THE ROLE OF SPECTRAL ANALYSIS IN TIME SERIES ANALYSIS

BY
EMANUEL PARZEN

TECHNICAL REPORT NO. 2
July 12, 1965

PREPARED UNDER CONTRACT Nonr-225(80)
(NR-042-234)
FOR
OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

THE ROLE OF SPECTRAL ANALYSIS IN TIME SERIES ANALYSIS

by

Emanuel Parzen

TECHNICAL REPORT NO. 2

July 12, 1965

PREPARED UNDER CONTRACT Nonr-225(80)

(NR-042-234)

FOR

OFFICE OF NAVAL RESEARCH

DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

The Role of Spectral Analysis in Time Series Analysis

by

Emanuel Parzen
Stanford University

1. Introduction

Statistical spectral analysis has several roles in time series analysis: (i) estimation; (ii) hypothesis testing and hypothesis suggesting; and (iii) description and reduction of data.

In any field where the properties of the phenomenon being studied can be characterized in terms of its behavior in the frequency domain one needs to estimate spectral density functions and other spectral characteristics associated with stationary multiple time series.

However, spectral analytic techniques seem to provide also a means of testing the fit of various models (the goodness-of-fit of a model can be discussed using sample spectra of the residuals from the fitted model) and suggesting possible models to fit (explanatory "variables" or "mechanisms" to be fitted to a time series are often suggested by sample spectra). As stated so lucidly by Herman Wold (1947), "empirical time series present such a host of widely different patterns that the hypotheses about their structure cannot adequately be brought together into a single parameter system." Consequently, an analysis of a time series is not accomplished by adopting a single

---

Prepared with the partial support of the Office of Naval Research. To be presented at the International Statistical Institute, September, 1965.

model, the parameters of which are estimated. Rather, it is best carried out by a process of increasing insight from successive analyses.

To a sample $\{X(t),\ t = 1, 2, \ldots, T\}$ of a time series, one can associate a function, called the sample spectral density function or periodogram, defined by

$$f_T(\omega) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} e^{-i\omega t} X(t) \right|^2, \quad -\pi \leq \omega \leq \pi .$$

The periodogram was introduced by Schuster to estimate the frequencies of strict periodicities in a time series satisfying the following assumptions: (i) it is not evolving but is oscillating about a constant level; (ii) it may be regarded as composed of a number of "strict" periodicities plus purely random fluctuations.

Since the model of strict periodicities plus random noise seems to occur rarely in practice, the Schuster periodogram often discovered spurious cycles. To remedy this, the notion of "disturbed" periodicity was introduced by means of autoregressive models and moving average models. The correlogram came to the fore, and periodogram analysis fell into disfavor, if not disrepute.

Autoregressive and moving average models are (under some additional assumptions) special cases of stationary time series. The problems of finding the order of a finite parameter scheme such as autoregressive and moving average models led time series analysts to adopt a "non-parametric" approach and first estimate the spectral density function of the observed time series under the assumption that it was a stationary time series with no strict periodicities. Techniques of spectral analysis

(using smoothed periodograms) came back into favor, when interpreted as estimates of the spectral density function of an underlying stationary time series.

But there is more to life than stationary time series with continuous spectra. Consequently, statisticians added the possibility of strict periodicities back to the model. Thus was born the so-called problem of mixed spectra (see Hext (1966) for a current survey).

Finally, the assumption that the observed time series is trend free is unnatural. If one adds trend to a stationary time series, one has a time series which can be regarded as derived from a stationary time series by a filtering process. A theory of Fourier analysis can be developed for such time series, and one might seek to estimate this spectrum. In our approach to empirical time series analysis, [see Parzen (1966)] the emphasis is on the use of spectra defined from samples rather than from populations or ensembles. Given an observed time series of finite length, or a time series derived from it, one defines various "sample spectral functions" such as windowed sample spectral density functions and distribution functions. Their properties can be determined for each possible model one desires to consider for the observed time series. Consequently, they can be used to form estimates of the parameters characterizing the model. Further, they can be used to determine an appropriate model by comparing the actual appearance of these spectral functions with their expected appearance under the various models; that model for which the correspondence is closest is considered the most likely.

Our aim in this paper is: (1) to summarize the basic formulas

employed in the empirical spectral analysis of a single time series, and (2) to show their applicability to the problem of analyzing and synthesizing "adaptive predictors" for time series. The empirical spectral analysis of multiple time series is discussed in Parzen (1965).

Other uses of spectral analysis are described in the excellent survey paper of Jenkins (1965).

2. Sample convolution function

In order to define windowed sample spectral density functions (or smoothed periodograms) it is convenient to first introduce the sample convolution function, denoted $R_T(\cdot)$, of an observed sample $\{X(t), t = 1, 2, \ldots, T\}$:

$$R_T(v) = \frac{1}{T} \sum_{t=1}^{T-v} X(t)\, X(t+v), \quad v = 0, 1, \ldots, T-1,$$

$$= R_T(-v), \quad v = -1, \ldots, -(T-1), \tag{1}$$

$$= 0, \quad \text{otherwise}.$$

The terminology "sample convolution function" is not standard but is introduced in this paper in order to reserve for other purposes the terms "sample correlation function" or "autocorrelation function" which are used by other authors. In the case that the time series of which $\{X(t), t = 1, 2, \ldots, T\}$ is a sample is known to have zero means and to be covariance stationary with covariance function $R(\cdot)$ then $R_T(v)$ provides a possible estimate of $R(v)$. Because of this, in previous writings [see Parzen (1964 a), (1964 b)] the author has called the function $R_T(\cdot)$ the sample covariance function. However, it is our belief that the computation of $R_T(\cdot)$ is of great value even for time

series which are not necessarily covariance stationary. It seems best therefore when introducing this function for the first time to give it a name which indicates its data-handling, rather than statistical, character; such a name is "sample convolution function." Similarly, the sample correlation function $\rho_T(v)$, defined below, is from a statistical point of view, an estimate of the true correlation function $\rho(v)$ of a covariance stationary time series with zero means, while from a data-handling point of view, it is just the convolution function multiplied by a scale factor so as to have value 1 at $v = 0$.

