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TWO EXTENSIONS OF STATISTICAL DECISION THEORY

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FOREWORD

The author wishes to acknowledge the contribution of Dr. Charles C. Ying, Rochester University, who authored the model presented in Section III of this report. The author also wishes to acknowledge the research orientation and advice he received from Dr. E. H. Shuford, Jr., Decision Sciences Laboratory.

PUBLICATION REVIEW AND APPROVAL

This Technical Documentary Report has been reviewed and is approved.

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ABSTRACT

The objective of the project was to develop broader formulations of the mathematical (statistical) theory of decisions. This final report presents two broad scope generalizations which have resulted from the project.

The first generalization discussed is a decision-making model which applies to the case of a not-well-informed decision maker with independent data sources. In this model, the inference about the prior distribution is determined from the solution of an adjunct decision problem, which specifies the minimum risk hypothesis in the light of the available information.

The second generalization presented is a model of multi-period decision making for both stationary and Markovian environments. In contrast to the model discussed in the above paragraph, this model does not assume independent data sources, i. e., that the observation processes are not affected by the actions of the decision maker.

TABLE OF CONTENTS

ILLUSTRATIONS

SECTION I

INTRODUCTION

The objective of the project was to develop broader formulations of the mathematical (statistical) theory of decisions. This final report presents two broad scope generalizations which have resulted from the project.

The first generalization discussed is a decision-making model which applies to the case of a not-well-informed decision maker with independent data sources. In this model, the inference about the prior distribution is determined from the solution of an adjunct decision problem, which specifies the minimum risk hypothesis in the light of the available information. This model, together with the corresponding averaging model of Reference 1, has been programmed and debugged so that in future work it will be possible to obtain information about the behavior of the model.

The second generalization presented is a model of multi-period decision making for both stationary and Markovian environments. In contrast to the model discussed in the above paragraph, this model does not assume independent data sources, i. e., that the observation processes are not affected by the actions of the decision maker. Stated in another way, in this generalization we are dealing with strongly sequential decision tasks in which the decision maker's actions can redesign the information system.

In addition to the decision-making models discussed in this report, the present contract has resulted in an additional report which has been separately submitted for publication as a Technical Documentary Report. This report is entitled "The Karhunen-Loève Expansion and Factor Analysis" and was written by Dr. Satosi Watanabe of Yale University.

The remainder of this report is divided into three sections. Section II discusses the decision-making model for the not-well-informed decision maker in a stationary environment with stationary and independent data sources. Section III discusses the multi-period decision-making model for both stationary and Markovian environments. In Section IV, problem areas uncovered by the research and potential areas for future research are discussed.

SECTION II

DECISION TASKS WITH INDEPENDENT STATIONARY DATA PROCESSES AND STATIONARY ENVIRONMENT

2.1 Case I: Known Environment and Known Data Processes

This, of course, is a well known case and is presented here as background. Let us assume that a decision maker has to select an act out of a collection of acts $\{A^k\}$ so to pursue some rational objective. The rational objective being the maximization of the expected utility of his decisions.

Let us assume that the utility of an act is a function of the state of nature X^i, i.e., that for each ordered pair (A^k, X^i) there is a utility scalar $u(A^k, X^i)$. Let us assume further that the decision maker is equipped with an observation (or information) system which outputs messages belonging to the collection $\{Y^j\}$. The information system will be characterized by the collection of conditional probabilities $\{P(Y^j | X^i)\}$.

We will also assume that the environment can be characterized by a probability distribution over the states $\{P(X^i)\}$. We are assuming, furthermore, that $\{P(Y^j | X^i)\}$ is not affected by the action of the decision maker (i.e., the decision maker is not redesigning his information system during operation) and that both $\{P(X^i)\}$ and $\{P(Y^j | X^i)\}$ are stationary.

The problem we wish to solve is to obtain the decision rule

$$A^{*j} = D(Y^j)$$

where A^{*j} is the act which maximizes the expected utility conditional on receiving the message Y^j from the information system. But we have that

$$E_{|Y^j}(A^k) = \sum_i u(A^k, X^i)\, P(X^i | Y^j)$$

Then A^{*j} is given by:

$$\nu(A^{*j}) = \sum_i u(A^{*j}, X^i)\, P(X^i | Y^j) = \max_k \sum_i u(A^k, X^i)\, P(X^i | Y^j)$$

by Bayes rule we have that

$$P(X^i | Y^j) = \frac{P(Y^j | X^i)\, P(X^i)}{P(Y^j)}$$

$$\nu(A^{*j}) = \max_k \sum_i u(A^k, X^i)\, P(X^i)\, P(Y^j | X^i)\, \frac{1}{P(Y^j)}$$

Let us define the matrices U, D, Q as follows:

$$\{U\}_{ki} = u(A^k, X^i)$$

$$\begin{aligned} \{D\}_{ij} &= 0 & i \neq j \\ &= P(X^i) & i = j \end{aligned}$$

$$\{Q\}_{ij} = P(Y^j | X^i)$$

Then

$$\nu(A^{*j}) = \max_k \sum_i \{UD\}_{ki} \{Q\}_{ij} \frac{1}{P(Y^j)}$$

That is,

$$\nu(A^{*j}) = \frac{1}{P(Y^j)} \max_{\text{row}} UD\,[Q]_j$$

where $[Q]_j$ is the j-th column of Q, and

$$\nu(A^{*j}) = \frac{1}{P(Y^j)} \max_{\text{row}} [UDQ]_j = \frac{1}{P(Y^j)} [UDQ]_j^*$$

where the * indicates that the largest component of the included vector is to be taken.

Since the decision rule yields the value of the index *j for each possible observation Y^j, it is readily seen that a convenient representation of the decision rule is

$$\delta[Y] = [UDQ]^{+}$$

where Y is the row vector listing all possible observations and the operator + substitutes each column of the matrix inside with the row number of its largest component.

The performance of the decision maker will then be modeled (in the long run) by

$$\nu(U, D, Q) = \sum_j P(Y^j)\,\nu(A^{*j}) = \sum_j [UDQ]^*_j = [UDQ]^* \xi$$

where ξ is a column vector of all ones conformable to the row vector $[UDQ]^*$.

The function $\nu(U, D, Q)$ allows us to set up a measure of effectiveness for information systems, in fact, if Q_0 is the system characterized by:

$$P(Y^j | X^i) = P(Y^j)$$

i. e., the null system, the effectiveness of Q is given by

$$\mu(Q) = \nu(U, D, Q) - \nu(U, D, Q_0)$$

In utilizing statistical decision theory to evaluate decision behavior, attention must be paid to the fact that the assumptions underlying the formal model in use should be empirically valid. This in turn generates an incentive to develop the formal theory for the assumptions of as many systems as possible.

