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JULY 1950

ON THE CHARACTERIZATION OF FIELDS OF DIABATIC FLOW

PART II

Calculations of Steady Diabatic Flow in One and Two Dimensions

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Project No. TB3-0110 of the Research and Development Division, Ordnance Department

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 720

Hicks/Hebrank/Kravitz/lbe
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ON THE CHARACTERIZATION OF FIELDS OF DIABATIC FLOW

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ABSTRACT

The purpose of this report is to describe specific calculations of diabatic flows in order that both the physical understanding of these flows and methods for their computation may be improved.

After a review of the previously formulated theory, one-dimensional radial and vortex flows are considered. They illustrate several differences between adiabatic and diabatic compressible flows such as absence of limit circles and non-minimal stream-tube area at sonic velocity. Two-dimensional (uniplanar) flows describable by a potential are computed next. For one form of potential flow, the exact partial differential equation for the potential is linear with constant coefficients and always of elliptic type, if certain indirect analytical restrictions are placed upon the flow and upon the variation of the rate of heat addition from point to point. Linearization of a wide variety of uniplanar potential flows, including those of elliptic, parabolic and hyperbolic type, is possible if the heat addition rate is not too large. Scaling laws may be based on this linearization.

A third type of diabatic flow, somewhat less general than those preceding, is analyzed in more detail. Here the flow is uniform at infinity but is perturbed by a localized heat source. The velocity, pressure and density variations are calculated over the field of flow. Since the largest perturbation is on the density, it is the density perturbation which limits the size of the heat source that still permits a first-order perturbation calculation.

The concluding section describes a formulation of the general equations for uniplanar flow in terms of the components, u, v of the Crocco vector $\underset{\sim}{W}$. The partial differential equation obtained, say for u, is always hyperbolic and quasi-linear if it is assumed that v(x,y) is specified first. The heat addition function is directly calculable once u and v are known.

LIST OF SYMBOLS

a velocity of sound

c_p specific heat at constant pressure

$D \equiv 1 - N^2 g + g_\ell$

D_o A constant value of D

$\underset{\sim}{e}_r$ radial unit vector

$\underset{\sim}{e}_\theta$ unit vector orthogonal to $\underset{\sim}{e}_r$

$g(N) \equiv V/\left[N^2 R T\right]^{1/2}$

$g_\ell \equiv d \ln g / d \ln N$

$H_2 \equiv \exp \int g \, d(\frac{N^2}{2})$

$\underset{\sim}{i}, \underset{\sim}{j}$ unit vectors in the x and y directions respectively

M Mach number

$\underset{\sim}{N}$ vector function chosen to represent an irrotational field of flow

n number of dimensions less one

$\underset{\sim}{n}$ unit vector normal to streamline

p pressure

Q heat added to the fluid per unit mass and time

$q_N \equiv Q/\left[2 c_p T (g R T)^{1/2}\right]$

$q_W \equiv Q/\left[V_t^3 (1 - W^2)\right]$

q_1 heat added per unit mass, per velocity squared, per unit length

R gas constant

r radial distance

S specific entropy

$\underset{\sim}{s}$ unit vector tangent to streamline

T static temperature

T_t stagnation temperature

p_t stagnation pressure

u', v' pertubation velocities in $\underset{\sim}{i}$ and $\underset{\sim}{j}$ directions (Section 4)

u, v components of the vector $\underset{\sim}{W}$ in the x and y directions respectively (Section 5)

$\underset{\sim}{V}$ fluid velocity,

V_t local value of the limiting velocity $\equiv (2\, c_p\, T_t)^{1/2}$

$\underset{\sim}{W} \equiv \underset{\sim}{V}/V_t$

x, y coordinates

$\beta \equiv (\gamma + 1)/(\gamma - 1)$

γ ratio of specific heats

θ angular coordinate in a polar coordinate system

Ξ a function of u and v containing no second partial derivatives

ρ density

ϕ_N potential for an irrotational $\underset{\sim}{N}$ field

ϕ_V^0 potential for irrotational $\underset{\sim}{V}/V_0$ field

$\underset{\sim}{\omega}_M, \underset{\sim}{\omega}_V, \underset{\sim}{\omega}_{V'}, \underset{\sim}{\omega}_W$ vorticity $\nabla \times \underset{\sim}{M}$, $\nabla \times \underset{\sim}{V}$, $\nabla \times \underset{\sim}{V}'$ and $\nabla \times \underset{\sim}{W}$

1 as a subscript, refers to a transformed coordinate system

$(')$ refers to a perturbation of a variable

INTRODUCTION

The study of diabatic (that is non-adiabatic) steady flows is required in order that phenomena associated with combustion in steadily moving gases may be thoroughly understood. Since the heat release during combustion has a strong influence upon the gas flow we consider a theoretical model for moving, burning gases in which only the dynamical effects of the heat release are considered, and the effects of viscosity, diffusion and of change in specific heat or in composition of the gas are neglected. The steady mean flow of a turbulent burning gas could, for example, be described by this model, or the pressure distribution along a boundary layer owing to combustion outside the boundary layer could be computed. Theory based on the model is thus concerned with steady diabatic flow of an idealized fluid just as classical aerodynamic theory was concerned with the flow of air outside boundary layers.

The direct problem in diabatic flow theory consists in determining characteristics of the flow pattern from knowledge of the heat source distribution. The inverse problem, in which calculation of heat sources follows from knowledge of the flow pattern, is easier mathematically and yet is of some help in planning the solution of the direct problem. In our earlier investigations of both direct and inverse problems of diabatic flow,[1,2] formal manipulations of the partial differential equations have been emphasized. The nature of the resulting physical and mathematical problems has been described, but no detailed solutions of the problems have been given. Accordingly, in this report we wish to discuss several explicit solutions of the equations for steady diabatic flow and shall derive from them a more intimate understanding of the relation between the physical problems of aerodynamic combustion and their mathematical solution. We note that in some cases only part of a calculated flow pattern has direct physical interest, but that part is worth calculating even by what appears to be artificial methods.

The differences, for each variety of fluid motion, between adiabatic and diabatic flow are noteworthy. In adiabatic flow the enthalpy of each fluid particle is constant and the entropy change of each particle is zero except across shocks. On account of this special thermodynamic behavior, adiabatic compressible flows are characterized physically, for example, by minimum area of stream tubes at sonic velocity and mathematically by the identification of subsonic and supersonic flow, respectively, with elliptic and hyperbolic equations. Also in this adiabatic case the two kinds of compressibility effects, namely those associated with high Mach number and those associated with density or temperature changes, are closely coupled. In diabatic flow, on the other hand, the enthalpy and entropy of fluid particles can vary owing either to addition of heat locally by combustion or by conduction

(1) B. L. Hicks, "Diabatic Flow of a Compressible Fluid", Quart. App. Math. 6, 221-237 (Oct. 1948) (Referred to later as D-1).

(2) B. L. Hicks, "On the Characterization of Fields of Diabatic Flow", BRL Report No. 633, (May 10, 1947). Part I - General Theory of Steady Diabatic Flow; or Quart. App. Math. 6, 407-416 (Jan 1949). (Referred to later as D-2).

of heat from the neighborhood of the fluid particles. One has therefore no a priori idea of the behavior of diabatic flows with regard to minimum area of stream tubes and cannot associate the magnitude of the Mach number with the mathematical type of the flow. Also the two manifestations of "compressibility" are now less tightly coupled since the density can change appreciably with but small change in velocity and at small Mach numbers. From the one-dimensional theory, however, we know that the effects of heating and of stream tube area variation are similar. For example, a transition from sub- to super-sonic flow may be accomplished in adiabatic flow by a converging-diverging nozzle. In diabatic flow it may be effected, in a duct of uniform area, by proper addition and abstraction of heat. Although one can, therefore, guess at some of the effects produced by heating in two- or three-dimensional flow, it is possible to gain a more general insight into diabatic flow in the higher number of dimensions by studying examples of two- or three-dimensional theory.

In Section 1 the basic equations developed in D-1, 2 for two- and three-dimensional diabatic flow are summarized. The simplest flows are then considered first (Section 2). These are flows which are one-dimensional in the sense that for them the local Mach number depends on but one space variable. The connection with both the one-dimensional or hydraulic approximation for flow in ducts and two- or three-dimensional diabatic flows can be exhibited as well as the relationship to various adiabatic flows. The irrotational uniplanar diabatic flows (Section 3) are next in order of simplicity because they are describable in terms of a single potential function. Special elliptic, parabolic and hyperbolic flows are discussed, and then a general treatment of irrotational flows is given, based on assumption of a slightly perturbed flow, which leads to similarity laws.

The third type of flow discussed (Section 4) is of the greatest basic importance. Here one asks, what are the effects of heat added locally in an unbounded gas, flowing steadily, whose velocity is uniform far upstream of the local source of heat. The equations are first linearized according to a perturbation scheme since even for low Mach numbers the original equations are still non-linear. It is then possible to understand how the effects of heat sources distributed in an arbitrary fashion throughout the field of flow could be built up by superposition of the elementary solutions. The construction of the elementary solution itself offers some difficulty if the appearance in the field of vortex filaments and of infinite changes in enthalpy is to be avoided.

From the consideration of these detailed problems one can formulate what appears to be a reasonable approach to general two- or three- dimensional problems in diabatic flow. This formulation has been examined particularly (Section 5) for rotational two-dimensional diabatic flows, where it amounts to specifying throughout the field of flow one component of the Crocco vector ($\underset{\sim}{W} = \underset{\sim}{V}/V_t$ where V_t is the local value of the limit-velocity), and specifying the second component and its normal derivative along some curve in the field. The differential equation to be solved is then always hyperbolic. After calculation of the second component of $\underset{\sim}{W}$ the nature of the heat sources and of other characteristics of the flow could be computed. This kind of procedure in which the heat sources are not specified first seems to introduce somewhat more tractable equations, without unduly restricting the flow pattern, in almost all cases of diabatic flow that have been studied whether in one or more dimensions. (See Section 5 and References 12 and 13 for discussion of new work along these lines.)

The theory developed in this paper has been presented in part at meetings of the American Physical Society in 1947-48 and in the Third Symposium on Combustion, Flame and Explosion Phenomena, (Williams and Wilkins, 1949) (212 - 222). The present report differs from the Symposium account only in additions and corrections that have been made, particularly in Sections 2 to 5.

1. SUMMARY OF BASIC THEORY

For an inviscid compressible fluid containing heat sources, the equations of steady flow are

$$\nabla p + \rho \underset{\sim}{V} \cdot \nabla \underset{\sim}{V} = 0 \tag{1.1}$$

$$\nabla \cdot \rho \underset{\sim}{V} = 0 \tag{1.2}$$

$$c_p \underset{\sim}{V} \cdot \nabla T_t = T \underset{\sim}{V} \cdot \nabla S = Q \tag{1.3}$$

where the symbols p, ρ, T, T_t, V, S and Q stand respectively for pressure, density, temperature, stagnation temperature, fluid velocity, specific entropy and heat added to the fluid per unit mass and time. The quantity (Q/V) then gives the heat added to unit mass in unit distance along a streamline. The stagnation temperature is a measure of the total energy of a fluid particle. For the perfect gas here considered the specific heat at constant pressure, c_p, is constant and the equation of state is

$$p - R \rho T = 0 \tag{1.4}$$

Although variation of specific heat and of gas constant R occur in combustion zones and the phenomena of diffusion and turbulence often play a role, those complications are neglected here, as in earlier diabatic flow theory, in order that the important effects of heat generation alone can be examined. It is noted that the term Q could include effects of heat conduction explicitly although this possibility is not examined further here.

As has been shown previously (cf. D-1,2) the equations are placed in their most generally useful form if transformation is made to Crocco vector* $\underset{\sim}{W}$ and stagnation pressure, p_t, in place of velocity vector and (static) pressure p through the equations

$$\underset{\sim}{V} = V_t \underset{\sim}{W} \tag{1.5}$$

$$p = p_t (1 - W^2)^{\gamma/\gamma - 1} \tag{1.6}$$

where V_t is the local value of the limiting velocity

$$V_t = (2\, c_p\, T_t)^{1/2} \tag{1.7}$$

and the stagnation temperature T_t is related to the (static) temperature T by

$$T = T_t (1 - W^2) \tag{1.8}$$

The transformed equations are

$$\nabla \log p_t = \frac{2\gamma}{\gamma - 1} \left[(1 - W^2)^{-1} \underset{\sim}{W} \times (\nabla \times \underset{\sim}{W}) - q_W \underset{\sim}{W} \right] \tag{1.9}$$

$$\nabla \cdot (1 - W^2)^{1/\gamma - 1} \underset{\sim}{W} = q_W (1 - W^2)^{1/\gamma - 1} \left(1 + \frac{\gamma + 1}{\gamma - 1} W^2\right) \tag{1.10}$$

$$\underset{\sim}{W} \cdot \nabla \log V_t = (1 - W^2)\, q_W \tag{1.11}$$

in which

$$q_W = Q/V_t^{\,3} (1 - W^2) \tag{1.12}$$

* The Crocco vector was first used by the senior author and his co-workers at the Cleveland Laboratory of the NACA in 1945. The vector $\underset{\sim}{W}$ suggested itself as a natural generalization of variables used in one-dimensional flow theory[(3)], of Crocco's language, and as a corollary of the Mach vector we had introduced in 1943[(4)]. Recently Munk and others[(5,6)] have also introduced the vector $\underset{\sim}{W}$ (their "reduced velocity vector") which they are now using in detailed studies of general adiabatic flow.

(3) B. L. Hicks, D. J. Montgomery and R. H. Wasserman, "On the One-Dimensional Theory of Steady Compressible Fluid Flow in Ducts with Friction and Heat Addition", J. App. Phys. 18, 891-902 (Oct. 1947).

(4) B. L. Hicks, P. E. Guenther and R. H. Wasserman, "New Formulation of the Equations for Compressible Flow", Quart. App. Math. 5, 357-361, (Oct. 1947)

(5) M. M. Munk and R. Prim, "The Multiplicity of Steady Gas Flows Having the Same Streamline Pattern", Proc. Nat. Acad. Sci. 33, 137-141 (May 1947).

(6) M. M. Munk and R. C. Prim, "On the Canonical Form of the Equations of Steady Motion of a Perfect Gas", J. App. Phs. 19, 957-958, (Oct. 1948).

