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INTERPRETATION OF THE FLUCTUATING ECHO FROM RANDOMLY DISTRIBUTED SCATTERERS: PART 3

Paul L. Smith, Jr.

McGILL UNIVERSITY
STORMY WEATHER GROUP

REPORT MW-39

Contract No. AF 19(628)-2489
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Task No. 667205
Scientific Report No. 1

DECEMBER 1964

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

This study was made and this report was written while Dr. Smith was a Fellow, at McGill University, of the National Science Foundation. Collaboration of the Stormy Weather Group and the publication of this report were supported by this contract.

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INTERPRETATION OF THE FLUCTUATING ECHO FROM RANDOMLY DISTRIBUTED SCATTERERS

PART 3

BY

Paul L. Smith, Jr.

ABSTRACT

That the radar echo from a weather target comprising a random array of scatterers exhibits pulse-to-pulse fluctuations is a well-established fact. The interpretation of the fluctuating echo is an important problem in radar meteorology. The problem is usually to estimate the long-term mean echo intensity by examining only a rather small number of echoes. This "observer's problem" is the principal subject of the present report.

The solution to the observer's problem is obtained as a probability distribution of the long-term mean echo intensity. This distribution becomes narrower and more sharply peaked as the number of independent echoes measured increases. The exact form of the distribution depends on the assumed a priori probability distribution; however, the dependence becomes negligible when the number of echoes is sufficiently large.

Averaging the echo intensities is the optimum method of processing the echoes; averaging intensity levels or amplitudes is less satisfactory. However, the loss of precision when intensity levels are averaged is small, and it may be offset by other advantages of the logarithmic scale. Measuring only the intensity level of the maximum echo gives better results than averaging when the number of echoes is small, and somewhat poorer results when the number is large.

INTERPRETATION OF THE FLUCTUATING ECHO

FROM RANDOMLY DISTRIBUTED SCATTERERS

PART 3

CONTENTS

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"PART 3"

This scientific report constitutes a continuation of the two MW reports on the interpretation of the fluctuating echo from randomly distributed scatterers. Part 1, by J. S. Marshall and W. Hitschfeld, and Part 2, by P. R. Wallace, of "Interpretation of the Fluctuating Echo from Randomly Distributed Scatterers" were published in 1951 as MW-4 and MW-6, respectively.

1. INTRODUCTION

The weather targets of interest in radar meteorology comprise enormous numbers of individual scatterers - cloud droplets, raindrops, snowflakes, or hailstones. Because the positions of these scatterers in the atmosphere are random, the mean intensity of the radar echo from a given point is just the sum of the intensities from the individual scatterers. Because of motions due to winds, turbulence, and the fall velocities of the hydrometeors, the scatterers move about in space and the radar echo fluctuates with time. The fluctuations normally appear as a pulse-to-pulse variation in the echo received from any point of the atmosphere. These fluctuations give weather echoes their familiar "incoherent" nature.

Interpreting the fluctuating echo from randomly distributed scatterers is a problem of great concern in radar meteorology. The equations relating meteorologically significant parameters such as rainfall rate to the radar observations are usually expressed in terms of the mean echo intensity. Therefore it is the mean intensity that one usually wishes to determine from the radar data. To determine the mean intensity exactly would require averaging an infinite number of independent echoes from the target. The time required for the hydrometeors to reshuffle themselves into a new and independent arrangement, and hence yield an independent echo, is typically several milliseconds. Hence the rate at which independent echoes can be obtained is limited. But the necessity of scanning in range, azimuth, and /or elevation severely limits the amount of time available for probing any one point of the atmosphere. The observer's problem is to derive a satisfactory estimate of the mean-of-many intensity from only a rather small number of independent echoes.

The general problem of interpreting the fluctuating echoes from randomly distributed scatterers was investigated some time ago at McGill University. This work was published in a pair of papers (Marshall and Hitschfeld, 1953; Wallace 1953) which were revisions of complementary scientific reports that had been given limited circulation (Marshall and Hitschfeld, 1951; Wallace 1951). These papers and reports will be referred to in the present report as MH and W, followed by a '51' or a '53' if the reference is specifically to the report or specifically to the paper. The analyses by Marshall and Hitschfeld and Wallace yielded many important and useful ideas concerning the interpretation of the fluctuating echoes. They also considered in some detail the requirements for independent echoes, and various methods of achieving independence. Parts of their analysis were hampered by the lack of a digital computer on which large-scale numerical simulations could be carried out. In addition, the physical implications of certain of their results are rather disturbing.

The purpose of the present analysis has been threefold: first, to re-examine the problem of interpreting the fluctuating echoes in an effort to obtain a better understanding of the problem and a more meaningful solution; second, to carry out a more extensive numerical simulation, on a digital computer, to obtain more comprehensive results than are available in the earlier papers; and third, to attempt to find better ways of processing radar echoes to derive information about the weather targets. Throughout this report, the observed echoes are assumed to be independent of one another; no consideration is given to the problem of achieving independence, or to the possibility of deriving useful information from partially-dependent echoes.

2. THE PROBABILITY DISTRIBUTIONS OF INDIVIDUAL ECHOES

The probability distributions for individual echoes from a random array of scatterers, derived by MH53, are given (in the notation of the present report) in Table 2-1. Some of these results were obtained earlier by Goldstein (1951).

For the echo amplitude, P(A)dA is the probability that the amplitude of the echo will fall between A and A + dA, and $\overline{A^2}$ is the mean-squared echo amplitude. Since the intensity of the echo is proportional to the square of the amplitude, $\overline{A^2}$ is proportional to the mean intensity.

For the echo intensity, P(I)dI is the probability that the intensity will fall between I and I + dI, and $\bar{I}$ is the mean intensity.

For the echo intensity level, P(L)dL, the probability that the intensity level L (= log I) will lie between L and L + dL, is

$$P(L)dL = \frac{m}{\bar{I}} \exp(mL - \frac{1}{\bar{I}} e^{mL})dL \qquad (2\text{-}3)$$

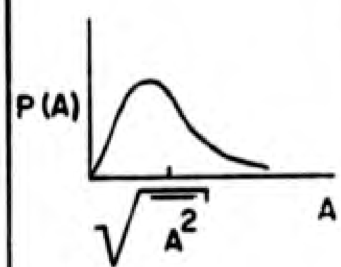
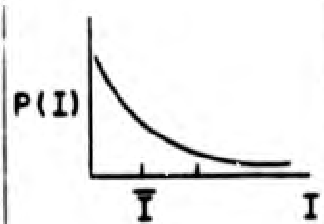
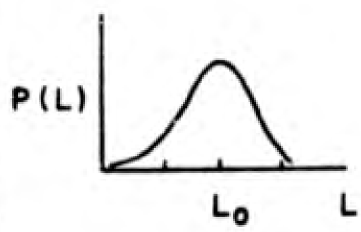
where m = ln 10 = 2.30258. If the target intensity level is defined as

$$L_o = \log \bar{I} \qquad (2\text{-}4)$$

then equation (2-3) takes on the form given in Table 2-1.

The above functions can be used to find the distribution of individual echoes from a weather target when the mean intensity $\bar{I}$ is known. But in practice the mean intensity is never known, so the relationships as given do not relate explicitly to the observer's problem of estimating $\bar{I}$. However, a knowledge of the above probability distributions is essential in the analysis of the observer's problem.

TABLE 2-1: PROPERTIES OF THE PROBABILITY DISTRIBUTIONS OF INDIVIDUAL ECHOES (after MH)

SIGNAL FUNCTION i	AMPLITUDE A	INTENSITY $I \propto A^2$	INTENSITY LEVEL $L = \log I$
PROBABILITY FUNCTION $P(i)$	(Equation 2-1) $\frac{2A}{\overline{A^2}} e^{-A^2/\overline{A^2}}$ 	(Equation 2-2) $\frac{1}{\bar{I}} e^{-I/\bar{I}}$ 	(Equation 2-5) $m \exp\left[m(L-L_o) - e^{m(L-L_o)}\right]$ $L_o = \log \bar{I}$ $m = \ln 10 = 2.3026$ 
MOST PROBABLE VALUE i_p	$\sqrt{\frac{\overline{A^2}}{2}}$	0	L_o
MEAN VALUE $\bar{i}$	$\frac{\sqrt{\pi \overline{A^2}}}{2}$ $= 0.886\sqrt{\overline{A^2}}$	$\bar{I}$	$\log(\bar{I} e^{-\gamma})$ $= \log(0.561\bar{I})$ $= L_o - 0.251$
STANDARD DEVIATION i	$\frac{1}{2}\sqrt{(4-\pi)\overline{A^2}}$ $= 0.463\sqrt{\overline{A^2}}$	$\bar{I}$	$\frac{\pi \log e}{\sqrt{6}} = 0.557$

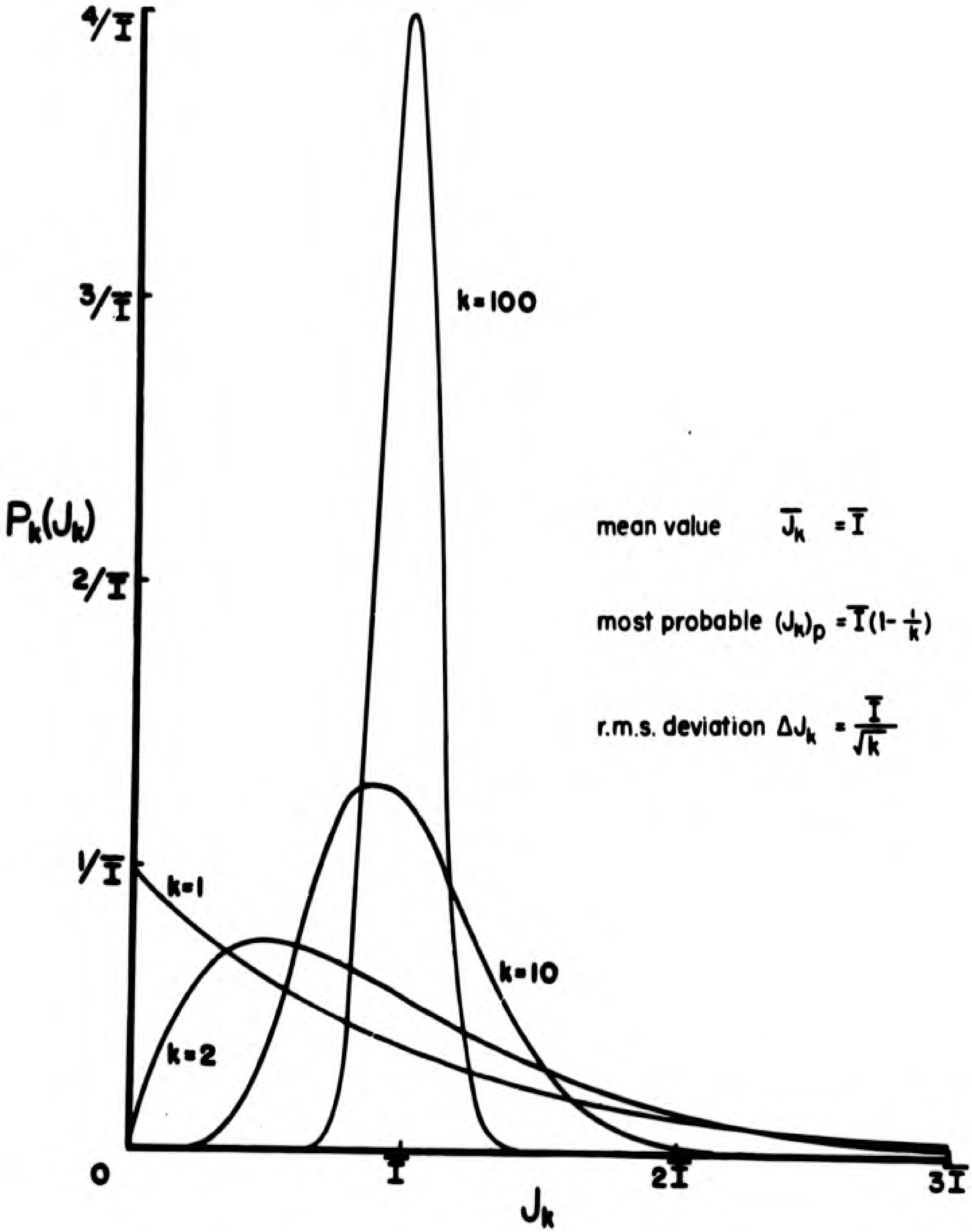


Fig. 3.1: The probability distribution $P(J_k)$ of the average J_k of k independent echo intensities, when the long-term mean echo intensity is $\bar{I}$ (after W51).

3. PROBABILITY DISTRIBUTIONS OF THE AVERAGE OF SEVERAL INDEPENDENT ECHOES

The widths of the various probability distributions of individual echoes suggest that the measurement of a single echo cannot lead to a very precise determination of the mean intensity. To obtain a reasonably precise estimate of $\bar{I}$, several independent echoes need to be measured. Since the quantity to be determined is the mean echo intensity $\bar{I}$, an obvious approach would be to average the echoes to obtain an estimate of $\bar{I}$. To aid in evaluating this approach it is useful to determine the probability distribution of the average value of several independent echoes.

The problem can be stated as follows: given a target comprising a random array of scatterers for which the mean echo intensity is known to be $\bar{I}$, determine the probability distribution of the average of k independent echoes from the target. It should be emphasized that this kind of probability distribution has limited usefulness in practical problems, because the true value of the mean intensity $\bar{I}$ is never known.

3.1 The probability distribution of average intensity

MH calculated the probability distribution for the average J_k of k independent echo intensity values, where

$$J_k = \frac{1}{k} \sum_{i=1}^{k} I_i \qquad (3\text{-}1)$$

They found the probability distribution to be

$$P(J_k)dJ_k = \frac{k^k}{(\bar{I})^k (k-1)!} J_k^{k-1} e^{-kJ_k/\bar{I}} dJ_k \qquad (3\text{-}2)$$

Curves for representative values of $P(J_k)$ are shown in Fig. 3.1 (after W51).

MH investigated the properties of the function $P(J_k)$ in some detail. They found that the most probable value of J_k, i.e. the mode of the distribution, is $\bar{I}(1-1/k)$. When $k = 1$, the most probable "average" value J_1 is zero; as k increases, the most probable value of J_k approaches the mean intensity $\bar{I}$. The standard deviation of $P(J_k)$, which is a measure of the width of the distribution, is $\bar{I}/\sqrt{k}$. Although the standard deviation decreases as k increases, the inverse square-root dependence suggests that the narrowing of the distribution with increasing k will be disappointingly slow; this has been borne out by experience. As k becomes large, the function $P(J_k)$ approaches the Gaussian form, as required by the Central Limit Theorem. This fact aids in extrapolating the results to very large k.

3.2 The probability distribution of average intensity level

Because of the wide dynamic range of echo intensities from weather targets, the intensities are often converted to a logarithmic scale by means of a logarithmic amplifier. The echoes are then expressed in terms of their intensity level L, where

$$L = \log I \tag{3-3}$$

It is therefore of value to determine the probability distribution $P_k(L_{av})$ for the average intensity level, defined by

$$L_{av} = \frac{1}{k} \sum_{i=1}^{k} L_i \tag{3-4}$$

The distribution of individual echo intensity levels given by equation (2-5) is so complicated that $P_k(L_{av})$ cannot be obtained in closed form.

MH51 undertook to determine $P_k(L_{av})$ by numerical methods. They generated, by a random-number process, simulated values of L, and then determined the distribution of the values of L_{av}. Because they had no digital computer available at the time, they used a sample of only 1000 independent values of L. Consequently, as k increased, the difficulties imposed by the limited size of their sample became increasingly troublesome.

In the present investigation, a similar numerical determination of $P_k(L_{av})$ has been carried out, using a digital computer to process a much larger sample. A description of the Monte Carlo technique used to simulate the individual echo intensity levels is given in Appendix A. It is possible to study a sample of enormous size with the aid of the computer. However, as the simulated echoes were generated, operations other than just averaging were carried out (specifically, the function $Q_k(L_o)$ discussed in Section 4.2 of this report was computed). The computation time required restricted the size of the sample somewhat; from 8,000 to 32,000 independent echoes were used in various instances. The results were tested for convergence by plotting frequency distributions of L_{av} at intermediate points in the computation. These tests indicated that only minor changes in the resulting distributions $P_k(L_{av})$ would occur if the sample size were further increased.

The results obtained from the computations comprise a group of frequency distributions of L_{av}. These distributions have been obtained for k = 2, 4, 8, 16, 32 and 64. The frequency distributions themselves are listed in Table B-1 of Appendix B; Fig. 3.2 shows the distributions in the form of histograms representing $P_k(L_{av})$.

