AFCRL -65-454

RAYLEIGH WAVES AT THE CONTINENTAL MARGIN

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Contract No. AF 19(628)-319

Project No. 8652

Task No. 865203

Scientific Report No. 6

Report Date: 10 June 1965

Prepared for

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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ARPA Order No. 180-62 and 292-62

Project Code No. 8100 4810 Task 2

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Julius Kane

a b s t r a c t

In this report, the first of a series on the analytic continuation of wave functions past discontinuities, we introduce an elementary procedure for the solution of problems involving the diffraction of vector fields. In particular, the report discusses the propagation of Rayleigh waves incident obliquely upon the continental margin. The crustal layering on either side of the coastline is modeled mathematically as a two-part boundary layer in such a fashion that the relevant reflection and transmission coefficients emerge as elementary algebraic expressions. The procedure that permits such a solution to be found is to introduce a diffraction analogue of the well-known procedure in electrical engineering: replace a transmission line network by a lumped parameter equivalent in a specified frequency or wave number domain.

I. INTRODUCTION

From the point of view of a geophysicist, mathematical analysis cannot begin to acquire practical importance unless the solution is in such a form that the answers to his questions can be promptly given. Unfortunately, the class of elastic wave diffraction problems which have tractable solutions usually describe oversimplified geophysical structures. On the other hand, when the theoretician studies complex problems, too often he concentrates on the mathematical details of the solution so that his calculations are often more impressive than useful.

One powerful mathematical technique, the Wiener Hopf method, has not had the application to geophysics that it should have. The difficulty in the application of this method is that it usually requires an involved transcendental kernel to be "split" into suitable analytic factors. In my experience with problems of this type, it has been my observation that very often this prolix calculation is of more mathematical than physical interest[1,2], and that usually the interesting parts of the solution depend in a very minor fashion with respect to this "split" function. Indeed, the necessity for this function - theoretic decomposition can often be dropped as a requirement from the analysis, and used as the basis of an approximation procedure to solve problems beyond the scope of the Wiener-Hopf theory[3]. Other investigators such as Koiter[4] and Carrier[5] have been aware of the relative unimportance of this complex factorization and have suggested the use of simpler substitute kernels which can be factored by inspection. While there is much merit in this approach, I feel that it is a poor philosophy to take the solution to an idealized problem and then approximate the answer. A preferable procedure is to formulate the problem in such a way that the mathematical difficulties are anticipated and avoided from the start. That is, a certain amount of intuition and insight into the nature of the problem can be used to guarantee that the subsequent analysis will lead to simple representations.

In the sequel, I should like to give an example of this philosophy by considering the propagation of oblique Rayleigh waves past a crustal discontinuity such as the land-sea boundary (Figure 1). As a side condition, an elementary character for the solution will be required. Of course, there will be a need to make certain assumptions concerning the fields to be calculated. However, we shall be explicit about these modifications and indicate how arbitrary improvements can be made in the analysis if necessary.

A crucial phase in this study will be the need to formulate appropriate boundary conditions to characterize the crustal