If a flow described by equations (1.9 - 1.11) were irrotational in the $\underset{\sim}{N}$ field (therefore admitting a potential $\phi_{\bar{N}}$) then heat addition must be such that (cf. D-1)

$$q_{\bar{N}} = q_{\bar{N}}(\phi_{\bar{N}}) \tag{1.13}$$

so that $q_{\bar{N}}$ is constant on surfaces normal to streamlines. Although this is a physically important and visualizable case, many other diabatic flows rotational in the $\underset{\sim}{N}$ field may also be derived (cf. D-2) from a potential function ϕ_N introduced with the more general transformation

$$\underset{\sim}{V} = \left[g(N)\ RT\right]^{1/2}\ \nabla\phi_N;\ |\underset{\sim}{N}| = N = |\nabla\phi_N| \tag{1.14}$$

in which $g(N)$ can be arbitrarily specified. The accompanying restriction on Q was found to be expressible as

$$(1 - N^2 g + g_\ell)\, q_N = \frac{1}{2} N \left(g_\ell - \frac{\gamma - 1}{\gamma} N^2 g\right) \nabla\cdot\underset{\sim}{S}$$

$$+ g^{-1}\left(1 - \frac{1}{\gamma} N^2 g\right)\left(1 + \frac{1}{2} g_\ell\right) F(\phi_N), \tag{1.15}$$

$$g_\ell = d \log g / d \log N$$

$$q_N = Q/2\ c_p\ (g\ R\ T)^{1/2} \tag{1.16}$$

We note that $q_{\bar{N}}$ and q_N have always the same sign as Q although their variation in the field of flow may be different owing to the variation of the factor $T^{3/2}\ g(N)$. The quantity $\nabla\cdot\underset{\sim}{S}$ is the fractional rate of change of stream-tube area along a streamline (4), $\underset{\sim}{S}$ being the unit vector $\underset{\sim}{N}/N$ tangent to the streamline at each point in the field of flow. The function $F(\phi_N)$ controls the variation of $(g\ T)$ along streamlines,

$$\frac{1}{2} g\ \underset{\sim}{N}\cdot\nabla \log (g\ T) = F(\phi_N) \tag{1.17}$$

fixes the variation of pressure throughout the field of flow,

$$\log p + \int g\ d\,\frac{1}{2} N^2 + \int F(\phi_N)\ d\ \phi_N = \text{constant} \tag{1.18}$$

and enters the partial differential equation which the potential ϕ_N must satisfy,

$$\sum_i \frac{\partial^2\phi_N}{\partial x_i^2}\left[N^2 - (1-D)\left(\frac{\partial\phi_N}{\partial x_i}\right)^2\right] - 2(1-D)\sum_{i>j}\frac{\partial\phi_N}{\partial x_i}\,\frac{\partial\phi_N}{\partial x_j}\,\frac{\partial^2\phi_N}{\partial x_i\,\partial x_j} =$$

$$= N^2\left(N^2 + g^{-1}\right) F(\phi_N) \tag{1.19}$$

The type of equation (1.19) is elliptic, parabolic or hyperbolic according to whether D is $>$, $=$, or <0. The function D depends only on N and its form is fixed by choice of $g(N)$ alone,

$$D = 1 - N^2 g + g\ell \qquad (1.20)$$

Because of the presence of two arbitrary functions, $F(\phi_N)$ and $g(N)$, irrotational $\underset{\sim}{N}$ fields are expected to be somewhat less restricted than they appear to be at first sight. A family of irrotational flows parameterized by $D = D_o$ = constant were described in D - 2 where expressions were given for $g(N)$ for all real values of D_o. It is conveninet to have available the corresponding expressions for $H_2 = \exp \int g \, d \frac{1}{2} N^2$, namely,

$$H_2 = \left| 1 - (N/K)^{1 + D_o} \right|^{-1} \qquad D_o \neq -1$$

$$= -\left[\log (N/K)\right]^{-1} \qquad D_o = -1 \qquad (1.21)$$

Various general properties of these equations and of the corresponding physical systems have been discussed in D-1, 2. We shall apply them here to the discussion of a number of special cases. In all of these cases, equation (1.11) shows that, in the motion of any fluid particle, the fractional rate of change of its total energy with respect to distance is simply $2\, Q_N\, (1 - N^2)/N$ while the fractional rate of change with respect to time is $2\, Q_N\, (1 - N^2)\, V_t$.

2. ONE-DIMENSIONAL FLOWS

As in other branches of fluid dynamics, an understanding of one-dimensional flows is essential before the more difficult problems of calculating multi-dimensional flows are attempted. Uniform, parallel, one-dimensionsional flow that is diabatic has been the subject of many papers (cf. for example, reference 3 and papers cited there) because of its direct applicability to jet propulsion and to detonation phenomena. We shall give here an introductory discussion of one-dimensional radial and vortex flows.

Radial Flows

The equation

$$\underset{\sim}{W} = W(r)\, \underset{\sim}{e}_r$$

describes a simple radial flow for which the magnitude of the Crocco vector $\underset{\sim}{W}$ depends only on the radial distance and the direction of the vector field is given by the radial unit vector, $\underset{\sim}{e}_r$. If (n + 1) indicates the number of dimensions (n = 1,2 correspond to line and point sources respectively) then the equations of motion, continuity and energy, equations (1.9 - 1.11) in W language become simply

$$\frac{\partial \log p_t}{\partial r} = \frac{-2\gamma}{\gamma - 1} W q_W \qquad (2.2)$$

$$r^{-n} \frac{\partial}{\partial r} r^n W(1 - W^2)^{1/\gamma - 1} = q_W (1 - W^2)^{1/\gamma - 1}\left(1 + \frac{\gamma+1}{\gamma-1} W^2\right) \qquad (2.3)$$

$$\frac{\partial \log V_t}{\partial r} = \frac{(1 - W^2)\, q_W}{W} \qquad (2.4)$$

Although adiabatic, compressible radial flows are restricted to be outside a limiting circle (or sphere) on which $W^2 = (\gamma - 1)/(\gamma + 1)$, $(M = 1)$, diabatic flows are not so restricted. For integration of equation (2.3) gives

$$r^n W(1 - W^2)^{\frac{1}{\gamma - 1}} = \int_{r_1}^{r} r^n q_W (1 - W^2)^{1/\gamma - 1}\left(1 + \frac{\gamma+1}{\gamma-1} W^2\right) d\, r + C_1 \qquad (2.5)$$

Both C_1 and the R.H.S of equation (2.5) are not negative. Since $W(1 - W^2)^{1/\gamma - 1}$ cannot exceed its maximum value,

$$C_2 = \left[\left(\frac{\gamma-1}{\gamma+1}\right)\right]^{1/2} \left[\frac{2}{\gamma+1}\right]^{\frac{1}{\gamma - 1}}$$

we obtain the inequalities

$$0 \leq \int_{r_1}^{r} r^n (1 - W^2)^{1/\gamma - 1}\left(1 + \frac{\gamma+1}{\gamma-1} W^2\right) q_W d\, r + C_1 \leq C_2 r^n \qquad (2.6)$$

If $q_W \equiv 0$, $C_1 > 0$, (adiabatic case), the flow is characterized by an inner limiting circle at $r^n = C_1/C_2$. For arbitrary functions $q_W(r)$ we would expect one or more circles limiting the flow on either the outside or the inside. (cf. discussion of $D = D_c$ flows later in this section.) For many functions $q_W(r)$ we should find no limiting circles. Thus a bounded function $q_W(r)$ can be found giving $W \propto r^{K_1}$ $(K_1 \geq 1)$ or giving $(1 - W^2) \propto r^{K_2}$ $(K_2 \geq \gamma - 1)$ at the origin. If W is to be different from zero at $r = 0$ then $q_W \to \infty$ there, and also $q_W \to \infty / r \to 0$ if $0 < K_1 < 1$ or if $0 < K_2 < \gamma - 1$. In order that no limiting circle exclude the origin it is necessary if $q_W < C_3 r^{K_3}$ near the origin either that K_3 exceed -1 or that, for $K_3 = -1$, $C_2 > 2\gamma C_3/n(\gamma - 1)$.

To illustrate some of these remarks we shall consider two types of radial flow. In equations (2.2 - 2.4) specification of any one of the four quantities, p_t, W, q_W, V_t as a function of r should in principle permit calculation of the others. Some combination of the variables might equally well be specified, for example $q_M = q_W \left[g(W)/g(M) \right]^{1/2}$. However, calculations are especially simple if $W(r)$ is specified and q_W, p_t, V_t computed subsequently, and this procedure will be adopted here. For our first example we shall therefore assume a uniplanar flow (n = 1) in which the form of $W(r)$ is

$$W = r(1 + r^2)^{-1/2} \quad ; \quad \left(M = \left(\frac{2}{\gamma - 1}\right)^{1/2} r\right) \tag{2.7}$$

corresponding to a flow extending from $r = 0$ to $r = \infty$ with a continuous transition through the velocity of sound at $r^2 = (\gamma - 1)/2$. The heating parameter q_W has, from equation (2.3), the value 2 at the origin and depends on r according to the equation

$$q_W = 2\left[1 - \left(\frac{3-\gamma}{2(\gamma-1)}\right) r^2\right]\left[1 + \frac{2\gamma}{\gamma - 1} r^2\right]^{-1}\left[1 + r^2\right]^{-1/2} \tag{2.8}$$

Owing to the divergence of the stream tubes, q_W passes through zero for a velocity greater than that of sound, namely for $W^2 = (\gamma - 1)/(\gamma + 1)$, $(M^2 = 4/(3 - \gamma))$, $r^2 = 2(\gamma - 1)/(3 - \gamma)$. At sonic velocity

$$q_W = \left[2(\gamma + 1)\right]^{-1/2} \tag{2.9}$$

in accordance with equation (3-3) of D-1. The r-dependence of V_t is found by integration to be

$$(V_t/V_{t\ max}) = \left(\frac{\gamma+1}{2}\right)\left[27(\gamma - 1)\right]^{1/2} r^2 (1 + r^2)^{1/2} (\gamma - 1 + 2\gamma r^2)^{-3/2} \tag{2.10}$$

in which V_t reaches its maximum value $V_{t\ max}$ where $q_W = 0$.(*) The velocity ratio $V/V_{t\ max}$ is obtained by multiplication with W. The other quantities can be obtained similarly.

In figure 1, a, b, c the computed characteristics of this diabatic source flow are illustrated for $\gamma = 1.400$. The net energy change of this fluid in passing from $r = 0$ to $r = \infty$ is measured by $\left[T_t(\infty)/T_{t\ max}\right] = \left[V_t(\infty)/V_{t\ max}\right]^2 = 0.708$. Thus only part of the energy added in accelerating the fluid from rest and zero temperature at $r = 0$ to maximum energy at $r = 0.707$ is removed in continuing the expansion to where $V = V_t$, $M = \infty$, $R = \infty$. The maximum rate of heat evolution occurs for

(*) Actually $V_{t\ max}$ could be different for each streamline, in analogy with the generalized adiabatic flows considered in references 4, 5, and 6.