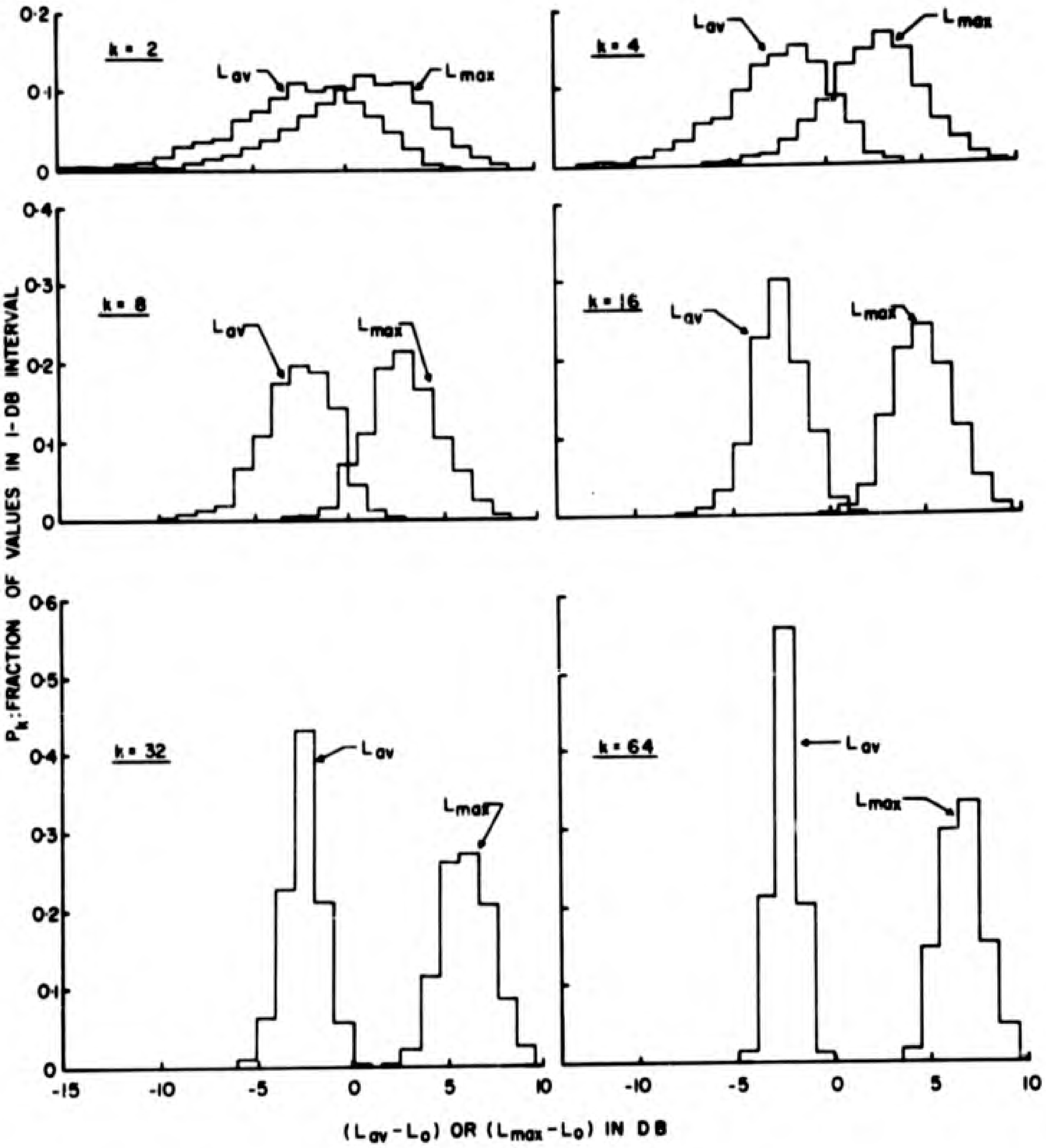


Fig. 3.2: Histograms representing the probability distributions $P_k(L_{av})$ and $P_k(L_{max})$ of the average L_{av} and the maximum L_{max}, respectively, of k independent echo intensity levels. The quantity L_o is related to the long-term mean echo intensity by $L_o = \log \bar{I}$.

Class intervals of 1 db were used for the computed values of L_{av} as well as for the simulated individual echoes. There is theoretical justification for decreasing the size of the class interval for L_{av} in proportion to $1/\sqrt{k}$, but this was not done in these computations.

Also shown in the same figure are histograms representing the distributions $P_k(L_{max})$ of the maximum echo within each group of k echoes. In other words, if the echoes are L_1, L_2, L_3 . . . L_k, then L_{max} is the largest of these echoes. The actual computed frequency distributions for L_{max} are listed in Table B-2 of Appendix B. These distributions are of interest in connection with a technique discussed in Section 4.2.7 of this report, in which the unknown target intensity level L_o is estimated from a measurement of the maximum echo L_{max} only.

The curves of Fig. 3.2 exhibit several interesting features. While the most probable value of a single echo is L_o, the most probable value of L_{av} shifts from L_o toward L_o- 2.5 db as k increases. This agrees with the result obtained by MH53 that the average of an infinite number of intensity levels is 2.5 db below L_o. Because of the coarse 1-db class intervals used in the computations, no meaningful functional dependence of the most probable value of L_{av} on k can be obtained from the results.

The most probable value of L_{max}, on the other hand, increases from L_o when $k = 1$ to approximately $L_o + 6.5$ db when $k = 64$. Again, because of the coarse class intervals, no attempt has been made to derive a functional relationship.

The standard deviation of the distribution $P_k(L_{av})$ shown in Fig. 3.3 decreases as $1/\sqrt{k}$. This behavior is in agreement with the theorem (Hoel, 1947) stating that the standard deviation of averages of k independent quantities should decrease as $1/\sqrt{k}$, regardless of the form of the distribution from which the k quantities are taken. The standard deviation of $P_k(L_{max})$ decreases at first more rapidly, and later more slowly, than $1/\sqrt{k}$.

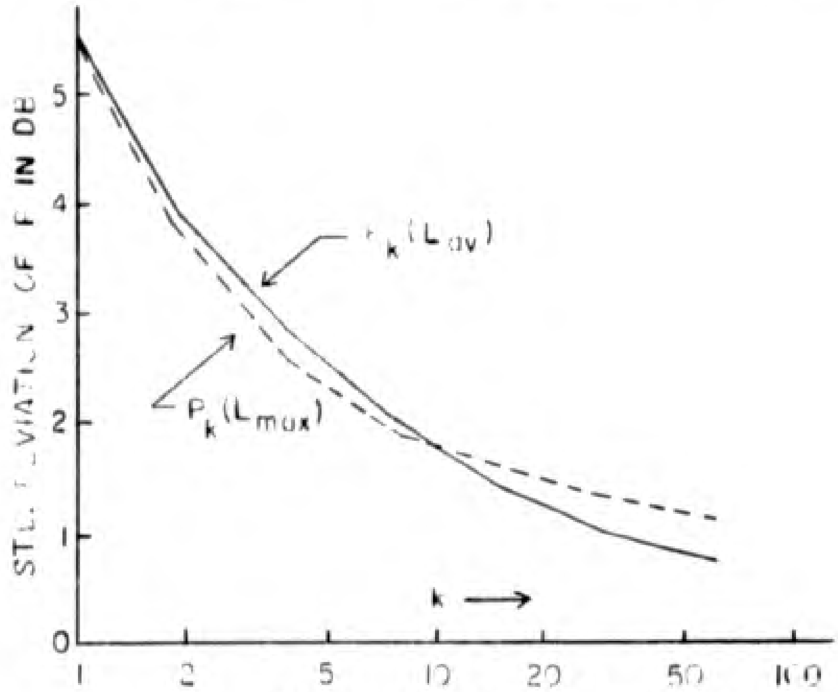


Fig. 3.3 Standard deviations of the distributions $P_k(L_{av})$ and $P_k(L_{max})$.

3.3 Comparison of the distributions of average intensity and average intensity level.

A comparison of the distribution of average intensity levels, $P_k(L_{av})$, with that of average intensities, $P(J_k)$, is of interest. To make such a comparison is somewhat difficult, because the logarithmic and linear scales are not easily compared. The method adopted by MH51 is to compare "confidence limits", representing boundaries between which some given percentage (say 50%) of the values of J_k or L_{av} would fall. These boundaries could be plotted on either linear or logarithmic scales of intensity, but since the accuracy of radar measurements is usually expressed in db, the logarithmic scale may be preferable.

The "confidence limits" for 50% and 95% of the values are used in Fig. 3.4 to compare the present results for $P_k(L_{av})$ with the values of $P(J_k)$ previously

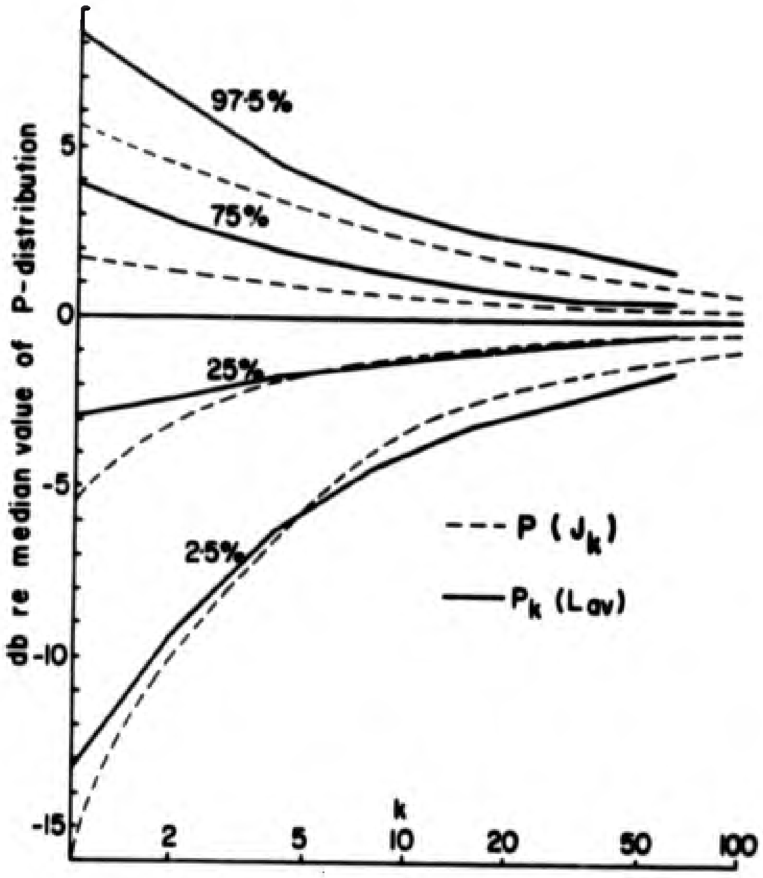


Fig. 3.4: Comparison of the distributions $P(J_k)$ and $P_k(L_{av})$ by means of confidence limits. The confidence limits are expressed in decibels with respect to the median value in the case of $P(J_k)$, and with respect to the average value in the case of $P_k(L_{av})$. For each distribution, 50% of the values fall between the 75% and 25% curves, and 95% fall between the 97.5% and 2.5% curves.

obtained by MH. The boundaries for $P_k(L_{av})$ were obtained from the frequency distributions in Table B-1 of Appendix B by linear interpolation within the 1-db class intervals. Because of the slope of the frequency distributions, the limits thus obtained are slightly broader than the true values. The average value of L_{av} (i.e. approximately -2.5 db) has been subtracted from each boundary before plotting, to simplify comparisons with $P(J_k)$.

It is evident in Fig. 3.4 that the spread of values of L_{av} is somewhat greater than the spread of values of J_k. Thus one may infer that averaging the echo intensities would lead to a more precise estimate of the mean intensity than averaging the intensity levels. However, this inference is qualitative at best; to make quantitative comparisons, an analysis of the "observer's problem" is required. Such an analysis is the subject of the next section of this report.

4. THE OBSERVER'S PROBLEM

The preceding section has been concerned with this problem: given a target with a known mean echo intensity $\bar{I}$, determine the probability distributions for various parameters of a group of k independent echoes from the target. In addition to the average value of the k observed signals, parameters of interest might include the maximum signal within the group, the median signal, and others. In any real situation, however, the mean echo intensity $\bar{I}$ is unknown. The problem facing the radar meteorologist is usually to estimate the value of $\bar{I}$ by analyzing a small number of independent echoes from the target. This problem can be termed the "observer's problem"; it is the central problem in the interpretation of the fluctuating echo from randomly distributed scatterers.

4.1 The observer's problem for echo intensities

The observer's problem was investigated by W (51,53) and MH53 for the case of echo intensities. Their solution is disturbing, however, because it implies that the measurement of a single echo intensity yields essentially no information about the mean echo intensity $\bar{I}$. The analysis to follow will demonstrate that this feature of their solution is a consequence of an unrealistic assumption made in their treatment.

The problem can be phrased as follows: given an observed set of independent echo intensities, say $I_1, I_2, I_3 \ldots I_k$, determine the mean echo intensity $\bar{I}$. The quantity $\bar{I}$ is the mean that would be obtained from an infinite number of independent echo intensity measurements, and not just the average J_k of the k observed intensities. Henceforth $\bar{I}$ will be referred to as the "target intensity". The essential problem is to estimate $\bar{I}$ from a finite number (k) of measurements.

Clearly, the observed set of echo intensities could have come from a target having $\bar{I}$ anywhere in the range from zero to infinity. However, the observed set of echoes is more likely to have come from some ranges of target intensity than others. The maximum information that can be derived from the observed set of k independent echoes will be the probability that this set of echoes came from targets having various values of $\bar{I}$. The description of the unknown target must be given in terms of a probability distribution for target intensity. The use of a single value, such as the most probable value of $\bar{I}$ or the mean value of $\bar{I}$, may be preferred, but these values are merely condensations of the information contained in the probability distribution.

4.1.1 The case k=2. - To calculate the distribution of target probabilities, consider first the case k = 2. Suppose that the two observed (and independent) values of echo intensity are I_1 and I_2. The probability of obtaining this particular pair of intensities if the mean echo intensity is known to be $\bar{I}$ is

$$\begin{aligned} p(I_1, I_2; \bar{I})dI_1 dI_2 d\bar{I} &= p(I_1;\bar{I})p(I_2;\bar{I})dI_1 dI_2 d\bar{I} \\ &= \frac{1}{\bar{I}} e^{-(I_1+I_2)/\bar{I}} dI_1 dI_2 d\bar{I} \end{aligned} \qquad (4\text{-}1)$$

Here $p(I_1;\bar{I})dI_1d\bar{I}$ denotes the probability of receiving an echo intensity in the range I_1 to $I_1 + dI_1$ when the target intensity is in the range $\bar{I}$ to $\bar{I} + d\bar{I}$, and so on. Equation (4-1) gives essentially the joint probability density for two independent echoes, the probability density for individual echoes being given by equation (2-2). Since the differential quantities dI_1, dI_2, $d\bar{I}$ make the equations rather cumbersome, they will be omitted from subsequent equations.

It should be understood that they must be reinserted in the equations to follow before the results can be interpreted as probabilities.

The relative probability that the two signals I_1, I_2 were actually received from a target having mean echo intensity $\bar{I}$ is

$$q_2(\bar{I}) = \mathcal{P}(\bar{I})\ p(I_1, I_2; \bar{I})$$

$$= \frac{\mathcal{P}(\bar{I})}{(\bar{I})^2}\ e^{-(I_1 + I_2)/\bar{I}} \qquad (4\text{-}2)$$

where $\mathcal{P}(\bar{I})$ is the <u>a priori</u> probability that the mean echo intensity of the unknown target is in fact $\bar{I}$. To obtain from equation (4-2) a numerical probability value, the result must be normalized by dividing by $\int_0^\infty q_2(\bar{I})d\bar{I}$, yielding

$$Q_2(\bar{I}) = \frac{\mathcal{P}(\bar{I})\ e^{-(I_1 + I_2)/\bar{I}}}{(\bar{I})^2 \int_0^\infty \frac{\mathcal{P}(\bar{I})}{(\bar{I})^2}\ e^{-(I_1 + I_2)/\bar{I}}\ d\bar{I}} \qquad (4\text{-}3)$$

$Q_2(\bar{I})$ then represents, apart from the differential factors, the probability that the unknown target intensity is $\bar{I}$. This quantity represents the maximum information about the unknown target that can be obtained from the two observed values of echo intensity.

The result contained in equation (4-3) has two important features. First, the observed intensity values appear only in the combination $(I_1 + I_2)$. The probability distribution of target intensity is thus a function only of the sum of the observed echo intensities, or equally well of the average echo intensity. The implication of this statement is that <u>averaging echo intensities entails no loss of information about the target</u>. If the individual echo intensities are added or averaged initially, one can still proceed to obtain the probability distribution given by equation (4-3). The second important feature is that the form of the probability distribution of target intensity depends on the function $\mathcal{P}(\bar{I})$. This function represents the a priori probability distribution of target intensities. In the absence of any knowledge concerning the frequency of occurrence of various values of $\bar{I}$, an arbitrary assumption must be made about $\mathcal{P}(\bar{I})$ to complete the evaluation of equation (4-3).