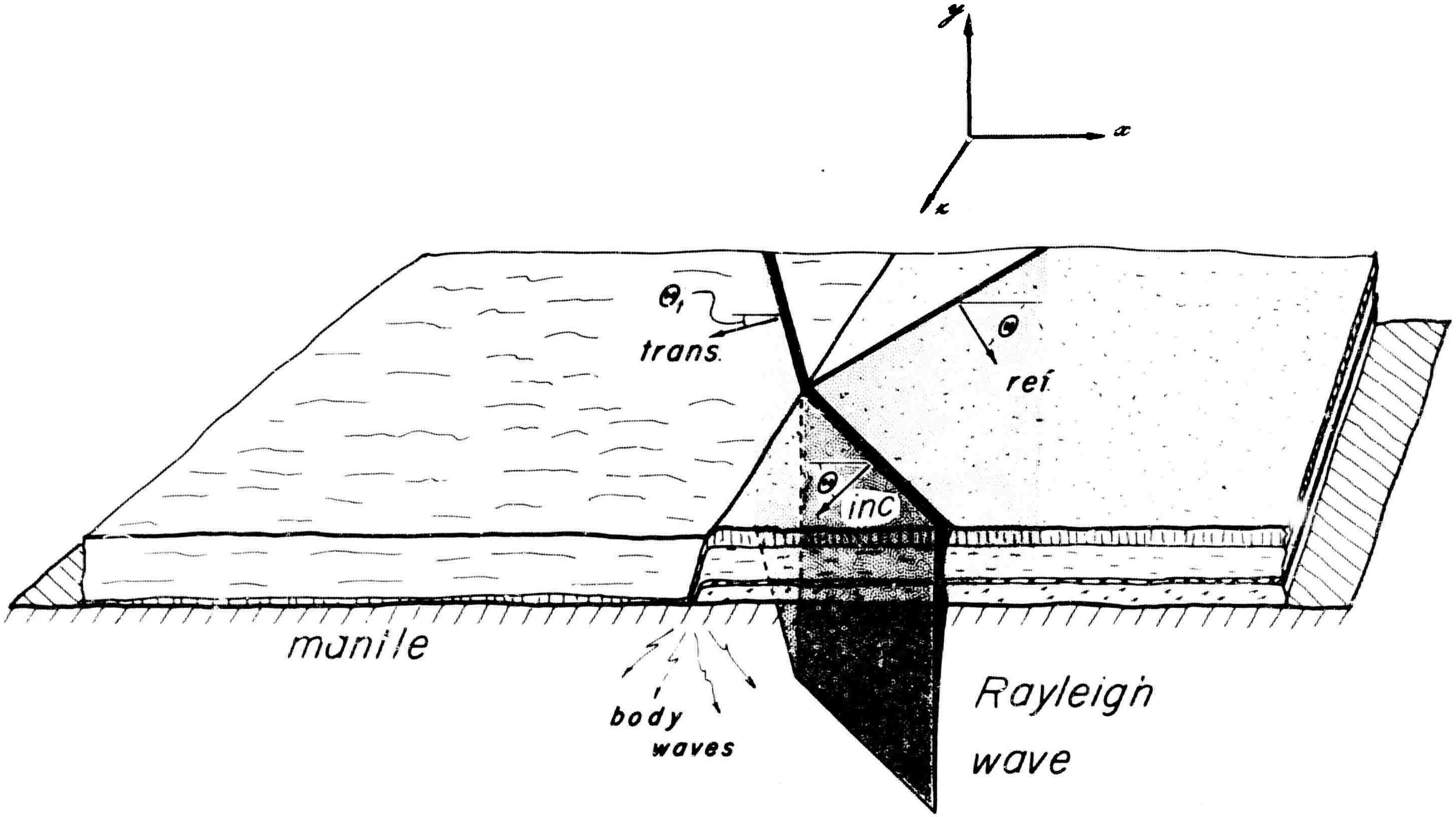

Figure 1: A Rayleigh wave is incident obliquely upon the land-sea interface at an angle. The transmitted wave is refracted by an angle $_t$. For purposes of illustration, the crustal thickness is greatly exaggerated and is considered to be a two-part boundary layer in the analysis.

layering of the mantle. In the simpler problem of electromagnetic wave propagation, Leontovich[6] has introduced the notion of an "impedance boundary condition" to model the physics at an interface. However, this idea while very useful does have its limitations. For one the solution to the associated boundary value problem is still rather complex, second the idea is valid only if the surface impedance is reasonably independent of the nature of the excitation. Although the impedance concept can be generalized to describe interfaces with more general properties[7,8,9], the intractable nature of the solution remains. A feature of the subsequent analysis will be a derivation of an approximate boundary condition for elastic wave diffraction whose merit is that its accuracy can be arbitrarily great without sacrificing simplicity in the form of the solution.

II. BASIC EQUATIONS

In the absence of body forces, small displacements $\mathbf{S}(u,v,w)$ of an elastic solid characterized by the Lamé paremeters λ,μ, and density ρ can be derived from a scalar potential Φ and a vector potential $\boldsymbol{\Psi}(\Psi_1,0,\Psi_3)$

(1) $$\mathbf{S} = \text{grad}\ \Phi + \text{curl}\ \boldsymbol{\Psi}\ (\Psi_1,0,\Psi_3)$$

(Three components in the vector potential are redundant for we can express say $\Psi_2 = f(\Psi_1,\Psi_3)$ so that without any loss of generality we can set $\Psi_2 = 0$.) For monochromatic vibrations, we can suppress a time factor $e^{-i\omega t}$ and it can be shown that Φ and the components Ψ_i satisfy the reduced wave equations

(2) $$(\nabla^2 + k_c^2)\ \Phi = 0, \qquad k_c^2 = \omega^2\rho/(\lambda + 2\mu)$$

$$(\nabla^2 + k_s^2)\ \Psi_i = 0, \qquad k_s^2 = \omega^2\rho/\mu$$

Simultaneously we should like a solution for the Rayleigh wave incident from either side of the crustal discontinuity. For this reason let v_L and v_R be the phase velocity of the Rayleigh wave on the left and right sides of the boundary, and let $k_L = \omega/v_L$, and $k_R = \omega/v_R$ designate the corresponding wave numbers. Also, Γ_L and Γ_R represent the required shear/compressional ratios for either side. In the notation of figure one, the variation of the incident Rayleigh wave in the mantle will have the form[10]

(3)
$$\begin{cases} \Phi_{inc} = e^{-ik_R\xi + ay} \\ \Psi_{inc} = (\sin\theta, 0, \cos\theta)\, i\Gamma_R e^{-ik_R\xi + by} \end{cases}$$

where

(4)
$$\xi = x\cos\theta + z\sin\theta,$$

(5)
$$a = (k_R^2 - k_c^2)^{\frac{1}{2}}, \qquad b = (k_R^2 - k_s^2)^{\frac{1}{2}},$$

and for the special case of a free elastic half space the coefficient Γ_R would be given by