The relations that exist between the sample convolution function and sample spectral density function of an observed time series are the same as those that exist between the covariance function and spectral density function of a covariance stationary time series. In particular, we note the following facts.

The sample convolution function $R_T(v)$ and the sample spectral density function $f_T(\omega)$ are both even functions of their arguments and are a Fourier transform pair:

$$R_T(v) = \int_{-\pi}^{\pi} \cos v\omega \, f_T(\omega) \, d\omega = 2 \int_0^{\pi} \cos v\omega \, f_T(\omega) \, d\omega ,$$

(2)

$$f_T(\omega) = \frac{1}{2\pi} R_T(0) + \frac{1}{\pi} \sum_{v=1}^{T-1} \cos v\omega \, R_T(v) .$$

The sample distribution function. Given an observed time series $\{X(t), t = 1, 2, \ldots, T\}$, the sample distribution function $F_T(\omega)$ is a function of ω in the interval $0 \leq \omega \leq \pi$ defined by

$$F_T(\omega) = 2\int_0^{\omega} f_T(\omega')\, d\omega'$$

(3)

$$= \frac{\omega}{\pi} R_T(0) + \frac{2}{\pi} \sum_{v=1}^{T-1} \frac{\sin v\omega}{v} R_T(v)$$

Conversely, $R_T(v)$ is the Fourier-Stieltjes transform of $F_T(\omega)$:

$$R_T(v) = \int_0^{\pi} \cos v\omega \, dF_T(\omega) \tag{4}$$

The spectral distribution function is a monotone increasing function of ω. Consequently it fluctuates much less than the sample spectral density function $f_T(\omega)$. This is both a virtue and a vice. Certain real effects which it is the aim of the investigation to discern will show up most clearly in the spectral density function whereas they may be overlooked in the spectral distribution function. On the other hand, certain specious effects may appear to show up in the spectral density function which on the basis of the spectral distribution function may be rejected as pure fluctuation.

Sample correlation function. The sample correlation function, denoted $\rho_T(\cdot)$, is defined by

$$\rho_T(v) = \frac{R_T(v)}{R_T(0)} \quad . \tag{5}$$

In words, $\rho_T(v)$ is the sample convolution function normalized to have value 1 at $v = 0$.

Normalized sample spectral density function. For ease of comparing

the sample spectral density functions arising from different time series, it seems best to compute and plot normalized versions of these functions. Since

$$R_T(0) = \int_{-\pi}^{\pi} f_T(\omega)\, d\omega \,, \tag{6}$$

the natural normalization of $f_T(\omega)$ is

$$\bar{f}_T(\omega) = \frac{f_T(\omega)}{R_T(0)} \tag{7}$$

which has the property that its integral form $-\pi$ to π equals 1. We call $\bar{f}_T(\omega)$ the normalized spectral density function; note that it is also the spectral density function of the sample correlation function,

$$\rho_T(v) = \int_{-\pi}^{\pi} e^{iv\omega}\, \bar{f}_T(\omega)\, d\omega \,. \tag{8}$$

3. Windowed sample spectral density and distribution functions

The windowed sample spectral density function, denoted $f_{T,M}(\omega)$, is defined by (for $-\pi \leq \omega \leq \pi$)

$$f_{T,M}(\omega) = \frac{1}{2\pi} \sum_{|v| \leq M} \cos v\omega\, k(\tfrac{v}{M})\, R_T(v) \tag{1}$$

where $R_T(\cdot)$ is the sample convolution function.

The windowed normalized sample spectral density function, denoted $\bar{f}_{T,M}(\omega)$, is defined by (for $-\pi \leq \omega \leq \pi$)

$$\bar{f}_{T,M}(\omega) = \frac{1}{2\pi} \sum_{|v| \leq M} \cos v\omega\, k(\tfrac{v}{M})\, \rho_T(v) \,. \tag{2}$$

where $\rho_{1T}(\cdot)$ is the sample correlation function.

The windowed normalized sample spectral distribution function, denoted $\overline{F}_{T,M}(\omega)$, is defined by (for $0 \leq \omega \leq \pi$)

$$\overline{F}_{T,M}(\omega) = 2\int_0^{\omega} \overline{f}_{T,M}(\omega')\, d\omega'$$

$$= \frac{\omega}{\pi} + \frac{2}{\pi} \sum_{v=1}^{M} \frac{\sin v\omega}{v}\, k(\tfrac{v}{M})\, \rho_T(v) \qquad (3)$$

Lag Windows. The function $k(\cdot)$ is known as the lag window of the windowed spectrum. In our work we use mainly the following lag window

$$\begin{aligned} k(u) &= 1 - 6u^2 + 6|u|^3 \ , && |u| \leq 0.5 \\ &= 2(1-|u|)^3 \ , && 0.5 \leq |u| \leq 1.0 \\ &= 0 \ , && |u| \geq 1 \ . \end{aligned} \qquad (4)$$

A kernel widely used in existing spectral analysis programs is one suggested by Tukey (see Blackman and Tukey (1958), p. 14):

$$\begin{aligned} k(u) &= \frac{1}{2}(1 + \cos \pi u) \ , && |u| < 1 \ , \\ &= 0 \ , && \text{otherwise} \ . \end{aligned} \qquad (5)$$

This lag window is not used in our work because the corresponding windowed spectrum is not necessarily non-negative (and the corresponding estimates of coherence are not necessarily between 0 and 1).

Truncation Points. The integer $M(< T)$ is called the truncation

point of the windowed spectrum since it represents the number of sample correlations of the T available actually used in computing the spectrum. It is wise to choose several truncation points in practice. In our computations, we usually choose three truncation points M_1, M_2, M_3 as percentages of T:

$$5\% \leq \frac{M_1}{T} \leq 10\%, \qquad 10\% \leq \frac{M_2}{T} \leq 25\%, \qquad 25\% \leq \frac{M_3}{T} \leq 75\% .$$

An alternative rule is:

$$5\%\ T \leq M_1 \leq 10\%\ T, \qquad 2M_1 \leq M_2 \leq 3M_2, \qquad 2M_2 \leq M_3 \leq 3M_2 .$$

Spectral Computation Number. There is a third choice to be made in forming the estimate $f_{T,M}(\omega)$, and this is the number of points on the interval 0 to π at which it will be computed. We adopt the attitude that $f_{T,M}(\omega)$ should be computed for equispaced frequencies

$$\omega = 0, \frac{\pi}{Q}, 2\frac{\pi}{Q}, \ldots, \pi$$

where Q is an integer to be chosen. We call Q the spectral computation number.