4.1.2 The distribution function $\mathcal{P}(\bar{I})$. - In the prior work of W53 and MH53, the a priori distribution function was implicitly assumed to be $\mathcal{P}(\bar{I})$ = constant. With this assumption, the evaluation of equation (4-3) is straightforward. The result is

$$Q_2(\bar{I})\Big|_{\mathcal{P}(\bar{I}) = \text{Constant}} = \frac{I_1 + I_2}{(\bar{I})^2} e^{-(I_1 + I_2)/\bar{I}} \tag{4-4}$$

This result should be compared with the result obtained by MH W. For the case k = 2, their solution to the observer's problem becomes

$$Q_2(\bar{I})\Big|_{\mathcal{P}(\bar{I}) = \text{Constant}} = \frac{2J_2}{(\bar{I})^2} e^{-2J_2/\bar{I}} \tag{4-5}$$

Since by definition $J_2 = (I_1 + I_2)/2$, equations (4-4) and (4-5) are identical. This substantiates the observation, made in the preceding section, that the probability distribution of target intensity will be the same whether the observed echo intensities are considered as individual entities or whether only their average value J_k is used.

The applicability of the results contained in equations (4-4) and (4-5) is limited, because of the assumption that $\mathcal{P}(\bar{I})$ = constant. This uniform distribution function is the simplest choice, but it is physically unrealistic. To demonstrate this, let the constant be C, so that $\mathcal{P}(\bar{I}) = C$. Then the a priori probability that the target intensity lies somewhere between zero and some arbitrary value $\bar{I}'$ is

$$\mathcal{P}(\text{Target intensity} \leq \bar{I}') = \frac{\int_0^{\bar{I}'} C d\bar{I}}{\int_0^{\infty} C d\bar{I}} = \frac{\bar{I}'}{\infty} = 0 \qquad (4\text{-}6)$$

for any finite value of $\bar{I}'$. But this means the probability that the target intensity is in any finite range, however large, is zero. Of course this condition is unacceptable, if for no other reason than that the echo power received by the radar cannot exceed the transmitted power, and hence must be finite in any case.

The implicit assumption that $\mathcal{P}(\bar{I})$ = constant led MH W to the conclusion that the measurement of a single echo intensity provides essentially no information about the target intensity. This conclusion is intuitively unsatisfactory, and the preceding paragraphs suggest that the unrealistic assumption $\mathcal{P}(\bar{I})$ = constant is the underlying cause of the unsatisfactory conclusion. To investigate this matter further, the analysis of the preceding sections will first be generalized to the case of an arbitrary number of observed echo intensity values.

4.1.3 The case of arbitrary k. - The preceding analysis can easily be extended to the case of an arbitrary value of k. For a group of k independent echo intensity observations, equation (4-1) must be replaced by

$$p(I_1, I_2 \ldots I_k; \bar{I}) = \frac{1}{(\bar{I})^k} e^{-\sum_{i=1}^{k} I_i / \bar{I}} \qquad (4\text{-}7)$$

As before, the relative probability that the observed group of signals came from a target having mean intensity $\bar{I}$ is

$$q_k(\bar{I}) = \frac{\rho(\bar{I})}{(\bar{I})^k} e^{-\sum_{i=1}^{k} I_i / \bar{I}} \qquad (4\text{-}8)$$

After normalization, the probability distribution of target intensity becomes

$$Q_k(\bar{I}) = \frac{\rho(\bar{I}) e^{-\sum_{i=1}^{k} I_i / \bar{I}}}{(\bar{I})^k \int_0^\infty \frac{\rho(\bar{I}) e^{-\sum I_i / \bar{I}}}{(\bar{I})^k} d\bar{I}} \qquad (4\text{-}9)$$

The observed echo intensities enter into $Q_k(\bar{I})$ only in the form $\sum_{i=1}^{k} I_i$. Thus for any k, the probability distribution for $\bar{I}$ depends essentially on the average echo intensity, and the intensities can be averaged without loss of information about the target.

When the a priori assumption that $\rho(\bar{I})$ = constant is introduced, the value of $Q_k(\bar{I})$ is found to be

$$Q_k(\bar{I}) \Big|_{\rho(\bar{I}) = \text{constant}} = \frac{\left[\sum I_i\right]^{k-1}}{(k-2)!} \frac{e^{-\sum I_i / \bar{I}}}{(\bar{I})^k} \qquad (4\text{-}10)$$

as long as $k \geq 2$. When k = 1, the normalization integral in the denominator of (4-9) fails to converge, so that $Q_1(\bar{I}) = 0$ for all finite values of $\bar{I}$. The result previously obtained by W53 for the distribution of target probabilities was

$$Q_k(I) \Big|_{\rho(I) = \text{constant}} = \frac{(kJ_k)^{k-1}}{(k-2)!} \frac{e^{-kJ_k / \bar{I}}}{(\bar{I})^k} \qquad (4\text{-}11)$$

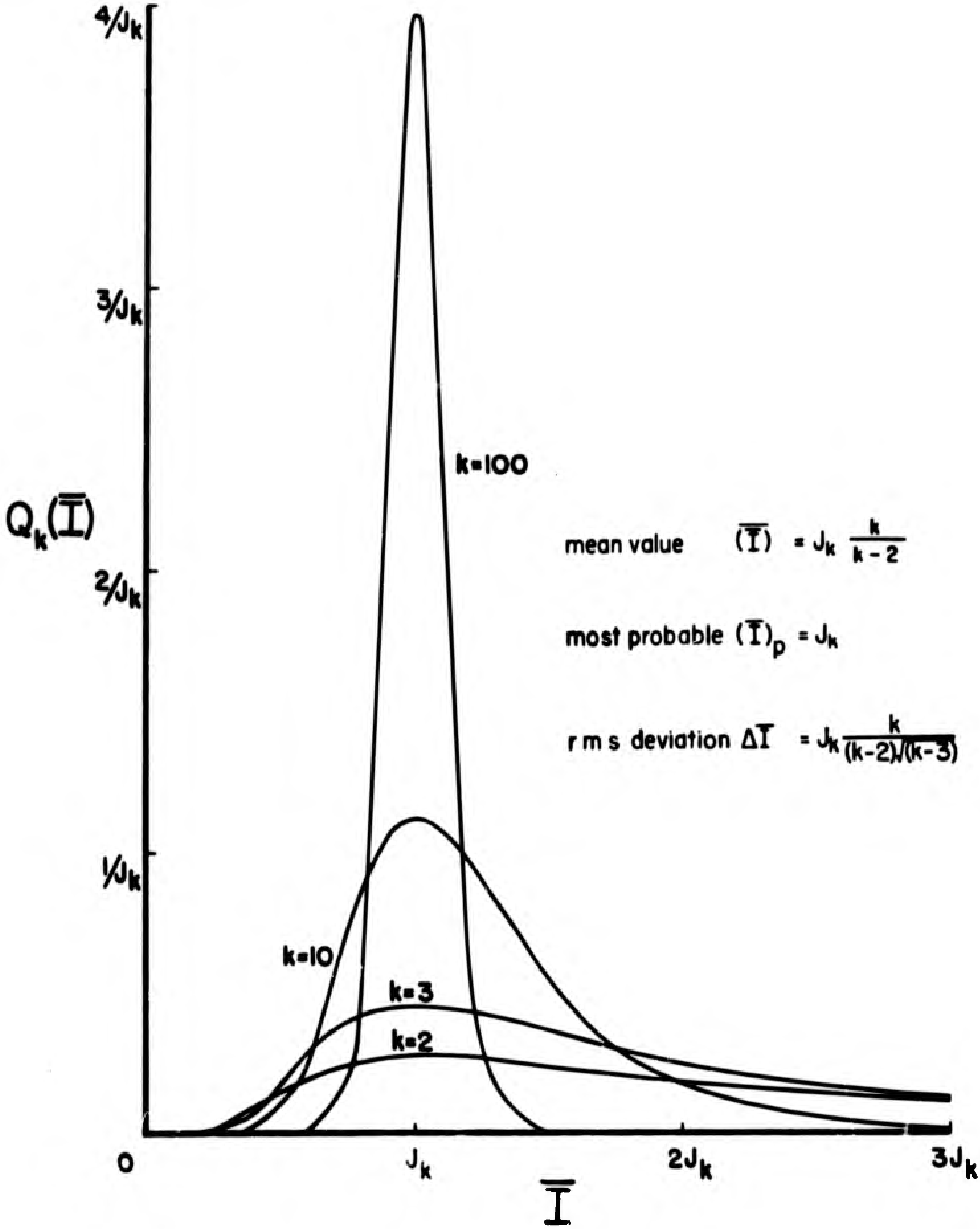


Fig. 4.1: The probability distribution $Q_k(\bar{I})$ of the target intensity $\bar{I}$, when the average of k measured echo intensities is J_k (after W51).

Since by definition $\sum I_i = k J_k$, equations (4-10) and (4-11) are identical. Representative values of the distribution function $Q_k(\bar{I})$ given by equation (4-11) are shown in Fig. 4.1 (after W51).

The failure of the normalization integral to converge when k = 1 led MH W to the conclusion that the measurement of a single echo intensity provides essentially no information about the target intensity. However, the failure of the integral to converge is a consequence of the assumed form of the a priori target distribution function $\mathcal{P}(\bar{I})$. Equation (4-6) shows that the assumption $\mathcal{P}(\bar{I})$ = constant is equivalent to assuming that the probability of having a finite target intensity $\bar{I}$ is vanishingly small. Thus it is not surprising that the receipt of a single echo of finite intensity fails to convince us that the target intensity is finite. As the number of observed echoes of finite intensity increases, the situation changes and when k exceeds 2 or 3 we finally recognize that the chance of $\bar{I}$ being infinite is very remote.

4.1.4 <u>The need for an alternative form for $\mathcal{P}(\bar{I})$</u>. - Evidently the results would be more satisfactory if a more realistic assumption about the a priori distribution $\mathcal{P}(\bar{I})$ were introduced. However, selecting a suitable assumption presents considerable difficulty. In the mathematical analysis, any function $\mathcal{P}(\bar{I})$ which decays (as $\bar{I} \rightarrow \infty$) at least as $1/(\bar{I})^a$ $(a > 0)$ will cause the normalization integral in (4-9) to converge when k = 1. This convergence would remove the anomaly that occurs for k = 1 when $\mathcal{P}(\bar{I})$ = constant. In addition to satisfying the mathematical requirements, it would be desirable to use for $\mathcal{P}(\bar{I})$ some function related to the actually observed frequency of occurrence of various target intensity values. Thus the best approach may be to study past records of echo intensity to establish a frequency-of-occurrence history from which a suitable form for $\mathcal{P}(\bar{I})$ can be derived.

The preceding analysis demonstrates the dependence of the form of the probability distribution of target intensity $Q_k(\bar{I})$ on the assumed form of $\mathcal{P}(\bar{I})$. However, it should be emphasized that the influence of $\mathcal{P}(\bar{I})$ on $Q_k(\bar{I})$ will be significant only for small values of k. Even when k = 4, the distribution $Q_4(\bar{I})$ is narrow enough that wide variations in the form of $\mathcal{P}(\bar{I})$ will have little effect. In physical terms, the assumption $\mathcal{P}(\bar{I})$ = constant means that all target intensities from zero to infinity are equally probable. In view of the 60-db dynamic range of precipitation echo intensities, one alternative assumption would be that all targets within a range of 10^6 are equally probable, with $\mathcal{P}(\bar{I}) = 0$ outside this range. In most cases the difference between these two assumptions will have negligible effect on $Q_k(\bar{I})$.

It should also be emphasized that introducing any other assumption than $\mathcal{P}(\bar{I})$ = constant will almost certainly reduce the width of the distribution $Q_k(\bar{I})$. Thus the standard deviations calculated by MH W for estimates of target intensity based on their values of $Q_k(\bar{I})$ are probably conservative. For a given value of k, the standard deviation will be somewhat less than their value $kJ_k/(k-2)\sqrt{k-3}$, but the difference will be negligible for values of k greater than about 4.

4.1.5 The assumptions $\mathcal{P}(\bar{I}) \propto 1/\bar{I}$ and $\mathcal{P}(\bar{I}) \propto 1/\sqrt{\bar{I}}$. - Two special forms of the function $\mathcal{P}(\bar{I})$ are of some interest, at least academically. The first is $\mathcal{P}(\bar{I}) \propto 1/\bar{I}$, which corresponds to assuming that all target intensity levels (intensities expressed on a logarithmic scale) are equally probable. This simple form of $\mathcal{P}(\bar{I})$ also permits analytical evaluation of equation (4-9), the result being

$$Q_k(\bar{I})\Big|_{\mathcal{P}(\bar{I}) \propto 1/\bar{I}} = \frac{(kJ_k)^k}{(k-1)!} \frac{e^{-kJ_k/\bar{I}}}{\bar{I}^{k+1}} \qquad (4\text{-}12)$$

This result is quite similar to that for $\mathcal{P}(\bar{I})$ = constant given in equation (4-11). Numerical values derived from (4-12) would be useful for comparison with values of $Q_k(L_o)$ obtained under the corresponding assumption $\mathcal{P}(L_o)$ = constant in Section 4.2 of this report.

It should be emphasized that the assumption $\mathcal{P}(\bar{I}) \propto 1/\bar{I}$ is no more realistic than the assumption $\mathcal{P}(\bar{I})$ = constant. When one attempts to find the constant of proportionality by normalizing the function $\mathcal{P}(\bar{I}) = C/\bar{I}$, the integral

$$\int_0^{\infty} \mathcal{P}(\bar{I})\, d\bar{I} = C \int_0^{\infty} \frac{1}{\bar{I}}\, d\bar{I} \qquad (4\text{-}13)$$

is found to diverge at both ends of the range. This behavior of $\mathcal{P}(\bar{I})$ can be interpreted to mean that there is some probability that the value of $\bar{I}$ is zero, some probability that it is infinite, and zero probability that it lies anywhere in between. This is hardly a more satisfactory assumption than was $\mathcal{P}(\bar{I})$ = constant. It does have the desirable property of causing the normalization integral in equation (4-9) to converge when k = 1, but its meaningfulness is still questionable.

The second special form of $\mathcal{P}(\bar{I})$ of interest is $\mathcal{P}(\bar{I}) \propto 1/(\bar{I})^{\frac{1}{2}}$. This form corresponds to the assumption that all target amplitudes are equally probable. In this case also, an analytical solution to equation (4-9) is possible. The result is

$$Q_k(\bar{I}) \Bigg|_{\mathcal{P}(\bar{I}) \propto 1/(\bar{I})^{\frac{1}{2}}} = \frac{(kJ_k)^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \frac{e^{-kJ_k/\bar{I}}}{\bar{I}^{k+\frac{1}{2}}} \qquad (4\text{-}14)$$

Again the result is similar to that given in equation (4-11). Also, it is again important to note that the assumption $\mathcal{P}(\bar{I}) \propto 1/\sqrt{\bar{I}}$ is unrealistic because the normalization integral diverges as $\bar{I} \to \infty$.

For reference purposes, Table 4-1 lists the various a priori assumptions that have been introduced in this report. Included in the table are the forms these assumptions take when echo amplitudes, intensities, or intensity levels are considered. In the table, L_o and A_o represent respectively the logarithm and the square root of the mean echo intensity $\bar{I}$.

Table 4-1: Various Forms of Several a priori Target Distribution Functions

Physical Statement of Assumption	Mathematical Form: Amplitudes	Mathematical Form: Intensities	Mathematical Form: Intensity Levels
All amplitudes equally probable	$\mathcal{P}(A_o) = \text{constant}$	$\mathcal{P}(\bar{I}) \propto 1/\sqrt{\bar{I}}$	$\mathcal{P}(L_o) \propto e^{mL_o/2}$
All intensities equally probable	$\mathcal{P}(A_o) \propto A_o$	$\mathcal{P}(\bar{I}) = \text{constant}$	$\mathcal{P}(L_o) \propto e^{mL_o}$
All intensity levels equally probable	$\mathcal{P}(A_o) \propto \frac{1}{A_o}$	$\mathcal{P}(\bar{I}) \propto \frac{1}{\bar{I}}$	$\mathcal{P}(L_o) = \text{constant}$

4.1.6 Numerical determination of $Q_k(\bar{I})$. – For the elementary forms of $\mathcal{P}(\bar{I})$ discussed above, and for a few others, analytical evaluation of equation (4-9) is possible. For other cases, particularly for distributions of $\mathcal{P}(\bar{I})$ that are obtained from empirical data, $Q_k(\bar{I})$ must be evaluated by numerical techniques. For a given value of k, the numerical procedure for determining $Q_k(\bar{I})$ is as follows. First the form to be assumed for the a priori probability distribution $\mathcal{P}(\bar{I})$ must be established. Then a hypothetical value of the observed average intensity J_k is selected. The relative probability $q_k(\bar{I})$ that the unknown target intensity is $\bar{I}$ can then be computed using equation (4-8).

