_1,F_2,\ldots,F_k$, especially F_k .

To obtain the MLE of $F_i(t)$, $0 \leq t < \infty$, $i=1,\ldots,k$, we use the procedure developed by Brunk (1955, 1958). If we fix t , the problem is similar to the one discussed by Barlow and Scheuer (1964) and by Ayer, Brunk, Ewing, Reid, and Silverman (1955). For $i=1,\ldots,k$ obtain the empirical distribution function $F_{in_i}(t) = m_i(t)/n_i$, where $m_i(t)$ = number of observations among $X_{i1},\ldots,X_{in_i}$ not exceeding t . Then the Brunk procedure yields for fixed t the MLE

$$(5.1) \qquad \hat{F}_i(t) = \max_{s\geq i} \min_{r\leq i} \frac{m_r(t)+\ldots+m_s(t)}{n_r+\ldots+n_s} \quad , \; i=1,2,\ldots,k \; .$$

To prove that each $\hat{F}_i(t)$ is monotonic increasing in t and so qualifies as a distribution function, we simply note that each $m_i(t)$ is monotonic increasing in t , implying by (5.1) that $\hat{F}_i(t)$ is also increasing in t . Thus $\hat{F}_i(t)$, $0 \leq t < \infty$, is the MLE of $F_i(t)$, $i=1,\ldots,k$.

5.2 Conservative Confidence Bounds on Final Reliability Ordinarily, to obtain a confidence bound on a parameter, we use a statistic with known distribution. Unfortunately, in the present case the relevant distributions are unknown. However, we can still obtain a conservative confidence bound using the following general theorem.

Theorem 5.1 Let (a) $\underline{Y}$ be an observation on a random variable (in general vector valued) having distribution function $G(\underline{y},\theta)$, θ a one dimensional parameter, (b) $\hat{\theta}(\underline{Y})$ be a one dimensional statistic based on the observed vector $\underline{Y}$, (c) $\rho(\hat{\theta}(\underline{Y}))$ be a $1-\alpha$ upper confidence bound on θ , where $\rho(u)$ is a decreasing function, (d) $\underline{X}$ be an observation on a random variable (in general vector valued) having distribution function $F(\underline{x},\theta)$, and (e) $\hat{\theta}(\underline{Y})$ be stochastically $\geq \hat{\theta}(\underline{X})$. Then $P[\rho(\hat{\theta}(\underline{X})) \geq \theta \mid F(\underline{x},\theta)] \geq 1-\alpha$; i.e., $\rho(\hat{\theta}(\underline{X}))$ is a conservative $1-\alpha$ upper confidence bound on θ , the parameter of the distribution F .

<u>Proof</u> First assume ρ continuous and strictly increasing. Let u be the inverse of the function ρ. (In case of ambiguity, let $u(x)$ be the largest value satisfying $\rho(u) = x$). The

$1-\alpha = P[\rho(\hat{\theta}(\underline{Y})) \geq \theta \mid G(\underline{y},\theta)] = P[\hat{\theta}(\underline{Y}) \leq u(\theta)]$; the first inequality follows from (c), the second holds since u is the inverse function of ρ and ρ is decreasing. By (e)

$P[\hat{\theta}(\underline{Y}) \leq u(\theta) \mid G(\underline{y},\theta)] \leq P[\hat{\theta}(\underline{X}) \leq u(\theta) \mid \bar{F}(\underline{x},\theta)] = P[\rho(\hat{\theta}(\underline{X})) \geq \theta \mid F(\underline{x},\theta)]$;

the last equality holds since u is the inverse of ρ and ρ is decreasing. Combining the statements above we obtain

$$P[\rho(\hat{\theta}(\underline{X})) \geq \theta \mid F(\underline{x},\theta)] \geq 1-\alpha .$$

If ρ is not continuous or strictly increasing, the same result may be obtained by limiting arguments.‖

Other cases of interest are covered in

<u>Corollary 5.2</u> (1) If ρ is an increasing function and $\hat{\theta}(\underline{X})$ is stochastically larger than $\hat{\theta}(\underline{Y})$, then the same conclusion follows.

(2) If ρ is a decreasing function, $\rho(\hat{\theta}(\underline{Y}))$ is a $1-\alpha$ lower confidence bound on the parameter θ of G, and $\hat{\theta}(\underline{X})$ is stochastically larger than $\hat{\theta}(\underline{Y})$, then $\rho(\hat{\theta}(\underline{X}))$ is a conservative $1-\alpha$ lower confidence bound on the parameter θ of F.

(3) If ρ is an increasing function, $\rho(\hat{\theta}(\underline{Y})$ is a $1-\alpha$ lower confidence bound on the parameter θ of G, and $\hat{\theta}(\underline{X})$ is stochastically smaller than $\hat{\theta}(\underline{Y})$, then $\rho(\hat{\theta}(\underline{X}))$ is a conservative $1-\alpha$ lower confidence bound on the parameter θ of F.