In the past Q has frequently been chosen to be equal to the truncation point M. One can prove a sampling theorem to the effect that the estimated spectrum (which is a function of ω, measured in cycles per unit of observation time, in the interval $0 \leq \omega \leq 0.5$) can be recovered from its value at M equally spaced points. However, this recovery cannot necessarily be done by linear interpolation. If the graph of the estimated spectrum is to be obtained by merely drawing

line segments connecting the computed values, one needs to compute the spectrum at Q equi-spaced frequencies, where Q should be at least $2M$ and perhaps should be $4M$ (note: further research is needed on this point).

If one uses 3 truncation points $M_1 < M_2 < M_3$, it has seemed reasonable to me to compute each spectrum at $Q = M_3$ points. However, one should choose Q (approximately equal to M_3) such that the frequencies which are multiples of π/Q are of physical interest. For economic time series of monthly data we usually choose Q to be a multiple of 12.

Spectral Window. The spectral window of the windowed spectrum defined by (2) is defined to be the function

$$K_M(\omega) = \frac{1}{2\pi} \sum_{|v| \leq M} e^{iv\omega} k(\tfrac{v}{M}) \tag{6}$$

For the lag window (4), it may be shown that

$$K_M(\omega) = \frac{3}{8\pi M^3} \left\{ \frac{\sin(M\omega/4)}{\frac{1}{2}\sin\frac{\omega}{2}} \right\}^4 \{1 - \tfrac{2}{3}(\sin\tfrac{\omega}{2})^2\} \tag{7}$$

This is an even function which integrates to 1, has maximum value

$$K_M(0) = \frac{3}{8\pi} M \tag{8}$$

and achieves its first zero at $\omega = 4\pi/M$. It is thus concentrated about $\omega = 0$ with a rectangular bandwidth $8\pi/3M$ in radians and $4/3M$ in cycles per unit time.

In order to understand the name "spectral window," we must first

note the relation that exists between the sample spectral density function $f_T(\omega)$ and the windowed spectrum $f_{T,M}(\omega)$:

$$f_{T,M}(\omega) = \int_{-\pi}^{\pi} K_M(\omega-\lambda)\, f_T(\lambda)\, d\lambda \tag{9}$$

since

$$f_{T,M}(\omega) = \frac{1}{2\pi} \sum_{|v| \leq M} e^{-iv\omega}\, k\left(\frac{v}{M}\right) \int_{-\pi}^{\pi} e^{iv\lambda}\, f_T(\lambda)\, d\lambda \tag{10}$$

Thus $f_{T,M}(\omega)$ is the convolution of $f_T(\omega)$ and $K_M(\omega)$. In other words, $f_{T,M}(\omega)$ is an averaging over the values of $f_T(\omega)$ when it is viewed through a window (or channel) of variable transmission properties given by $K_M(\omega)$.

A useful approximation to $K_M(\omega)$ can be obtained by introducing the Fourier transform

$$K(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu\omega}\, k(u)\, du \tag{11}$$

which we call the <u>spectral window generator</u>. It may be shown that <u>approximately</u>

$$K_M(\omega) = M\, K(M\omega) \tag{12}$$

since <u>exactly</u>

$$K_M(\omega) = M \sum_{j=\infty}^{\infty} K(M(\omega-2\pi j)) \tag{13}$$

From this basic formula we obtain the formula (7) for $K_M(\omega)$. For the lag window (4), the spectral window generator may be shown to be

$$K(\omega) = \frac{3}{8\pi}\left\{\frac{\sin(\omega/4)}{\omega/4}\right\}^4 \tag{14}$$

Then

$$MK(M\omega) = \frac{3}{8\pi M^3}\left\{\frac{\sin(M\omega/4)}{\omega/4}\right\}^4 \tag{15}$$

approximates $K_M(\omega)$ given by (7).

Rather than giving a theoretical discussion of the properties of windowed sample spectra we illustrate their use by analyzing a time series which has been extensively discussed from the point of view of forecasting. This is a monthly series of international airline passenger bookings, 1949-1961; compare Brown (1963), p. 429 and Barnard (1963).

It is to be emphasized that there is not a uniquely best way in which spectral analytic ideas enter into time series analysis once one drops the assumption that one is dealing with a stationary time series. The attitudes to data analysis presented in this paper should be used in conjunction with other attitudes such as computing time varying spectra.

4. Analysis of an empirical time series

The monthly time series of international airline passengers has the characteristic features of many social and economic time series; in particular, there is an upward trend and a seasonal variation. In

figure 1 we give plots of three windowed sample spectral density functions (computed for truncation points 72, 36, and 18 on a series of length 144). The existence of trend is evidenced by peaks at frequency 0, while the existence of seasonal variation is evidenced by peaks at the seasonal frequencies .083, .167, .25, .33, and .42 cycles per month.