In principle, the quantity $q_k(\bar{I})$ must be computed over the range of $\bar{I}$ from zero to infinity. However, the computations can be terminated when a value of $\bar{I}$ is reached such that contributions from higher values to the end results are negligible. The normalization integral in equation (4-9) is then computed by numerical integration of $q_k(\bar{I})$. Dividing the values of $q_k(\bar{I})$ by the value of the normalization integral yields the desired result $Q_k(\bar{I})$.

For the cases discussed in the preceding section, the form of $Q_k(\bar{I})$ will be the same, regardless of the value of J_k selected. Thus, using just one value of J_k, the results already obtained analytically could be obtained numerically. For other forms of $\mathcal{P}(\bar{I})$, however, the form of $Q_k(\bar{I})$ will depend on the value of J_k selected. Thus in general, for each value of k, a whole family of curves corresponding to various values of J_k would be needed to display $Q_k(\bar{I})$, instead of the single curve shown in Fig. 4.1.

No numerical computation of $Q_k(\bar{I})$ using the method outlined here has been made as yet. For such a computation to be meaningful, a meaningful form for the _a priori_ distribution function $\mathcal{P}(\bar{I})$ is needed. Further investigation is needed to determine a suitable form for $\mathcal{P}(\bar{I})$.

4.2 The observer's problem for echo intensity levels

Because of the wide dynamic range of echo intensities from weather targets, it is often easier to measure, instead of the actual echo intensity I, the intensity level L given by

$$L = \log I \qquad (4\text{-}15)$$

Logarithmic amplifiers are widely used for this purpose. The problem of determining the unknown target intensity $\bar{I}$ when the observed echoes are expressed on a logarithmic scale is thus of great practical importance.

For the case of intensity levels, the observer's problem can be expressed as follows: given a group of measured echo intensity levels $L_1, L_2, L_3 \ldots L_k$, determine the unknown target intensity $\bar{I}$. It is convenient to express the target intensity on a logarithmic scale as well, so that the objective becomes to estimate the "target intensity level"

$$L_o = \log \bar{I} \tag{4-16}$$

Here the symbol L_o is used in preference to $\bar{L}$, since L_o cannot be interpreted (as can $\bar{I}$) as the mean of an infinite number of observed echoes.

The analysis for intensity levels is quite similar to that for the case of intensities. The maximum information about the target that can be derived from the observed group of echoes will again be a probability distribution, this time of target intensity level, designated by $Q_k(L_o)$. For intensity levels, however, the complicated form of the distribution function for individual echoes given in equation (2-5) prevents the evaluation of $Q_k(L_o)$ in closed form. Therefore the use of numerical techniques will be required to obtain useful results.

4.2.1 <u>The case k = 2</u>. - As before, the simple case k = 2 will be considered first. The problem is to determine L_o, given two observed echo intensity levels L_1 and L_2. The probability of receiving this particular pair of echoes when the target intensity level is known to be L_o is

$$\begin{aligned} p(L_1,L_2;L_o) &= p(L_1;L_o)\, p(L_2;L_o) \\ &= \frac{m^2}{\bar{I}^2} \exp\left\{ m(L_1 + L_2) - \frac{e^{mL_1} + e^{mL_2}}{\bar{I}} \right\} \\ &= m^2 \exp\left\{ m(L_1 + L_2 - 2L_o) - e^{m(L_1 - L_o)} - e^{m(L_2 - L_o)} \right\} \end{aligned} \tag{4-17}$$

The relative probability that the observed pair of echoes has actually come from a target L_o is then

$$q_2(L_o) = \mathcal{P}(L_o)\, m^2 \exp\left\{ m(L_1 + L_2 - 2L_o) - e^{-mL_o}\left(e^{mL_1} + e^{mL_2}\right)\right\} \qquad (4\text{-}18)$$

where $\mathcal{P}(L_o)$ is the a priori probability that the unknown target intensity level is in fact L_o. To obtain a numerical probability value, equation (4-18) must be normalized by dividing by the quantity $\int_0^\infty q_2(L_o)\, dL_o$. The result is

$$Q_2(L_o) = \frac{\mathcal{P}(L_o) \exp\left\{ m(L_1 + L_2 - 2L_o) - e^{-mL_o}\left(e^{mL_1} + e^{mL_2}\right)\right\}}{\int_0^\infty \mathcal{P}(L_o) \exp\left\{ m(L_1 + L_2 - 2L_o) - e^{-mL_o}\left(e^{mL_1} + e^{mL_2}\right)\right\} dL_o} \qquad (4\text{-}19)$$

$Q_2(L_o)$ represents, apart from the omitted differential factors, the probability that the unknown target intensity level is L_o. Thus $Q_2(L_o)$ comprises the maximum information about the unknown target that can be extracted from the observed pair of echoes.

Aside from difficulties involving the function $\mathcal{P}(L_o)$, the normalization integral in equation (4-19) is already too complicated to be evaluated in closed form. Thus quantitative results for $Q_2(L_o)$ can only be obtained by numerical techniques.

In the case of intensities, the observed echoes entered into $Q_2(\bar{I})$ only in the combination $(I_1 + I_2)$. As a result, the observed echo intensity values can be averaged without loss of information about the target. In the present instance, however, the value of $Q_2(L_o)$ cannot be expressed in terms of the combination $(L_1 + L_2)$ alone. Consequently, averaging the observed echo intensity levels would involve discarding part of the information needed to determine $Q_k(L_o)$. Therefore any attempt to determine the unknown target intensity after averaging echo intensity levels will give poorer results than could be obtained after averaging the intensities themselves. Stated another way, averaging the intensity levels (logarithms of intensity) is equivalent to taking the geometric mean of the intensities. Since the arithmetic mean leads to the best possible determination of the target intensity, it is reasonable to expect that the geometric mean will be less satisfactory. This does not mean that equation (4-19) is inferior to equation (4-9) as an expression of the probability distribution. These two equations incorporate the same input information, and when applied correctly they will give identical results.

One important point is that the obvious assumption $\mathcal{P}(L_o)$ = constant is not equivalent to the assumption made previously that $\mathcal{P}(\bar{I})$ = constant. As indicated in Table 4-1, $\mathcal{P}(L_o)$ = constant corresponds to the form $\mathcal{P}(\bar{I}) \propto 1/\bar{I}$. Therefore values obtained for $Q_k(L_o)$ when assuming $\mathcal{P}(L_o)$ = constant cannot be compared with results obtained for $Q_k(\bar{I})$ while assuming $\mathcal{P}(\bar{I})$ = constant. Of course as k increases the influence of the a priori distribution function diminishes, so that comparisons for large values of k will be valid regardless of the assumed a priori distribution functions.

4.2.2 The case of arbitrary k. - The extension of the preceding analysis to other values of k is straightforward. For the probability of receiving a particular group of signals L_1, L_2, L_3 . . . L_k when the target intensity level is known to be L_o, equation (4-17) is replaced by

$$p(L_1, L_2 \ . \ . \ . \ L_k; L_o) = m^2 \exp\left\{ m\left[\sum_{i=1}^{k} L_i - kL_o\right] - e^{-mL_o}\sum_{i=1}^{k} e^{mL_i}\right\} \quad (4\text{-}20)$$

The relative probability that the unknown target is actually L_o is

$$q_k(L_o) = \wp(L_o)\, m^2 \exp\left\{ m\left[\sum_{i=1}^{k} L_i - kL_o\right] - e^{-mL_o} \sum_{i=1}^{k} e^{mL_i} \right\} \qquad (4\text{-}21)$$

After normalization, the probability that the unknown target intensity level is in fact L_o becomes

$$Q_k(L_o) = \frac{\wp(L_o) \exp\left\{ m\left[\sum_{i=1}^{k} L_i - kL_o\right] - e^{-mL_o} \sum_{i=1}^{k} e^{mL_i} \right\}}{\int_0^{\infty} \wp(L_o) \exp\left\{ m\left[\sum_{i=1}^{k} L_i - kL_o\right] - e^{-mL_o} \sum_{i=1}^{k} e^{mL_i} \right\} dL_o} \qquad (4\text{-}22)$$

As was noted for $k = 2$, the result cannot be expressed in terms of the sum (or of the average) of the observed intensity levels alone. Thus any attempt to determine the unknown target intensity level L_o by first averaging the observed echoes will be less accurate than the direct application of equation (4-22). However, because of the difficulty of using (4-22), in contrast with the simplicity of averaging, it is worth inquiring into the degree to which the averaging technique is inferior. To examine this question, it is necessary to turn to numerical techniques.

4.2.3 Procedure for numerical computation of $Q_k(L_o)$. - There are several possible ways of investigating the properties of the function $Q_k(L_o)$ numerically. The procedure used here consists of two essentially separate parts. The first part comprises the generation of simulated echo intensity levels by the computer, playing the role of the unknown weather target. For this part of the computation, the Monte Carlo routine described in Appendix A can be used. The second part then comprises the analysis of the simulated "observed" echoes by the computer, this time playing the role of the observer at the radar.

The computation proceeds as follows: First the target intensity level is assumed to be 0 db (representing the class interval $-0.5 \text{ db} \leq L_o \leq +0.5 \text{ db}$ in the present instance). Groups of k simulated independent echo intensity levels are then generated using the Monte Carlo routine.

In the second part of the computation, the simulated echo intensity levels are analyzed by the computer acting as the observer. In practice the observer does not know the actual target intensity level, so the value assumed in the first part of the program when generating the simulated echoes cannot be revealed during the second part. The observer's problem is to estimate L_o from the observed echoes alone. To do this most accurately, equation (4-22) must be used to compute the function $Q_k(L_o)$.

Direct evaluation of $Q_k(L_o)$ from (4-22) is extravagant of computation time, however, because the computation of the various exponentials is slow. It is preferable to employ the basic definition of the (unnormalized) probability distribution of target intensity levels:

$$q_k(L_o) = \rho(L_o)\, p(L_1;\, L_o)\, p(L_2;\, L_o) \ldots p(L_k;\, L_o) \qquad (4\text{-}23)$$

The probability distribution for individual intensity levels, given in equation (2-5), is a function only of $(L - L_o)$. Thus a table giving values of $p(L_i;\, L_o)$ for various values of $(L_i - L_o)$ can be entered in the computer memory.* For any particular group of echoes, it is then a simple matter to look up the necessary values of $p(L_i;\, L_o)$ and multiply them together as required by equation (4-23). Normalization to obtain $Q_k(L_o)$ can then be accomplished by numerical integration of $q_k(L_o)$.

* The table actually used for this purpose is quite similar to Table A-1 in Appendix A.

Now in general the probability distribution $Q_k(L_o)$ given by equation (4-22) will be different for each different group of "observed" echoes. However, a remarkable feature revealed by the computations is that when $\mathcal{P}(L_o)$ = constant (or more generally when the form of $\mathcal{P}(L_o)$ is exponential) the shape of the distribution $Q_k(L_o)$ is the same, regardless of the specific values of the "observed" echoes $L_1, L_2 \ldots L_k$. An analytical proof of this fact is given in the next section of this report. For the present, however, the inference is that under the assumption $\mathcal{P}(L_o)$ = constant (or an exponential) only one group of k echoes is needed to determine the general form of $Q_k(L_o)$. Thus extensive numerical computation is not really necessary, since the group of echoes could be taken as $L_1 = L_2 = \ldots = L_k$ and $Q_k(L_o)$ could be calculated manually. (In fact, this is exactly how the results shown in Figs. 4.3 and 4.4 were obtained). In defense of the extended computations performed, it should be pointed out that the similarity of $Q_k(L_o)$ for various groups of echoes only became apparent after examining the results of the computations.

For other forms of $\mathcal{P}(L_o)$, a more complete numerical simulation will be necessary. A large number of groups of echoes must be simulated when the target is assumed to be 0 db, and in addition the simulation must also be performed for other values of the target intensity level. Since the form of $Q_k(L_o)$ will in general be different for each simulated group of echoes, a wide range of values would have to be covered in the simulation. It is evident that display of all the results will be impossible; all that can be hoped for is to obtain representative values of $Q_k(L_o)$.

4.2.4 Results for k = 2. - The results of the computations are illustrated by the typical values of $Q_2(L_o)$, obtained with $\mathcal{P}(L_o)$ assumed constant, shown

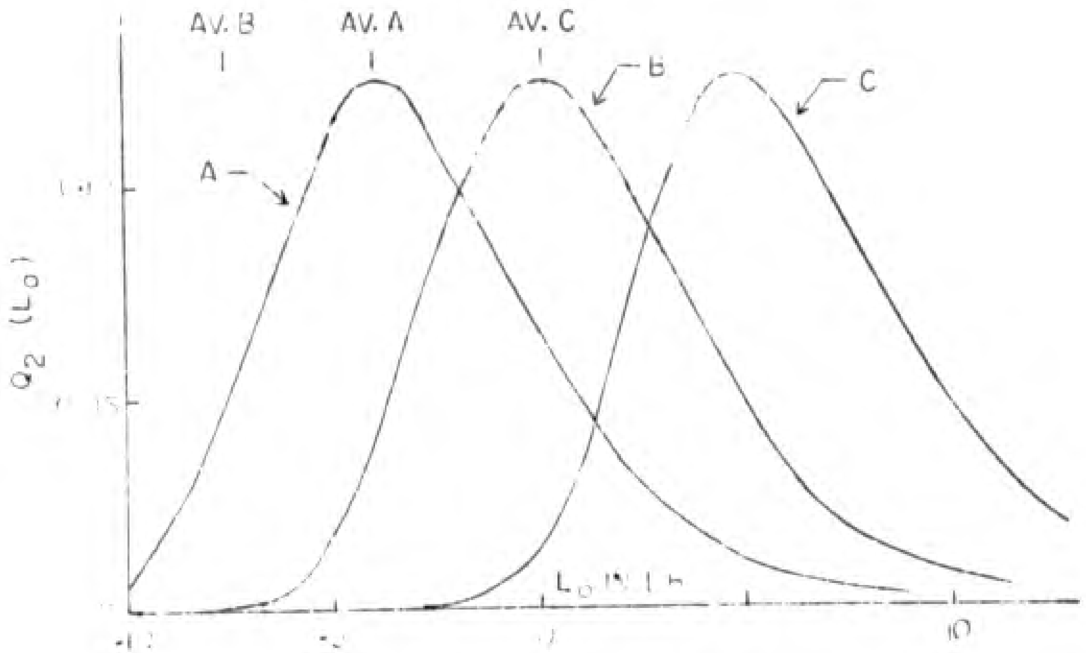


Fig. 4.2: Typical probability distributions $Q_2(L_o)$ of the target intensity level L_o when the intensity levels of two independent echoes are measured. The two echoes in each case are: Curve A, -4 db and -4 db (average -4 db); Curve B, -18 db and +3 db (average -7.5 db); Curve C, -8 db and +8 db (average 0 db).

in Fig. 4.2. Each pair of "observed"echoes was obtained with a simulated target intensity level of 0 db (this fact was of course unknown to the "observer"). For comparison, the average of the two observed intensity levels is also indicated in the figure for each case.

When the two observed echoes are equal, as in curve A, their average value gives a good estimate of the most probable target intensity level. When the two echoes are separated by several db, however, their average value is a less satisfactory estimate of the most probable value of L_o. In some cases such as curve C, the average value is nearer 0 db than is the mode of $Q_2(L_o)$, but this must not be interpreted to mean that averaging gives a better estimate of the target intensity level than does $Q_2(L_o)$ itself. For example, on any occasion when the observed echoes are -8 db and +8 db, the unknown target intensity level is much more likely to be +5 db than 0 db.

The curves of Fig. 4.2 substantiate the previous assertion that when $\wp(L_o)$ = constant is assumed, the form of $Q_2(L_o)$ is the same regardless of the specific values of the observed echoes. This fact is certainly not obvious from an inspection of equation (4-22). Therefore, that equation was investigated more thoroughly in an effort to obtain a general proof of the assertion. Such a proof was obtained when $\wp(L_o)$ = constant; a similar proof is possible when $\wp(L_o)$ takes an exponential form such as e^{nL_o}.