(6)
$$\Gamma_R = \frac{\left[1 - (v_R/v_c)^2\right]^{\frac{1}{2}}}{1 - \frac{1}{2}(v_R/v_s)^2}$$

The z-variation of all fields will be of the form $e^{+ik_R z \sin\theta}$, i.e.

$$
\begin{aligned}
&\Phi(x,y,z) = \varphi(x,y) \quad e^{+ik_R z \sin\theta} \\
(7)\qquad & \\
&\Psi_i(x,y,z) = \psi_i(x,y) \quad e^{+ik_R z \sin\theta}, \qquad i = 1,3
\end{aligned}
$$

so that $\varphi(x,y)$ and the $\psi_i(x,y)$ satisfy the reduced two-dimensional wave equations

$$
\begin{aligned}
&\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + p^2\varphi = 0, \qquad p^2 = k_c^2 - k_R^2 \sin^2\theta, \\
(8)\qquad & \\
&\frac{\partial^2\psi_i}{\partial x^2} + \frac{\partial^2\psi_i}{\partial y^2} + q^2\psi_i = 0, \qquad q^2 = k_s^2 - k_R^2 \sin^2\theta.
\end{aligned}
$$

Note that for angles of incidence such that $\sin\theta > v_R/v_s$ both p^2 and q^2 will be negative.

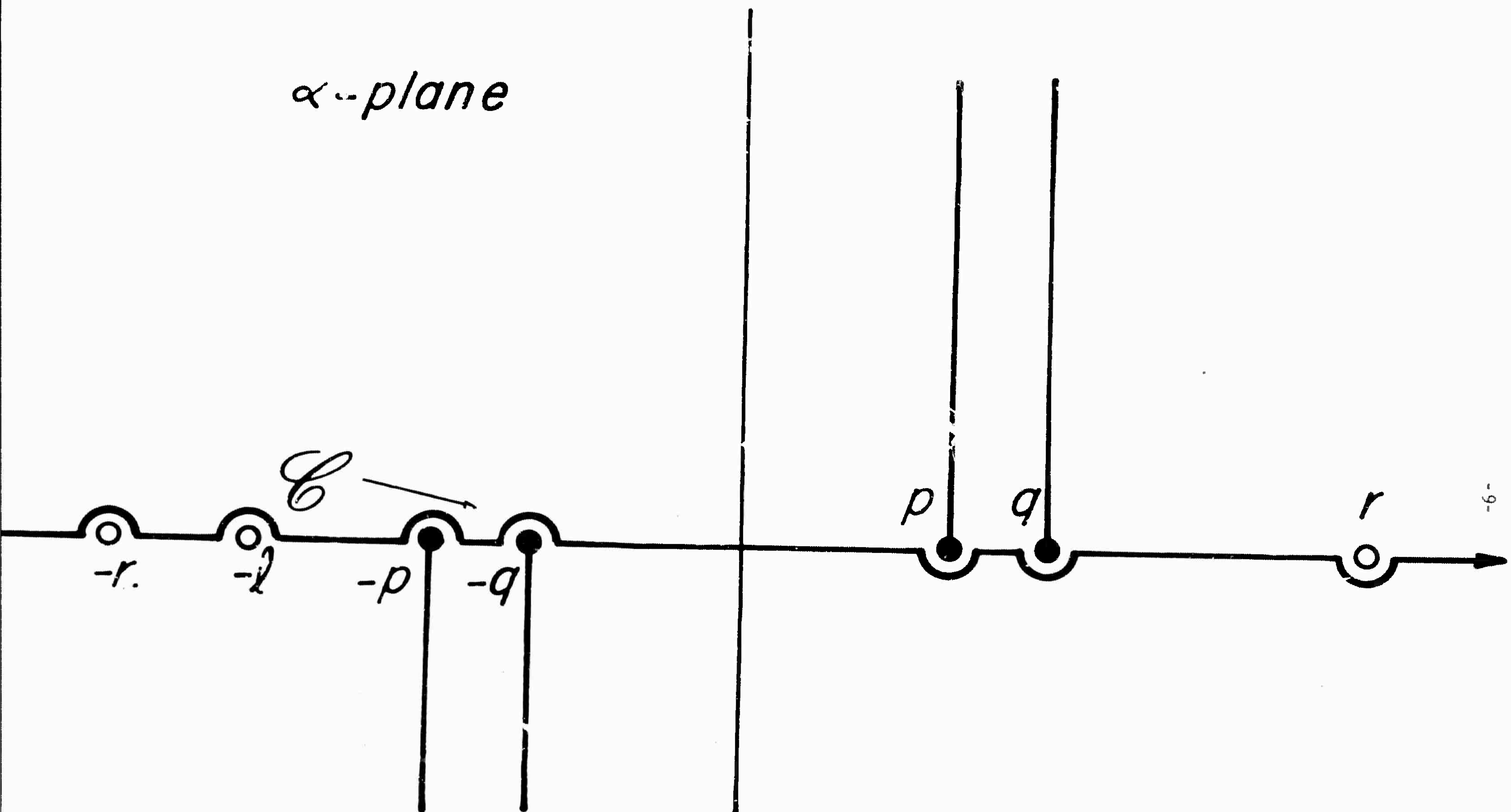

Figure 2: The complex α-plane. The contour follows the real axis except for the indicated pole and branch point deformations. The pole at ℓ does not play any role in the analysis and is not shown.