In order to gain insight into the structure of a time series, we often seek to find the coefficients of the minimum mean square error linear predictor $\hat{X}(t)$ of the time series $X(t)$ of the form

$$\hat{X}(t) = a_1 X(t-1) + \ldots + a_m X(t-m) \tag{1}$$

There are three ways in which one can fit an autoregressive scheme to data: (i) one can specify the order m and estimate the coefficients a_i by solving the system of linear equations

$$E[\hat{X}(t)\, X(t-i)] = E[X(t)\, X(t-i)], \quad i = 1, 2, \ldots, m ; \tag{2}$$

(ii) one can take the possible order m to be some large number (such as 50 months) but admit only those lags whose coefficients a_i are "significantly" different from zero; (iii) one can take the possible order m to be some large number but solve for the coefficients a_i in order of decreasing contribution to the residual sum of squares, and use only a specified number of coefficients. Applying these procedures to the airline passenger series we find the following results: (i) if one fits a 13th order scheme,

$$
\begin{aligned}
\hat{X}_1(t) = \quad & 0.985 \quad X(t-1) \\
& + 0.248 \quad X(t-11) \\
& - 0.346 \quad X(t-13) \\
& - 0.134 \quad X(t-6) \\
& + 0.190 \quad X(t-12) \\
& + 0.068 \quad X(t-2) \\
& - 0.080 \quad X(t-3) \\
& + 0.050 \quad X(t-4) \\
& - 0.007 \quad X(t-8) \\
& + 0.019 \quad X(t-5) \\
& - 0.008 \quad X(t-7) \\
& + 0.012 \quad X(t-10) \\
& - 0.011 \quad X(t-9)
\end{aligned} \tag{3}
$$

The coefficients are written in order of decreasing contribution to mean square prediction error. Next let us seek only the coefficients making a "significant" contribution to the residual sum of squares; one would then fit a first order predictor

$$\hat{X}_2(t) = 0.986 \quad X(t-1) \; . \tag{4}$$

Finally let us arbitrarily choose to fit the best fitting 3 terms; one obtains the predictor

$$
\begin{aligned}
\hat{X}_3(t) = \quad & 0.957 \quad X(t-1) \\
& + 0.257 \quad X(t-11) \\
& - 0.224 \quad X(t-13)
\end{aligned} \tag{5}
$$

The means and variances of the original series and the residual series are as follows:

Series	Mean	Variance
$X(t)$	280	$1.43 \quad 10^4$
$\epsilon_1(t) = \hat{X}(t) - X_1(t)$	5.36	$5.13 \quad 10^2$
$\epsilon_2(t) = \hat{X}(t) - X_2(t)$	5.39	$1.11 \quad 10^3$
$\epsilon_3(t) = \hat{X}(t) - X_3(t)$	4.81	$6.28 \quad 10^2$

Next one examines the spectra of the residual series $\epsilon(t) = X(t) - \hat{X}(t)$. The windowed sample spectral density function and spectral distribution function of $\epsilon_3(t)$ is plotted in the top half of figures 2 and 3, respectively; the trend in the original $X(t)$ series has been eliminated, but the seasonal peaks remain. The spectra of $\epsilon_1(t)$ and $\epsilon_2(t)$ are similar except that $\epsilon_2(t)$ has stronger seasonal peaks.

We next repeat the autoregressive model fitting procedures on the residual $\epsilon_2(t)$ and $\epsilon_3(t)$. We find (applying procedure 2 to $\epsilon_2(t)$)

$$\hat{\epsilon}_2(t) = 0.831 \quad \epsilon_2(t-12) \tag{6}$$

while (applying procedure 3 to $\epsilon_3(t)$)

$$\begin{aligned} \hat{\epsilon}_3(t) = {} & 0.657 \quad \epsilon_3(t-12) \\ & + 0.141 \quad \epsilon_3(t-24) \\ & - 0.080 \quad \epsilon_3(t-20) \end{aligned} \tag{7}$$

In order to interpret the properties of the residuals $\eta(t) = \epsilon(t) - \hat{\epsilon}(t)$ let us compare them with the forecasting errors

given by Barnard (1963) in his comparison of the "adaptive forecasting" and Box-Jenkins method Data for 1949 and 1950 were used to provide initial values, so that the forecast errors were given only for the 10 years, 1951-1960. For comparison we computed the spectra of the forecasting errors $\hat{\eta}_2(t)$ and $\hat{\eta}_3(t)$ over this 10-year period.

Series of forecasting errors, 1951-1960	Mean	Variance
Adaptive forecasting method	.28	$1.78\ 10^2$
Box-Jenkins method	- .36	$1.83\ 10^2$
$\eta_2(t) = \epsilon_2(t) - \hat{\epsilon}_2(t)$	1.52	$2.05\ 10^2$
$\eta_3(t) = \epsilon_3(t) - \hat{\epsilon}_3(t)$	1.76	$1.73\ 10^2$

The spectra of the forecasting errors arising from adaptive forecasting and the Box-Jenkins method are very different! They both are far from the spectrum of white noise, but the adaptive forecasting errors are predominantly low frequency while the Box-Jenkins forecasting errors are predominantly high frequency; their sample windowed spectral density functions and spectral distribution functions are plotted in figures 4 and 5, respectively, for a Parzen window and in figures 6 and 7 for a Tukey window.

In the bottom half of figures 2 and 3 we plot the sample spectrum of $\eta_3(t)$; it is essentially the spectrum of white noise.

The foregoing considerations lead to both a model and a forecasting formula for $X(t)$. Let U_r be the r-th backward shift operator, $U_r X(t) = X(t-r)$, and let I be the identity operator. Define

$$P_1 = 0.957\ U_1 + 0.257\ U_{11} - 0.274\ U_{13}\ ,$$

(8)

$$P_2 = 0.657\ U_{12} - 0.080\ U_{20} + 0.141\ U_{24}\ .$$

Then there is a white-noise series $\eta(t)$ such that

$$(I-P_2)\ (I-P_1)\ X(t) = \eta(t) \tag{9}$$

Therefore a predictor of $X(t)$ is given by

$$\hat{X}(t) = \{P_1 + P_2 - P_2P_1\}\ X(t) \tag{10}$$

More generally, let $\hat{X}(t+v)$ denote the predictor of $X(t+v)$ given values of the time series up to time t. Then

$$\hat{X}(t+v) = \{P_1 + P_2 - P_2P_1\}\ \hat{X}(t+v)\ ; \tag{11}$$

note that $$\hat{X}(s) = X(s) \quad \text{if} \quad s \leq t\ .$$

The aim of the foregoing discussion has been to show one important use of empirical spectral analysis; given an operator (such as $I - P_1$) the properties of the time series $(I - P_1)\ X(t)$ can be studied without regard to the procedure by which one formed the operator.