The point to be proved is essentially that $Q_2(L_o)$ for any one pair of echoes can be transformed into $Q_2(L_o')$ for any other pair of echoes by a simple translation along the L_o - axis. To prove this, consider first the case of two identical echoes $L_1 = L_2 = L$, for which

$$Q_2(L_o)\Big|_{\wp(L_o) = \text{constant}} = \frac{\exp\left\{2m(L - L_o) - 2e^{m(L - L_o)}\right\}}{(\text{N.I.})} \tag{4-24}$$

where (N.I) represents the normalization integral. Now consider any other pair of echoes L_1, L_2 for which

$$Q_2(L_o)\Big|_{\wp(L_o) = \text{constant}} = \frac{\exp\left\{m(L_1 + L_2 - 2L_o') - e^{-mL_o'}\left(e^{mL_1} + e^{mL_2}\right)\right\}}{(\text{N.I.})} \tag{4-25}$$

To prove the point, it suffices to show that (4-25) can be transformed into (4-24) by the translation $L_o' = L_o + x$. Such a translation can in fact be made, by taking

$$x\Big|_{k = 2} = \log\left[\cosh\frac{m(L_1 - L_2)}{2}\right] \tag{4-26}$$

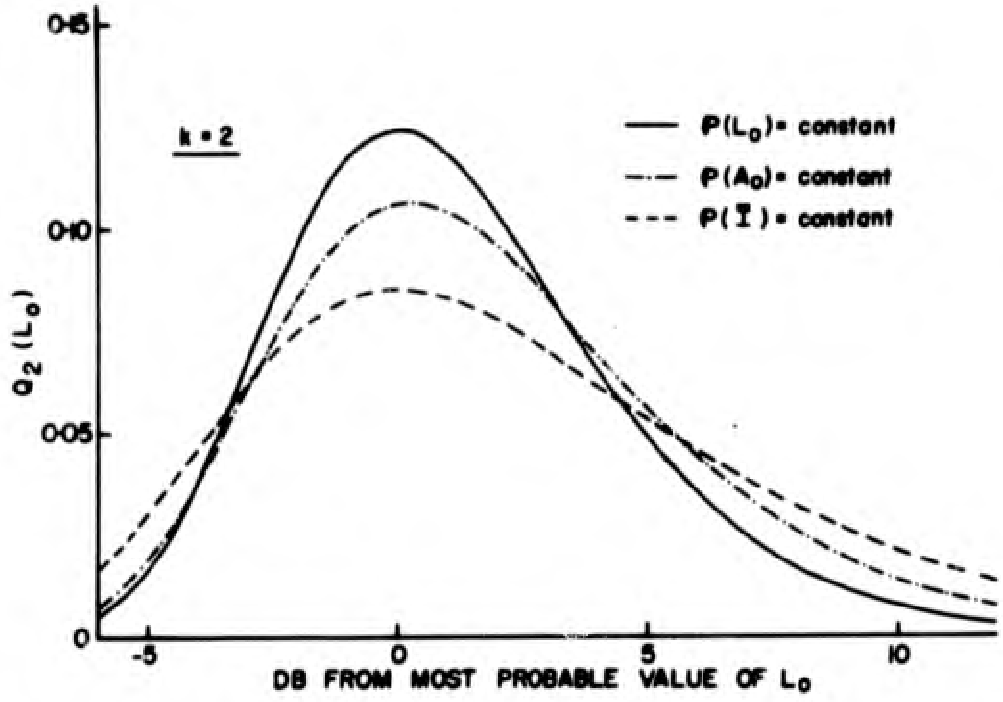


Fig. 4.3: The probability distribution $Q_2(L_o)$ of the target intensity level L_o, for different forms of the assumed a priori probability distribution $\mathcal{P}(L_o)$. Each curve has been plotted with the most probable value of L_o designated as 0 db.

The point just proven means that, as long as one is willing to assume $\mathcal{P}(L_o)$ = constant (or more generally $\mathcal{P}(L_o) \propto e^{nL_o}$), the form of the probability distribution $Q_2(L_o)$ will be independent of the actual values of the observed echoes. Since the location of the mode or of any other characteristic point of the distribution is not fixed in relation to the observed echoes or their average value, this knowledge might seem to be of limited value. However, the mode of the distribution can easily be positioned by adding x as given by equation (4-26) to the average value of the two signals.

Of course, the probability distribution of target intensity level does change when the assumed form of the a priori distribution $\mathcal{P}(L_o)$ changes. This behavior is illustrated in Fig. 4.3. This figure shows that the distribution $Q_2(L_o)$ is narrowest when the assumption $\mathcal{P}(L_o)$ = constant (i.e. all intensity levels equally probable) is made. The distribution is somewhat wider when $\mathcal{P}(A_o)$ = constant (all amplitudes equally probable) is assumed, and it is

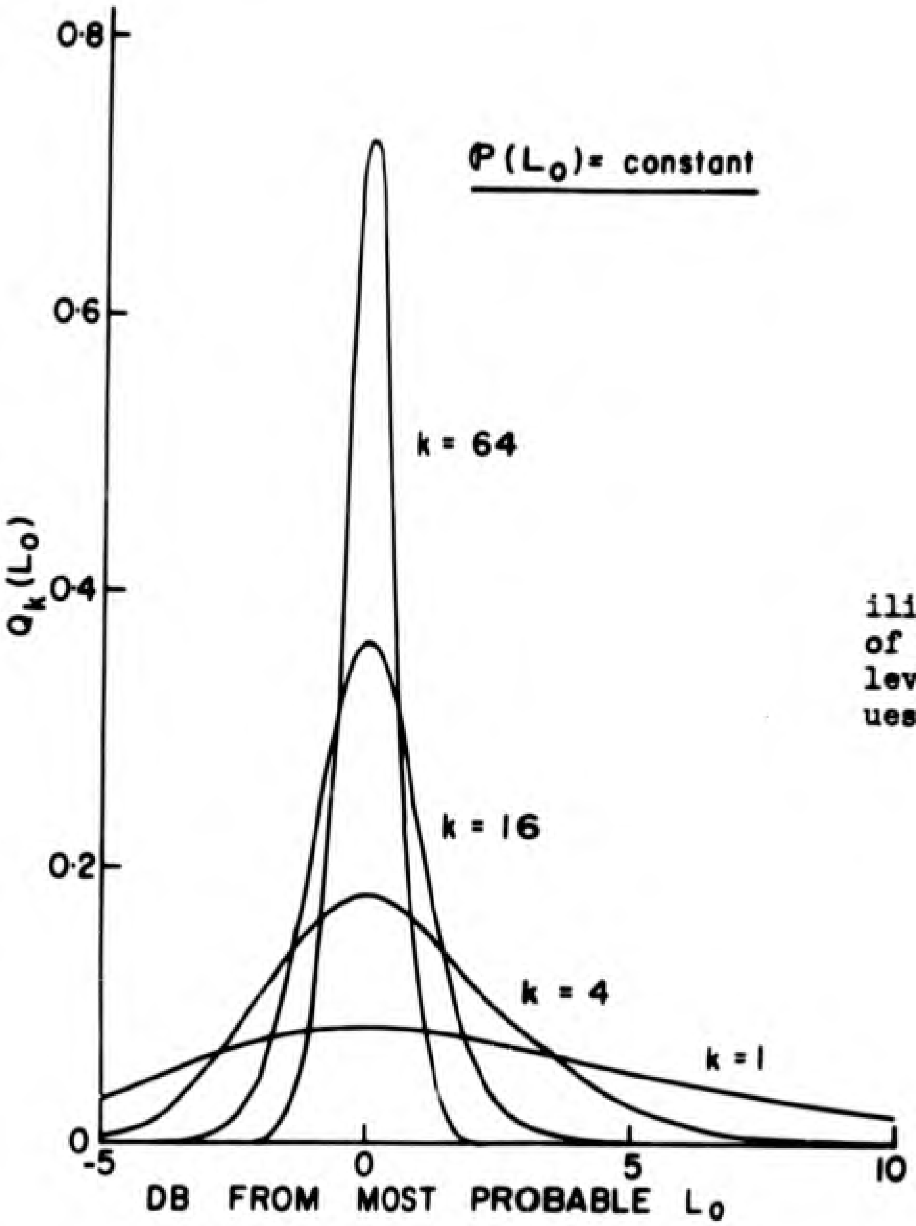


Fig. 4.4: The probability distribution $Q_k(L_o)$ of the target intensity level L_o, for different values of k with $\mathcal{P}(L_o)$ = const.

widest when $\mathcal{P}(\bar{I})$ = constant (all intensities equally probable) is assumed. These curves support the contention that assuming $\mathcal{P}(\bar{I})$ = constant probably leads to the broadest form for the target probability distribution.

4.2.5 Results for other values of k. - The results of the computations for other values of k are similar in many respects to those for k = 2. The function $Q_k(L_o)$ has been calculated for k = 1,2,4,8,16,32 and 64 under each of the three different assumptions listed in Table 4-1. Typical results obtained for $\mathcal{P}(L_o)$ = constant are shown in Fig. 4.4. The decreasing width of the curves as k increases implies increasing precision in the estimate of L_o.

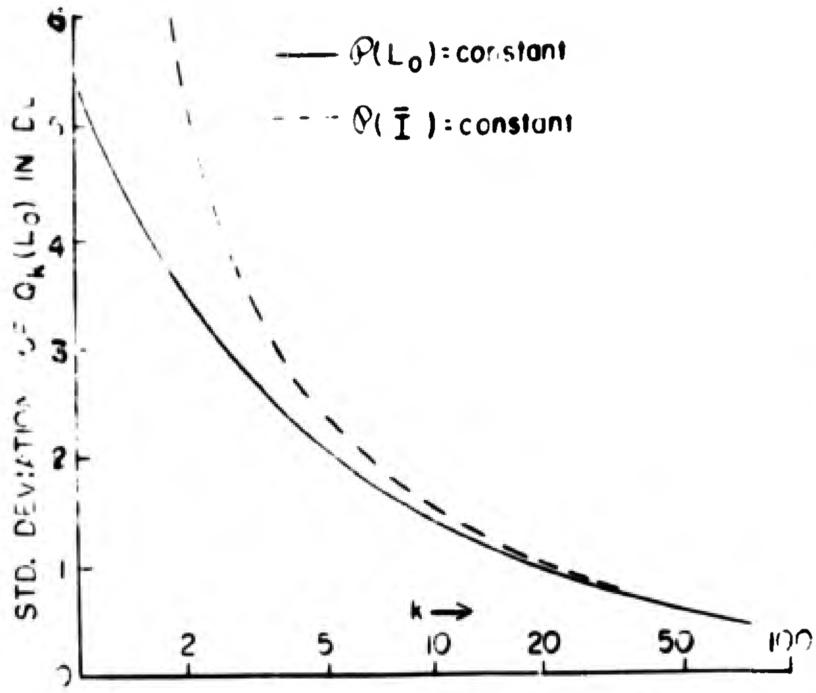


Fig. 4.5: Standard deviation of the probability distribution $Q_k(L_o)$, for the assumptions $\rho(L_o)$ = const. and $\rho(\bar{I})$ = const. When $\rho(L_o)$ = const., the std. dev'n follows a $1/\sqrt{k}$ law.

Standard deviations of the $Q_k(L_o)$ curves are shown as a function of k in Fig. 4.5; also shown are the standard deviations of $Q_k(L_o)$ under the assumption $\rho(\bar{I})$ = constant. It is interesting to note that under the assumption $\rho(\bar{I})$ = constant, the function $Q_k(L_o)$ is zero everywhere when k = 1. This agrees with the difficulties encountered by MH W when trying to determine $Q_1(\bar{I})$, while assuming $\rho(\bar{I})$ = constant.

As was true for k = 2, the shape of the curves of $Q_k(L_o)$ for other values of k are independent of the specific values of the observed echoes. That this is true can be proved as long as $\rho(L_o)$ = constant or $\rho(L_o) \propto e^{nL_o}$ is assumed. The proof follows the same argument that led to equation (4-26) when k = 2; for any value of k, the translation distance x turns out to be

$$x = \log \left[\sum_{i=1}^{k} e^{m(L_i - L_{av})} \right] - \log k \qquad (4\text{-}27)$$

where L_{av} is the average of the observed intensity levels.

As k increases, the diminishing effect of the assumed <u>a priori</u> distribution $\rho(L_o)$ on the resulting $Q_k(L_o)$ is evident in the results of these calculations.

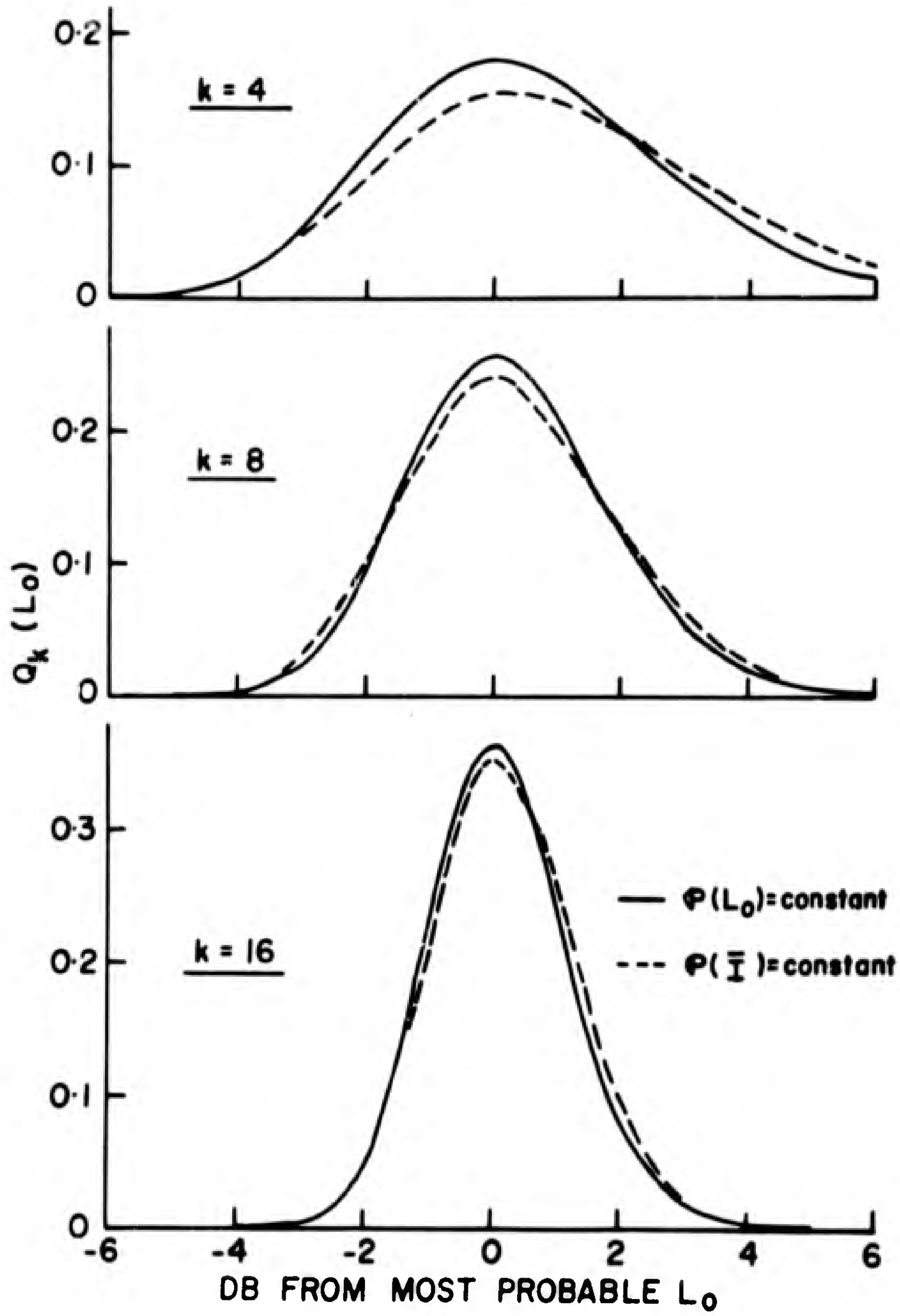


Fig. 4.6: The probability distribution $Q_k(L_o)$, of the target intensity level L_o, for various values of k. Curves are shown for both $\mathcal{P}(L_o)$ = const. and $\mathcal{P}(\bar{I})$ = const; the distributions for $\mathcal{P}(A_o)$ = const. would fall between those shown.

Typical curves of $Q_k(L_o)$ under each of the assumptions $\mathcal{P}(L_o)$ = constant and $\mathcal{P}(\bar{I})$ = constant are shown in Fig. 4.6; curves for $\mathcal{P}(A_o)$ = constant would fall in between the other two. As k increases, the curves for the different forms of $\mathcal{P}(L_o)$ become more nearly identical. When k = 8, the differences among the curves have become practically negligible.

4.2.6 <u>The observer's problem when the intensity levels are averaged</u>. - As an alternative to the complicated procedure of calculating $Q_k(L_o)$, estimating the target intensity level by averaging the observed intensity levels appears attractive. It has already been noted that in averaging the observed intensity levels part of the information needed to determine $Q_k(L_o)$ would be discarded. Thus estimates of L_o obtained after averaging will of necessity be inferior to estimates obtained directly from $Q_k(L_o)$. A problem of great practical importance is the degree to which the precision of the estimate suffers when averaging of the intensity levels is employed.

To determine the loss in precision, the observer's problem must first be solved in the following form: given an observed value L_{av} for the average of k independent echo intensity levels, determine the unknown target intensity level L_o. It is clear again in this case that the observed value L_{av} could have come from a wide range of possible target intensity levels. Therefore the solution to the problem must again be expressed as a probability distribution for target intensity level, which will be designated $Q_k^{av}(L_o)$. The form of this distribution may be anticipated to be similar to, but rather broader than $Q_k(L_o)$.