A completely general solution of equations (8) can be written down at once by separation of variables as

$$\varphi(x,y) = \frac{1}{2\pi i} \int_C A(\alpha) \exp[i\alpha x - i(p^2 - \alpha^2)^{\frac{1}{2}} y]\, d\alpha$$

(9)

$$\psi_i(x,y) = \frac{1}{2\pi i} \int_C B_i(\alpha) \exp[i\alpha x - i(q^2 - \alpha^2)^{\frac{1}{2}} y]\, d\alpha, \qquad i = 1,3$$

where $A(\alpha)$ and $B_i(\alpha)$ are kernels to be found, and the contour C follows the real axis except for the standard pole and branch point deformations (figure 2).

III. THE BOUNDARY CONDITIONS

Either the oceanic or the continental crustal structure can be considered to be a shallow transition layer that continues the field within the mantle to the free boundary at the earth's surface. The details of the transition become more important for the shorter period Rayleigh waves. However, the group velocity dispersion curves for Rayleigh waves traveling oceanic or continental paths intersect at about seventeen seconds. This implies that for waves of period greater than ten seconds we can treat such a crustal structure as a two-part boundary layer whose coefficients have different values on the land and sea sides of the coastline. For this purpose, we need some way of relating the free surface conditions to the crust-mantle interface.

A. Exact Formulation

At a free surface, the normal and tangential stresses must vanish or

$$\sigma_{yy} = \lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + 2\mu \frac{\partial v}{\partial y} = 0,$$

$$\sigma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \tag{10}$$

$$\sigma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0,$$

which in terms of φ and ψ_z become

$$\sigma_{yy} = -\lambda k_c^2 \varphi + 2\mu \left[\frac{\partial^2\varphi}{\partial y^2} + i k_R \sin\theta \frac{\partial\psi_1}{\partial y} - \frac{\partial^2\psi_3}{\partial x\partial y}\right] = 0$$

$$(11) \qquad \sigma_{xy} = 2\frac{\partial^2\varphi}{\partial x\partial y} + i k_R \sin\theta \frac{\partial\psi_1}{\partial x} + \frac{\partial^2\psi_3}{\partial y^2} - \frac{\partial^2\psi_3}{\partial x^2} = 0$$

$$\sigma_{yz} = i k_R \sin\theta \left[2\frac{\partial\varphi}{\partial y} + i k_R \sin\theta\, \psi_1 - \frac{\partial\psi_3}{\partial x}\right] - \frac{\partial^2\psi_1}{\partial y^2} = 0$$

By means of the Haskell[11] - Thomson[12] matrix method, these equations at the surface can be transformed to an equivalent set which refer the boundary conditions at the free surface to the crust-mantle interface. In this fashion the details of the crustal layering on either side of the boundary will be introduced into the analysis. However, the boundary conditions so obtained will be no simpler than the set (11), and in fact a good deal more complex. Although the program we have just described would lead to a soluble Wiener-Hopf problem, the complexities of the analysis would render this procedure rather prolix for the information desired.

B. The Simplified Formulation

The important features of the exact formulation described in part A can be incorporated in a much simpler approach. The conditions we need are those that characterize the boundary layer in such a fashion that the geometric acoustics poles mirror the crustal layering correctly. We can assume that we know the dispersion relations for either side of the boundary, this is equivalent to making use of the Haskell - Thomson matrix analysis in an indirect fashion.

The dispersion relations define the $v_{R,L}$ and $\Gamma_{R,L}$ of a Rayleigh wave uniquely. If we refer to the explicit representation (3) we see that at any characteristic depth say $y = 0$ the Rayleigh wave will satisfy the set of equations

$$k_{R,L}\Gamma_{R,L}\Phi - \frac{\partial\Psi_1}{\partial z} - \frac{\partial\Psi_3}{\partial x} = 0$$

(12)
$$\frac{\partial\Phi}{\partial z} + \frac{k_{R,L}}{\Gamma_{R,L}}\Psi_1 = 0$$

$$\frac{\partial\Phi}{\partial x} + \frac{k_{R,L}}{\Gamma_{R,L}}\Psi_3 = 0$$

These equations are analogues of the telegrapher's equations in electromagnetic theory, and are used to describe the propagation characteristics of Rayleigh waves in a layered half space. They describe the surface loading of the mantle in an exact fashion for Rayleigh wave propagation. That is, they define the correct velocity $v_{R,L}$, and the shear/compressional ratio $\pm \Gamma_{R,L}$ together with its proper change of sign when the direction of propagation is reversed. The surface parameter $\Gamma_{R,L}$ is depth dependent and transforms according to the law

$$\Gamma_{R,L}\Big|_{y = y_o} = \frac{\exp\left[(k^2_{R,L} - k^2_c)^{\frac{1}{2}} y_o\right]}{\exp\left[(k^2_{R,L} - k^2_s)^{\frac{1}{2}} y_o\right]} \Gamma_{R,L}\Big|_{y = 0} \tag{13}$$

With this observation, the propagation of a Rayleigh wave past a step discontinuity can be described as a special case of the subsequent analysis if the step discontinuity is interpreted as a jump in Γ according to the law (13), without any discontinuity in v (Figure 3).