References

G. A. Barnard (1963), "New methods for quality control," Jour. Royal Stat. Soc. A, Vol. 126, pp 255-258.

R. G. Brown (1963), Smoothing, Forecasting, and Prediction of Discrete Time Series, Prentice Hall: Englewood Cliffs, New Jersey.

G. Hext (1966), "A new approach to time series with mixed spectra," Ph. D. thesis, Statistics Department, Stanford University.

G. M. Jenkins (1965), "Some examples of and comments on spectral analysis," Proceedings IBM Scientific Computing Symposium on Statistics, IBM: White Plains, N. Y.

E. Parzen (1964a), "An approach to empirical time series analysis," Radio Science, Vol. 68D, pp 937-957.

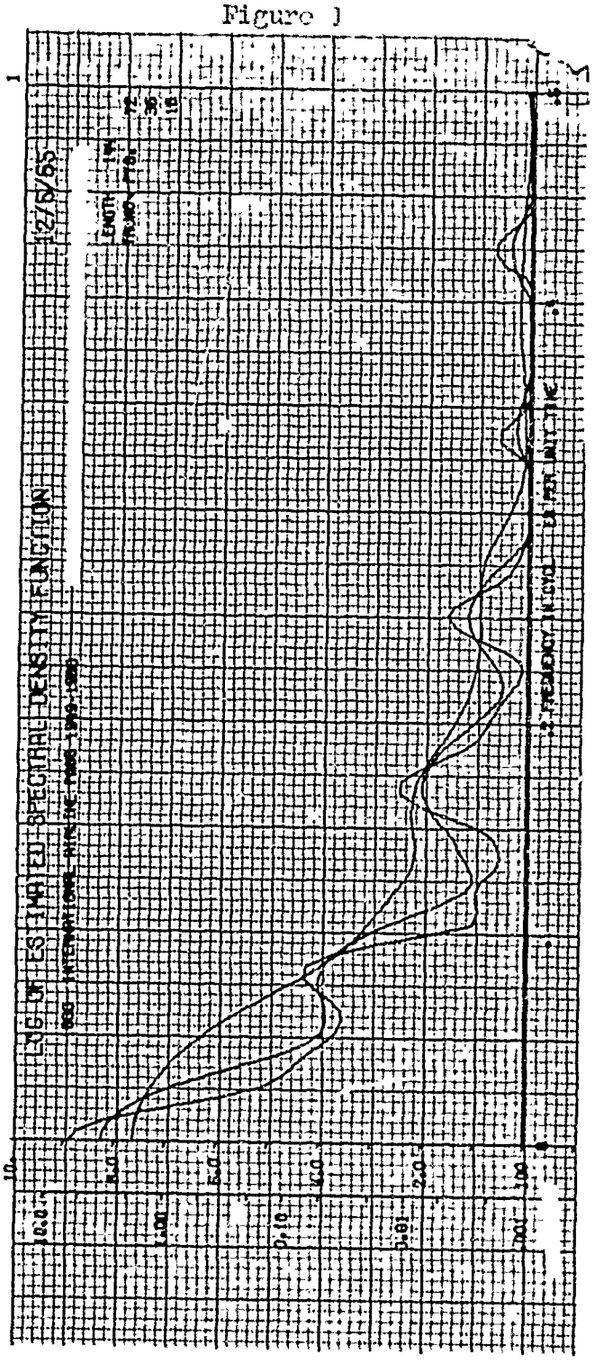
E. Parzen (1964b), "On statistical spectral analysis," Proceedings of Symposia in Applied Mathematics, Vol. XVI, pp 221-246. American Mathematical Society: Providence, R.I.

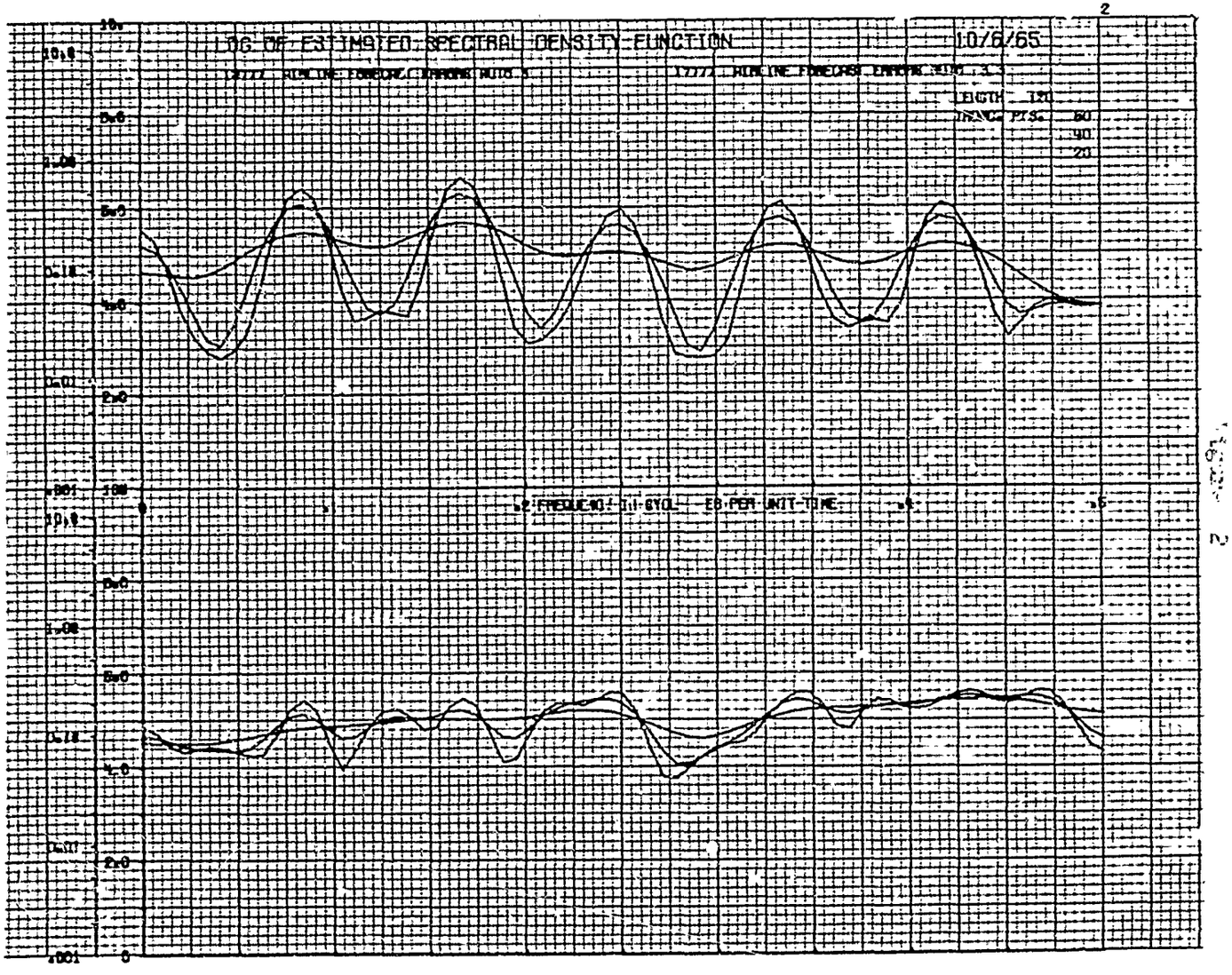
E. Parzen (1965), "Empirical Multiple Time Series Analysis," paper presented at the Fifth Berkeley Symposium on Mathematical Statistics and Probability , July, 1965.