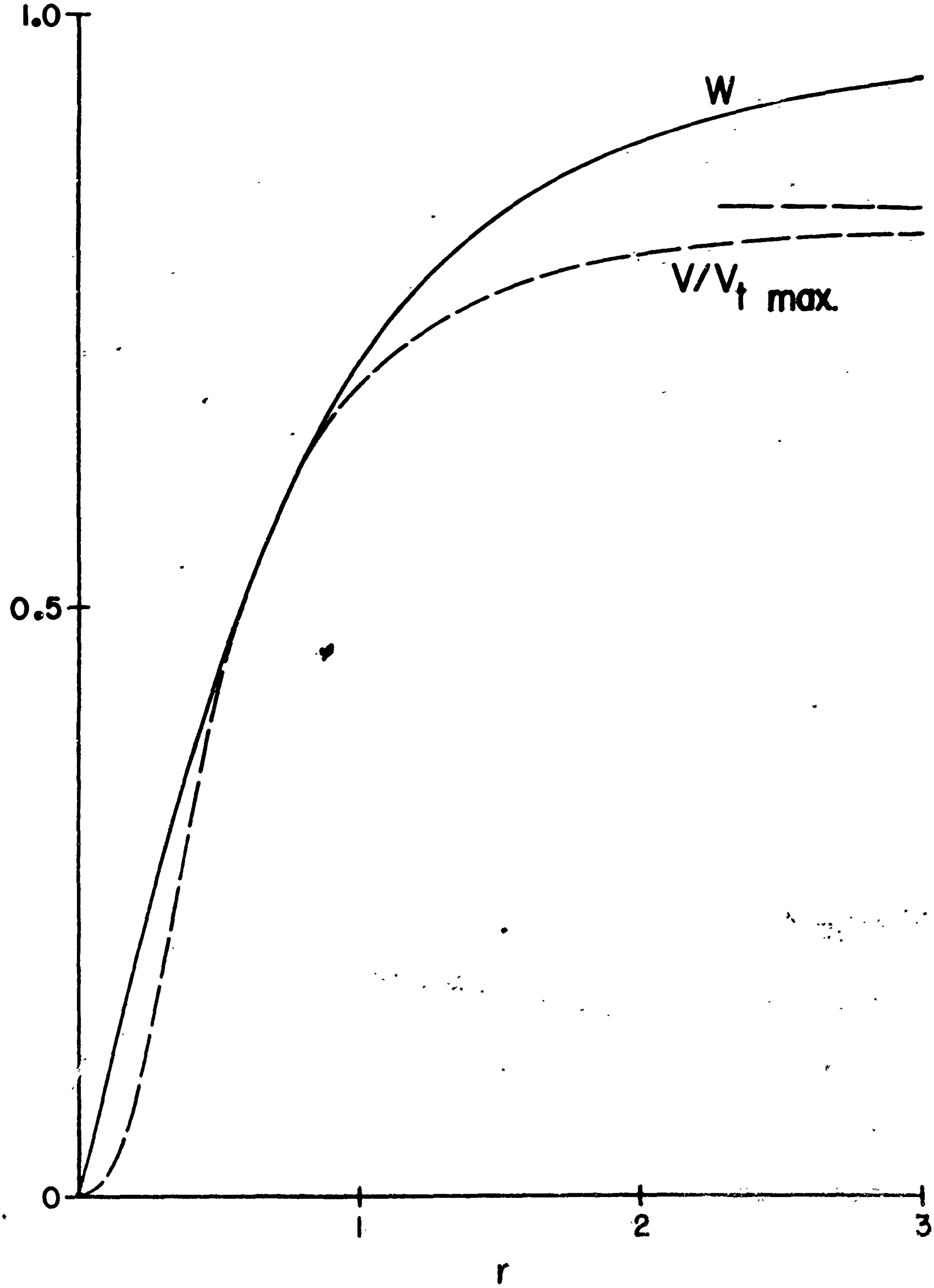


FIG. 1a. RADIAL DIABATIC FLOW

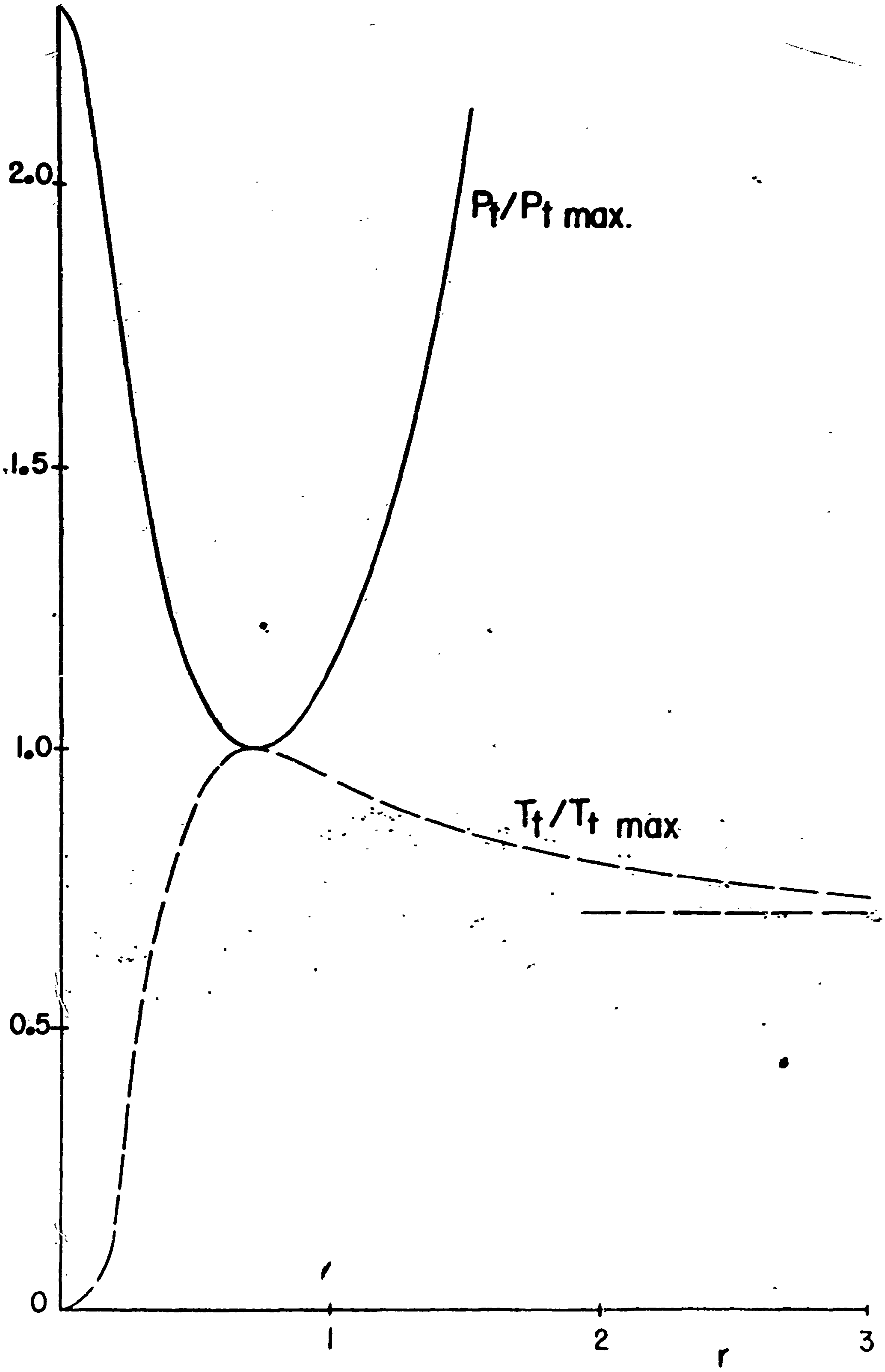


FIG. Ib. RADIAL DIABATIC FLOW

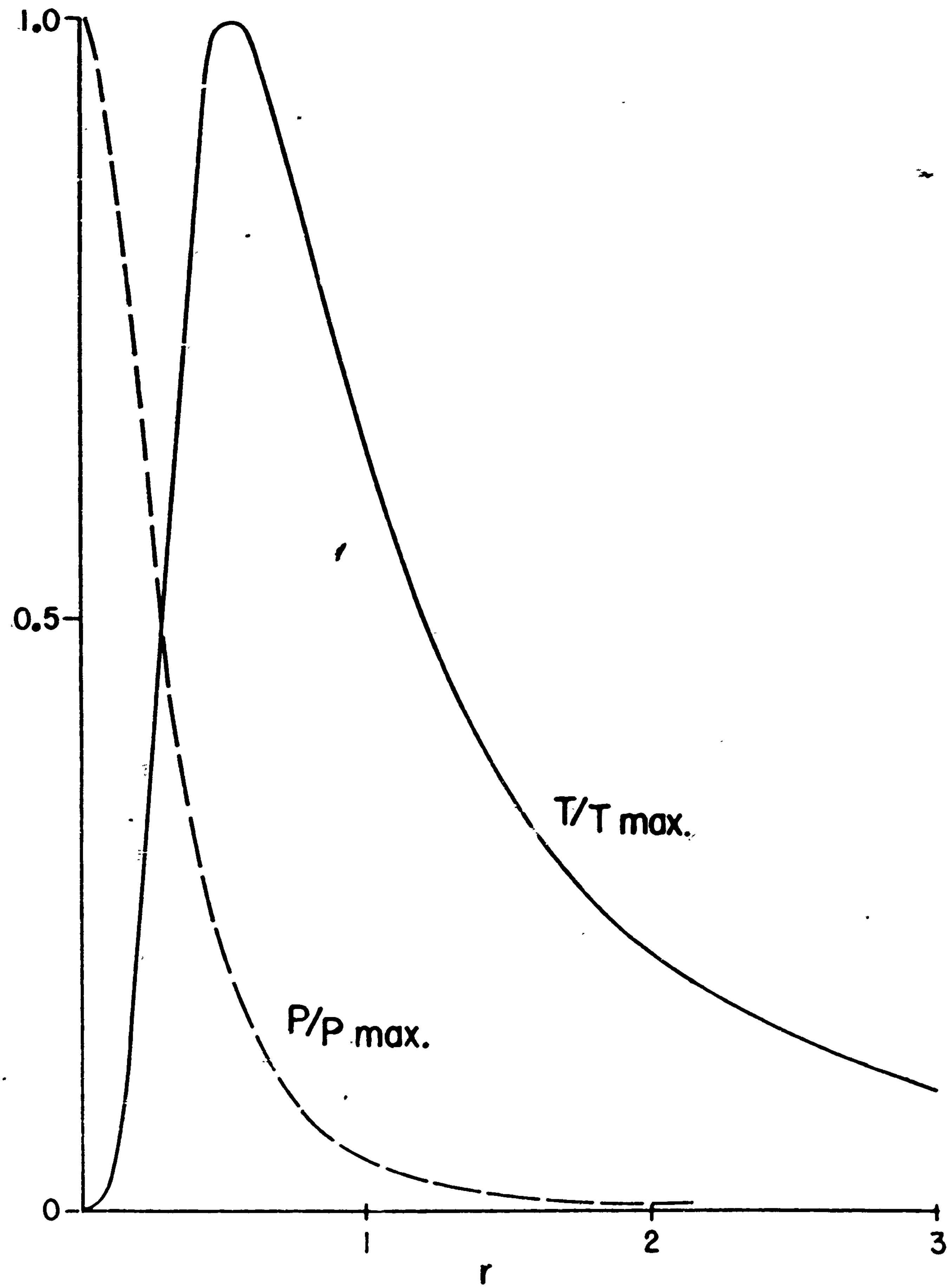


FIG. Ic. RADIAL DIABATIC FLOW

$r \sim 0.36$ ($M \sim 0.80$) where it equals (for $V_{t\,max} = 10^3$ unit lengths/sec) $\sim 2 \times 10^3\ sec^{-1}$, expressed as fraction released per second of total heat added between $r = 0$ and $r = 0.707$. The value of M at $r = 0.707$ where $Q = 0$, is 1.58. For larger values of r, $Q < 0$ but approaches zero rapidly as $r \to \infty$. The stagnation pressure, Figure 1b, which reaches a minimum at $r = 0.707$, increases indefinitely as $r \to \infty$, illustrating the extreme sensitivity of hypersonic flow to heating. Thus our example shows the possibility, already discussed in the hydraulic approximation,(3) of obtaining supersonic flow with combustion even though the nozzle has no converging part. One could have, for example, increased M beyond 1.58 by adiabatic expansion instead of by the exothermic expansion calculated. Such a radial flow could possibly be set up experimentally in a nozzle consisting of two parallel discs between which combustible mixture is fed through tubes on the axis of the discs.

The second case of radial flow to be considered originates in the use of potential functions in two and three dimensional fields (cf. Sections 1 and 3). It is instructive there (see D-2) to study flows of constant type ($D = D_o$ = constant) leading to a family of functions $g_N(N, D_o)$ parameterized by D_o. Specifications of the same g_N functions for purely radial flow amounts in equations (2.2 - 2.4) to specification of a combination of the variables q_W and W and will illustrate again the differences between adiabatic and diabatic flow. We note that the second arbitrary function $F(\phi_N)$, which occurs in equations (1.16, 18), must also be given, if g is to be determined. For the present simple illustrative case we place $F(\phi_N) \equiv 0$ which corresponds (see D-2, Section 3) in more general cases to irrotational $\underset{\sim}{V}$ flow in which (V/N) can vary between streamlines. (In the next section irrotational flows for which $F(\phi_N) \neq 0$ will be computed.)

We can again consider two and three-dimensional radial diabatic flows simultaneously. From the earlier theory (D-2), ϕ_N must satisfy the equation

$$r^{-n} \frac{d}{dr}\left(r^n N^{D_o - 1} \frac{d\phi_N}{dr}\right) = 0 \tag{2.11}$$

so that

$$(N/K) = r^{-n/D_o} \tag{2.12}$$

where K is the the integration constant and $N^2 g$ becomes (D-2)

$$\begin{aligned} N^2 g &= (1 + D_o)(r^{n(D_o + 1)/D_o} - 1) \qquad & D_o \neq -1 \\ &= \left[- n \log r\right]^{-1} & D_o = -1 \end{aligned} \tag{2.13}$$

Now for real flows, $N^2 g$ must be > 0. From equation (3.1) we see that r is correspondingly restricted.

$$r^{1/D_0} > 1 \qquad (2.14)$$

for all values of D_0. Since the value of D_0 determines the type of flow in non-radial cases according to its sign, we can carry over the terminology and say that for

$$\text{elliptic flows, } D_0 > 0;\ r > 1$$
$$\text{hyperbolic flows, } D_0 < 0;\ r < 1 \qquad (2.15)$$

($D_0 = 0$ is not considered since it implies, for $F = 0$, that streamlines are parallel.) Thus a second departure from adiabatic flow behavior presents itself, for now we have exhibited flows which can be wholly within as well as wholly without the circle $r = 1$. Also where $r = 1$, $(N/K) = 1$ and $N^2 g = \gamma M^2$ becomes infinite for all D_0 ($\neq 0$) whereas in the adiabatic case $M = 1$ on the circle limiting the flow internally.

Vortex Flows

If

$$\underset{\sim}{W} = W(r,\theta)\, \underset{\sim}{e}_\theta \qquad (2.16)$$

the equations now are (cf. (D-1), equations (2.5), (3.2) with $\nabla \cdot \underset{\sim}{S} = 0$), in cylindrical polar coordinates,

$$r^{-1} \frac{\partial \log p_t}{\partial \theta} = -\left(\frac{2\gamma}{\gamma-1}\right) W q_W \qquad (2.17)$$

$$\frac{\partial \log p_t}{\partial r} = \left(\frac{2\gamma}{\gamma-1}\right) \frac{\omega_W W}{\left[1 - W^2\right]} \qquad (2.18)$$

$$r^{-1} \frac{\partial W}{\partial \theta} = q_W (1 + \beta W^2)(1 - \beta W^2)^{-1}(1 - W^2), \quad \beta = (\gamma + 1)/(\gamma - 1) \qquad (2.19)$$

$$(2\, r)^{-1} \frac{\partial \log T_t}{\partial \theta} = (1 - W^2)\, q_W / W \qquad (2.20)$$

These equations show that for an adiabatic vortex flow, the velocity, stagnation pressure and stagnation temperature are independent of θ. For an irrotational adiabatic $\underset{\sim}{W}$ field, p_t is everywhere the same, and Wr = constant, thus leading to an inner limiting circle. For a rotational adiabatic flow $\underset{\sim}{W} = W(r)\, \underset{\sim}{e}_\theta$ and equation (2.18) can be integrated,

$$\log p_t = \frac{\gamma}{\gamma - 1} \int \frac{r^{-2}\, d(W^2 r^2)}{(1 - W^2)} \tag{2.21}$$

The functional dependence of W and ω_W upon r is now not restricted so that flows in which there are no limiting circles are easy to construct. For example with (compare equation (2.7) et. seq.) $W^2 = r^2/(1 + r^2)$ again

$$\omega_W = (2 + r^2)/(1 + r^2)^{3/2} \tag{2.22}$$

and

$$\log (p_t/p_{t_o}) = \frac{\gamma}{\gamma - 1} \left[\log (1 + r^2) + r^2 \right] \tag{2.23}$$

Thus a vorticity that is everywhere finite can permit adiabatic vortex flows without limit circles.

In diabatic vortex flows of the type described by equation (2.16) W and p_t can depend upon θ. However, we notice that the difference between $\log p_t$ and a certain function of W

$$\begin{aligned} f_1(W) &= \frac{-2\gamma}{\gamma - 1} \int W(1 - \beta W^2)(1 + \beta W^2)^{-1}(1 - W^2)^{-1}\, dW \\ &= -\ln \left\{ \left[1 + \beta W^2\right] \left[W^2 - 1\right]^{1/\gamma - 1} \right\} \end{aligned} \tag{2.24}$$

must be independent of θ or

$$\log p_t = f_1(W) + f_2(r) \tag{2.25}$$

We can summarize the properties of the radial and vortex flows thus:

(i) rotational or irrotational adiabatic radial flows have limit circles owing to the form of the continuity equation;

(ii) adiabatic or diabatic <u>irrotational W</u> vortex flows have limit circles owing to the form of the irrotationality condition;

(iii) flows without limit circles can be constructed by adding heat to radial flows or vortex flows or by changing the vorticity pattern of vortex flows.

3. IRROTATIONAL N FLOWS IN TWO-DIMENSIONS

If the vector $\underset{\sim}{N} = \underset{\sim}{V} \left[g(N)\, R\, T \right]^{-1/2}$ is irrotational then its potential ϕ_N must satisfy equation (1.19) and the heating function q_N is given by equation (1.15). Since these two equations involve two arbitrary functions, $g(N)$ and $F(\phi_N)$, the nature of N flow patterns, even though irrotational, and the corresponding distribution of heat sources can still be quite varied. In order to illustrate the nature of the flows we shall choose certain functions $g(N)$, $F(\phi_N)$ and give approximate solutions of equation (1.19) with appropriate boundary or initial conditions.

Elliptic Flows

Equation (1.19) is simplified if $D \equiv 1$ which implies that the flow is always elliptic (cf. D-2) and that

$$g = 2(k^2 - N^2)^{-1} \tag{3.1}$$

The equations become, with the added assumption $F(\phi_N) = \phi_N$

$$q_N(1 - N^2/k^2) = \frac{1}{2} k^2 \left[1 - (\gamma+2) N^2/\gamma k^2\right] \phi_N + N^3 \nabla \cdot \underset{\sim}{S}/\gamma k^2 \tag{3.2}$$

$$\nabla^2 \phi_N = \frac{1}{2} (k^2 + N^2) \phi_N \tag{3.3}$$

A uniplanar elliptic flow described by these equations is the first case of irrotational $\underset{\sim}{N}$ flow we shall treat.