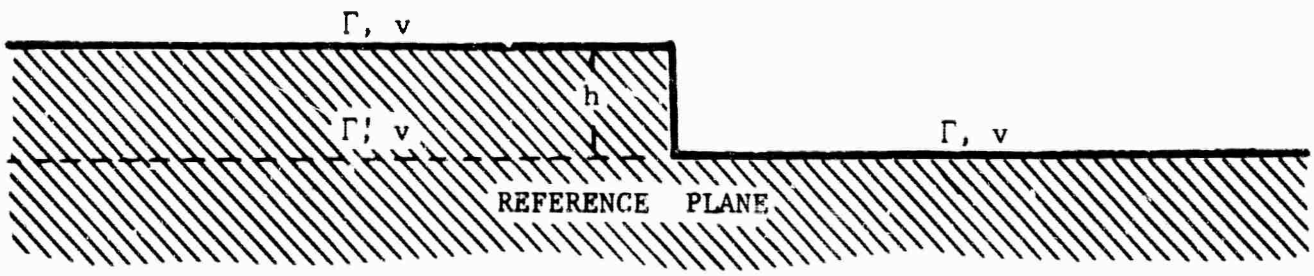

Figure 3: An elastic solid with a small step discontinuity can be thought of as a half space with different values of Γ on either side of the transition if the analysis is referred to the reference plane indicated.

Of course there are any number of other simplified boundary conditions equations that the Rayleigh wave will satisfy, but the equations in the set (12) have several important advantages.

(a) They are simple and uncomplicated.

(b) They are isotropic in that they do not depend upon the Rayleigh wave's direction of propagation in the xz-plane.

(c) They avoid the use of the transverse operator $\partial/\partial y$ which would have the effect of introducing radicals in the transform plane. This deliberate avoidance of the irrational factors will enormously simplify the subsequent function-theoretic analysis.

(d) They characterize the crustal layering in a direct fashion for they introduce the parameters $v_{R,L}$ and $\Gamma_{R,L}$ defined by the relevant period equations.

(e) It can be shown that they imply conservation of energy. That is, the rays which are reflected from a boundary at which these synthetic conditions are imposed will have the same energy as the incident plane wave.

C. Discussion

For those familiar with electrical engineering practice, an analogy between formulations A and B may prove to be useful. In simplified terms, the Haskell matrix method is a vector analogue of transforming an impedance through a transmission line network where the details of the crustal structure characterize the network parameters. In symbolic terms, the exact formulation can be considered to be the shunt transmission line networks of figure 3A which loads a transmission line representation of the mantle. As is well known, the impedance characteristics of such networks are involved transcendental functions of frequency. Within any significant frequency interval however, a transmission line network can be replaced by an equivalent network consisting of lumped parameter elements. Such networks have much simpler impedance functions which are rational functions of the frequency. In diffraction problems, wave number plays a role analagous to frequency, and the substitute set of equations (12) can be considered to be the mathematical equivalent of a lumped circuit representations of the shunt transmission line networks (11) in a region of the wave number plane centered about the Rayleigh pole.

From a mathematical point of view, the nature of the solution to a Wiener-Hopf problem is that it analytically continues a wave function past a discontinuity. If two distinct but reasonably equivalent sets of bounda y conditions are used to characterize each interface then the corresponding fields will also agree rather well. Koiter's and Carrier's justification of the use of substitute kernels in the transform plane has the immediate corollary of validating the use of simplified boundary conditions to describe the physics, since the kernel is in effect the transform realization of the boundary condition operators. Our use of the set (12) as boundary conditions is however a preferable procedure for it is more direct and we can justify it on physical grounds by using the Haskell method backwards. That is, we can transform the conditions in (12) from the crust-mantle interface to the free surface. Naturally, they will not imply that the surface stresses vanish except for Rayleigh waves. However, any incident ray whose wave number is close to the Rayleigh wave number will not induce appreciable stresses at the surface. Indeed, for P or SV waves striking the free surface at grazing angles of incidence, the equations in (12) describe the physics exactly. Essentially all the diffracted rays are tangential to the surface discontinuity and for waves of this type the stress mismatch at the surface will be small. In any event, more terms can be added to the simplified boundary conditions after the fashion of the analysis in [7,8,9] to get any desired degree of surface matching.