Of the assumptions of the traditional model presented above, the two which are most likely not to be verified in the experimental situation are:

. that the decision maker knows the prior distribution

. that the decision maker knows the statistical properties of his observation processes.

We are then interested in developing the theory so as to remove one or both of these assumptions substituting them with weaker ones.

A decision task which results from weakening one or more of the assumptions of another decision task will be said to be a degradation of the original task (Ref. 1).

This note is concerned with the degradation of the decision task presented above which occurs when the decision maker is assumed to know a density function over the space of priors rather than the prior itself. In other words, the decision maker is assumed to have some uncertainty on which prior actually prevails. We first consider this degraded task from the viewpoint of Reference 1, generalizing its approach to include any independent stationary data generating process and any density over the space of priors. Next, we introduce a new formal model for the same degraded task; in this model the uncertainty about the prior distribution of the basic decision task is not eliminated by averaging over the space of priors, but is eliminated by selecting that prior distribution which minimizes the subjective risk of mis-inference. This model should predict behavior which is more conservative than the averaging model when the decision maker performance is strongly sensitive to the prior distribution assumed.

2.2 Case II: Known Data Sources, Known $g_t(\pi)$

2.2.1 Averaging Model

Let us indicate with π the vector $\{P(X^1), P(X^2), \ldots, P(X^n)\}$ and with $g_t(\pi)$ the density function which prevails at time t over the space of π distributions (N-dimensional simplex). Instead of assuming that the decision maker knows which π applies, we shall assume that he knows $g_t(\pi)$ over the space of π-distributions, $g_t(\pi)$ of course will be transformed from instant to instant to reflect the learning the decision maker undergoes by observing the environment. The problem, then, is to find the decision rule under these circumstances.

In this case, we can compute the expected utility of an act A^k given that Y^j has been observed and that π is assumed to be the prior distribution as follows:

$$\nu(A^k)_{Y^j,\pi} = E_{|Y^j,\pi}(A^k) = \sum_i u(A^k, X^i)\, P(X^i | Y^j, \pi)$$

By Bayes rule we have

$$P(X^i | Y^j, \pi) = \frac{P(Y^j | X^i, \pi)\, P_\pi(X^i)}{P_\pi(Y^j)}$$

where

$P_\pi(X^i) \quad = \quad \pi_i$

$P_\pi(Y^j) \quad =$ the marginal probability of Y^j when π is assumed to be the prior distribution.

$P(Y^j | X^i, \pi) \quad = \quad P(Y^j | X^i)$ since the data sources are independent from the characteristics of the environment.

Then

$$\nu(A^k)_{Y^j, \pi} = \sum_i u(A^k, X^i)\, P_\pi(X^i)\, P(Y^j | X^i) \frac{1}{P_\pi(Y^j)}$$

If in addition to the matrices U and Q defined above, we define the matrices D_π and Λ_π as follows:

$$\begin{aligned} \{D_\pi\}_{ij} &= 0 && i \neq j \\ &= P_\pi(X^i) = \pi_i && i = j \end{aligned}$$

$$\begin{aligned} \{\Lambda_\pi\}_{ij} &= 0 && i \neq j \\ &= P_\pi(Y^j) && i = j \end{aligned}$$

We have

$$\nu(A^k)_{Y^j, \pi} = \sum_i \{UD_\pi\}_{ki}\, \{Q\Lambda_\pi^{-1}\}_{ij} = \{UD_\pi Q\Lambda_\pi^{-1}\}_{kj}$$

i. e., the expected value of the act A^k, given that Y^j is observed and the prior π is held, is the kj-th element of the matrix

$$UD_\pi Q\Lambda_\pi^{-1}$$

The decision rule is a transformation of the message Y^j into an optimal act, thus it requires the specification of one input; namely, Y^j and not Y^j and π. It follows thus that in order to compute the decision rule, we have to eliminate π. In the averaging model, one computes the quantities

$$\nu(A^k)_{Y^j} = E[\nu(A^k)_{Y^j,\pi}]$$

i.e., by expecting over π and then uses these quantities as the basis for the computation of the decision rule.

The optimal act upon receipt of Y^j is given by

$$\nu(A^{*j}) = \max_k \nu(A^k)_j = \max_k \int g_t(\pi)\nu(A^k)_{Y^j,\pi} d\pi = \max_{\text{row}} \int g_t(\pi)\,[UD_\pi Q\Lambda_\pi^{-1}]_j\, d\pi$$

If

$$\int g_t(\pi)\,[UD_\pi Q\Lambda_\pi^{-1}]d\pi = E_t[UD_\pi Q\Lambda_\pi^{-1}]$$

the decision rule is simple

$$\delta(Y) = |E_t[UD_\pi Q\Lambda_\pi^{-1}]|^+$$

The learning of the decision maker would be reflected by his updating the $g_t(\pi)$, using Bayes rule, i.e.,

$$g_{t+1}(\pi) = \frac{P_r(Y^j|\pi)\,g_t(\pi)}{\int P_r(Y^j|\pi')g_t(\pi')d\pi'}$$

where $g_{t+1}(\pi)$ is the posterior density on the π-distributions after having observed Y^j. $P_r(Y^j|\pi)$, i.e., the probability of Y^j given that the prior π holds, is nothing else but the marginal probability of Y^j computed using that π as the prior. That is,

$$P_r(Y^j|\pi) = \sum_i P(Y^j|X^i)\,\pi_i = \sum_i Q_{ij}\pi_i = \{\pi Q\}_j$$

i. e., the j-th component of the row vector πQ, π being a row vector. Thus,

$$g_{t+1}(\pi) = \frac{\{\pi Q\}_j}{\int \{\pi' Q\}_j \; g_t(\pi')\, d\pi} \; g_t(\pi) = \frac{\{\pi Q\}_j}{\{\bar{\pi} Q\}_j} \; g_t(\pi)$$

where $\bar{\pi}$ is the mean vector W.R.T. $g_t(\pi)$ of the population of π vectors.

2.2.2 Double Decision Task Model

Let us consider the decision task in this case to be composed of two parts: a) make a decision about which π prevails in the environment; b) make a decision about which act is optimal in the light of the so chosen π, the utility structure and the data source characteristics. The first decision, the one about π, has also to be optimal in a Bayesian sense.

Let us, then, formalize the first decision. For simplicity, we will assume that there is a discrete population of π's and π^m represents a generic member of such a population. The $\{\pi^m\}$ is then the collection of states of the world for the first decision task. The messages about the world are still the Y^j since the decision maker has to use the same data sources for both decision tasks. The acts are the $\{\alpha^n\}$ where α^n is the act of selecting π^n as the value of π to be used in the second decision task (the basic task).

In order to obtain the decision rule for the first task, we have to have:

- A prior distribution over $\{\pi^m\}$
- A utility structure for the ordered pairs (α^n, π^m)
- A model of the information system which supports the first task.