The proof in each case is similar to that of Theorem 3.1.

Returning to the model described in Section 5.1, let us set up a conservative confidence bound (a) on $\bar{F}_k(t_0)$, the reliability at the final stage corresponding to fixed t_0 and (b) on the entire distribution function F_k.

(a) Let X' be a binomial random variable representing the number of successes in $n = \sum_{i=1}^{k} n_i$ independent trials, where the probability of success on each individual trial is $\bar{F}_k(t_0)$. Then X' is stochastically larger than $X = \sum_{i=1}^{k} [n_i - m_i(t_0)]$ since X' is the sum of n independent binomial random variables each of which is stochastically larger than the n corresponding binomial random variables comprising X. Thus using Corollary 5.2 (3) we may obtain a conservative $1-\alpha$ lower confidence bound for $\bar{F}_k(t_0)$ as follows:

1. Find the value $p(x)$ satisfying $\sum_{i=0}^{x} \binom{n}{i} p^i q^{n-i} = 1-\alpha$, where $q = 1-p$, $x = \sum_{i=1}^{k} (n_i - m_i(t_0))$.
2. Then $P[\bar{F}_k(t_0) \geq p(x)] \geq 1-\alpha$; i.e., $p(x)$ is a conservative $1-\alpha$ lower confidence bound for $\bar{F}_k(t_0)$, the reliability at the last stage of development.

(b) To obtain a conservative confidence bound on the entire distribution function F_k, we first show

Lemma 5.3 Let $X_{i1}, \ldots, X_{in_i}$ be independent observations from F_i, $i=1,\ldots,k$, with $F_1(t) \geq \ldots \geq F_k(t)$ for all t, and $\hat{F}(t)$ be the empirical distribution formed from all the observations $X_{i1}, \ldots, X_{in_i}$, $i=1,\ldots,k$. Let $X'_{i1}, \ldots, X'_{in_i}$, $i=1,\ldots,k$, be independent observations from $F_k(t)$ and $\hat{F}'(t)$ be the empirical distribution formed from $X'_{i1}, \ldots, X'_{in_i}$, $i=1,\ldots,k$. Then given any function $u(t)$,

$$P[\hat{F}(t) \geq u(t), -\infty < t < \infty] \geq P[\hat{F}'(t) \geq u(t), -\infty < t < \infty]$$

Proof First assume each F_i continuous and strictly increasing. Define $X''_{ij} = F_i^{-1} F_k(X'_{ij})$, $j=1,\ldots,n_i$; $i=1,\ldots,k$.

Then the set of random variables X''_{ij}, $j=1,\ldots,n_i$; $i=1,\ldots,k$, has the same joint distribution as the set X_{ij}, $j=1,\ldots,n_i$; $i=1,\ldots,k$.

Moreover, since $F_i \geq F_k$, then $X''_{ij} \leq X'_{ij}$, $j=1,\ldots,n_i$, $i=1,\ldots,k$. Thus for each t , $\hat{F}''(t) \geq \hat{F}'(t)$, where $F''(t)$ is the empirical distribution formed from X''_{ij} , $j=1,\ldots,n_i$, $i=1,\ldots,k$. The conclusion of Lemma 5.3 follows from a generalization of the Lemma on page 73 of Lehmann (1959).

If any F_i is not continuous or strictly increasing, the same result may be obtained by limiting arguments.||

Now we may form a conservative $1-\alpha$ upper confidence bound on the entire distribution function F_k as follows. Let $\hat{F}(t)$ be the empirical distribution formed from <u>all</u> n observations X_{ij}, $j=1,\ldots,n_i$; $i=1,\ldots,k$. From Birnbaum and Tingey (1951), we may find $\epsilon_{n\alpha}$ satisfying $P[H_n(t) + \epsilon_{n\alpha} \geq H(t), -\infty < t < \infty] = 1-\alpha$, where $H(t)$ is a distribution function, and $H_n(t)$ is the corresponding empirical distribution function based on a sample of size n from $H(t)$. The value $\epsilon_{n\alpha}$ is independent of $H(t)$. Then we may claim $P[\hat{F}(t) + \epsilon_{n\alpha} \geq F_k(t), 0 \leq t < \infty] \geq 1-\alpha$. That is, $\hat{F}(t) + \epsilon_{n\alpha}$ is a conservative $1-\alpha$ upper confidence bound on the entire distribution function $F_k(t)$. This is an immediate consequence of Lemma 5.3.

In a similar manner we may obtain MLE's and conservative confidence bounds in a variety of other reliability improvement models. A forthcoming paper by Barlow, Proschan, and Scheuer will give details.

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