The constant $k/\sqrt{2}$ can be removed by a change of scales in the (x,y) plane*

$$\underset{\sim}{r}_1 = k\,\underset{\sim}{r}/\sqrt{2} \tag{3.4}$$

which also transforms $\underset{\sim}{N} = \nabla\phi_N$

$$\underset{\sim}{N} = \underset{\sim}{N}_1\, k/\sqrt{2} \tag{3.5}$$

The equations for q_N and ϕ_N become

$$2(1 - \frac{1}{2} N_1^2)\, q_N/k^2 = \left[1 - (\gamma+2) N_1^2/2\gamma\right]\phi_N + N_1^3 \nabla_1 \cdot \underset{\sim}{S}/2\gamma \tag{3.6}$$

and

$$\nabla_1^2 \phi_N = (1 + \frac{1}{2} N_1^2)\, \phi_N \tag{3.7}$$

the subscript 1 referring to the new coordinate vector $\underset{\sim}{r}_1$.

* The parameter k can be removed by a scale change in all the irrotational flows parameterized by D_o.

In order to complete the specification of our problem we adjoin to (3.7) the boundary condition

$$\phi_N = \frac{1}{2}\ \phi_1\ (1 + \cos\theta) \qquad r_1 = r_1' > 1 \qquad (3.8)$$

$$\frac{\partial \phi_N}{\partial r_1} = 0 \qquad r_1 = 1 \qquad (3.9)$$

corresponding to a diabatic flow around the circle $r_1 = 1$ with prescribed values for ϕ_N upon the larger circle $r_1 = r_1'$. Other boundary conditions and choices for the functions $F(\phi_N)$ have been discussed briefly elsewhere(7). We believe that useful results may be obtained if $\partial\phi/\partial y$ is taken to vanish on $r_1' \gg 1$ corresponding to a flow that is uniform in direction at large distances from the cylindrical obstacle $r_1 \leq 1$. Then when the extension to values of D_0 other than 1 is made by scaling rules the flow would remain uniform in direction at large distances. Also Dr. Dimsdale of these Laboratories has shown that $F(\phi_N)$ in the more general elliptic flow equation

$$\nabla^2 \phi_N = (1 + \overline{\nabla\phi_N}^2)\ F(\phi_N) \qquad (3.10)$$

can be so chosen that exact linearization is possible. Thus he makes the transformation

$$\psi = \int^{\phi_N} dt\ \exp\left(-\int^{t} F(\sigma)\ d\sigma\right) \qquad (3.11)$$

which leads to

$$\nabla^2 \psi = \Psi(\psi) = F(\phi_N)\ \exp\left(-\int^{\phi_N} F(s)\ ds\right) \qquad (3.12)$$

The condition for linearity of $\Psi(\psi)$ is that

$$F'(\phi_N) - \left[F(\phi_N)\right]^2 = A = \text{constant} \qquad (3.13)$$

According to the value of A, there are then three possibilities

$$\text{(i)}\quad A = 0;\ F(\phi_N) = -(\phi_N + \phi_0)^{-1};\quad \psi = \alpha\,(\phi_N + \phi_0)^2 + \beta;\ (3.14)$$

$$\Psi(\psi) = -2\alpha$$

---

(7) B. L. Hicks, "Simple Diabatic Flows of Elliptic Type", Phys. Rev. 72, 179(A), (July 15, 1947).

(ii) $A = \alpha^2$; $F(\phi_N) = \alpha \tan\left[\alpha(\phi_N + \phi_0)\right]$; (3.15)

$$\psi = \beta \sin\left[\alpha(\phi_N + \phi_0)\right] + \psi_0; \quad \Psi(\psi) = \alpha^2(\psi - \psi_0)$$

(iii) $A = -\alpha^2$; $F(\phi_N) = -\alpha \coth\left[\alpha(\phi_N + \phi_0)\right]$; (3.16)

$$\psi = \beta \cosh\left[\alpha(\phi_N + \phi_0)\right] + \psi_0; \quad \Psi(\psi) = -\alpha^2(\psi - \psi_0)$$

For these three cases calculation of diabatic flow fields can be based, in principle, wholly upon analytical computations. Further study of these cases might well be rewarding since, for example, exact calculation of transonic diabatic fields would be possible. Our further treatment here will be based upon approximate linearization of equation (3.7). The parameter ϕ_0 fixes the scale of ϕ_N. If ϕ_0 is small enough, max N_1 will be small compared to one and equation (3.7) becomes

$$\nabla_1^2 \phi_N = \phi_N \tag{3.17}$$

whose solution, for the boundary conditions equations (3.8, 3.9) is

$$\begin{aligned}\phi_N(r_1, \theta) = {} & \tfrac{1}{2}\phi_1\left[K_1(1)I_0(r_1) + I_1(1)K_0(r_1)\right]\left[K_1(1)I_0(r_1') + I_1(1)K_0(r_1')\right]^{-1} \\ & + \tfrac{1}{2}\phi_1 \cos\theta\left\{\left[I_0(1) - I_1(1)\right]K_1(r_1) + \left[K_0(1) + K_1(1)\right] I_1(r_1)\right\} \\ & \left\{\left[I_0(1) - I_1(1)\right]K_1(r_1') + \left[K_0(1) + K_1(1)\right] I_1(r_1')\right\}^{-1}\end{aligned} \tag{3.18}$$

Calculations have been made for $r_1' = 2$ and $\phi_0 = 0.2864$ which insures that max $N_1 = 0.203$ (corresponding to max $M = 0.242$). The accurate evaluation of $\nabla_1 \cdot \underset{\sim}{S}$ in equation (3.6) requires here, as elsewhere, computation, of the expression

$$\begin{aligned} N^3\nabla \cdot \underset{\sim}{S} = {} & N^2\nabla^2\phi - \phi_r^2\,\phi_{rr} - 2r^{-2}\phi_r\,\phi_\theta\,\phi_{r\theta} \\ & - r^{-4}\,\phi_\theta^2\,\phi_{\theta\theta} + r^{-3}\,\phi_r\,\phi_\theta^2 \end{aligned} \tag{3.19}$$

(or its equivalent).

It would be difficult to obtain reasonable accuracy if the calculation of $\nabla \cdot \underset{\sim}{S}$ were based upon measurement of the divergence of streamlines drawn by graphical interpolation or if it were based upon values of ϕ_N and its derivatives determined from say elaxation calculations that might be required in a non-linear problem.

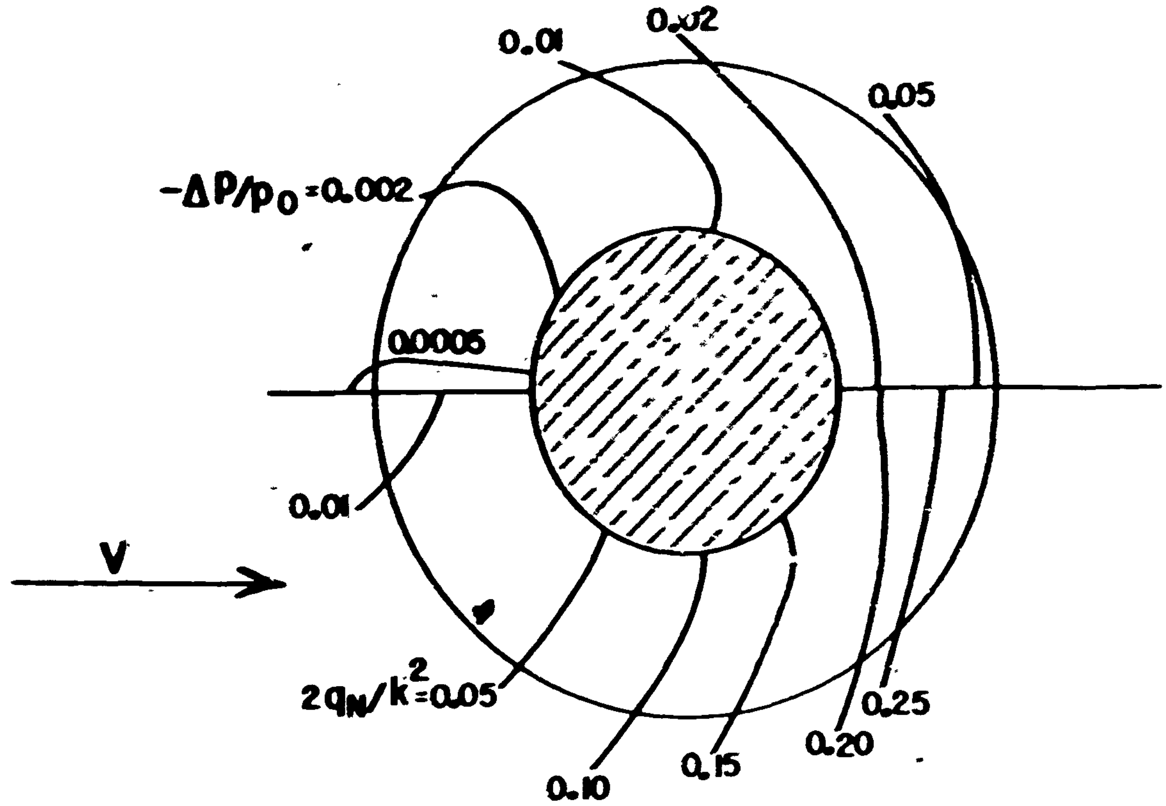


FIG. 2b. ELLIPTIC DIABATIC FLOW

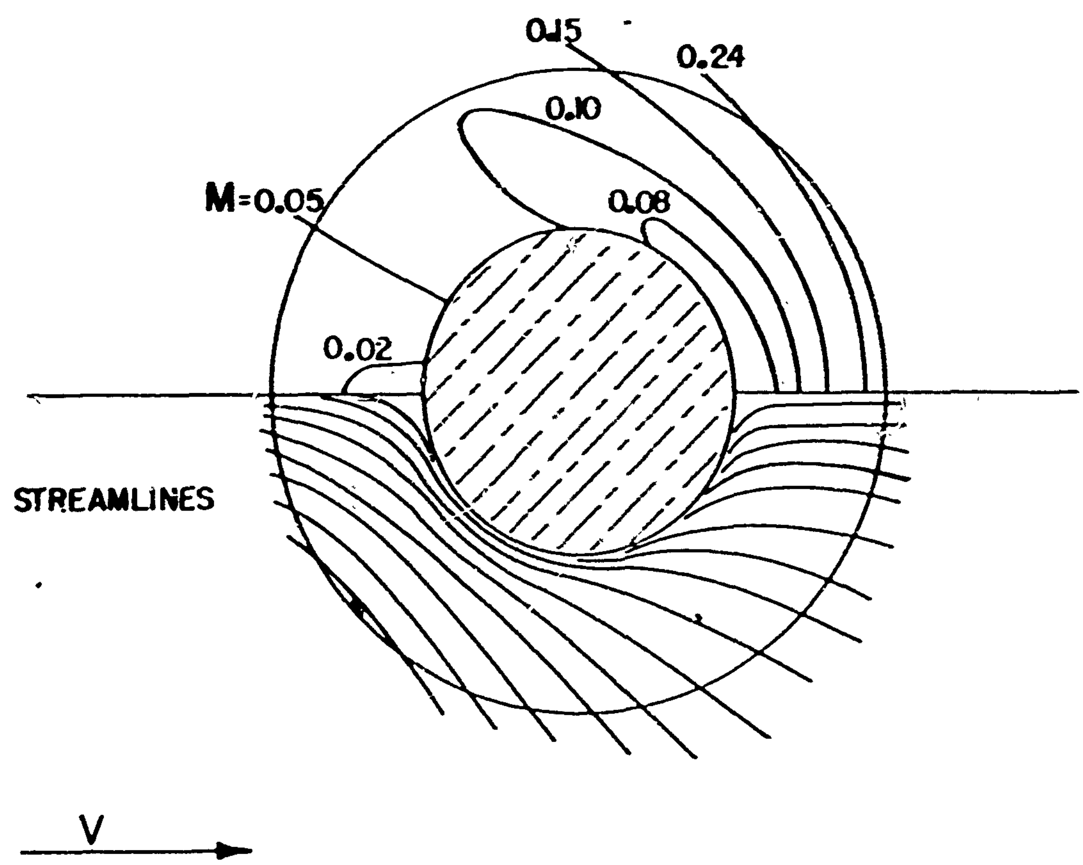


FIG. 2a. ELLIPTIC DIABATIC FLOW

In figures 2 a, b, the streamline pattern and lines of constant M, $\Delta p/p_0$ and $2q_M/k^2$ are shown, p_0 being approximately the value of p at $r_1' = 2.0$, $\theta = \pi$. Owing to the symmetry of the flow about the y axis, it is only necessary to give these curves for either half-plane.

Hyperbolic and Parabolic Flows

Examples of wholly hyperbolic or parabolic flows can be easily constructed where a Glauert - Prandtl type of linearized treatment is permissible. In this way we shall obtain some characteristics of diabatic flows in curved ducts which will serve to introduce an approximate general similarity treatment of irrotational diabatic flows. Let us consider first a wholly hyperbolic flow mentioned in D-2 (Section 8) which has the property that the flow velocity is everywhere sonic ($M \equiv 1$) and that

$$q_M = \frac{1}{2} M \nabla \cdot \underset{\sim}{S} \tag{3.20}$$

This flow is derived by choosing $g = \gamma/M^2$ for which $D_0 = -(\gamma + 1)$. The linearized differential equation for ϕ_M (in two dimensions) becomes, if $F(\phi_M) = \phi_M$ (cf. D-2, equation (8.4)),

$$-(\gamma + 1) \frac{\partial^2 \phi_M}{\partial x^2} + \frac{\partial^2 \phi_M}{\partial y^2} \doteq \frac{\gamma+1}{\gamma} M_0^2 \phi_M \tag{3.21}$$

An elementary solution of this hyperbolic equation is

$$\phi_M = M_0 x \cosh \left(\sqrt{\frac{\gamma+1}{\gamma}} M_0 y\right) \tag{3.22}$$

satisfying the boundary conditions

$$\left.\begin{aligned} \phi_M &= M_0 x \\ \frac{\partial \phi_M}{\partial y} &= 0 \end{aligned}\right\} \text{ on } y = 0 \tag{3.23}$$