Let's begin with the model for the information system. The relevant information system is represented by the set of conditional probabilities $\{P(Y^j | \pi^m)\}$ since the π^m's are the states and the Y^j's are the available messages. We have seen above that $P(Y^j | \pi^m)$ is the marginal probability of Y^j computed assuming that $\pi = \pi^m$ and that

$$P(Y^j | \pi^m) = [\pi^m Q]_j$$

If Π is the matrix such that its m-th row is the vector π^m, then we have that

$$\{\Omega\}_{mj} = \sum_e \{\Pi\}_{me} \{Q\}_{ej} = \sum_e \pi_e^m P(Y^j | X^e)$$

thus,

$$\Omega = \Pi Q$$

Thus the model for the information system of the first task is obtained by premultiplying the model for the basic observation processes by a matrix whose rows are the various possible priors of the basic decision task.

The suitable prior distribution for the first task can be obtained from the density function $g_t(\pi)$. For example, if the state π^m is said to obtain in the region R_m, then the prior probability of the state π^m is

$$\int_{R_m} g_t(\pi)\, d\pi = G(\pi^m)$$

This distribution can reflect the learning of the decision maker from one instant to the next through the simple Bayesian learning process discussed above in connection with the averaging model.

Finally, we can determine which is the proper utility structure for the first decision task. Let us indicate with $L(\alpha^n, \pi^m)$ the utility of assuming that π^n is the case when actually π^m is true. This utility can be computed by considering the difference which assuming π^n instead of the true π^m will make in the expected return of the decision maker in connection with the basic decision task. Clearly

$$L(\alpha^n, \pi^m) = \{[UD_{\pi^m} Q]^{*\pi^n} - [UD_{\pi^m} Q]^*\} \xi$$

where the operation $[\;]^{*\pi^n}$ selects the column components which would have been the largest in the expression $UD_{\pi^n} Q$. Since in general these components will not be the largest of their column for the expression $UD_{\pi^m} Q$, the quantity $L(\alpha^n, \pi^m)$ is zero or negative.

Then the decision rule is to select π^n according to

$$[LD_G\Omega]^+$$

(The rows of the $LD_G\Omega$ matrix being associated with the various π^m's and the columns with the various messages.)

The π^n so obtained would then be used to select the act for the basic decision task according to the decision rule

$$[UD_{\pi^n}Q]^+$$

2.2.3 Computer Programs of the Models

A general purpose FORTRAN model which incorporates the formal models discussed in Sections 2.2.1 and 2.2.2 has been written. To clarify the computations required by the two models, a brief description of the control structure of the computer program is presented in the flow chart presented as Figure 1. As it can be seen from the flow chart, one can select between the two formal models by operating the console switch No. 1. A comparison of the behavior of the two models in the face of the same observation series can be carried out quite easily. The program and all its subroutines have already been written and debugged and listings of the computer programs are presented in the Appendix. Data will be collected on the behavior of these models for a very simple decision task. The task which will be used in exercising the models is described in the next section.

2.2.4 A Numerical Example

Assume that an object (for example, an enemy ship) is in an area divided into four regions, as illustrated in Figure 2.

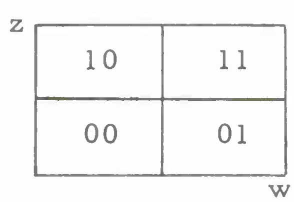

Figure 2

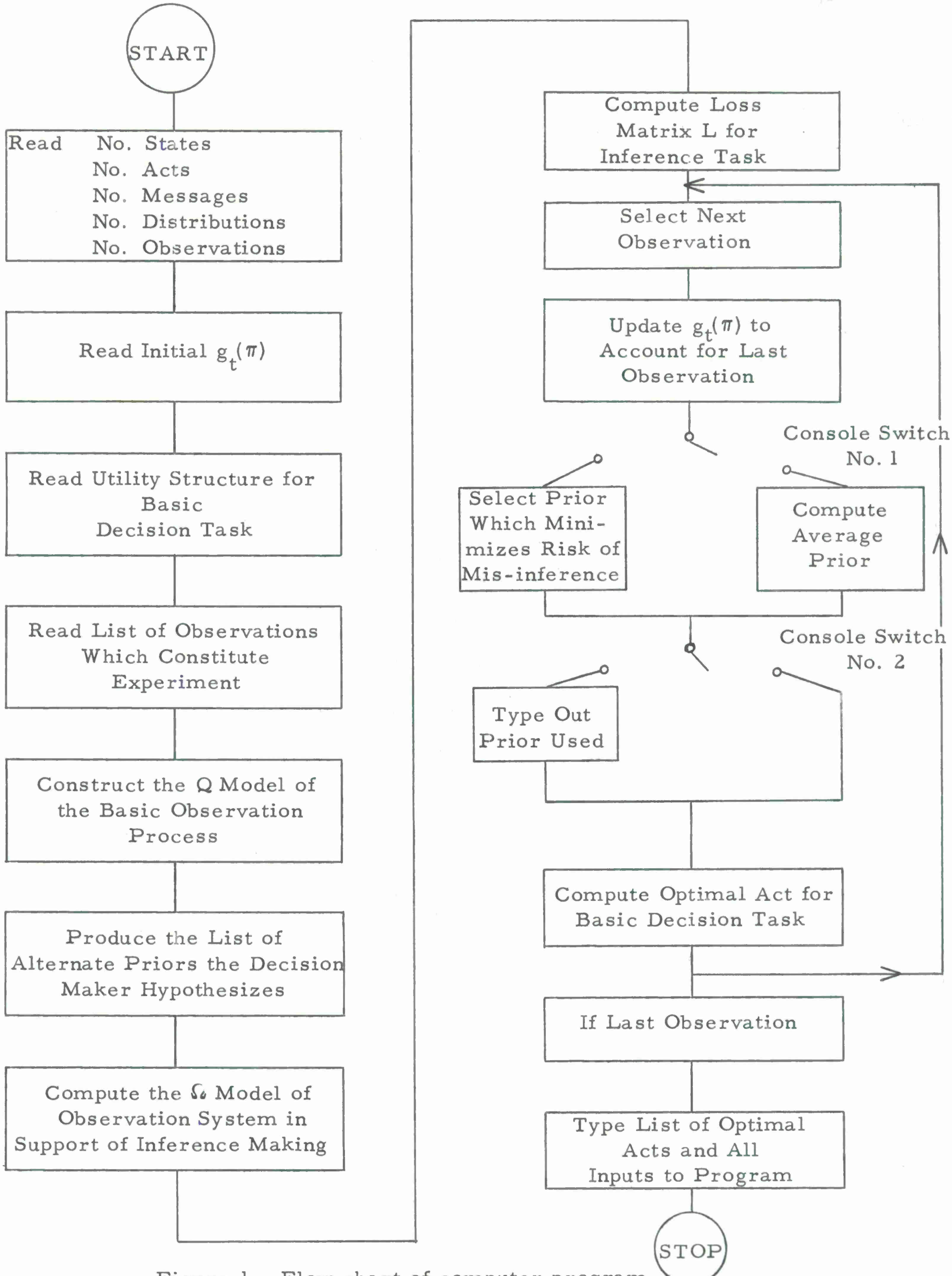

Figure 1. Flow chart of computer program.