The determination of $Q_k^{av}(L_o)$ relies entirely on numerical techniques. The Monte Carlo routine described in Appendix A is used to generate large numbers of simulated independent echo intensity levels. For a given value of k, the procedure used to determine $Q_k^{av}(L_o)$ is as follows. With the target intensity level initially assumed to be 0 db (representing the class interval -0.5 db L_o +0.5 db) a group of k independent intensity levels $L_1, L_2 \ldots L_k$ is generated. For these simulated echoes the average intensity level

$$L_{av} = \frac{1}{k} \sum_{i=1}^{k} L_i \qquad (4\text{-}28)$$

is calculated, and the result classified in the proper class interval. In the computations, class intervals of 1 db, centered on half-integer values, were used. Thus if L_{av} = 2.20 db, the appropriate class would be 2.0 db $\leq L_o <$ 3.0 db, and the nominal value assigned would be 2.5 db.

The determination of L_{av} is then repeated for a large number N(0) of groups of k signals, to obtain a frequency distribution of the various values of L_{av}. The number of values of L_{av} generated must be proportional to the *a priori* probability that the target intensity level is L_o = 0 db. That is N(0) = a $\mathcal{P}(0)$, where

$$\mathcal{P}(0) = \int_{-0.5\ db}^{+0.5\ db} \mathcal{P}(L_o)\ dL_o \qquad (4\text{-}29)$$

and "a" is a constant of proportionality large enough to ensure that the total size of the sample used will be adequate.

This procedure yields a frequency distribution of L_{av} for the assumed value of L_o. This distribution should be identical in form to the probability distribution $P(L_{av})$ for average intensity levels from a known target intensity level. In fact, the frequency distribution required can be (and has been) obtained without additional computation by calculating $N(0)P(L_{av})$ from the values of $P(L_{av})$ given in Section 3.

To continue the analysis, a new target intensity level of 1 db (representing the class interval 0.5 db $\leq L_o <$ 1.5 db) is assumed, and a new frequency distribution for L_{av} obtained in the same way. This time, however, the total number of values of L_{av} needed is N(1 db), where

$$N(1\ db) = a\,\mathcal{P}(1\ db) = a\int_{0.5\ db}^{1.5\ db} \mathcal{P}(L_o)\ dL_o \qquad (4\text{-}30)$$

The necessary frequency distribution is obtained by taking N(1 db)$P(L_{av})$ when $L_o = 1$ db.

Now the probability distribution of individual echo intensity levels $p(L_i;L_o)$ is a function only of the difference $(L_i - L_o)$. The same will be true of the probability distribution of average intensity levels, $p(L_{av};L_o)$. Therefore the new frequency distribution of L_{av}, with $L_o = 1$ db, can be obtained directly from $P(L_{av})$ just as was done with $L_o = 0$ db.

This procedure must in principle be repeated for all values of $L_o \rightarrow \infty$, and for $L_o \rightarrow -\infty$ as well. However, it will normally be possible to terminate the calculations at some point where contributions from higher (or lower) values of L_o to the determination of $Q_k^{av}(L_o)$ can be neglected.

The procedure just described will yield a large array of numbers, which can be conveniently arranged in the matrix depicted in Table 4-2. The numbers shown in the matrix correspond to the case k = 4 with $\mathcal{P}(L_o)$ assumed constant. The complete matrix can be filled in column by column simply by multiplying the probability distribution $P(L_{av})$, given in Section 3 of this report, by the appropriate value of $N(L_o)$ and entering the results in the proper rows of the matrix. Each column of the matrix comprises the frequency distribution of the values of L_{av} obtained with the value of the target intensity level L_o given at the head of the column. Each row of the matrix is a frequency distribution showing the number of occasions on which, with the observed value of L_{av} being as indicated, the target intensity level was $L_o = \ldots -1,0,1,2,3$ db etc.

Table 4-2: Display of the results of the numerical analysis in the form of a matrix. Numbers in the matrix were obtained for $k = 4$ and $\rho(L_o)$ = constant. Any row of the matrix represents $Q_k^{av}(L_o)$ in the form of a frequency distribution.

Values of L_{av} (db) \ Values of L_o (db)	. . .	-2	-1	0	1	2	3	4	
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	
-1.5	. . .	176	261	303	279	258	192	121	. . .
-0.5	. . .	107	176	261	303	279	258	192	. . .
+0.5	. . .	28	107	176	261	303	279	258	. . .
1.5	. . .	14	28	107	176	261	303	279	. . .
2.5	. . .	1	14	28	107	176	261	303	. . .
3.5	. . .	0	1	14	28	107	176	261	. . .
4.5	. . .	0	0	1	14	28	107	176	. . .
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	
COLUMN TOTAL		$\overline{N(-2)}$	$\overline{N(-1)}$	$\overline{N(0)}$	$\overline{N(1)}$	$\overline{N(2)}$	$\overline{N(3)}$	$\overline{N(4)}$	

Each row of the matrix therefore represents a frequency distribution which, when normalized by dividing each number in the row by the row total, is just $Q_k^{av}(L_o)$. This rather involved procedure has thus generated a solution to the observer's problem of estimating the unknown target intensity level L_o, when an observed value of L_{av} is given.

When the matrix in Table 4-2 is examined, certain features are readily apparent. Let the row index be i (numbered from L_{av} = +0.5 db) and the column index be j (numbered from $L_o = 0$ db). Then the matrix element (i,j+r) is identical to the element (i-r,j). Consequently the ith row of the matrix can be obtained as the mirror image of the jth column about the element i = j. (This turns out to be true <u>only</u> when $\wp(L_o)$ = constant). In effect, any row of the matrix can be obtained from any column. Thus all the rows represent frequency distributions that are identical in form. The form of $Q_k^{av}(L_o)$ when $\wp(L_o)$ = const. is therefore independent of the specific value of L_{av}. This is consistent with the fact, previously noted, that the form of $Q_k(L_o)$ is independent of the specific values of the observed echoes when $\wp(L_o)$ = constant. Moreover, the distribution $Q_k^{av}(L_o)$ for the case $\wp(L_o)$ = constant is just a mirror image of the distribution $P(L_{av})$, given in Section 3 of this report, about the line $L_{av} = L_o$.

For other assumed forms of $\wp(L_o)$, the matrix becomes more complicated. In general, the ratio of the element (i,j+r) to the element (i-r,j) will be the ratio of N(j+r)/N(j), that is essentially the ratio of the <u>a priori</u> probabilities $\wp(L_o)$ for the two columns. If this ratio depends only on r (and not on j,) each row of the matrix will give a distribution $Q_k^{av}(L_o)$ of the same form. That is to say, the form of $Q_k^{av}(L_o)$ in such cases will be independent of the specific value of L_{av}. The ratio N(j+r)/N(j) depends on r alone only if $\wp(L_o)$ is exponential in form, i.e. if $\wp(L_o) \propto e^{nL_o}$. For other forms of $\wp(L_o)$, each row of the matrix will give a different form of the distribution of target probability $Q_k^{av}(L_o)$. $Q_k^{av}(L_o)$ will no longer be independent of the observed value of L_{av}. In such cases the results of this analysis cannot be expressed in any convenient, compact way.

The frequency distributions in the rows of the matrix can also be used to assign a specific estimated value to the target intensity level, for any observed average intensity level. In the example shown in Table 4-2, the most probable value of the target intensity level is 1.5 db greater than the observed average (and not 2.5 db greater, as might be supposed). That is, the most probable value of L_o is L_{av} + 1.5 db, for the conditions on which Table 4-2 is based. However, because the distributions $Q_k^{av}(L_o)$ are skewed,

Table 4-2: (Reproduced from page 41)

Values of L_o (db) / Values of L_{av} (db)	. . .	-2	-1	0	1	2	3	4	
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	
-1.5	. . .	176	261	303	279	258	192	121	. . .
-0.5	. . .	107	176	261	303	279	258	192	. . .
+0.5	. . .	28	107	176	261	303	279	258	. . .
1.5	. . .	14	28	107	176	261	303	279	. . .
2.5	. . .	1	14	28	107	176	261	303	. . .
3.5	. . .	0	1	14	28	107	176	261	. . .
4.5	. . .	0	0	1	14	28	107	176	. . .
		⋮	⋮	⋮	⋮	⋮	⋮	⋮	
COLUMN TOTAL		$\overline{N(-2)}$	$\overline{N(-1)}$	$\overline{N(0)}$	$\overline{N(1)}$	$\overline{N(2)}$	$\overline{N(3)}$	$\overline{N(4)}$	

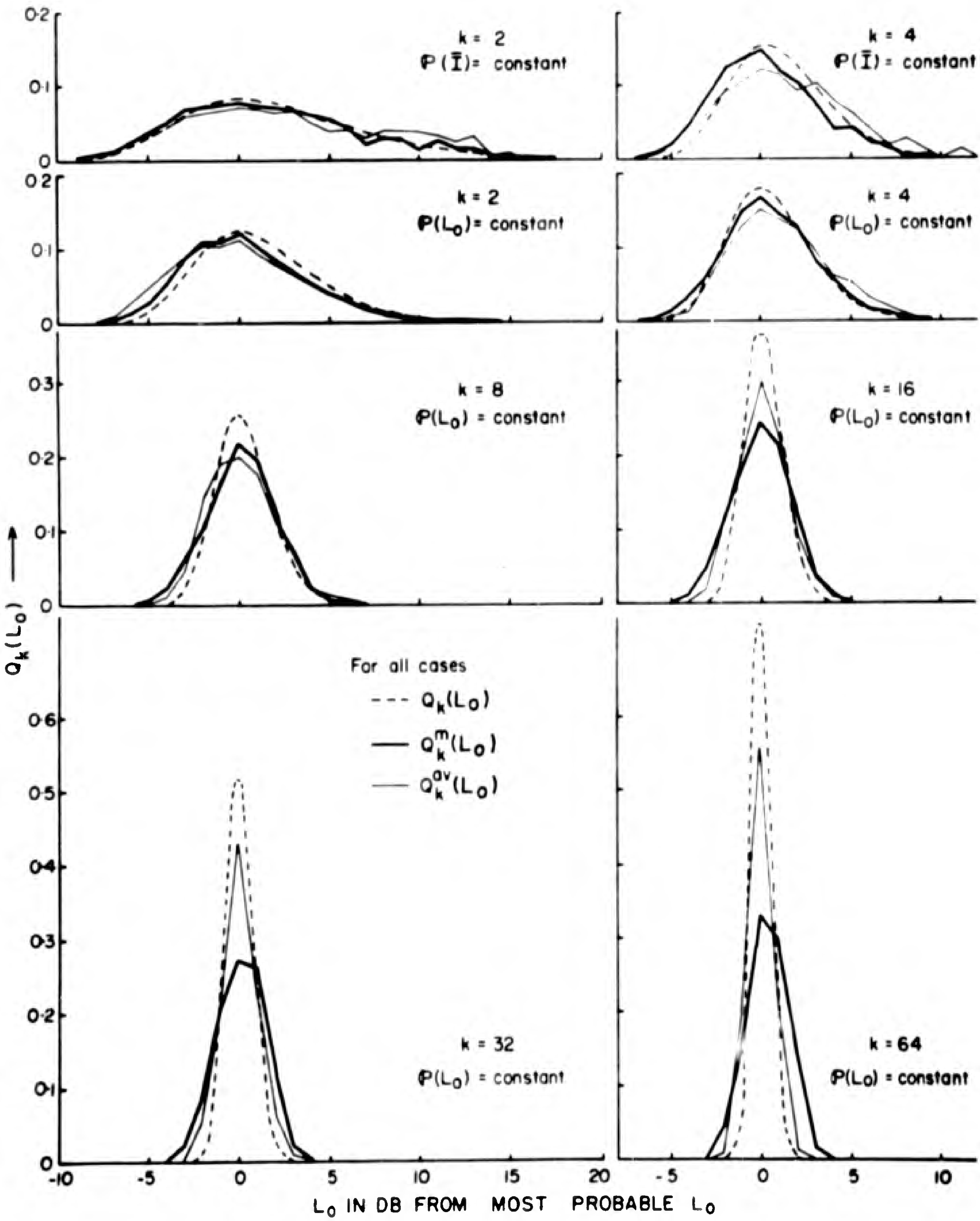


Fig. 4.7 Comparison of the probability distributions $Q_k(L_o)$, $Q_k^{av}(L_o)$ and $Q_k^m(L_o)$. For k = 2 and k = 4, curves are shown for both $\wp(L_o)$ = const. and $\wp(\bar{I})$ = const; for larger values of k, the differences between the curves would be insignificant.

the most probable value is not necessarily the best estimate of L_o. If the criterion of minimum mean-squared error is applied, then for an observed L_{av} the best estimate derivable from $Q_k^{av}(L_o)$ will be the mean value of L_o. When $\mathcal{P}(L_o)$ = constant is assumed, the mean value turns out to be $L_o = L_{av} + 2.5$ db for any k. It should be understood that the value 2.5 db is a consequence of the assumption that $\mathcal{P}(L_o)$ = constant, and will not apply in all other cases. Of course, when k is large the specific form of $\mathcal{P}(L_o)$ becomes unimportant; thus the best estimate of L_o approaches L_{av} + 2.5 db as k increases, whatever the form of $\mathcal{P}(L_o)$.

Typical probability distributions $Q_k^{av}(L_o)$ are illustrated in Fig. 4.7. For k = 2 and k = 4, curves corresponding to each of the assumptions $\mathcal{P}(L_o)$ = constant and $\mathcal{P}(\bar{I})$ = constant are shown. For the assumption $\mathcal{P}(A_o)$ = constant, the curves would be intermediate between the two cases illustrated. Fig. 4.7 also shows typical probability distributions $Q_k(L_o)$ and $Q_k^m(L_o)$, for purposes of comparison; the function $Q_k^m(L_o)$ is derived from the maximum observed echo, and is discussed in Section 4.2.7 of this report. For k = 1, of course, the curves $Q_1(L_o)$, $Q_1^{av}(L_o)$ and $Q_1^m(L_o)$ must be identical, since the observed echo, the average value, and the maximum value are one and the same.

Fig. 4.7 indicates that the average intensity level L_{av} gives a less precise estimate of the target intensity level than does $Q_k(L_o)$ itself. However, in no case is the difference very great; standard deviations for each of the distributions $Q_k(L_o)$, $Q_k^{av}(L_o)$ and $Q_k^m(L_o)$ are shown as a function

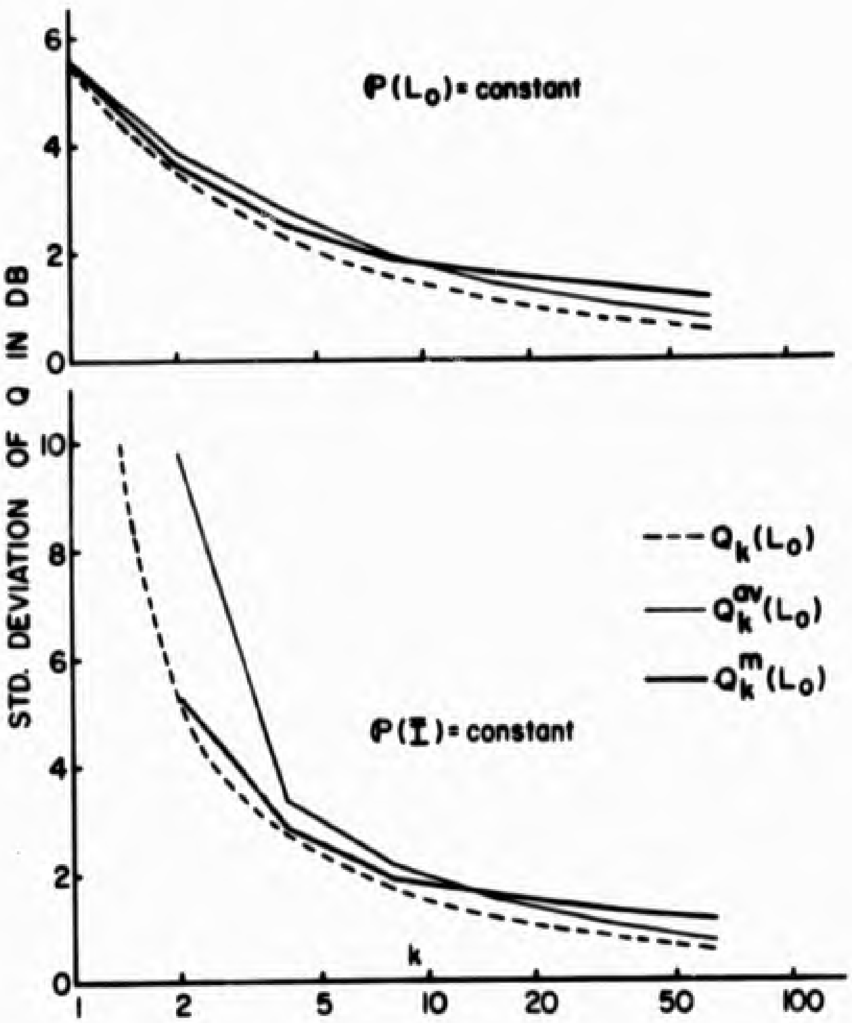


Fig. 4.8 Standard deviations of the distributions $Q_k(L_o)$, $Q_k^{av}(L_o)$ and $Q_k^{m}(L_o)$ shown in Fig. 4.7, corresponding to the assumptions $\mathcal{P}(L_o)$=const. and $\mathcal{P}(\bar{I})$=const. The two sets of curves are nearly identical for $k > 8$.