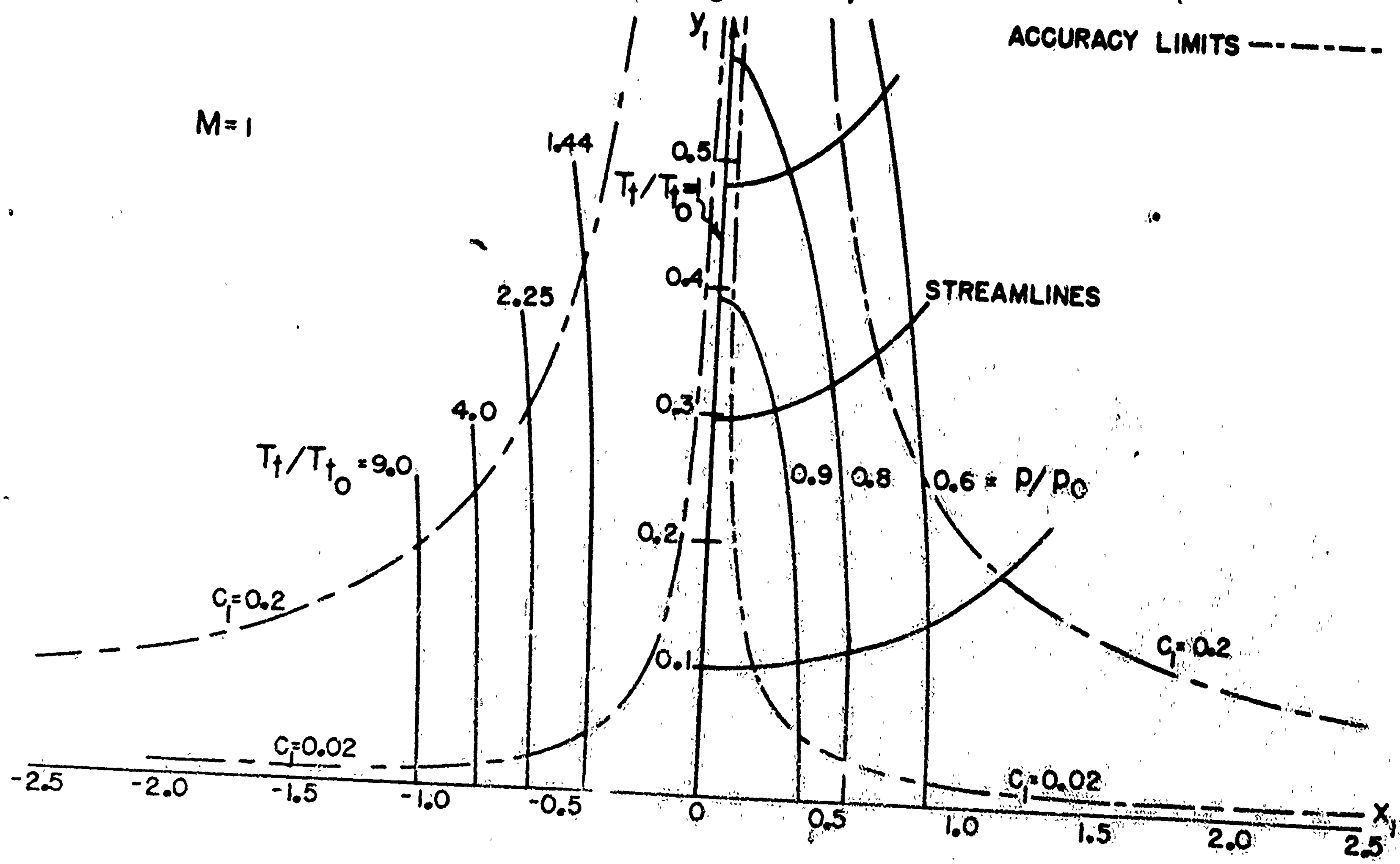
The linearization is valid when $|\partial\phi_M/\partial y| \ll |\partial\phi_M/\partial x| \sim N \sim N_0$ or roughly in the region $|[(\gamma + 1)/\gamma]^{1/2} M_0 y| \ll 1$, $|[(\gamma + 1)/\gamma]^{1/2} N_0^2 xy| \ll 1$. The expression in Cartesian coordinates that is analogous to equation (3.19) is

$$M^3 \nabla \cdot \underset{\sim}{S} = \phi_y^2 \phi_{xx} - 2\phi_x \phi_y \phi_{xy} + \phi_x^2 \phi_{yy} \tag{3.24}$$

which in our present approximation becomes

$$\nabla \cdot \underset{\sim}{S} = (\gamma + 1)\gamma^{-1/2} M_0 x_1 (1 - 2 \tanh^2 y_1) \tag{3.25}$$

FIG. 3. HYPERBOLIC DIABATIC FLOW
$D_O = -(\gamma+1)$



in which the scale changes

$$
\begin{aligned}
x_1 &= \sqrt{\frac{1}{\gamma}}\, N_0\, x \\
y_1 &= \sqrt{\frac{\gamma+1}{\gamma}}\, N_0\, y
\end{aligned}
\tag{3.26}
$$

have been made.

The heating factor q_N is given by

$$
q_N = \frac{1}{2} N_0^{\,2} (\gamma + 1) \gamma^{-1/2}\, x_1 \cosh y_1 \,(1 - 2 \tanh^2 y_1)
\tag{3.27}
$$

The pressure variation is found to be (cf. D-2 equation (8.6))

$$
\log (p\, N^{\gamma}) + \frac{1}{\gamma}\, \phi_N^{\,2} = \text{constant}
\tag{3.28}
$$

Therefore

$$
\log (p/p_0) = -\gamma \left(2 x_1 \cosh y_1 + \frac{1}{2} x_1^{\,2} \cosh^2 y_1\right)
\tag{3.29}
$$

Temperatures (or velocities $\propto T^{1/2}$) can also be computed. The results are illustrated in figure 3. The regions of validity of this solution are indicated by the accuracy limit curves in the figure which are loci of constant values c_1 of $x_1 \tanh y_1$. Thus at all points beneath the $c_1 = 0.2$ curves, the y component of $\underset{\sim}{N}$ is not greater than ~ 30% of the x component.

A more general analysis of the hyerbolic equation (3.30) can be based on the Riemann function for the equation which leads to the expression

$$
\begin{aligned}
\phi_N(x_1, y_1) = \frac{1}{2}\Big[f(x_1 + y_1) + f(x_1 - y_1)\Big] &+ \frac{y_1}{2} \int_{x_1 - y_1}^{x_1 + y_1} f(\zeta_1)\, \frac{I_1(Z)}{Z}\, d\zeta_1 \\
&+ \frac{1}{2} \int_{x_1 - y_1}^{x_1 + y_1} g(\zeta_1)\, I_0(Z)\, d\zeta_1
\end{aligned}
\tag{3.30}
$$

where

$$
z = \left[y_1^{\,2} - (x_1 - \zeta_1)^2 \right]^{1/2}
\tag{3.31}
$$

corresponding to the boundary conditions

$$
\left.
\begin{aligned}
\phi_N &= f(x_1) \\
\frac{\partial \phi_N}{\partial y_1} &= g(x_1)
\end{aligned}
\right\} \quad \text{on } y_1 = 0
\tag{3.32}
$$

The functions $I_0(z)$ and $I_1(z)$ are modified Bessel functions. (If $F(\phi_N) = -\phi_N$ the solution may be obtained by replacing $I_1(z)$ by $[-J_1(z)]$ and $I_0(z)$ by $J_0(z)$ in equation (3.30).) The expression for ϕ_N can be transformed to read

$$\phi_N(x_1,y_1) = \frac{1}{2}\left[f(x_1 + y_1) + f(x_1 - y_1)\right]$$
$$+ \frac{y_1}{2}\int_0^{y_1}\left[f(x_1 + \sqrt{y_1^2 - z^2}) + f(x_1 - \sqrt{y_1^2 - z^2})\right]\frac{I_1(z)}{\sqrt{y_1^2 - z^2}}\, dz$$
$$+ \frac{1}{2}\int_0^{y_1}\left[g(x_1 + \sqrt{y_1^2 - z^2}) + g(x_1 - \sqrt{y_1^2 - z^2})\right]\frac{z\, I_0(z)}{\sqrt{y_1^2 - z^2}}\, dz \qquad (3.33)$$

In this form it is clear how the nature of the solution differs from that for a simple wave. Integration of the values of $\phi_N = f$ as well as of $(\partial\phi_N/\partial y_1) = g$ along the support curve $y_1 = 0$ are included and the values are weighted with the greatest weighting near the endpoints, $(x_1 \pm y_1)$, of the region of influence for the point (x_1, y_1). The solution cannot satisfy the linearization condition for all x and y because of the divergence of $I_0(z)$ and $I_1(z)$ as exp z for large z. The heating factor and pressure satisfy the same equations (3.20), (3.28) as before. We also notice that change of N_0 can be absorbed as a scaling factor in each of these cases of hyperbolic flow. It would be interesting to work through some case such as $(\partial\phi_N/\partial x) \propto (1 - \tanh x)$ because, since q_N then need not be negative for $x < 0$, a greater part of the field would correspond to flow with (exothermic) combustion. Asymmetrical flows would occur if $(\partial\phi_N/\partial y) \neq 0$ on $y = 0$.

As a second example of duct flow consider the wholly parabolic flows which occurs only when $D \equiv 0$. Unlike the special hyperbolic flow just considered, in this parabolic flow the Mach number varies. In the Glauert-Prandtl approximation, the partial differential equation for ϕ_N is in this case (cf. D-2 equation (7-3) with $F(\phi_N) = \phi_N$).

$$\frac{\partial^2\phi_N}{\partial y^2} = k\, N_0\,\phi_N \qquad (3.34)$$

whose solution is

$$\phi_N = f(x)\cosh(\sqrt{k\,N_0}\, y) = k^{-1}\ \nabla\cdot \underset{\sim}{S} \qquad (3.35)$$

for the boundary conditions

$$\left.\begin{aligned} \phi_N &= f(x) \\ \frac{\partial \phi_N}{\partial y} &= 0 \end{aligned}\right\} \text{ on } y = 0 \qquad (3.36)$$

and is a valid approximation in the regions where $|f(x)| \sim N_o$, $|f(x)\ y| \ll (N_o/k)^{1/2}$. The special solution equation (3.31) of the previous hyperbolic equation was of the same form. However, the coordinate scale changes are now different and the scaling of the heating factor and the pressure are also changed. Thus corresponding to equation (3.35) and with the y scale change $y_1 = (k\ N_o)^{1/2}\ y$ we find (cf. equation (7-5), D-2)

$$q_N = \left[\eta_1(N_o)\ f(x) + \eta_2(N_o)\ f''(x)\right] \cosh y_1 \qquad (3.37)$$

where $\eta_1(N_o)$ and $\eta_2(N_o)$ are determinable functions of N_o. (If N_o in equation (3.37) were replaced by the value of N calculated from equation (3.35), q_N and p/po might be calculated with slightly greater accuracy in some instances.) The pressure variation is given by (cf. D-2, equation (7-6))

$$(p/po) = (k - N_o)\ \exp(-\tfrac{1}{2}\ \phi_N^{\ 2}) \qquad (3.38)$$

For this parabolic flow it would be possible to specify a function $f(x)$ for which q_N will generally be positive. For a specific illustration, however, we will take $f(x) = x$. Then

$$q_N = k^{1/2}\ N_o^{\ 3/2}\left(1 - \frac{\gamma+1}{2\gamma}\ \frac{N_o}{k}\right) x_1\ \cosh y_1 \qquad (3.39)$$

where $x_1 = (k\ N_o)^{1/2}\ x$. The streamlines, region of validity of this solution, and values of M, (p/po), (T_t/T_{to}) and V/V_o are shown in figures 4 a, b.

Similarity Rules

A general linearized treatment of irrotational diabatic flows can be given which leads to similarity rules for these flows. Like most linearization treatments, the method cannot make adequate allowance for large perturbations of density and temperature and therefore is limited in its range of usefulness. Our development starts from the partial

FIG. 4b. PARABOLIC DIABATIC FLOW

$D_0 = 0$

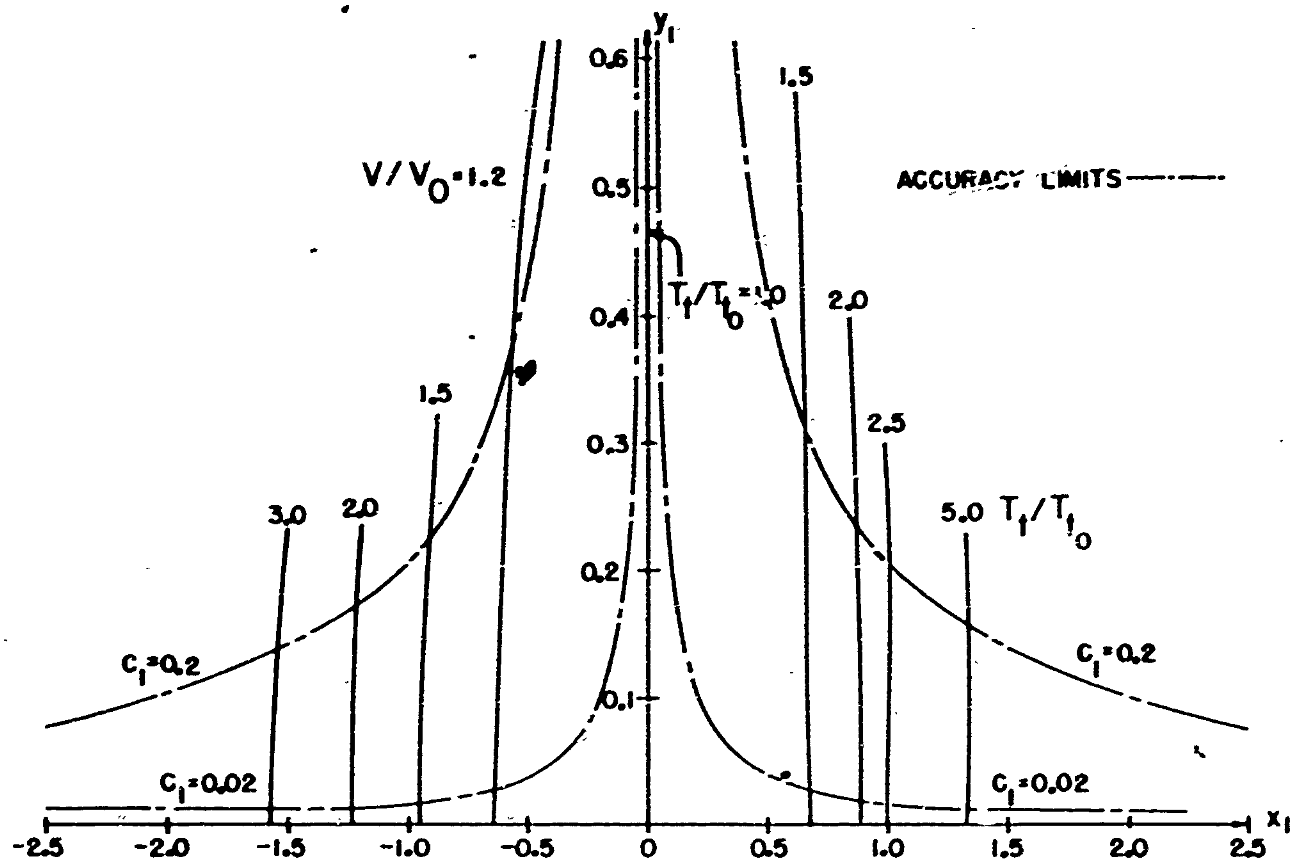
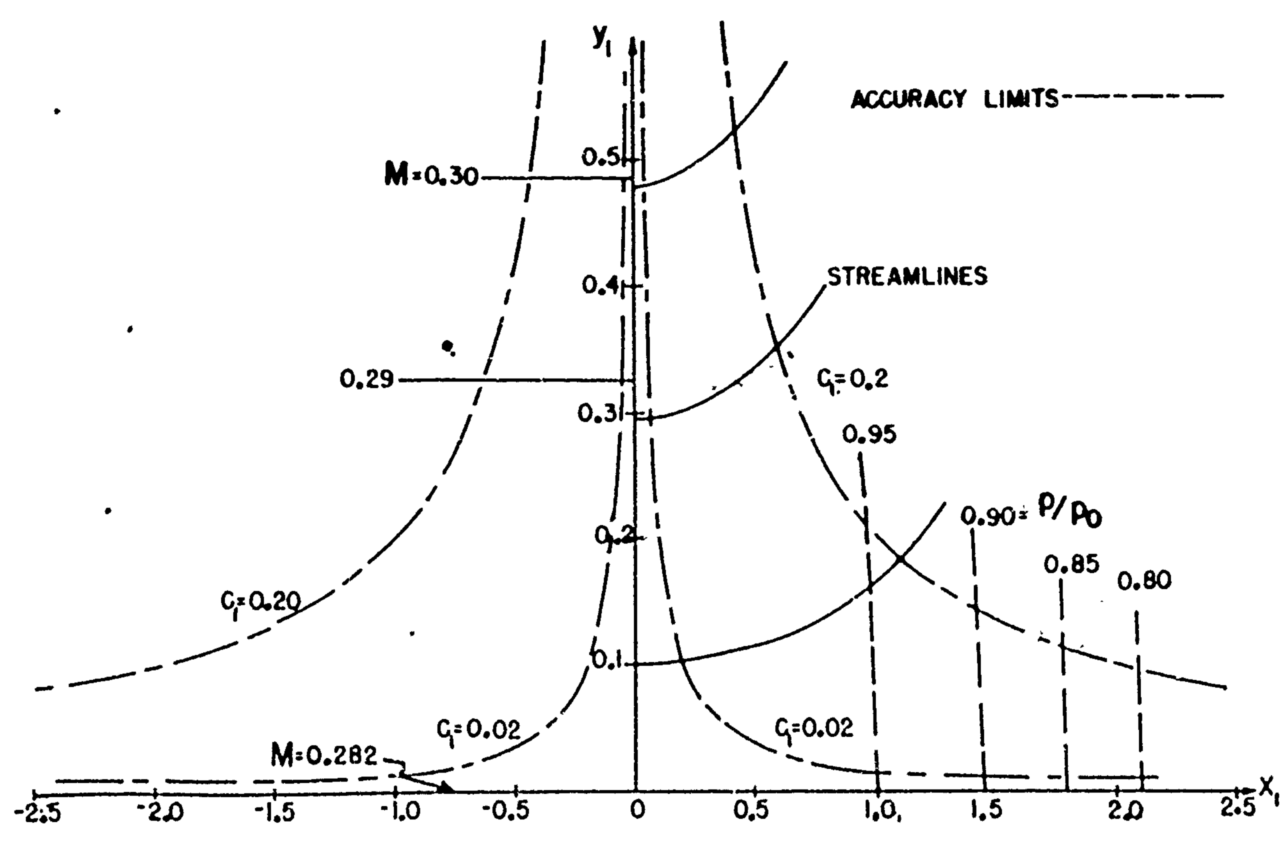