Let us assume that the sensors available are capable of identifying which intervals on the z and w axis "contain" the ship. The w-axis sensor is assumed to give a correct response with probability q_1 and the z-axis sensor with probability q_2. The action of the decision maker is to place a weapon (bomb) in one of the four regions.

From the above, it is clear that there are four states of the world:

X^1 = Ship is in 00 region.

X^2 = Ship is in 01 region.

X^3 = Ship is in 10 region.

X^4 = Ship is in 11 region.

and four messages:

Y^1 = Ship is reported in 00 region.

Y^2 = Ship is reported in 01 region.

Y^3 = Ship is reported in 10 region.

Y^4 = Ship is reported in 11 region.

and four acts:

A^1 = Place bomb in region 00.

A^2 = Place bomb in region 01.

A^3 = Place bomb in region 10.

A^4 = Place bomb in region 11.

The utility structure which we will assume models the destructive capabilities of the weapon specifies a return of two units of utility if the bomb is placed in the <u>same</u> region where the ship is located. If the bomb is placed in a region adjacent to where the ship is located, the utility is only one-half the previous value. Finally, if the bomb is placed in a non-adjacent region, the return is null. The utility matrix is then:

$$\begin{array}{c} \\ A^1 \\ A^2 \\ A^3 \\ A^4 \end{array} \begin{array}{c} \begin{array}{cccc} X^1 & X^2 & X^3 & X^4 \end{array} \\ \left| \begin{array}{cccc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{array} \right| \end{array} = U$$

The information system used by the decision maker is modeled by:

$$
\begin{array}{c|cccc|}
 & X^1 & X^2 & X^3 & X^4 \\
Y^1 & q_1\ q_2 & q_2(1-q_1) & q_1(1-q_2) & (1-q_1)(1-q_2) \\
Y^2 & q_2(1-q_1) & q_1\ q_2 & (1-q_1)(1-q_2) & q_1(1-q_2) \\
Y^3 & q_1(1-q_2) & (1-q_1)(1-q_2) & q_1\ q_2 & q_2(1-q_1) \\
Y^4 & (1-q_1)(1-q_2) & q_1(1-q_2) & q_2(1-q_1) & q_1\ q_2
\end{array} = Q
$$

The above Q model reflects the assumption that the two axis sensors are mutually statistically independent.

If we now characterize the collection of alternate prior distributions that the decision maker is willing to consider and a distribution over them, we will have all the inputs for both decision-making models.

To take the simplest case we can assume that the decision maker characterizes the environment weakly, i.e., ignoring any statistical dependency which may exist between the two coordinates of a ship location. In such a case, the prior distributions that the decision maker may entertain are of the form:

$$p(X^1) = (1-p_1)(1-p_2)$$

$$p(X^2) = (1-p_2)p_1$$

$$p(X^3) = (1-p_1)p_2$$

$$p(X^4) = p_1\ p_2$$

where p_1 and p_2 are respectively the probabilities that the ship w coordinate and z coordinate fall in the second interval of the corresponding axis.

We will allow the number p_1 and p_2 to assume four equidistant values obtaining 16 different priors. The 16 priors will be assumed at the onset to be equally likely. Thus we have defined the matrix Π and the vector $G(\pi^m)$ which were the only two missing inputs for our models.

It should be pointed out that the structure of the computer program described in Section 2. 2. 3 allows extremely flexible redefinition of the decision task. Thus, one could easily modify the sensor model to represent, for example, two isotropic sonar buoys placed in the region 00 and 11 by simply utilizing the following Q matrix:

$$\begin{array}{c} \\ Y^2 \\ Y^4 \\ Y^6 \end{array} \begin{array}{c} \begin{array}{cccc} X^1 & X^2 & X^3 & X^4 \end{array} \\ \left| \begin{array}{cccc} r^2 & q^2 & q^2 & 1-(q^2+r^2) \\ q^2 & 1-2q^2 & 1-2q^2 & q^2 \\ 1-(q^2+r^2) & q^2 & q^2 & r^2 \end{array} \right| \end{array} = Q$$

Here the sensors are assumed to have three distinct readouts: same region as that of buoy; one of the two regions adjacent to buoy; region furthest from buoy. Also we are assuming that a misreading between adjacent readouts can occur with probability q and between non-adjacent readouts with probability r; furthermore, multiple misreadings are assumed to occur in a mutually independent manner.

SECTION III

A MODEL OF MULTI-PERIOD DECISION MAKING

We will begin this section by considering multi-period decision making in stationary environments. These are environments described by probability distributions which are time invariant. Let X^m be a state of the world, A^i an act, θ^j an outcome. Then the environment can be described by the three-dimensional array of probability numbers

$$\{p(\theta^j | A^i, X^m)\}$$

The problem we want to solve is given: A prior distribution at some time t over the states of the world $\{p_t(X^m)\}$ and a set of utility numbers $\{u_{ij} = u(A^i, \theta^j)\}$, which indicate the utility which accrues to the decision maker if (his) act A^i is followed by the outcome θ^j, determine the optimal act for the present time which will lead to maximizing present and future payoffs.

The approach followed is to reduce the sequential decision-making problem to a static one by computing an "equivalent" utility matrix $U' = U + V$, where $U = \{u_{ij}\}$, and $V = \{v_{ij}\}$ is the maximum expected utility return over the remaining decision points which can accrue once the initial act A^i is followed by the outcome θ^j. In other words, v_{ij} is the maximum expected future return which could ensue if at the initial instant the pair (A^i, θ^j) obtained.

Once the equivalent utility matrix has been obtained, one can select the optimal act as follows: The expected value of the i-th act is

$$v(A^i) = \sum_j p_t(\theta^j | A^i) \, u'_{ij}$$

where $p_t(\theta^j | A^i)$ is given by

$$p_t(\theta^j | A^i) = \sum_k p(\theta^j | A^i, X^k) \; p_t(X^k) \qquad (1')$$

Then one selects the act which maximizes such value, i.e., the A^{i_o} such that

$$\sum_j p_t(\theta^j | A^{i_o}) \, u'_{i_o j} = \max_i \sum p_t(\theta^j | A^i) u'_{ij} \tag{1}$$

would solve our problem if the quantities $\{u'_{ij}\}$ where available.