E. Parzen (1966), Empirical Time Series Analysis: An Annotated Computer Program, Holden Day: San Francisco.

H. O. Wold (1947), "A short cut method of distinguishing between rigid and disturbed periodicity," Proceedings of the International Statistical Conferences, Vol. III, pp 302-306.

Figure 1


LOG OF ESTIMATED SPECTRAL DENSITY FUNCTION
12/5/65
LENGTH
FREQ. INCR.
CYCLES PER UNIT TIME


LOG OF ESTIMATED SPECTRAL DENSITY FUNCTION
10/6/65
LENGTH 120
FREQUENCY IN CYCLES PER UNIT TIME

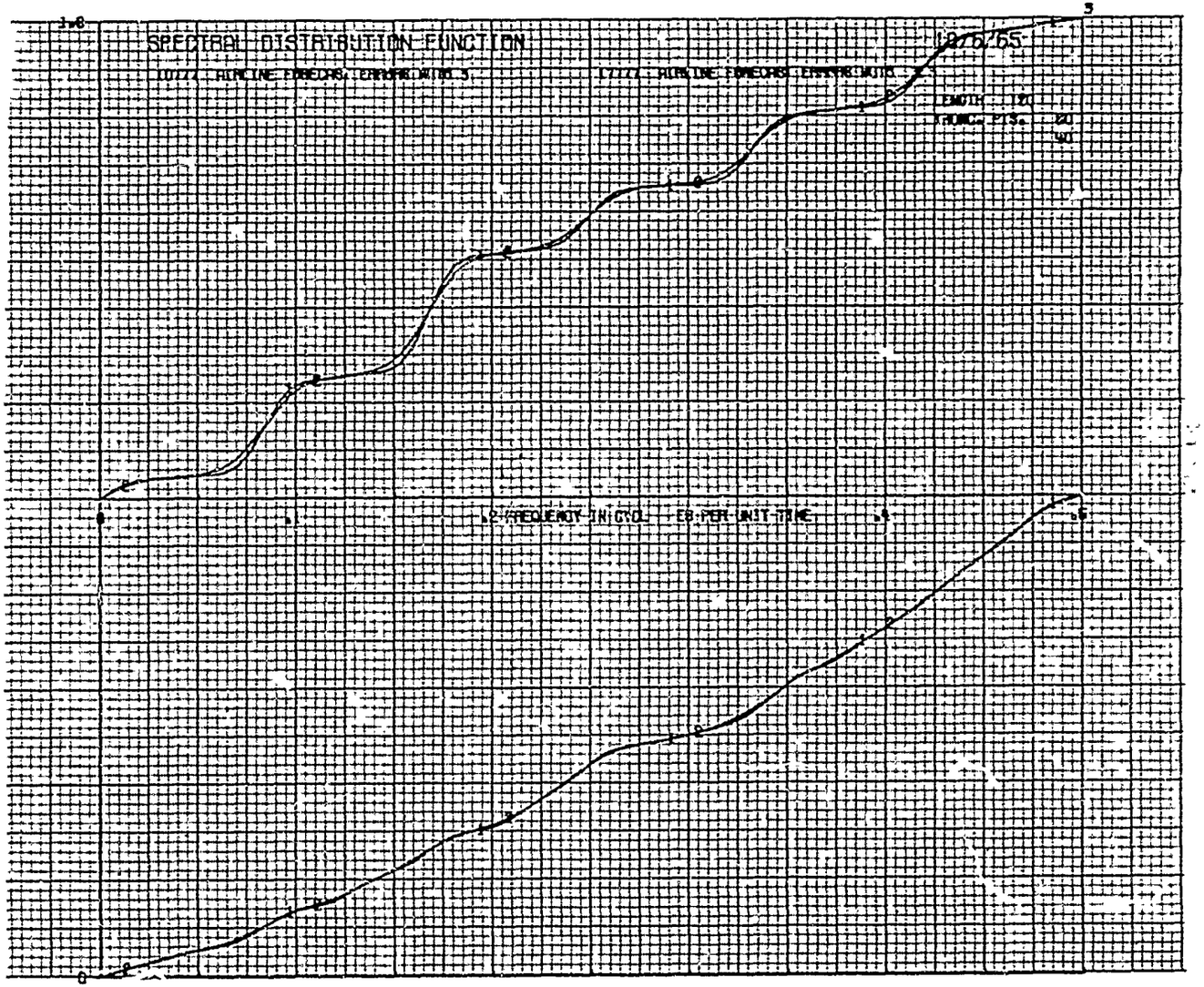

SPECTRAL DISTRIBUTION FUNCTION
1976/65
EMOTH (2)
FROMC. PTS. (2)
(40)
3.0
5
0
1
2
3
4
5
6
FREQUENCY IN (C/D) - EB PER UNIT TIME