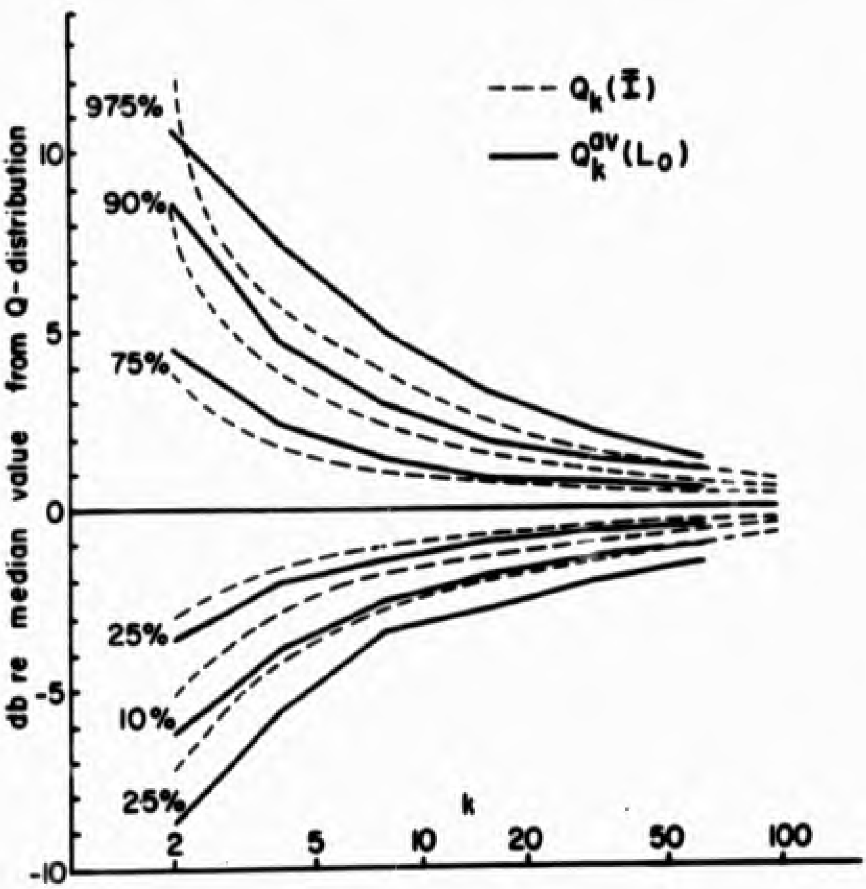


Fig. 4.9 Comparison of the probability distributions $Q_k(\bar{I})$ and $Q_k^{av}(L_o)$ by means of confidence limits. The values shown correspond to the a priori assumption $\mathcal{P}(\bar{I})$ = const. The confidence limits are expressed in db with respect to the median values of $\bar{I}$ and L_o respectively. For each case, 50% of the values would fall between the 75% and 25% curves, and 95% would fall between the 97.5% and 2.5% curves.

of k in Fig. 4.8. The loss in precision resulting from averaging the intensity levels is a modest penalty to pay for the great simplification achieved in processing the observations. Another interesting point revealed by Figs. 4.7 and 4.8 is that, at least up to $k = 8$, estimates of L_o derived by taking the maximum observed echo are as good or better than those derived from averaging the echoes. This fact may be of importance when methods of implementing the various techniques in operational equipment are considered.

The assumption $\mathcal{P}(\bar{I})$ = constant was used by MH W in their determination of $Q_k(\bar{I})$. Therefore a comparison, under this assumption, of the precision with which the target intensity level can be estimated after averaging the observed echo intensity levels with the precision of the estimate obtained when the echo intensities themselves are averaged is of interest. Such a comparison can most easily be made by comparing "confidence limits"* of the two distributions $Q_k(\bar{I})$ and $Q_k^{av}(L_o)$. The respective confidence limits are shown in Fig. 4.9 as a function of k; the confidence limits obtained by MH for $Q_k(\bar{I})$ have been expressed in decibels above or below the median value of $\bar{I}$, and the confidence limits for $Q_k^{av}(L_o)$ have also been taken about the median value.

The curves in Fig. 4.9 show that averaging the intensity levels leads to a less precise estimate of the target intensity level than that obtainable by averaging the intensities themselves. For $k = 1$, neither $Q_k(\bar{I})$ nor $Q_k^{av}(L_o)$ give a meaningful result under the assumption

* The term "confidence limits" is somewhat misleading here, since the degree of confidence in the estimate of L_o depends in large measure on the degree of confidence in the assumption $\mathcal{P}(\bar{I})$ = constant. As pointed out in Section 4.1.3, this confidence cannot be very great.

that $\rho(\bar{I})$ = constant. The superiority of $Q_k(\bar{I})$ is consistent with the previously noted fact that in averaging intensity levels part of the information needed to determine the distribution $Q_k(L_o)$ is discarded. The same general conclusions would be expected to hold for other assumed forms of $\rho(\bar{I})$, although confidence limits have not been determined for other cases.

4.2.7 <u>The observer's problem when maximum intensity levels are measured</u>. - In considering possible ways of processing the observed echoes to derive an estimate of the mean intensity $\bar{I}$, Professor J. S. Marshall suggested that measuring only the maximum echo might be a useful technique. The basis of this suggestion is easily understood by considering the hypothetical probability distribution P(L) for individual echoes, shown in Fig. 4.10. If k is large, there would be a high probability that one of the echoes would be very close to the sharp upper boundary of the distribution. Therefore, if X is subtracted from the measured maximum echo a good estimate of L_o should result.

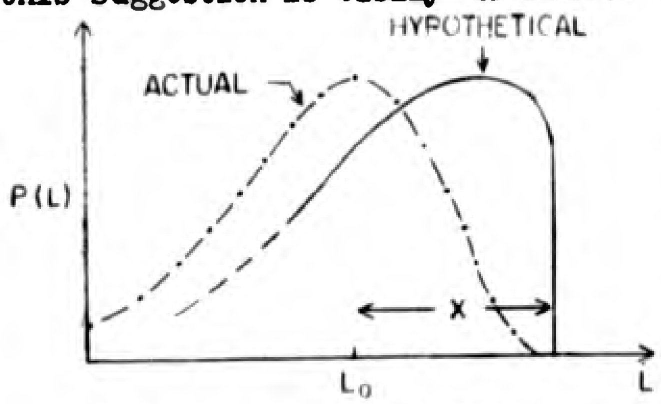


Fig. 4.10: A hypothetical probability distribution P(L) of echo intensity levels, having a sharp upper limit, and the actual distribution.

Of course the actual probability distribution P(L), also shown in Fig. 4.10, does not have such a sharp upper boundary, but the technique is worth considering. The frequency distributions obtained for L_{max} in Section 3.2 suggest that the idea may indeed be useful. For small values of k the distribution of L_{max} values is somewhat narrower than the distribution of L_{av} values. As k increases, however, the width of the L_{max} distribution decreases slowly so that, at large k, $P_k(L_{av})$ is narrower. It appears then that at least for small k, the technique suggested by Professor Marshall would be attractive. Practical considerations of ease of implementation may extend its usefulness to larger values of k as well.

The observer's problem when only the value of the largest of the k observed intensity levels is measured can be stated as follows: given the value L_{max} of the largest of k observed independent echo intensity levels, determine the unknown target intensity level L_o. As before, the solution to the problem must be expressed as a probability distribution which will be designated as $Q_k^m(L_o)$.

The procedure used to determine $Q_k^m(L_o)$ is almost identical to that used to determine $Q_k^{av}(L_o)$. The only difference is that for each group of k simulated echo intensity levels, the value determined is L_{max} instead of L_{av}. The sequence of operations leading to a matrix array similar to Table 4-2, from which $Q_k^m(L_o)$ is then obtained, is otherwise exactly the same as for $Q_k^{av}(L_o)$. Furthermore, the general features of $Q_k^{av}(L_o)$, such as its dependence on the assumed form of $\mathcal{G}(L_o)$, are also characteristic of $Q_k^m(L_o)$. When $\mathcal{G}(L_o)$ = constant is assumed, the mirror-image property can be used again to derive $Q_k^m(L_o)$ from the $P(L_{max})$ distributions given in Section 3.

The significant results with regard to $Q_k^m(L_o)$ have already been mentioned in connection with Fig. 4.7. Up to k = 8, the distribution $Q_k^m(L_o)$ gives estimates of L_o that are at least as precise as those given by $Q_k^{av}(L_o)$. When $k \geq 16$ the situation is reversed and the distribution $Q_k^{av}(L_o)$ is narrower than $Q_k^m(L_o)$. Since computations have not been carried out for values of k between 8 and 16, the exact "cross-over" value of k where $Q_k^{av}(L_o)$

begins to give more precise estimates of L_o is not known.

It is obvious that the best estimate of the target intensity level is not L_{max}, but rather some smaller value. For example, when $\vartheta(L_o)$ = constant is assumed, the most probable value of L_o is L_{max} -5 db, when k = 16. As in the case of $Q_k^{av}(L_o)$, the distributions $Q_k^m(L_o)$ are skewed and the most probable value is not necessarily the best estimate of L_o. The mean value of L_o minimizes the mean-squared error in the estimate, and may therefore be preferable. Accordingly, the mean values of L_o when $\vartheta(L_o)$ = constant is assumed are shown in Fig. 4.11.

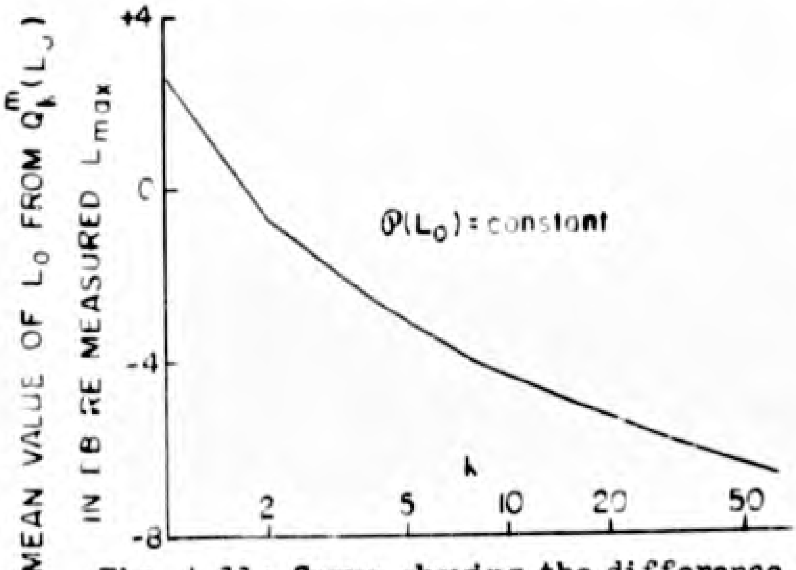


Fig. 4.11: Curve showing the difference between the observed value of L_{max} and the "best estimate" of the target intensity level L_o, as a function of k.

Returning to Fig. 4.7, the distributions $Q_k^m(L_o)$ are seen in every case to be broader than the corresponding distributions $Q_k(L_o)$ obtained by taking account of the values of each individual observed intensity level when estimating L_o. The distribution $Q_k(L_o)$ represents a fundamental limitation on the precision with which L_o can be estimated from the available information. No operations that may be performed on the observed echoes can lead to a narrower distribution indicating greater precision of the estimate of L_o. As the results presented in this report show, the general tendency of any operations made to simplify the observed data is to decrease the precision of the estimate.

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In deciding how to process observed intensity level values, then, a compromise has to be made between ease of processing and loss of precision.

4.3 The observer's problem for echo amplitudes

MH also considered the measurement of the echo amplitudes. For amplitudes, the observer's problem is as follows: given a set of observed echo amplitudes $A_1, A_2, A_3 \ldots A_k$, determine the unknown target intensity $\bar{I}$. It is convenient to represent the target intensity in terms of the "target amplitude" A_o where

$$A_o^2 = \overline{A^2} \propto \bar{I} \tag{4-31}$$

The constant of proportionality is just the ratio between the square of the amplitude and the intensity. The observer's problem is then to estimate the unknown target amplitude A_o.

The analysis of the observer's problem for echo amplitudes is similar to the analysis for intensities and intensity levels. When $k = 2$, for example, the probability of obtaining the echoes A_1, A_2, when the target amplitude is known to be A_o is, from equation (2-1),

$$p(A_1;A_o)p(A_2;A_o) = \frac{4A_1A_2}{A_o^4} e^{-(A_1^2 + A_2^2)/A_o^2} \tag{4-32}$$

The relative probability that the observed echoes were received from a target having target amplitude A_o is

$$q_2(A_o) = \mathcal{P}(A_o)p(A_1;A_o)p(A_2;A_o) \tag{4-33}$$

where $\mathcal{P}(A_o)$ is the **a priori** probability that the target amplitude will be A_o. The probability that the unknown target amplitude is in fact A_o is obtained by normalizing $q_2(A_o)$ to give

$$Q_2(A_o) = \frac{\mathcal{P}(A_o) e^{-(A_1^2 + A_2^2)/A_o^2}}{A_o^4 \int_0^\infty \frac{\mathcal{P}(A_o)}{A_o^4} e^{-(A_1^2 + A_2^2)/A_o^2} dA_o} \tag{4-34}$$

For any arbitrary value of k, the corresponding result is

$$Q_k(A_o) = \frac{\mathcal{P}(A_o) e^{-\sum_{i=1}^{k} A_i^2/A_o^2}}{A_o^{2k} \int_0^\infty \frac{\mathcal{P}(A_o) e^{-\sum A_i^2/A_o^2}}{A_o^{2k}} dA_o} \tag{4-35}$$

This result resembles the form of $Q_k(I)$ given in equation (4-9). An evaluation of equation (4-35) in closed form can be obtained for each of the forms of $\mathcal{P}(A_o)$ listed in Table 4-1. The respective results are:

$$Q_k(A_o)\Bigg|_{\mathcal{P}(A_o) = \text{constant}} = \frac{2\left[\sum_{i=1}^{k} A_i^2\right]^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2}) A_o^{2k}} e^{-\sum_{i=1}^{k} A_i^2/A_o^2} \tag{4-36}$$

$$Q_k(A_o)\Bigg|_{\mathcal{P}(A_o)\,\propto\,A_o} = \frac{2\left[\sum_{i=1}^{k} A_i^2\right]^{k-1}}{(k-2)!\,A_o^{2k-1}}\, e^{-\sum_{i=1}^{k} A_i^2/A_o^2} \tag{4-37}$$

$$Q_k(A_o)\Bigg|_{\mathcal{P}(A_o)\,\propto\,1/A_o} = \frac{2\left[\sum_{i=1}^{k} A_i^2\right]^{k}}{(k-1)!\,A_o^{2k+1}}\, e^{-\sum_{i=1}^{k} A_i^2/A_o^2} \tag{4-38}$$

The properties of these functions have not been investigated. Since the sum of the observed echo amplitudes, $\sum_{i=1}^{k} A_i$, does not appear explicitly in the equations, there is nothing to suggest that averaging the observed amplitudes would lead to a useful estimate of the target amplitude. However, the calculations made by MH51 indicated that averaging the amplitudes might lead to a better estimate of the target intensity than averaging the echo intensity levels. Of course, neither method will yield an estimate as good as that obtainable by averaging the echo intensities themselves. An investigation of the properties of the function $Q_k(A_o)$ and of the results obtainable by averaging echo amplitudes would be very useful.

5. CONCLUSIONS

The interpretation of the pulse-to-pulse fluctuations in the radar echo from a weather target is a problem of paramount importance in radar meteorology. The observer's problem is usually to estimate the long-term mean intensity of the echo from a target by examining only a finite (and usually rather small) number of echoes. This report has been concerned primarily with an investigation of this "observer's problem". The interpretation of the fluctuating echo was investigated by earlier workers, but several important additional findings have emerged from the present investigation.

The most important conclusions of the present report are as follows:

1) The solution to the observer's problem is basically a probability distribution of the long-term mean echo intensity (called the "target intensity"). A single value, such as the most probable target intensity or the median target intensity, may be easier to use, but such a value is merely a condensation of the information contained in the probability distribution.

2) If the intensities of k independent echoes from the target are measured, the probability distribution of the target intensity is a function of the average of the observed echo intensities. Consequently, averaging the echo intensities entails no loss of information about the target. Averaging the intensities is thus an optimum method of processing the echoes.