Let us illustrate, then, how to compute $\{u'_{ij}\}$. For simplicity, we will consider a case in which only two periods need be considered. In applying this formulation to a specific problem, one has to determine the value n such that the $\{u'_{ij}\}$ computed over n steps, as well as over n+1 steps, are "sufficiently" equal. Such n is the minimum amount of future which needs to be considered for the specific sequential problem on hand.

If two periods need to be considered, then since $u'_{ij} = u_{ij} + v_{ij}$, v_{ij} represents the maximum expected return in the second step given that in the first step (A^i, θ^j) obtained. But such a quantity is simply

$$v_{ij} = \max_l \sum_k p^{(i,j)}_{t+1} (\theta^k | A^l) \, u(A^l, \theta^k) \tag{2}$$

where the quantity $p^{(i,j)}_{t+1}(\theta^k | A^l)$ is the probability of θ^k (at t+1) given A^l (at t+1) given that at t, (A^i, θ^j) occurred. This probability is computed as follows:

$$p^{(i,j)}_{t+1} (\theta^k | A^l) = \sum_m p(\theta^k | A^l, X^m) \, p^{(i,j)}_{t+1} (X^m) \tag{3}$$

where $p^{(i,j)}_{t+1} (X^m)$ is the posterior distribution of the states after the occurrence of the (A^i, θ^j) pair and $p(\theta^k | A^l, X^m)$ is a member of the original three-dimensional array which describes the environment. The distribution $p^{(i,j)}_{t+1} (X^m)$ is in turn given by:

$$p^{(i,j)}_{t+1}(X^m) = p_t(X^m | \theta^j, A^i) \tag{4'}$$

where

$$p_t(X^m \mid \theta^j, A^i) = \frac{p(\theta^j \mid A^i, X^m)p_t(X^m)}{\sum_l p(\theta^j \mid A^i, X^l)p_t(X^l)} \tag{4}$$

Thus the complete solution of the two-period sequential decision problem in a stationary environment proceeds as follows: Using (4) and (4') (Bayes Theorem), one obtains the posterior distribution over the states after observing the outcome of the first act (thus this model accounts for the learning performed by the decision maker upon acting and observing the consequences (outcome) of his act).

Next, using (3) and (2) one is in a position to compute the "optimal-future-return" v_{ij} of the pair (A^i, θ^j). Finally, using (1) and (1'), one can select the optimal act A^{io} for now. This process can be repeated as soon as the outcome is observed by simply replacing $p_t(X^k)$ with the $p_{t+1}^{(i,j)}(X^k)$ where i and j correspond to the act that was chosen and the outcome which actually occurred. Thus we obtain the selection of the optimal act for each decision point in time, each selection taking into account only the returns to be expected at the decision point and the one just beyond it.

Extensions of this schema to futures of more than two adjacent decision points is conceptually straight forward, but computationally cumbersome.

3. 1 Markovian Environments

If the multi-period decision making takes place in a non-stationary environment, it is no longer possible to describe the environment with the probability array $\{p(\theta^j \mid A^i, X^m)\}$ since this array is independent of time. Let us consider the special class of Markovian environments. An environment is said to be Markovian if the probability array which controls the outcomes depends only on the act and outcome which took place in the previous instant. It then follows that a Markovian environment is described by the array

$$\{p(\theta_t^h \mid \theta_{t-1}^k, A_{t-1}^i, A_t^j, X_t^m)\}$$

whose individual entry represents the probability that given that the present (time t) act and state are respectively A^j and X^m and given that the previous instant (time t−1) was characterized by act A^i and outcome θ^k, the outcome θ^h will occur at t (now).

If one realizes that the only difference which is introduced by the environment being Markovian is that one can no longer talk about a fixed model of the environment $\{p(\theta^h | A^j, X^m)\}$ but has to model the environment instant by instant with an array

$$\{p_t^{(k,i)}(\theta^h | A^j, X^m)\}$$

where

$$\{p_t^{(k,i)}(\theta^h | A^j, X^m) = p(\theta_t^h | \theta_{t-1}^{(k)}, A_{t-1}^{(i)}, A_t^j, X_t^m)\}$$

where the brackets around k and i indicate that these indices are fixed in the array, then it is extremely easy to generalize the procedure given for stationary environments to the case of Markovian environments. The generalized procedure is given briefly below:

To solve completely the two-period decision problem in a Markovian environment, one proceeds as follows, using

$$p_{t+1}^{(j,h)|(k,i)}(X^m) = p_t(X^m | \theta_t^h, A_t^j, \theta_{t-1}^k, A_{t-1}^i) \tag{4b'}$$

$$p_t(X^m | \theta_t^h, A_t^j, \theta_{t-1}^k, A_{t-1}^i) = \frac{p(\theta_t^h | \theta_{t-1}^k, A_{t-1}^i, A_t^j, X_t^m)\, p_t(X^m)}{\sum_l p(\theta_t^h | \theta_{t-1}^k, A_{t-1}^i, A_t^j, X_t^l)\, p_t(X^l)} \tag{4b}$$

one obtains the posterior distribution over the states.

Equation (4b), which is Bayes Theorem, models the learning process of the decision maker who observing the datum θ_t^h under the conditions $\theta_{t-1}^k, A_{t-1}^i, A_t^j$ learns how to better discriminate the hypothesis $\{X^m\}$.

The optimal future return for the pair (A_t^j, θ_t^h) is computed as follows:

First obtain

$$p(\theta^r_{t+1} \mid \theta^h_t, A^j_t, A^q_{t+1}) = \sum_m p(\theta^r_{t+1} \mid \theta^h_t, \ A^j_t, \ A^q_{t+1}, \ X^m_{t+1}) p^{(j,h)|(k,i)}_{t+1} (X^m) \qquad (3b)$$

Notice that in (3b) one is using the array

$$\{p(\theta^r_{t+1} \mid \theta^{(h)}_t, A^{(j)}_t, A^q_{t+1}, X^m_{t+1})\}$$

and not the array

$$\{p(\theta^h_t \mid \theta^{(k)}_{t-1}, A^{(i)}_{t-1}, A^j_t, X^m_t)\}$$

This of course is due to the time variant character of the environment. Once (3b) is used, one can compute v_{jh} with:

$$v_{jh} = \max_q \sum_r p(\theta^r_{t+1} \mid \theta^h_t, A^j_t, A^q_{t+1}) u(\theta^r, A^q) \qquad (2b)$$

Finally using

$$u'_{jh} = u_{jh} + v_{jh} \qquad (1b'')$$

and

$$p(\theta^h_t \mid \theta^k_{t-1}, A^i_{t-1}, A^j_t) = \sum_m p(\theta^h_t \mid \theta^k_{t-1}, A^i_{t-1}, A^j_t, X^m_t) p_t(X^m) \qquad (1b')$$

and

$$\sum_h p(\theta^h_t \mid \theta^k_{t-1}, A^i_{t-1}, A^{j_o}_t) u'_{j_o h} = \max_j \sum_h p(\theta^h_t \mid \theta^k_{t-1}, A^i_{t-1}, A^j_t) u'_{jh} \qquad (1b)$$

we can select the optimal act A^{j_o} for the time t which is optimal when its expected consequences are considered over only two future decision points. Again extensions to more than two future decision points are quite straightforward but become very cumbersome computationally.