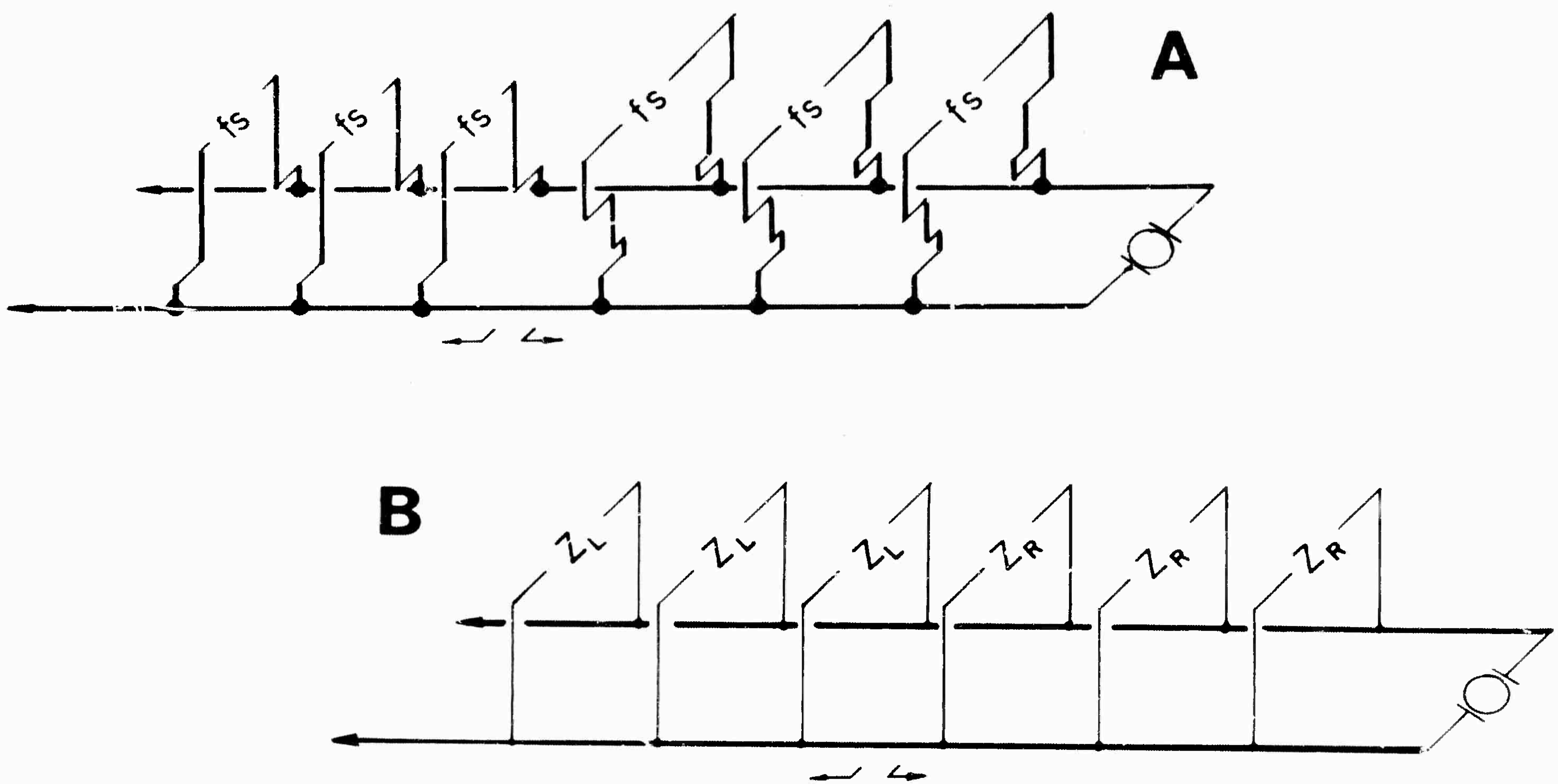

Figure 4: (A) The crustal layering on either side of the land sea interface can be thought of as a continuous distribution of shunt transmission line networks on either side of the discontinuity if the exact Haskell-Thomson matrix formulation is used.

(B) The simplified procedure is analagous to replacing the transmission line shunts by lumped parameter equivalents which have similar characteristics in the relevant region of the wave number plane.

IV. SOLUTION

In terms of $\varphi(x,y)$ and $\psi_i(x,y)$ the synthetic boundary conditions (12) at the crust-mantle interface can be written

(14) $$k_{R,L}\Gamma_{R,L}\varphi - iS\psi_1 - \frac{\partial\psi_3}{\partial x} = 0, \qquad y = 0$$

(15) $$iS\varphi + \frac{k_{R,L}}{\Gamma_{R,L}}\psi_1 = 0, \qquad y = 0$$

(16) $$\frac{\partial\varphi}{\partial x} + \frac{k_{R,L}}{\Gamma_{R,L}}\psi_3 = 0, \qquad y = 0$$

where

(17) $$S = \pm k_R \sin\theta,$$

is a coefficient that always involves k_R owing to the suppressed z-variation.