Figure 4

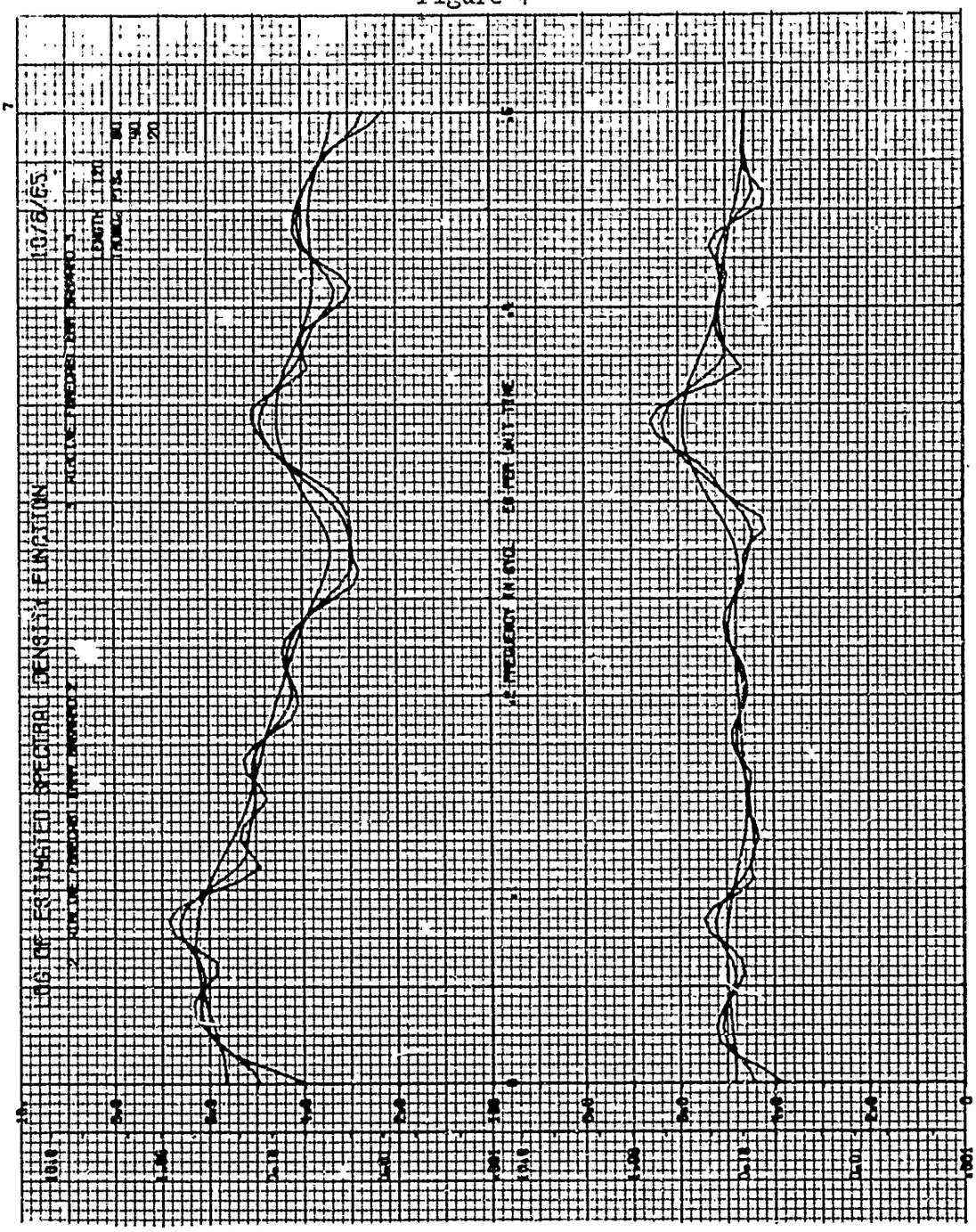

LOG OF ESTIMATED SPECTRAL DENSITY FUNCTION
10/8/65
FREQUENCY IN CYC. PER UNIT TIME

Figure 5

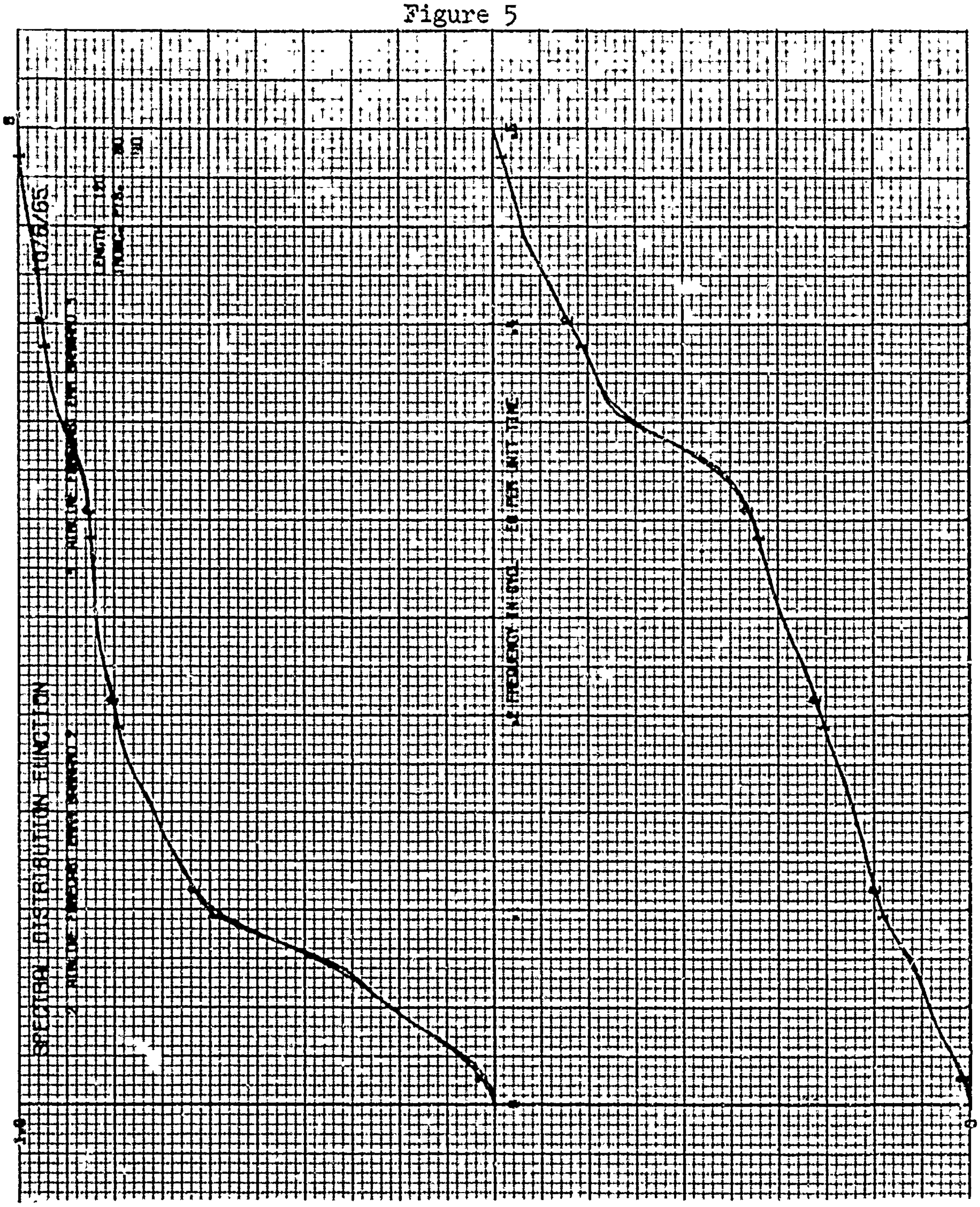