SECTION IV

PROBLEM AREAS UNCOVERED BY THE RESEARCH AND POTENTIAL EXTENSIONS

The decision-making model presented in the first part of this report is concerned with extending decision theory to the case where the decision maker does not know the statistical properties of the stationary environment in which he is assumed to operate. The model replaces the prior distribution with an ensemble of hypothetical distributions, i. e., with a space of prior distributions and an associated density function.

This density function represents the state of knowledge of the decision maker concerning the statistical properties of his environment. Thus a uniform density function would reflect a state of no information and a Dirac's density function would correspond to the state of perfect statistical information. Traditional decision theory is concerned only with the above two extreme cases. This information state is, of course, subject to modification under the impact of empirical evidence supplied by the observation system. The model incorporates empirical evidence by a suitable Bayesian learning submodel. The learning model used is appropriate for a stationary environment.

A natural extension of the theory is the incorporation of learning models for time-variant environments. In the following paragraphs, we would like to briefly describe how such an extension could be obtained in the case of the piece-wise-stationary environments, which are used in the experiments reported by A. Rapoport in Reference 2. An environment is said to be piece-wise-stationary if its statistical properties change only at a discrete instant in time, such points in time will be referred to as time markers.

Let us assume the simplest piece-wise-stationary environment, namely, an environment with a single time marker. The ensemble of hypotheses for such an environment is the collection of vectors $\{\pi_b, t, \pi_a\}$ and a suitable density function over them. The symbol π_b indicates the prior distribution which holds before the marker, while π_a is the prior distribution which applies after the marker and t is the value of the time marker.

The empirical information will update the density function on the hypothesis space via the mechanism of Bayes rule. The selection of the appropriate hypothesis has to be done by solving a decision theoretic schema which incorporates the differential cost structure for pairs of hypothesis. It is through this kind of mechanism that a more rapid discounting of past observations should result in time-variant environments. Mechanisms which rely on modifications of Bayes theorem suggested on intuitive grounds are not acceptable because there is no guarantee that they will result in an internally consistent formal system. We recommend that the above sketched out extension of the theory be developed in order to derive a normative theory for the experimental situations presented in Reference 2.

The multi-period decision-making model discussed in Section III can also be formulated for time-variant environments with a finite history. For decision tasks with an invariant utility structure and a well-informed decision maker, the multi-period model is the most general decision-making model. Consequently, further work on this model should not be concerned with generalizing it, but rather with the discovery of specialized cases which yield powerful algorithms for the exercise of the model.

REFERENCES

1. E. H. Shuford, Jr., "Some Bayesian Learning Processes," In Shelly, M. W. and Bryan, G. L. (Eds.), Human Judgments and Optimality. New York: Wiley, 1964. pp 127-152.

2. A. Rapoport, "Sequential Decision Making in a Computer Controlled Task," Journal of Math. Psycho., Vol. I, No. 2, pp. 351-374. July 1964.