Let $\mathbf{R}(\alpha)$ denote the hermitian matrix (r_{ij}) where the coefficients r_{ij} are identified as polynomial components of the realization of the differential operations (14) - (17) in the transform plane when x is positive, viz

$$\text{(18)} \qquad \mathbf{R}(\alpha) = \begin{pmatrix} k_R\Gamma_R & -iS & -i\alpha \\ iS & k_R/\Gamma_R & 0 \\ i\alpha & 0 & k_R/\Gamma_R \end{pmatrix}$$

Likewise, let $\mathbf{L}(\alpha)$ be the matrix counterpart of (19) but with parameters k_L and Γ_L appropriate for the left boundary. The right matrix $\mathbf{R}$ is singular when

$$\text{(19)} \qquad \alpha = \pm r = \pm k_R \cos\theta ,$$

and the left matrix $\mathbf{L}$ is singular when

$$\text{(20)} \qquad \alpha = \pm \ell = \pm (k_L^2 - k_R^2 \sin^2\theta)^{\frac{1}{2}} .$$

These points correspond to the geometric acoustics poles for the right and left boundaries respectively. The relationships (19) and (20) are seen to define the correct angles of refraction at the interface.

We introduce the unknown vector

$$(21)\qquad \mathbf{x} = \begin{pmatrix} A(\alpha) \\ B_1(\alpha) \\ B_3(\alpha) \end{pmatrix}$$

and a constant vector $\mathbf{m}$ describing the mismatched components of the incident wave upon the left boundary

$$(22)\qquad \mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix},$$

where

$$m_1 = k_L\Gamma_L - k_R\Gamma_R$$

$$(23)\qquad m_2 = i(k_L\Gamma_R/\Gamma_L - k_R) \sin \theta$$

$$m_3 = i(k_L\Gamma_R/\Gamma_L - k_R) \cos \theta$$

A. Function - Theoretic Method

Vajnstein[13], Karp[14], and Clemmow[15] have introduced a rather general method for solving dual integral equations. The VKC procedure can be generalized to solve vector problems with either the exact or simplified boundary conditions.

A plus (or minus) superscript on a matrix or vector quantity will indicate that all of its components are analytic in the upper (or lower) α-plane above (or below) the contour of integration C. Unless otherwise specified, these quantities vanish uniformly at infinity in their relevant half-plane of regularity. By Jordan's lemma, the boundary condition on the right will be satisfied provided that

$$R(\alpha)\,x = f^{+}(\alpha) \tag{24}$$

where $f^{+}(\alpha)$ is unknown as yet. On the left boundary we need satisfy

$$L(\alpha)\,x + \frac{m}{\alpha + r} = g^{-}(\alpha) \tag{25}$$

The term $m/(\alpha + k_R)$ represents a collection of poles whose residue contributions represent the driving effect of the incident Rayleigh wave upon the left boundary. In order to solve these equations, we substitute (24) into (25) and obtain

$$K(\alpha)\;f^{+}(\alpha) + \frac{m}{\alpha + r} = g^{-}(\alpha) \tag{26}$$

where the matrix to be "split" is

$$K(\alpha) = L(\alpha)\;[\,R(\alpha)\,]^{-1} \tag{27}$$

For the simplified boundary conditions the entries in the matrix kernel $\mathbf{K}(\alpha)$ are to be rational functions of the transform variable α so that by elementary means we can split $\mathbf{K}(\alpha)$ as

$$(28) \qquad \mathbf{K}(\alpha) = [\mathbf{N}^{-}(\alpha)]^{-1}\,\mathbf{P}^{+}(\alpha)$$

For the exact formulation, however, contour integration would be required to perform the factorization. In any event, the matrix factors $\mathbf{P}^{+}(\alpha)$ and $\mathbf{N}^{-}(\alpha)$ are regular in their respective half planes, however unlike the vectors $\mathbf{f}^{+}(\alpha)$ or $\mathbf{g}^{-}(\alpha)$ they do not vanish at infinity but approach a constant value.

If we add and subtract the same term, we can rearrange equation (26) into the form

$$(29) \quad \mathbf{P}^{+}(\alpha)\,\mathbf{f}^{+}(\alpha) + \frac{\mathbf{N}^{-}(-r)}{\alpha + r} = \mathbf{N}^{-}(\alpha)\,\mathbf{g}^{-}(\alpha) + \frac{1}{\alpha + r}\,[\mathbf{N}^{-}(-r) - \mathbf{N}^{-}(\alpha)]\,m$$

Each side of this equation is regular in a half plane and defines the analytic continuation of an entire function. Owing to the assumed growths of $\mathbf{f}^{+}(\alpha)$ and $\mathbf{g}^{-}(\alpha)$ this entire function vanishes uniformly at infinity and must thus be the null constant. Hence we can set each side of (29) to zero and obtain

$$f^{+}(\alpha) = -\frac{[\mathbf{P}^{+}(\alpha)]^{-1}\,\mathbf{N}^{-}(-r)\,\mathbf{m}}{\alpha + r}$$

(30)

$$g^{-}(\alpha) = -\frac{[\mathbf{N}^{-}(\alpha)]^{-1}}{\alpha + r}\,[\mathbf{N}^{-}(-r) - \mathbf{N}^{-}(\alpha)]\,\mathbf{m}$$

With the aid of the first of these expressions, the desired solution $\mathbf{X}$ is seen to be

(31) $$\mathbf{X} = -\frac{[\mathbf{R}(\alpha)]^{-1}\,[\mathbf{P}^{+}(\alpha)]^{-1}\,\mathbf{N}^{-}(-r)\,\mathbf{m}}{\alpha + r}$$

B. Algebraic Method

For rational kernels the general procedure described in Section A is rather abstruse and a more direct approach can yield the solution in a simple algebraic fashion; at the same time the transparency of the analysis will offer an insight into the mathematics that will suggest a generalization that can be used to solve more complex problems.