Figure 6

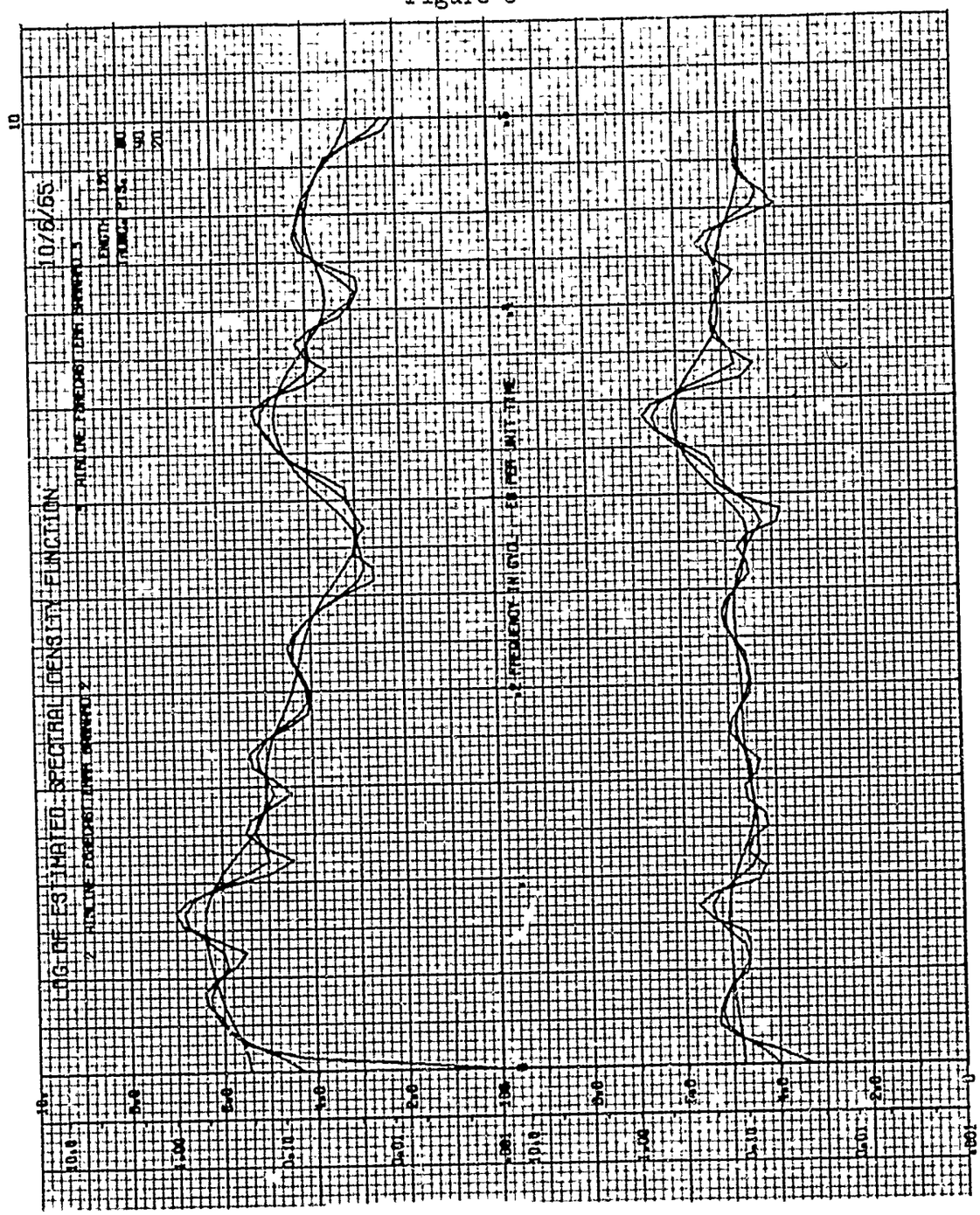

LOG OF ESTIMATED SPECTRAL DENSITY FUNCTION
10/6/65
FREQUENCY IN CYCL. ES PER UNIT TIME

Figure 7