FIG. 4a. PARABOLIC DIABATIC FLOW

$D_0 = 0$



differential equation

$$D(N_o) \frac{\partial^2 \phi_N}{\partial x^2} + \frac{\partial^2 \phi_N}{\partial y^2} = \left[N_o^2 + \frac{1}{g(N_o)} \right] F(\phi_N) \qquad (3.40)$$

which follows from equation (1.18) where $|\partial \phi_N/\partial_y| \ll |\partial \phi_N/\partial_x|$, $N \sim N_o$. The function $F(\phi_N)$ depends only ϕ_N and not on N. Accordingly all effects of choice of g(N) and of free stream value of $N = N_o$ can be absorbed by the scale changes.

$$x_1^2 = \pm \left[N_o^2 + \frac{1}{g(N_o)} \right] D_o^{-1} x^2 \qquad (D_o \neq 0) \qquad (3.41)$$

$$y_1^2 = \left[N_o^2 + \frac{1}{g(N_o)} \right] y^2 \qquad (3.42)$$

giving the equation

$$\pm \frac{\partial^2 \phi_N}{\partial x_1^2} + \frac{\partial^2 \phi_N}{\partial y_1^2} = F(\phi_N) \qquad (3.43)$$

in which the $\pm$ sign corresponds to the elliptic and hyperbolic cases, $D_o >$ or $D_o < 0$. When $D_o = 0$ (the parabolic case) the scale change (3.42) gives

$$\frac{\partial^2 \phi_N}{\partial y_1^2} = F(\phi_N) \qquad (3.44)$$

Boundary conditions are usually representable in the form

$$\alpha(x,y)\phi_N + \beta(x,y) = \lambda(x,y) \frac{\partial \phi_N}{\partial x} + \mu(x,y) \frac{\partial \phi_N}{\partial y} \qquad (3.45)$$

on curves Γ_i. With the scale changes this becomes

$$\alpha_1(x_1,y_1,N_o)\phi_N + \beta_1(x_1,y_1,N_o) = \lambda_1(x_1,y_1,N_o) \frac{\partial \phi_N}{\partial x_1} + \mu_1(x_1,y_1,N_o) \frac{\partial \phi_N}{\partial y_1} \qquad (3.46)$$

where now

$$\alpha(x,y) = \alpha_1(x_1,y_1,N_o); \ \beta(x,y) = \beta_1(x_1,y_1,N_o) \qquad (3.47)$$

$$D_o^{-1/2} \lambda(x,y)/\lambda_1(x_1,y_1) = \frac{\mu(x,y)}{\mu_1(x_1,y_1)} = \left[(1 + N_o^2 g(N_o))/g(N_o) \right]^{-1/2}$$

Thus whenever one can write down a solution of the system equations (3.46) and (3.43) or equation (3.44) for general values of the functions $\alpha_1, \ldots, \mu_1$, then solutions of the system equations (3.40), (3.45) can be constructed for any value of N_o and for any form of the function $g(N_o)$, so long as the same sign of D_o is kept (or the value zero in the parabolic case). It is most likely that general solutions of the system (3.46) and (3.43) or (3.44) will be available if the system is completely linearized. Suppose for example that $F(\phi_N) = \phi_N$. Our previous discussion of various cases shows that three basic solutions for example, equation (3.18) in the elliptic case, equation (3.30) in the hyperbolic case and the expression

$$\begin{aligned} &\phi_N = f(x) \cosh y_1 + g(x) \sinh y_1 \\ &\left(\phi = f(x), \frac{\partial \phi_N}{\partial y} = g(x), \text{ on } y = 0\right) \end{aligned} \tag{3.48}$$

in the parabolic case permit immediate calculation of the flow pattern, q_N etc. for all values of N_o just by making the scale changes, equations (3.41) and (3.42). Thus all elliptic and hyperbolic flows (with $F(\phi_N) = \phi_N$) and with the boundary conditions given in equations (3.8), (3.9) and (3.32) respectively can be derived, no matter what the function $g(N_o)$, from equations (3.18) and (3.30) in the Glauert-Prandtl approximation. The equation for q_N is

$$q_N \sim q_1 \left[N_o, g(N_o)\right] \left\{\phi_{y_1 y_1} + q_2 \left[N_o, g(N_o)\right] F(\phi_N)\right\} \tag{3.49}$$

in which q_1, q_2 are functions of N_o that are determinable once $g(N_o)$ is given.

We thus find the following similarity laws for linearized irrotational diabatic flow (in addition to equations (3.41, 3.42))

$$M(N_o')/M(N_o) = \left[N_o'^2 g(N_o')/N_o^2 g(N_o)\right]^{1/2} \tag{3.50}$$

$$\frac{N(N_o')}{N(N_o)} \sim \left[\frac{\partial \phi_N}{\partial x_1(N_o')} \Big/ \frac{\partial \phi_N}{\partial x_1(N_o)}\right] = \frac{x_1(N_o)}{x_1(N_o')} \tag{3.51}$$

$$\left[\frac{\partial \phi_N}{\partial y_1(N_o')}\right] \Big/ \left[\frac{\partial \phi_N}{\partial y_1(N_o)}\right] = \frac{y_1(N_o)}{y_1(N_o')} \tag{3.52}$$

$$p(N_o', \phi_N)/p(N_o, \phi_N) = H_2(N_o)/H_2(N_o') \tag{3.53}$$

where (cf. D-2 equation (3-8), $H_2(N) = \exp \int g(N)\, d\, N^2/2$, and

$$\frac{q_N\left[N_o',\left(\partial^2\phi_N/\partial y_1^2/\phi_N\right)\right]}{q_N\left[N_o,\left(\partial^2\phi_N/\partial y_1^2/\phi_N\right)\right]} = \frac{q_2\left[N_o',\ g(N_o')\right] + \dfrac{\partial^2\phi_N}{\partial y_1^2}/\phi_N}{q_2\left[N_o,\ g(N_o)\right] + \dfrac{\partial^2\phi_N}{\partial y_1^2}/\phi_N} \qquad (3.54)$$

Equation (3.53) could be used to calculate pressure forces on bodies in diabatic flow. The transformation of N_o and g would of course distort the body being considered just as in Glauert-Prandtl transformations for adiabatic flow. We note that for all irrotational diabatic flows

$$\left(1 + \frac{\gamma-1}{2\gamma} N^2 g\right) \underset{\sim}{N} \cdot \nabla \log T_t = 2q_N \qquad (3.55)$$

If T_{to} is the free-stream value of the stagnation temperature then

$$\log (T_t/T_{to}) = 2\int_s \left[q_N\, d\phi_N/N^2\left(1 + \frac{\gamma-1}{2\gamma} N^2 g\right)\right] \qquad (3.56)$$

where the integral is taken along a streamline. The similarity law here, in the Glauert-Prandtl approximation is

$$\left[\frac{\partial}{\partial x}\log T_t(N_o')/\frac{\partial}{\partial x}\log T_t(N_o)\right] = q_N(N_o'\ldots)N_o\left(1 + \frac{\gamma-1}{2\gamma} N_o^2\, g(N_o)\right)/$$

$$q_N(N_o)N_o'\left(1 + \frac{\gamma-1}{2\gamma} N_o'^2\, g(N_o')\right) \qquad (3.57)$$

Once N, p, T_t are known then of course M, V, p_t, T, and Q can be computed. We further note that the vorticity in $\underset{\sim}{V}$ language $|\nabla \times \underset{\sim}{V}|$ is, in two dimensional irrotational flow, simply equal to $(-\frac{1}{2}\,\partial \log g\, T/\partial\eta)$. The accuracy of the various approximate expressions we have given may be expected to be different for the different flow patterns and also for the various equations.

4. PERTURBATION OF UNIFORM FLOW BY A LOCAL SOURCE OF HEAT[8]

Among the simpler rotational diabatic flows only one has been treated fully by us, that of an almost uniform flow deflected by one

(8) B. L. Hicks, "Perturbation of Steady Uniform Flow by Localized Sources of Heat", Phys. Rev., 73, 636 (L) (March 15, 1948).

local heat source of the form $Q \propto e^{-\alpha r_1^2}$.[1] More general rotational problems can be discussed by examining perturbation due to heat sources in non-uniform flows, by superposition of more than one heat source in almost uniform flow, or by relaxation or characteristic parameter methods for high Mach numbers (cf. basic equations - Section 1). We hope to report some of these advanced calculations at a later time.

We had expressed a doubt in the Madison paper that a unique elementary source of heat, analogous to a fluid source, could be defined for diabatic flow. Comparison of the results to be discussed in this Section with the calculations of reference (9) lead to the conclusion that difficulties such as infinite jump in enthalpy or appearance of vortex sheets could be ignored in a first treatment if the total heat per second added to the fluid is kept constant as the spatial extent of the heat source is reduced to make it a line or point source. (We are indebted to Professor Tsien for sending us a prepublication copy of his abstract.) It must be recognized, however, that the infinite velocities connected with an elementary line source make it partially unsuitable for a perturbation theory. We will therefore present here the perturbation calculation for the smoothed source $\exp(-\alpha r_1^2)$.

The unperturbed flow is uniform and of velocity V_o in the direction of the positive x axis. In the perturbed flow let

$$\underset{\sim}{V} = V_o \underset{\sim}{i} + \underset{\sim}{V}' = (V_o + u')\underset{\sim}{i} + v'\underset{\sim}{j} \tag{4.1}$$

$$p = p_o + p' \tag{4.2}$$

$$\rho = \rho_o + \rho' \tag{4.3}$$

$$T_t = T_{to} + T_t' \tag{4.4}$$

in which the subscript zero refers to the unperturbed state and the prime to the perturbed state.* Each of the quantities V_o, p_o, ρ_o is constant while, in the case considered here, $\underset{\sim}{V}'$, p' and ρ' are functions of (x, y). The first order perturbation equations are derived from equations (1.1) to (1.3). Thus

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[1] The effects of a line source of heat have been described further in recent first order perturbation theory (9, 10).

(9) H. S. Tsien and M. Beilock, "Heat Source in a Uniform Flow", J. Aero. Sc. 16, 756, (Dec., 1949).

(10) B.L. Hicks, "An Extension of the Theory of Diabatic Flow", Phys. Rev. 77, 286 (b) (Jan. 15, 1950).

* For this first elementary but fundamental computation it seemed appropriate to use the V language as being more familiar. For more complex flows or in more accurate treatments, the conclusions in D-2 (Section 2) suggest that the W language may be more useful.

$$\nabla p' + \rho_o V_o \frac{\partial V'}{\partial x} = 0 \quad (4.5)$$

$$\nabla \cdot \underset{\sim}{V'} + V_o \frac{1}{\rho_o} \frac{\partial \rho'}{\partial x} = 0 \quad (4.6)$$

$$c_p V_o \frac{\partial T_t'}{\partial x} = Q \quad (4.7)$$

Using equations (4.5) and (4.6), equation (4.7) can be re-written to first order terms as

$$(1 - M_o^2) \frac{\partial u'}{\partial x_1} + \frac{\partial v'}{\partial y_1} = (\gamma - 1) M_o^2 Q/V_o^2 \quad (4.8)$$

It can be shown that $\omega'_v = | \nabla \times \underset{\sim}{V'} |$ is of second order. We can therefore intorduce a potential ϕ'_v for which $\underset{\sim}{V'} = V_o \nabla \phi'_v$ and after making the scale changes $x = x_1 | 1 - M_o^2 |^{1/2}$, $y = y_1$ $(M_o \neq 1)$ arrive at the basic equations for this type of flow

$$\pm \frac{\partial^2 \phi_v'}{\partial x_1^2} + \frac{\partial^2 \phi_v'}{\partial y_1^2} = (\gamma - 1) M_o^2 Q/V_o^3 \quad (4.9)$$

The action of the heat source as an effective fluid source (cf. D-1) is here illustrated. We note that the R.H.S. of equation (4.9) is simply $2q_M$, in the notation of D-2, evaluated in the free stream where the velocity of sound is a_o.

Let us treat a subsonic case $(M_o < 1)$ and take Q to be a function only of r_1, namely

$$Q = 2\alpha \, q_1 \, a_o^3 \, e^{-\alpha r_1^2} \quad (4.10)$$

or as used in the Wisconsin Symposium paper $Q = 2\alpha \, q_o V_{t_o}^3 \, e^{-\alpha r_1^2}$.
where α is a parameter of compactness and q_1 measures the intensity of the heat release.*

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* We can show, if the perturbation velocity u' is small compared to V_o, how q_1 can be calculated and given a practical meaning. When u' is small, the integral in equation (4.21) can be replaced by $\int_{-\infty}^{\infty} (Q/V_o) \, dx$ which is easily evaluated. For larger u' the same development still gives a qualitative idea of the meaning of q_1.