The matrices $\mathbf{L}$ and $\mathbf{R}$ are such that the only singularities we can have in the solution $\mathbf{X}$ are poles in the complex plane at $\alpha = \pm r, \pm \ell$. We know that the vector $\mathbf{X}$ must have entries which are rational functions of α. The poles of $\mathbf{X}$ can only arise at $\alpha = \pm r$ (incident and reflected waves), and at $\alpha = -\ell$ (transmitted wave) for the residue wave at $\alpha = + \ell$ is a non-physical contribution. Hence, $\mathbf{X}$ must be inversely proportional to $(\alpha - r)(\alpha + r)(\alpha + \ell)$, and we can immediately concentrate attention on the numerators of its components. That is, $\mathbf{X}$ must be of the form

$$(32) \qquad \mathbf{X} = \frac{1}{(\alpha - r)\ (\alpha + r)\ (\alpha + \ell)} \begin{pmatrix} a_1 + a_2(\alpha - r) \\ i\Gamma_R \sin\theta [a_1 + a_3(\alpha - r)(\alpha - a_4/a_3)] \\ i\Gamma_R \cos\theta [a_1 + a_5(\alpha - r)] \end{pmatrix}$$

where the a_i are five constants to be determined. The order of the polynomial entries in $\mathbf{X}$ is easily established if we anticipate that we need use Jordan's lemma after multiplication by either the operator $\mathbf{R}$ or $\mathbf{L}$. The five unknown constants a_i can and must be so chosen that all boundary conditions are obeyed. For x positive, we need satisfy

$$(33) \qquad \frac{1}{2\pi i} \int_C \mathbf{R}\mathbf{X} e^{i\alpha x}\, d\alpha = 0, \qquad x > 0$$

and the form of (32) is such that it automatically meets this constraint, since the only residue contribution in the upper half plane is a reflected Rayleigh wave.

For the boundary condition on the left side we need satisfy

(34) $$\frac{1}{2\pi i} \int_C \left(\mathrm{L}\, \mathrm{X} + \frac{\mathrm{m}}{\alpha + r} \right) e^{i\alpha x} \, d\alpha = 0, \qquad x < 0$$

and by Jordan's lemma this requirement will be et if the residue coefficients at $\alpha = -r, -\ell$ are null. That is we need satisfy

(35) $$(\alpha + r) \ \mathrm{L}(-r)\, \mathrm{X}(-r) + \mathrm{m} = 0,$$

and

(36) $$(\alpha + \ell) \ \mathrm{L}(-\ell)\, \mathrm{X}(-\ell) = 0,$$

Equation (36) is a homogeneous relation and can be satisfied by specifying two constants to make the residue contribution proportional to a transmitted Rayleigh wave that matches the left-hand conditions, viz,

(37)
$$\Gamma_R \sin \theta [a_1 + a_3 (\ell + r) (\ell + a_4/a_3)] = -\Gamma_L \sin \theta_t [a_1 - a_2(\ell + r)]$$
$$-\Gamma_R \cos \theta [a_1 - a_5 (\ell + r)] = +\Gamma_L \cos \theta_t [a_1 - a_2(\ell + r)]$$

The three equations in (35) represent the left boundary's reaction that cancels the incident fields and together with the two in (37) we determine the five constants a_i uniquely.

V. DISCUSSION OF THE SOLUTION

The elementary character of the solution (31) or (36) permits a ready evaluation of the scattered fields. For $y = 0$ and x positive the entire field is a collection of residue contributions which can be grouped into an incident Rayleigh wave, and a reflected Rayleigh wave with a relative amplitude

$$(38) \qquad \mathcal{R} = \frac{a_1}{2r\,(r + \ell)}$$

Our convention will be to describe the coefficient of the compressional potential of the Rayleigh wave as its amplitude. Although our solution includes scattered body waves, their contribution is absent for $y = 0$ owing to our formulation.

On the left side of the boundary, the incident Rayleigh wave is cancelled by pole contributions as it should. The remaining residue terms are seen to represent a transmitted Rayleigh wave of amplitude

$$(39) \qquad \mathcal{T} = \frac{a_1 - a_2(r + \ell)}{2r(\ell - r)}$$

This wave travels in a direction θ_t which is given by

$$\sin \theta_t = \frac{v_L}{v_R} \sin \theta ,$$

a familiar expression for the refraction of a ray at an interface. In general both phase velocities v_L and v_R depend upon frequency, and as a consequence, the angle θ_t is frequency dependent. Thus the diffraction of a Rayleigh pulse incident obliquely upon a crustal interface will sort the transmitted spectral components into different directions after the fashion of an optical prism.

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