APPENDIX

This appendix contains listings of the executive routine and the subroutines for the computer program described in the flow chart presented as Figure 1 in the report.

```
C     DECISION MODELS 1,2    16 DEC 1964
C     PARTIALLY INFORMED DECISION MAKING MODELS 1,2 EXECUTIVE PROGRAM
C     THIS PROGRAM ALLOWS THE COMPUTATION OF THE OPTIMUM STRATEGY
C     FOR DECISION MAKING IN A STATIONARY ENVIRONMENT WITH STATIONARY
C     AND INDEPENDENT DATA SOURCES AND NOT WELL INFORMED DECISION MAKER
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
C    1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(20),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
 99   FORMAT (5I4)
      READ 99,N,NA,ND,NM,NO
 98   FORMAT (8 F6.3)
      READ 98,(G(IG),IG=1,8)
      READ 98,(G(IG),IG=9,16)
      READ 98,((U(IU,JU),IU=1,4),JU=1,2)
      READ 98,((U(IU,JU),IU=1,4),JU=3,4)
C97   FORMAT(NO I2)
 97   FORMAT (20 I2)
      READ 97, NY
      CALL QSUB
      CALL PISUB
      DO 100 IPI=1,ND
      DO 100 JQ=1,NM
      OME(IPI,JQ)=0
      DO 100 KQ=1,N
 100  OME(IPI,JQ)=OME(IPI,JQ)+PI(IPI,KQ)*Q(KQ,JQ)
      IF (SENSE SWITCH 1)311,312
 311  CALL CSUB
 312  DO 104 I=1,NO-1
      CALL UPDATE
      IF(SENSE SWITCH 1) 110,111
 110  CALL PRIOR
      GO TO 113
 111  DO 112 IPR=1,N
 112  PR(IPR)=PIAV(IPR)
 113  IF(SENSE SWITCH 2) 102,103
C101  FORMAT($SELECTED PRIOR$/NF6.3)
 101  FORMAT($ SELECTED PRIOR$/4F6.3)
 102  TYPE 101,PR
 103  CALL DECIS
 104  CONTINUE
C202  FORMAT($PROGRAM INPUTS$///$N=$,I2,3X,$NA=$,I2,3X,$ND=$,I2,$NM=$,
C     I2,3X,$NO=$,I2//$Q MATRIX$//NM(N F6.3)//$PI MATRIX$/
C     N (ND/2F6.3/6X,ND/2F6.3//)//$ORIGINAL G DISTRIBUTION$/
C     ND/2F6.3/6X,ND/2F6.3//$LIST OF OBSERVATIONS $/NOI2/)
 202  FORMAT($PROGRAM INPUTS$///$N=$,I2,3X,$NA=$,I2,3X,$ND=$,I2,$NM=$,
     1I2,3X,$NO=$,I2//$Q MATRIX$//4(4F6.3/)//$PI MATRIX $/
     14( 8  F6.3/6X, 8  F6.3//)//$ORIGINAL G DISTRIBUTION$/
     18   F6.3/6X, 8  F6.3//$LIST OF OBSERVATIONS $/20I2/)
```

```
      TYPE 202,N,NA,ND,NM,NO,Q,PI,G,NY
C313  FORMAT (/$UTILITY MATRIX$/N(NA F6.3/))
 313  FORMAT(/$UTILITY MATRIX$/4(4 F6.3/))
      TYPE 313,U
C203  FORMAT($DECISION MAKER STRATEGY$///$LAST OBSERVATION$,6X,NO-1I2/
C     $PRESENT OBSERVATION$,3X,NO-1 I2/$SELECTED ACT$,10X,NO-1  I2)
 203  FORMAT($DECISION MAKER STRATEGY$///$LAST OBSERVATION$,6X, 19I2/
     1$PRESENT OBSERVATION$,3X, 19  I2/$SELECTED ACT$,10X,  19  I2)
      TYPE 203,(NY(I),I=1,NO-1),(NY(I),I=2,NO),(KACT(I),I=1,NO-1)
      END



      SUBROUTINE QSUB
C     THIS IS THE Q MODEL OF EXAMPLE 1
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
C    1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(20),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
 150  FORMAT (2F6.3)
      READ 150, Q1,Q2
      Q(1,1)=Q1*Q2
      Q(1,2)=Q2*(1.0-Q1)
      Q(1,3)=Q1*(1.0-Q2)
      Q(1,4)=(1.0-Q1)*(1.0-Q2)
      Q(2,1)=Q2*(1.0-Q1)
      Q(2,2)=Q1*Q2
      Q(2,3)=(1.0-Q1)*(1.0-Q2)
      Q(2,4)=Q1*(1.0-Q2)
      Q(3,1)=Q1*(1.0-Q2)
      Q(3,2)=(1.0-Q1)*(1.0-Q2)
      Q(3,3)=Q1*Q2
      Q(3,4)=Q2*(1.0-Q1)
      Q(4,1)=(1.0-Q1)*(1.0-Q2)
      Q(4,2)=Q1*(1.0-Q2)
      Q(4,3)=Q2*(1.0-Q1)
      Q(4,4)=Q1*Q2
C     QSUB INPUTS
 201  FORMAT (// $ Q1 AND Q2 VALUES $//F6.3,3X,F6.3)
 207  TYPE 201,Q1,Q2
      RETURN
      END



      SUBROUTINE PISUB
C     DISTRIBUTION THAT RESULTS IF TWO INDEPENDENT PROB AXIS ARE
```

```
C     ASSUMED TO ACQUIRE 4 EQUAL PROBABILITIES
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
C    1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(20),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
C     DIMENSION P1(NPV),P2(NPV)
      DIMENSION P1(4),P2(4)
 128  FORMAT (8F6.3, I3)
      READ 128,P1,P2,NPV
      ND=NPV*NPV
      DO 130 IPI=1,ND
      DO 130 JPI=1,N
 130  PI(IPI,JPI)=0
      DO 131 IP2=1,NPV
      DO 131 IP1=1,NPV
      IPI=IP1+NPV*(IP2-1)
      PI(IPI,1)=(1.0-P1(IP1))*(1.0-P2(IP2))
      PI(IPI,2)=P1(IP1)*(1.0-P2(IP2))
      PI(IPI,3)=(1.0-P1(IP1))*P2(IP2)
 131  PI(IPI,4)=P1(IP1)*P2(IP2)
C     PISUB INPUTS
C200  FORMAT(//$P1 AND P2 VALUES $//NPV F6.3/NPV F6.3)
 200  FORMAT (// $ P1 AND P2 VALUES $//4F6.3/4F6.3)
      TYPE 200,P1,P2
      RETURN
      END



      SUBROUTINE CSUB
C     THIS ROUTINE COMPUTES THE LOSS RESULTING FROM HYPOTHESIZING PI
C     RATHER THAN PJ AS THE PRIOR
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
C    1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(20),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
C     DIMENSION DPI(N,N),DPJ(N,N),C(NA,N),CJ(NA,NM ,CI(NA,NM),EI(NM),EJ(NM)
      DIMENSION DPI(4,4),DPJ(4,4),C(4,4),CJ(4,4),CI(4,4),EI(4),EJ(4)
      DO 11 IPL=1,ND
      DO 11 JPL=1,ND
      DO 12 L=1,N
      DO 12 K=1,N
      DPI(L,K)=0
      DPI(L,L)=PI(IPL,L)
      DPJ(L,K)=0
 12   DPJ(L,L)=PI(JPL,L)
      DO 1 M=1,NA
```

```
      DO 1 K=1,N
      C(M,K)=0
      DO 1 L=1,N
1     C(M,K)=C(M,K)+U(M,L)*DPJ(L,K)
      DO 2 M=1,NA
      DO 2 K=1,NM
      CJ(M,K)=0
      DO 2 I1=1,N
2     CJ(M,K)=CJ(M,K)+C(M,I1)*Q(I1,K)
      VJ=0
      DO 5 K=1,NM
      EJ(K)=CJ(1,K)
      DO 4 L=2,NA
      IF(EJ(K)-CJ(L,K)) 3,3,4
3     EJ(K)=CJ(L,K)
4     CONTINUE
5     VJ=VJ+EJ(K)
      DO 6 M=1,NA
      DO 6 K=1,N
      C(M,K)=0
      DO 6 L=1,N
6     C(M,K)=C(M,K)+U(M,L)*DPI(L,K)
      DO 7 M=1,NA
      DO 7 K=1,NM
      CI(M,K)=0
      DO 7 I1=1,N
7     CI(M,K)=CI(M,K)+C(M,I1)*Q(I1,K)
      VI=0
      DO 10 K=1,NM
      EI(K)=CI(1,K)
      EJ(K)=CJ(1,K)
      DO 9 L=2,NA
      IF(EI(K)-CI(L,K))8,8,9
8     EI(K)=CI(L,K)
      EJ(K)=CJ(L,K)