From equation (4.22), the enthalpy increase from $x = -\infty$ to $x = +\infty$ along the streamline $y = 0$ is

$$c_p \Delta T_t = c_p \Delta T = c_p T_o \cdot 2(\gamma - 1)(\pi\alpha)^{1/2}(1 - M_o^2)^{1/2} M_o^{-1} q_1 \quad (4\text{-}1)$$

Footnote continued from page 39

If $k_1 = (\Delta T_t/T_o)$ is used as a measure of the heating value of the combustible mixture then in terms of k_1

$$q_1 = \left[M_o/2\pi^{1/2}(\gamma - 1)(1 - M_o^2)^{1/2}\right](k_1 \alpha^{-1/2}) \tag{4-ii}$$

and is thus directly proportional to k_1 and to the scale defined by the compactness parameter α. The heat output of a combustion chamber is usually expressed in terms of the loading factor β, the rate of energy release per unit volume and pressure in the chamber. If t_o is the time necessary for combustion to be completed, then it may be shown that

$$\beta = \left[\gamma/(\gamma - 1)\right](k_1/t_o) \tag{4-iii}$$

or if β' is the loading given in the usual practical unit BTU/ft^3-atm-hr and t_o is in seconds.

$$\beta' = 3.44 \times 10^4 \; (k_1/t_o) \tag{4-iv}$$

The two measures of heat release, q_1 and β, can be related if we say that the distance $\Delta x = (V_o t_o)$ necessary to complete combustion is $k_2 \alpha^{-1/2}$ where k_2 is of the order of two. We then find

$$q_1 = \left[k_2 M_o/2\pi^{1/2}\gamma(1 - M_o^2)^{1/2}\right](\beta/\alpha V_o) \tag{4-v}$$

For q_1 in ft, α in ft^{-2}, β' in the practical units and with $k_2 \sim 2$, $M_o(1 - M_o^2)^{-1/2} = 0.130$, $\gamma = 1.40$, $V_o = 150$ ft/sec, we obtain

$$q_1 \sim 3.5 \times 10^{-8}(\beta'/\alpha) \tag{4-vi}$$

Therefore, for a combustion chamber for a turbo-jet engine for example with $\beta' = 2 \times 10^6$ BTU/ft^3-atm-hr, $\alpha = 0.44$ ($\Delta x = 3$ ft), q_1 is 0.16, which corresponds to (1/6) of the stoichiometric value of 0.96 calculated from (4-ii) for the same values of $M_o(1 - M_o^2)^{-1/2}$, γ and α. Equivalent results are obtained if the volume of the combustion space rather than its longitudinal dimension Δx is connected with α.

The value $\alpha = 1$ is used in the plots to be discussed later. If a different value of α is of interest, the x and y scales must be replaced by $\alpha^{1/2}x$ and $\alpha^{1/2}y$, q_1 by $(q_1 \alpha)$, T' by $(\sqrt{\alpha}\,T')$, ρ' by $(\sqrt{\alpha}\,\rho')$ and u' by $(\sqrt{\alpha}\,u')$.

The total heat added to the fluid per unit time and thickness (the Q of reference (9)) is thus

$$2\pi\int_0^\infty \rho_{avg} Q\, r\,dr = (1-M_o^2)^{1/2} \rho_{avg} 2\pi \int_0^\infty Q\, r_1 dr_1 = 2\pi (1-M_o^2)^{1/2} q_1 a_o^3 \rho_{avg}$$

$$= 2\pi\gamma\sqrt{1-M_o^2}\, q_1\, p_o\, a_o \tag{4.11}$$

According to equation (4.10), q_1 is a function of r_1 only and equation (4.9) reduces to

$$r_1^{-1} \frac{\partial}{\partial r_1}\left(r_1 \frac{\partial \phi_{v'}}{\partial r_1}\right) = 2(\gamma-1)\,\alpha M_o^{-1}\, q_1\, e^{-\alpha r_1^2} \tag{4.12}$$

One integration gives

$$V_o \frac{\partial \phi_{v'}}{\partial r_1} = (\gamma - 1)\, a_o\, q_1\, r_1^{-1}(1 - e^{-\alpha r_1^2}) \tag{4.13}$$

which differs from the usual (fluid) point source in two-dimensions in that there is no singularity at $r_1 = 0$. As a consequence it would be possible to insure the validity of the perturbation treatment in all parts of the flow for not too large values of q_1. It is important to note that $\nabla\phi$ with respect to the r_1 space has radial components only, but that the $\nabla\phi$ with respect to the physical space is the perturbation velocity vector which is not symmetrical about the heat source location in either the physical or the r_1 space. However, for Mach numbers which are very close to zero the perturbation velocity deviation from symmetry in both the physical and r_1 space is very small.

Because of the irrotationality of V', integration of equation (4.5) for the perturbation pressure p' is immediate, and the vanishing of u' at infinity yields

$$p' = -\rho_o V_o u' \tag{4.14}$$

This simply expresses the pressure variation needed to effect the change in momentum in the x direction associated with the heating, other components of the momentum being negligible. Finally, elimination of $\nabla \cdot \underset{\sim}{V'}$ between equations (4.6) and (4.8) followed by integration leads to

$$-\frac{\rho'}{\rho_o} = M_o a_o^{-1} u' \tag{4.15}$$

$$+ (\pi\alpha)^{1/2}(\gamma-1)(1-M_o^2)^{1/2} M_o^{-1} q_1 e^{-\alpha y_1^2}\left[1 + \mathrm{erf}(\alpha^{1/2} x_1)\right]$$

where $\rho' \rightarrow 0$ as $x_1 \rightarrow -\infty$. The expressions for the perturbed quantities in the case of a line source are obtained by letting $\alpha \rightarrow \infty$.

If $M_o = 0.20$, $\alpha = 1.0$, $\gamma = 1.40$, and $q_1 = 0.098$, (corresponding to $k_1 = (\Delta T_t/T_o) = 0.68$) then $\max |(V'/V_o)| < 0.13$, and to this extent the linearization is justified. Streamlines for this case (M_o^2 neglected compared to unity) are shown in figure 5a together with lines of constant velocity, (or of perturbed static pressure) and also lines of constant perturbed temperature (either static or stagnation within the accuracy of the perturbation treatment).

Owing to the low Mach number the static and stagnation pressure changes are small. In figure 5b the variation of these pressure changes, p'/p_o and p'_t/p_o, along the x axis are shown. Perhaps the most interesting aerodynamic effect illustrated is the slowing down of the fluid upstream of the heated region as though the fluid were approaching a real obstacle. For larger values of q_1 there would be a stagnation point and actual reversal of the flow.

The calculation of p'_t for the Madison paper was based on the simple formula

$$\left(\frac{p_t'}{p_o}\right) = \left(\frac{p'}{p_o}\right) + \frac{\gamma M_o^2}{2}\left[\left(\frac{V}{V_o}\right)^2\left(1 + \frac{\rho'}{\rho_o}\right) - 1\right] \qquad (4.16)$$

and is the curve I for $q_1 = 0.098$ in figure 5b. Since recovery of stagnation pressure is impossible so long as no heat is abstracted from the flow, the apparent recovery must arise from some inaccuracy in the calculation. In order to check this point a more careful calculation of p'_t was made by integration numerically equation (1.9) which, for the present case, reduces to

$$\frac{\partial \ln p_t}{\partial x} = \frac{-2\gamma M_o\sqrt{1-M_o^2}}{\sqrt{\frac{\gamma-1}{2}}}\, q_1 e^{-\alpha r_1^2}(1-W^2)\left(\frac{T_o}{T}\right)^2\left[1 + \frac{(\gamma-1)q_1}{M_o}\left(\frac{1-e^{-r_1^2}}{r_1}\right)\right] \qquad (4.17)$$

in which W and (T_o/T) can be computed from the formulae already given for the velocity, pressure and density variations. Values of p_t obtained in this way are plotted as Curve II (for $q_1 = 0.098$) in figure 5b. Comparison of I and II shows the large inaccuracy in equation (4.16), which is undoubtedly due to the large density variation along the x axis. (cf. figure 5c.) As a further check the calculations were repeated for $q_1 = 0.01115$ which is small enough so that equation (4.16) should be reasonably accurate. The p'_t curves for this value of q_1 for the two

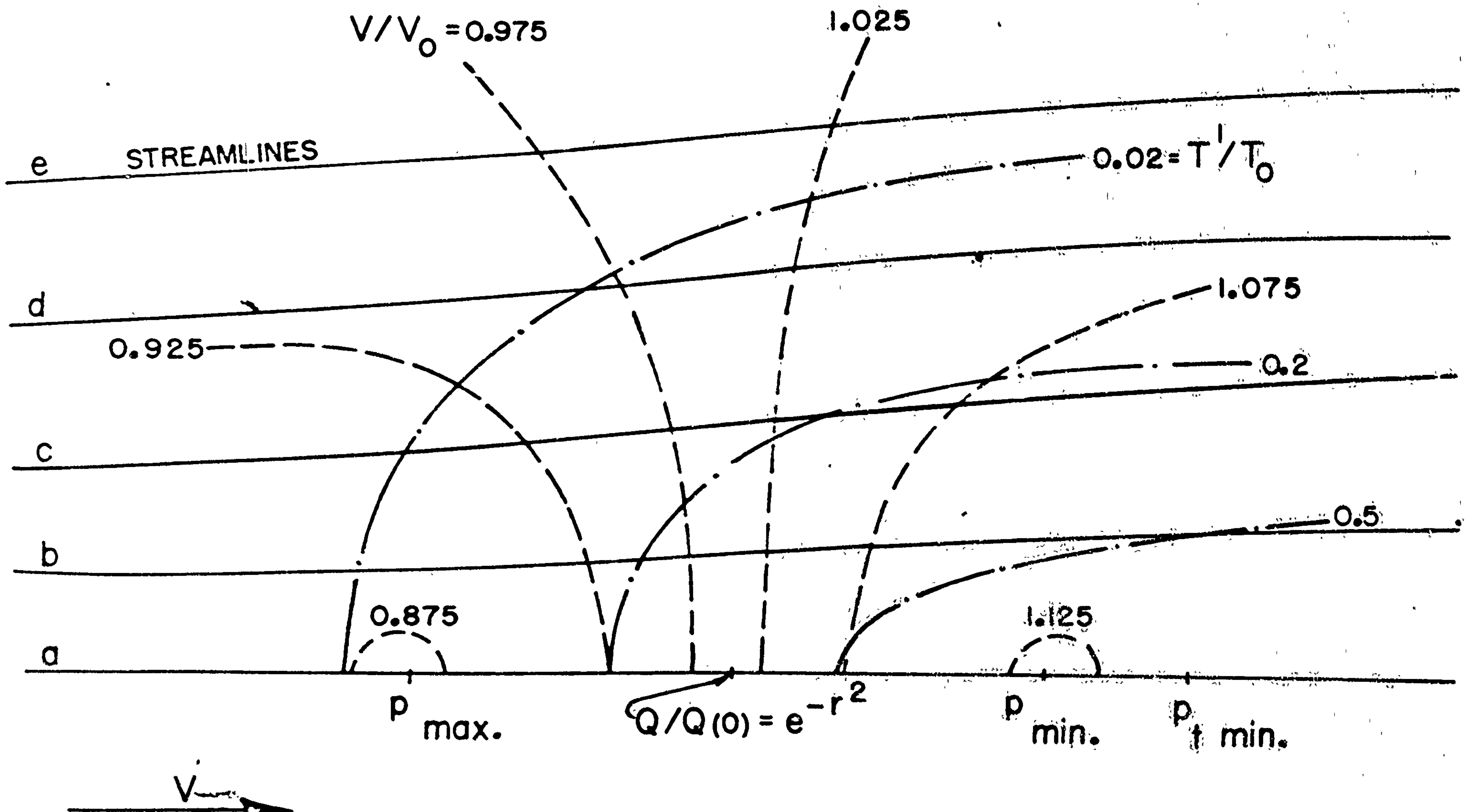

FIG. 5a. ALMOST UNIFORM FLOW

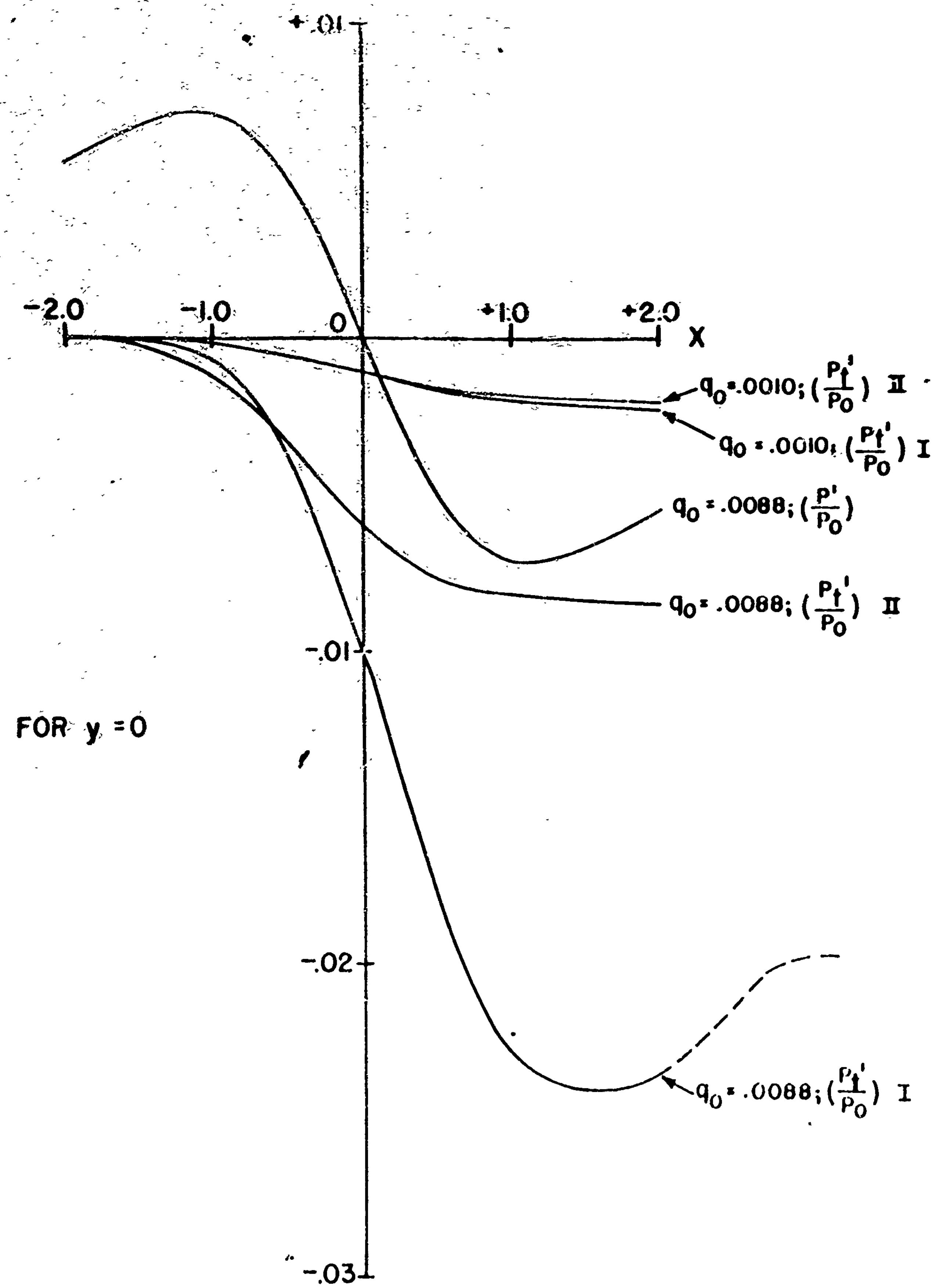

FIG. 5b - ALMOST UNIFORM FLOW

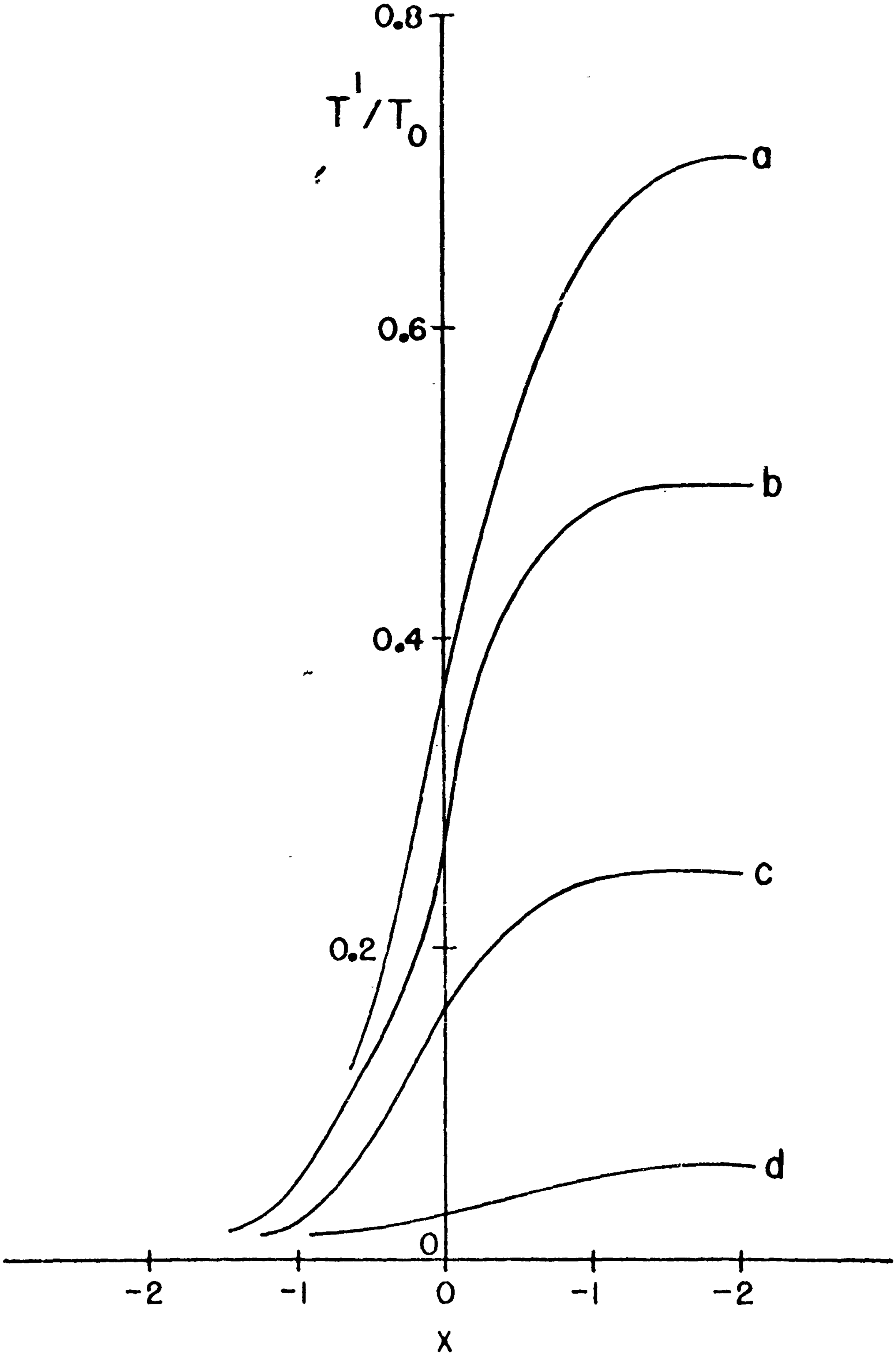

FIG. 5c. ALMOST UNIFORM FLOW

methods of calculation, as seen from figure 5b, do indeed lie close together. From equation (4.16) the asymptotic expression (x → + ∞ . is

$$\left(\frac{p_t'}{p_o}\right)_{asymp} = -\gamma(\gamma-1)\sqrt{\pi}\, M_o(1 - M_o^2)^{1/2} q_1 \qquad (4.18)$$

A more accurate asymptotic expression can be derived. The definition of p_t together with the fact that $p \to p_o$ as $x \to \infty$ gives

$$\left(\frac{p_t'}{p_o}\right)_{asymp} = (1 + \frac{\gamma-1}{2} M_\infty^2)^{\gamma/\gamma-1} - (1 + \frac{\gamma-1}{2} M_o^2)^{\gamma/\gamma-1} \qquad (4.19)$$

Since also $V \to V_o$ as $x \to \infty$

$$M_\infty^2 = M_o^2 (T_o/T_\infty) \qquad (4.20)$$

From the energy equation (1.3)

$$\int_{-\infty}^{\infty} (Q/V)\, dx = c_p(T_{t\infty} - T_{to}) = c_p(T_\infty - T_o) \qquad (4.21)$$

Making the approximation that $V \sim V_o$ in the integral we find that

$$\frac{T_\infty}{T_o} = 1 + 2(\gamma-1)\sqrt{\pi}\sqrt{1 - M_o^2}\, q_1 M_o^{-1} \sqrt{\sigma} \qquad (4.22)$$

where the denominator is $\left[1 - \left(\frac{\rho'}{\rho_o}\right)_{asymp}\right]$ obtainable from equation (4.15). Combination of equations (4.19, 20, 22) permits calculation of values of (p_t'/p_o) which are in error only because of the small difference between V and V_o. When $M_\infty^2 \ll 1$, this combination leads to the simple expression

$$\left(\frac{p_t'}{p_o}\right)_{asymp} = -\frac{\gamma}{2} M_o^2 \left(1 - \frac{T_o}{T_\infty}\right) \qquad (4.23)$$

which reduces to equation (4.18) when T'/T_o and $\rho'/\rho_o \ll 1$.

The variation of temperature or density along streamlines is approximately as shown in figure 5c, the letters a, b, c, d, on the curves referring to streamlines in figure 5a. The density or temperature changes along the axis amount to 70% of the unperturbed values and the corresponding entropy increase is about 1/2 c_p. Since max $\partial \log T_t/\partial s \sim 2q_w V_{to}$ the maximum fractional time rate of change of energy of a fluid particle is (for V_{to} = 500 unit lengths/sec) equal to about 200 per sec. Thus much lower burning rates than were involved in the special case of radial flow of Section 2 produce appreciable perturbations of the uniform fields of flow.

In order to illustrate the effect of Mach number let us take M_o = 0.80 instead of 0.20. Both the heat source function and the perturbation velocity, being functions of r_1, will be distorted since $r_1 = \left[(1 - M_o^2)^{-1} x^2 + y^2\right]^{1/2}$. The magnitude of the apparent fluid source is reduced equation (4.12) by the factor $.2\sqrt{1-(.2)^2}/.8\sqrt{1-(.8)^2} = 0.41$ and the maximum velocity perturbation is increased by a factor of $(1-.04)/(1-.64) = 2.7$ for a given total amount of heat added to the fluid per unit time. The accuracy of the perturbation calculation as a whole is thus about the same for $M_o = 0.8$ as for $M_o = 0.2$ for small values of q_1.

5. ON THE COMPUTATION OF GENERAL UNIPLANAR DIABATIC FLOWS

The mathematical structure of the basic diabatic flow equation in W language, equations (1.9 - 1.11) has immediate physical implications. If we regard W as the quantity to be determined, then we see that equation (1.10) looks like an equation of continuity for a medium possessing apparent fluid source proportional to the heating factor q_W. The equation of motion, equation (1.9), however, shows that the only important dynamical quantity is the stagnation pressure p_t which acts as a potential for the difference between vorticity and heating terms. In fact variations of p_t along streamlines are associated only with heating (between shock fronts) and variations perpendicular to streamlines with the distribution of the (reduced) vorticity $\underset{\sim}{\omega}_W = \nabla \times \underset{\sim}{W}$. The vorticity and heating effects are actually coupled, both because of the continuity equation and because p_t acts as a potential for them. This can be seen in detail after p_t is eliminated by taking the curl of equation (1.9). Since we shall only concern ourselves here with uniplanar flow, the curl of equation (1.9) yields the one equation (cf. D-1)

$$2\left(1 + \frac{1}{\gamma-1} W^2\right) q_W \omega_W + \omega_W \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right) \log \left[\omega_W / (1 - W^2)^{\gamma/\gamma-1}\right] = (1 - W^2)\, q_W \left(u \frac{\partial}{\partial y} - v \frac{\partial}{\partial x}\right) \log q_W \qquad (5.1)$$

where u and v are the components of the vector $\underset{\sim}{W}$.* But q_W is known from the equation of continuity in terms of W. Therefore we can obtain finally an equation in u, v and their derivatives only

$$-v\left(1 - \frac{\gamma+1}{\gamma-1} u^2 - v^2\right) u_{xx} + 2u\left(1 + \frac{2}{\gamma-1} v^2\right) u_{xy} + v\left(1 + u^2 + \frac{\gamma+1}{\gamma-1} v^2\right) u_{yy}$$
$$-u\left(1 + \frac{\gamma+1}{\gamma-1} u^2 + v^2\right) v_{xx} - 2v\left(1 + \frac{2}{\gamma-1} u^2\right) v_{xy} + u\left(1 - u^2 - \frac{\gamma+1}{\gamma-1} v^2\right) v_{yy} + \cdots = 0 \qquad (5.2)$$

* Note that u and v here bear no direct relation to the u', v' of Section 4.

where Ξ contains no second partials. This equation must be satisfied by the components u, v of the Crocco vector W for all steady uniplanar diabatic flows. Functions u(x,y), v(x,y) which satisfy it will determine the nature of the heat source function $q_W(x,y)$ through the continuity equation. We can then formulate a general method of attacking diabatic flow problems if, as in the case of irrotational diabatic flows, we first specify partially the flow pattern W and then calculate q_W. (11,12,13)**
It is recalled that specification of q_W first may lead to such undesired complications as slip lines or discontinuous temperature jumps in the flow but that (cf. Introduction) calculation of an indirect problem is generally easier than of the direct problem.

Let us suppose that u is given as a function of (x,y) throughout the region R and let us inquire what is involved in finding v(x,y). Equation (5-2) is then a second order partial differential equation for v that may be shown to be always hyperbolic. Consequently there will be no difficulty with change of type of the equation either in the field of flow or in going from one problem to another. Such equations of constant type occurred as special cases in irrotational flow (cf. Section 3). One needs to specify v and its normal derivative along curves Γ_k in R in order to permit solution of the equation for v within some sub-region of R that is partially bordered by Γ_k. We note that the constant hyperbolic type of equation (5-2) is not an obvious advantage in the adiabatic case (i.e., where q(x,y) is specified to be $\equiv 0$), because in general it is there still necessary to satisfy a second equation in v which does change type. However, it may be that some linearized adiabatic problems could best be handled by finding a solution of a related diabatic flow problem containing parameters which are varied in such a way that max $|q_W|$ or $\int |q_W|\, dx\, dy$ or some other expression of the heating factors' importance is minimized.

We can examine the equation (5-2) further in the limiting cases of small W ("incompressible" flow) and of the almost uniform flow approximation.

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(11) B.L. Hicks, "On the Calculation of Steady Diabatic Flows", Phys. Rev. 74, 1230(A) (Nov. 1, 1948).

(12) V.P. Starr, "A Mathematical Theory of Convection", J. Meteor. 6, 188-192 (June, 1949).

(13) G.W. Platzmann, "An Example of a General Integral of the Hydrodynamical Equations", Private communication.

** Professor Starr has shown that initial specification of the momentum distribution leads to a partial differential equation for the specific volume which is second-order but always hyperbolic and linear and calculates a specific example. Professor Platzmann has generalized these results. We are indebted to these authors for sending us pre-publication copies of their papers.

For $\Psi \ll 1$, the equation becomes

$$(-vu_{xx} + 2uu_{xy} + vu_{yy}) + (-uv_{xx} - 2vv_{xy} + uv_{yy}) = 0 \qquad (5.3)$$

Supposing again that $u(x,y)$ is known, the characteristics of this partial differential equation always exist since it is hyperbolic. The two families of characteristics intersect orthogonally. Non-linearity enters only owing to the coefficient v of the term v_{xy}. In our earlier formulation of the theory of low-speed uniplanar diabatic flow (D-1, equation (1.16)) the nonlinearity entered in the more complicated coupling equation, $2\Psi^{-1} q_\Psi \; \omega + \frac{\partial \omega}{\partial s} = \frac{\partial q_\Psi}{\partial n}$.

It might be instructive to take computed values of $u(x,y)$ and $v(x,o)$ from the flow of Section 4 and recalculate v, q_Ψ, Q, p, etc., throughout the flow field as an improvement of the linearized treatment. In this check, the coefficient v of v_{xy} could be initially assumed to have the value calculated in Section 4 in starting an iterative solution for $v(x,y)$. Less ambitiously, the terms in equation (5.1) could be computed and their sum compared to zero as a check of the linearized treatment of Section 4.

For flow which is almost uniform, $u = u_o + u'$, $v = v'$, $|u'|$, $|v'| \ll u_o$ = const., a scale change leads to the simpler equation

$$v'_{x_1x_1} - v'_{y_1y_1} = \Xi_1' \qquad (5.4)$$

with $x_1^{\,2} = (1 + \frac{\gamma+1}{\gamma-1} u_o^{\,2})^{-1} x^2$, $y_1^{\,2} = (1 - u_o^{\,2})^{-1} y^2$. The function Ξ_1' is a linear combination of the form

$$\Xi' = A_1(u_o)u'_{x_1}u'_{y_1} + A_2(u_o)u'_{x_1}v'_{x_1} + A_3(u_o)u'_{y_1}v'_{y_1} + A_4(u_o)v'_{x_1}v'_{y_1} \qquad (5.5)$$

which normally would be small compared to individual terms in the L. H. S. of equation (5.4). We note that in this Glauert-Prandtl approximation the partial differential equation for v' has been wholly linearized except for the presence of the (small) term in $v'_{x_1}v'_{y_1}$.

It appears that here, as for irrotational flows, the almost uniform or Glauert-Prandtl type of treatment will lead most quickly to approximate calculations of interesting flows. We note that in the case of all three equations (5.2, 3.4) the solution need not be obtained to as high a degree of accuracy as must a potential function, for the latter must be differentiated in order to give values for u, v, p, T, etc., which are the primary variables of interest.

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