9     CONTINUE
10    VI=VI+EJ(K)
      IF(SENSE SWITCH 4)301,11
301   TYPE 300,DPI,DPJ,C,CJ,VJ,CI,VI
300   FORMAT($ DPI$/4(4F6.3/)//$DPJ$/4(4F6.3/)//$C$/4(4F6.3/)//$CJ$/
     14(4F6.3/)//$VJ=$,F6.3//$CI$/4(4F6.3/)//$VI=$,F6.3)
11    PL(IPL,JPL)=VI-VJ
      IF(SENSE SWITCH 3)303,304
C302  FORMAT($COST OF MISINFERENCE$/// ND(ND/2F6.3/6X,ND/2F6.3//))
302   FORMAT($COST OF MISINFERENCE$/// 16( 8 F6.3/6X, 8 F6.3//))
303   TYPE 302,PL
304   RETURN
      END
```

```
      SUBROUTINE UPDATE
C     THIS SUBROUTINE UPDATES THE DISTRIBUTION G(PI) TO ACCOUNT FOR
C     THE RECEIPT OF THE MESSAGE IDENTIFIED BY NY(I).
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
C    1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(20),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
C     DIMENSION QJ(N), PIQJ(ND)
      DIMENSION QJ(4),PIQJ(16)
      JQ=NY(I)
      DO 50 IQ=1,N
 50   QJ(IQ)=Q(IQ,JQ)
      DO 51 IPI=1,ND
      PIQJ(IPI)=0
      DO 51 KPI=1,N
 51   PIQJ(IPI)=PIQJ(IPI) + PI(IPI,KPI)*QJ(KPI)
      DENO=0
      DO 52 JG=1,ND
 52   DENO=DENO+G(JG)*PIQJ(JG)
      DO 53 IPI=1,ND
 53   G(IPI)=(PIQJ(IPI)*G(IPI))/DENO
      DO 54 IDG=1,ND
      DO 54 JDG=1,ND
      DG(IDG,JDG)=0
 54   DG(IDG,IDG)=G(IDG)
      DO 55 JPI=1,N
      PIAV(JPI)=0
      DO 55 IPI=1,ND
 55   PIAV(JPI)=PIAV(JPI)+G(IPI)*PI(IPI,JPI)
      IF(SENSE SWITCH 3)306,307
C305  FORMAT($LAST OBSERVATION=$,I2//$G NEW DISTR$//2(ND/2F6.3/)/
C    1/$AVERAGE PRIOR$/N F6.3)
 305  FORMAT($LAST OBSERVATION=$,I2//$G NEW DISTR$//2(  8 F6.3/)/
     1/$AVERAGE PRIOR$/4 F6.3)
 306  TYPE 305, JQ,G,PIAV
 307  RETURN
      END


      SUBROUTINE PRIOR
C     THIS SUBROUTINE SELECTS THAT COLUMN OF THE MATRIX PL-DG-OME
C     WHICH CORRESPONDS TO THE NEXT OBS MESSAGE AND SELECTS THE PRIOR
C     WHICH CORRESPONDS TO ITS LARGEST ENTRY
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
```

```
C     1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(2∅),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
C         DIMENSION DL(ND,ND),DLOJ(ND)
      DIMENSION DL(16,16),DLOJ(16)
      DO 21 IL=1,ND
      DO 21 JD=1,ND
      DL(IL,JD)=∅
      DO 21 KLD=1,ND
 21   DL(IL,JD)=DL(IL,JD)+PL(IL,KLD)*DG(KLD,JD)
      DO 22 IL=1,ND
      JOM=NY(I+1)
C     JOM SELECTS COLUMN OF OMEGA THUS OF L-DG-OMEGA, NY(I) LISTS
C     THE IDENTIFIERS OF THE MESSAGES MAKING UP A GIVEN EXPERIMENT.
C     NY(I+1) IS THUS THE IDENTIFIER OF TH PRESENT OBSERVATION.
      DLOJ(IL)=∅
      DO 22 JD=1,ND
 22   DLOJ(IL)=DLOJ(IL)+DL(IL,JD)*OME(JD,JOM)
      AMAXIL=DLOJ(1)
      DO 25 IP=1,N
 25   PR(IP)=PI(1,IP)
      DO 24 IL=2,ND
      IF(AMAXIL-DLOJ(IL))23,23,24
 23   AMAXIL=DLOJ(IL)
      DO 26 IP=1,N
 26   PR(IP)=PI(IL,IP)
 24   CONTINUE
      IF(SENSE SWITCH 3)3∅9,31∅
C3∅8  FORMAT($PRESENT OBSERVATION=$,I2//$COST OF INF$/2(ND/2F6.3/))
 3∅8  FORMAT($PRESENT OBSERVATION=$,I2//$COST OF INF$/2(  8 F6.3/))
 3∅9  TYPE 3∅8,JOM,DLOJ
 31∅  RETURN
      END



      SUBROUTINE DECIS
C     THIS SUBROUTINE COMPUTES THE DECISION RULE (UDQ)+ AND THE
C     OPTIMAL ACT FOR THE ACTUALLY OBSERVED Y.IT CAN OUTPUT DECIS  RULE
      COMMON I,N,NA,ND,NM,NO,DG,G,PL,KACT,NY,OME,PIAV,PI,PR,Q,U
C     DIMENSION DG(ND,ND),G(ND),PL(ND,ND),KACT(NO-1),NY(NO),OME(ND,NM)
C    1PIAV(N),PI(ND,N),PR(N),Q(N,NM),U(NA,N)
      DIMENSION DG(16,16),G(16),PL(16,16),KACT(19),NY(2∅),OME(16,4)
     1PIAV(4),PI(16,4),PR(4),Q(4,4),U(4,4)
C     DIMENSION D(N,N),UD(NA,N),UDQ(NA,NM),AMAX(NM),KRULE(NM)
      DIMENSION D(4,4),UD(4,4),UDQ(4,4),AMAX(4),KRULE(4)
      DO 3∅ ID=1,N
      DO 3∅ JD=1,N
      D(ID,JD)=∅
```

```
30    D(ID,ID)=PR(ID)
      DO 31 IU=1,NA
      DO 31 JD=1,N
      UD(IU,JD)=0
      DO 31 KU=1,N
31    UD(IU,JD)=UD(IU,JD)+U(IU,KU)*D(KU,JD)
      DO 32 IU=1,NA
      DO 32 JQ=1,NM
      UDQ(IU,JQ)=0
      DO 32 KD=1,N
32    UDQ(IU,JQ)=UDQ(IU,JQ)+UD(IU,KD)*Q(KD,JQ)
      DO 34 JQ=1,NM
      KRULE(JQ)=1
      AMAX(JQ)=UDQ(1,JQ)
      DO 34 IU=2,NA
      IF(AMAX(JQ)-UDQ(IU,JQ))33,33,34
33    AMAX(JQ)=UDQ(IU,JQ)
      KRULE(JQ)=IU
34    CONTINUE
      IF(SENSE SWITCH 3)36,37
35    FORMAT($COST OF ACT$/ 4( 4 F6.3/)//$KRULE$/ 4 I2)
C35   FORMAT($COST OF ACT$/NM(NA F6.3/)//$KRULE$/NM I2)
36    TYPE 35,UDQ,KRULE
37    JQ=NY(I+1)
C     KACT(I) IS TAKEN OBSERVING NY(I+1)
      KACT(I)=KRULE(JQ)
      RETURN
      END
```

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Electronic Systems Division, Air Force Systems Command, USAF, L. G. Hanscom Field, Bedford, Massachusetts 01731

13. ABSTRACT

The objective of the project was to develop broader formulations of the mathematical (statistical) theory of decisions. This final report presents two broad scope generalizations which have resulted from this project.

The first generalization discussed is a decision-making model which applies to the case of a not-well-informed decision maker with independent data sources. In this model, the inference about the prior distribution is determined from the solution of an adjunct decision problem, which specifies the minimum risk hypothesis in the light of the available information.

The second generalization presented is a model of multi-period decision making for both stationary and Markovian environments. In contrast to the model discussed in the above paragraph, this model does not assume independent data sources, i.e., that the observation processes are not affected by the actions of the decision maker.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Decision Making						
Statistical Analysis						
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