3) The probability distribution of the target intensity depends on the *a priori* probability distribution of target intensity assumed before any echoes are received. This means that the observer's problem has no unique solution; the solution depends on the initial *a priori* assumption. The conclusion reached by earlier workers that a single echo provides essentially no information about the target is a consequence of the particular

a priori assumption that all target intensities are equally probable. In general, other a priori assumptions do not lead to the same conclusion.

4) When the number of independent echoes is sufficiently large, the influence of the a priori assumption on the resulting probability distribution of target intensity is small. The results of the present analysis suggest that k = 8 is adequate to render the effect of the a priori assumption negligible.

5) If, instead of the echo intensities, the echo intensity levels (or amplitudes) are measured, the solution to the observer's problem is a probability distribution that does not depend on just the average intensity level (or average amplitude). Consequently, when intensity levels (or amplitudes) are averaged, some of the information needed to obtain the probability distribution is discarded. As a result, estimates of the target intensity derived from average echo intensity levels will be less precise than those derived from the average intensities; this fact is borne out by examples considered in the report. However, the loss in precision is rather small, and it may be a modest price to pay for the well-known advantages of the logarithmic scale of intensities.

6) As compared with averaging echo intensity levels, measuring only the maximum of k observed intensity levels yields an estimate of the target intensity that is more precise for small k ($k \leq 8$) and somewhat less precise for larger k. This fact may be of practical importance because of the relative ease of implementing means for measuring the maximum echo.

A number of points brought out in this report would merit further study. For example, the a priori distributions assumed in the report were chosen primarily for mathematical convenience. It would be useful to introduce a physically meaningful a priori distribution, based for example on past observations of rainfall rates. Another problem worthy of further investigation is the use of average echo amplitudes to estimate the target intensity. Earlier workers found that averaging amplitudes was somewhat better than averaging intensity levels, but this question has not been investigated in this report. The problem of threshold-crossing techniques also should be re-examined in the light of the findings of the present analysis. Other problems deserving further attention will no doubt suggest themselves to the readers.

APPENDIX A: THE MONTE CARLO TECHNIQUE USED TO SIMULATE ECHO INTENSITY LEVELS

Because of the complicated form of the probability distribution of echo intensity levels, as given by equation (2-5), numerical techniques are required to investigate the properties of groups of independent echoes. In these numerical investigations, a method of generating simulated independent radar echoes is required. The simulated echoes must conform to the probability distribution of equation (2-5), but they must appear in a random sequence. To generate simulated echoes meeting these requirements, the Monte Carlo technique described in this appendix was used.

It was first necessary to establish intensity level classes, and to calculate the probability that an echo would fall in each respective class. Choosing the class intervals, i.e. intervals within which all echoes will be assumed to have the same intensity level, is largely a matter of judgement. When wide classes are used, the number of samples needed in the numerical investigation will be small, but very coarse approximations of the desired frequency distributions will be obtained. Narrowing the classes to improve the detail in the frequency distributions increases the number of samples needed to smooth out the statistical fluctuations in the results. In the present investigation, the class intervals were chosen as 1 decibel.

The probability that an echo will fall in any particular class is just the integral of P(L), as given by equation (2-5), over the appropriate class interval. For example, the probability that the echo intensity level will be L' db, or more precisely that it will fall within the range $L' \pm 0.5$ db, is

$$P(L = L') = \int_{L' - 0.5\ \text{db}}^{L' + 0.5\ \text{db}} P(L)\, dL \qquad \text{(A-1)}$$

APPENDIX B: TABLE B-1 FREQUENCY DISTRIBUTION OF L_{av}

L_{av}	k = 2	k = 4	k = 8	k = 16	k = 32	k = 64
-21.5	0	-	-	-	-	-
-20.5	1	-	-	-	-	-
-19.5	1	-	-	-	-	-
-18.5	5	-	-	-	-	-
-17.5	6	-	-	-	-	-
-16.5	10	-	-	-	-	-
-15.5	15	0	-	-	-	-
-14.5	19	2	-	-	-	-
-13.5	24	0	-	-	-	-
-12.5	23	4	-	-	-	-
-11.5	40	9	-	-	-	-
-10.5	48	8	0	-	-	-
-9.5	76	21	2	-	-	-
-8.5	127	42	9	0	-	-
-7.5	155	65	14	3	-	-
-6.5	205	109	20	10	0	-
-5.5	267	121	68	34	10	0
-4.5	313	192	110	93	63	7
-3.5	379	258	177	229	227	107
-2.5	452	279	200	303	432	279
-1.5	415	303	192	197	211	101
-0.5	429	261	146	108	56	6
0.5	357	176	46	21	1	0
1.5	280	107	13	2	0	-
2.5	192	28	3	0	-	-
3.5	115	14	0	-	-	-
4.5	31	1	-	-	-	-
5.5	13	0	-	-	-	-
6.5	2	-	-	-	-	-
7.5	0	-	-	-	-	-
Total No. of Samples	4000	2000	1000	1000	1000	500
Av. Value of L_{av}	-2.81	-2.68	-2.63	-2.59	-2.56	-2.52
Standard Deviation (db)	3.91	2.81	1.94	1.41	1.00	0.77

TABLE B-2 FREQUENCY DISTRIBUTION OF L_{max}

L_{max}	k = 2	k = 4	k = 8	k = 16	k = 32	k = 64
-21	0	-	-	-	-	-
-20	1	-	-	-	-	-
-19	0	-	-	-	-	-
-18	0	-	-	-	-	-
-17	0	-	-	-	-	-
-16	1	-	-	-	-	-
-15	3	-	-	-	-	-
-14	4	-	-	-	-	-
-13	8	-	-	-	-	-
-12	6	-	-	-	-	-
-11	13	-	-	-	-	-
-10	18	-	-	-	-	-
-9	16	-	-	-	-	-
-8	40	0	-	-	-	-
-7	64	1	-	-	-	-
-6	88	6	-	-	-	-
-5	125	10	-	-	-	-
-4	165	23	-	-	-	-
-3	216	28	0	-	-	-
-2	286	65	4	-	-	-
-1	345	113	6	0	-	-
0	423	164	18	1	-	-
1	493	256	73	12	0	-
2	445	292	113	37	2	-
3	450	335	197	126	23	0
4	353	294	219	213	116	9
5	214	198	169	244	263	73
6	123	115	106	191	274	149
7	67	67	63	114	208	167
8	26	26	25	48	87	76
9	7	7	7	13	25	24
10	0	0	0	1	2	2
Total No. of Samples	4000	2000	1000	1000	1000	500
Av. Value of L_{max}	0.44	2.46	3.94	4.99	5.89	6.62
Standard Deviation (db)	3.68	2.56	1.90	1.61	1.33	1.13

APPENDIX C: FORTRAN SOURCE LIST

```
      DIMENSION LS(41,2),ANTIL(41),L(64),IFREQL(41),IFRLAV(41),IFRIAV(10
     11),P(41),AL(64),Q(81),SQ(81),QQ(81),IFRIAP(81),IFRIAM(81),IFRIAW(8
     21),NL(64),IFRMAL(41),IFRW(81)
      READ(5,201)(LS(I,1),I=1,41)
      READ(5,202)(ANTIL(I),I=1,41)
      READ(5,204)(P(I),I=1,41)
  201 FORMAT(20I4)
  202 FORMAT(10F8.0)
  204 FORMAT(16F5.4)
      DO1I=1,41
    1 LS(I,2)=I-31
      K=8
      X=0.1
      DO5 M=1,41
      IFREQL(M)=0
      IFRMAL(M)=0
    5 IFRLAV(M)=0
      DO31 M=1,101
   31 IFRIAV(M)=0
      DO23 M=1,81
      IFRIAM(M)=0
      IFRIAP(M)=0
   23 IFRIAW(M)=0
      DO37 LLL=1,5
      DO19 NNN=1,20
      DO18 MMM=1,10
      DO2 J=1,K
      CALL RANDR(X)
      IR=5000.+5000.*X
      DO3 M=1,41
      IF(IR.LE.LS(M,1)) GOTO4
    3 CONTINUE
    4 L(J)=LS(M,2)
      IFREQL(M)=IFREQL(M)+1
    2 AL(J)=ANTIL(M)
      AVL=0.
      AVI=0.
      DO6 J=1,K
      AVL=AVL+FLOAT(L(J))
    6 AVI=AVI+AL(J)
      AK=K
      V=AVL/AK
      LV=V+31.99
      IFRLAV(LV)=IFRLAV(LV)+1
      MV=L(1)
      DO32 J=1,K
      IF(L(J).GT.MV)MV=L(J)
   32 CONTINUE
      MV=MV+31
      IFRMAL(MV)=IFRMAL(MV)+1
      AVI=AVI/AK+0.05
      IV=10.*AVI+1.0
```

```
      IFRIAV(IV)=IFRIAV(IV)+1
      DO7 J=1,K
    7 NL(J)=L(J)+71
      S2Q=0.
      DO8 J=1,81
      Q(J)=1.
      M=J
      MM=M+40
      DO10 JJ=1,K
      IF(NL(JJ).LT.M.OR.NL(JJ).GT.MM) GOTO11
   10 CONTINUE
      DO9 N=M,MM
      DO9 JJ=1,K
      IF(NL(JJ).NE.N) GOTO9
      NJ=N-J+1
      Q(J)=Q(J)*P(NJ)
    9 CONTINUE
      GOTO12
   11 Q(J)=0.
   12 S2Q=S2Q+Q(J)*FLOAT(J-41)
      IF(J.NE.1) GOTO14
      SQ(1)=Q(1)
      GOTO8
   14 SQ(J)=SQ(J-1)+Q(J)
    8 CONTINUE
      QMAX=Q(1)
      DO15 J=1,81
      IF(Q(J).LE.QMAX) GOTO15
      QMAX=Q(J)
      IAP=J-41
   15 CONTINUE
      S1=SQ(81)/2.
      DO16 J=1,81
      IF(SQ(J).GE.S1) GOTO17
   16 CONTINUE
   17 IAM=J-41
      S=S2Q/SQ(81)
      JAW=S+41.49999
      JARW=S+41.99999
      JAP=IAP+41
      JAM=IAM+41
      IFRW(JARW)=IFRW(JARW)+1
      IFRIAW(JAW)=IFRIAW(JAW)+1
      IFRIAP(JAP)=IFRIAP(JAP)+1
      IFRIAM(JAM)=IFRIAM(JAM)+1
      MV=MV-31
      WRITE(6,302)
      WRITE(6,301)(L(J),J=1,K)
   18 WRITE(6,303)V,AVI,IAP,IAM,S,MV
  301 FORMAT(1HJ,16I6)
  302 FORMAT(1HK,11HVALUES OF L)
  303 FORMAT(1HJ,6HL(AV)=,F6.2,5X,7H I(AV)=,F5.2,5X,7H A(OP)=,I4,5X,7H A
     1(OM)=,I4,5X,7H A(OW)=,F6.2,5X,7H LMAX =,I4)
      DO20 J=1,81
```

```
   20 QQ(J)=Q(J)/SQ(81)
   19 WRITE(6,304)(QQ(J),J=1,81)
  304 FORMAT(1HK/(1HJ,9F12.5))
      WRITE(6,305)K
  305 FORMAT(3H1K=,I2)
      WRITE(6,306)
  306 FORMAT(43HKFREQUENCY DISTRIBUTION OF L(AV) AND L(MAX))
      WRITE(6,329)
  329 FORMAT(1HK,10X,1HL,10X,6HL(MAX))
      DO21 J=1,41
      LAV=J-31
   21 WRITE(6,307)LAV,IFRMAL(J)
  307 FORMAT(1HJ,10X,I4,10X,I4,10X,I4)
      WRITE(6,328)
  328 FORMAT(1H1,10X,1HL,10X,5HL(AV))
      DO43 J=1,41
      XJ=FLOAT(J)-31.5
   43 WRITE(6,299)XJ,IFRLAV(J)
  299 FORMAT(1HJ,10X,F7.1,10X,I4)
      WRITE(6,308)
  308 FORMAT(32H1FREQUENCY DISTRIBUTION OF I(AV)//1HK,10X,19HI(AV)
     1   FREQ)
      DO22J=1,101
      AV=FLOAT(J-1)*0.1
   22 WRITE(6,309)AV,IFRIAV(J)
  309 FORMAT(1HJ,9X,F5.2,10X,I4)
      WRITE(6,310)
  310 FORMAT(44H1FREQUENCY DISTRIBUTION OF A(OP),A(OM),A(OW)//1HK,10X,47
     1HAO           A(OP)            A(OM)            A(OW))
      DO30 J=1,81
      IA=J-41
   30 WRITE(6,311)IA,IFRIAP(J),IFRIAM(J),IFRIAW(J)
  311 FORMAT(1HJ,9X,I4,9X,I5,10X,I5,10X,I5)
      DO37 J=1.81
      XJ=FLOAT(J)-41.5
   37 WRITE(6,318)XJ,IFRW(J)
  318 FORMAT(1HJ,F7.1,I10)
      WRITE(6,312)K
  312 FORMAT(3H1K=,I2)
      WRITE(6,313)
      DO99 J=1,41
      LJ=J-31
   99 WRITE(6,314)LJ,IFREQL(J)
  314 FORMAT(1HJ,8X,I4,9X,I4)
  313 FORMAT(28HKFREQUENCY DISTRIBUTION OF L//1HJ,10X,1HL,10X,4HFREQ)
      CALL EXIT
      END
```

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MW-31: The quantitative display of radar weather patterns on a scale of grey, by T. H. Legg, June 1960.

MW-32: Weather-radar attenuation systems from raingauge statistics, by P. M. Hamilton and J. S. Marshall, January 1961.

For each 1-db class, values of P(L) were calculated (by referring to tables of the exponential function) at each end of the class interval and also at the center, i.e. at L' - 0.5 db, L', and L' + 0.5 db. The value of the integral in equation (A-1) was then approximated by Simpson's rule. The resulting probabilities are shown in the second column of Table A-1. As a check on the values, equation (2-5) was plotted on graph paper and the area within each class was determined by counting squares. The differences between values of P(L = L') determined in the two ways were all less than the error inherent in the square-counting technique.

Next a range of 4-digit numbers was assigned to each class, the range being in proportion to the probability that an echo will fall within the class. The 4-digit numbers assigned to the respective classes are listed in the third column of Table A-1. For example, the probability of the echo intensity level being L_o - 10 db is, according to Table A-1, 0.02089. Therefore, out of the 10,000 possible 4-digit numbers, 209 (from 0845 through 1053 inclusive) were assigned to the class representing intensity levels of L_o - 10 db.

To simulate an echo, the computer first generated a 4-digit "random number", using a standard routine for generating pseudo-random numbers. Then it looked up this 4-digit number in Table A-1 to determine the corresponding echo intensity level. Since the sequence of 4-digit numbers generated by the computer is random, the order of appearance of the various echo intensity level values will also be random. A sample sequence of echoes

TABLE A-1: PROBABILITY DISTRIBUTION OF ECHO INTENSITY LEVEL FOR 1-db CLASS INTERVALS* AND THE CORRESPONDING 4-DIGIT NUMBERS

ECHO INTENSITY LEVEL L RELATIVE TO L_o (db)	$P(L) \times 10^5$	4-DIGIT NUMBERS
-30	23	0000-0001
-29	29	0002-0004
-28	36	0005-0008
-27	46	0009-0013
-26	58	0014-0019
-25	73	0020-0026
-24	92	0027-0035
-23	115	0036-0047
-22	145	0048-0061
-21	182	0062-0079
-20	229	0080-0102
-19	287	0103-0131
-18	360	0132-0167
-17	452	0168-0212
-16	566	0213-0269
-15	708	0270-0340
-14	883	0341-0428
-13	1100	0429-0538
-12	1368	0539-0675
-11	1693	0676-0844
-10	2089	0845-1053
-9	2562	1054-1309
-8	3121	1310-1621
-7	3770	1622-1998
-6	4506	1999-2449
-5	5314	2450-2980
-4	6162	2981-3596
-3	6994	3597-4295
-2	7729	4296-5068
-1	8259	5069-5894
0	8460	5895-6740
+1	8217	6741-7562
+2	7467	7563-8309
+3	6239	8310-8933
+4	4693	8934-9402
+5	3095	9403-9711
+6	1731	9712-9884
+7	788	9885-9963
+8	277	9964-9991
+9	71	9992-9998
+10	14	9999
TOTAL:	100003	

* This table was also used in computing the function $q_k(L_o)$ discussed in Section 4.2. Actually, to avoid floating-point traps in the computation, it was necessary to multiply each value of P(L) by 40 before entering the table into the computer. Since only <u>relative</u> probabilities are needed in calculating $q_k(L_o)$, this caused no difficulties. A probability of zero was assigned to all values of L less than -30 db or greater than +10 db re L_o. Actually, about one echo in a thousand will be less than L_o-30 db; virtually none will be greater than L_o+10 db.

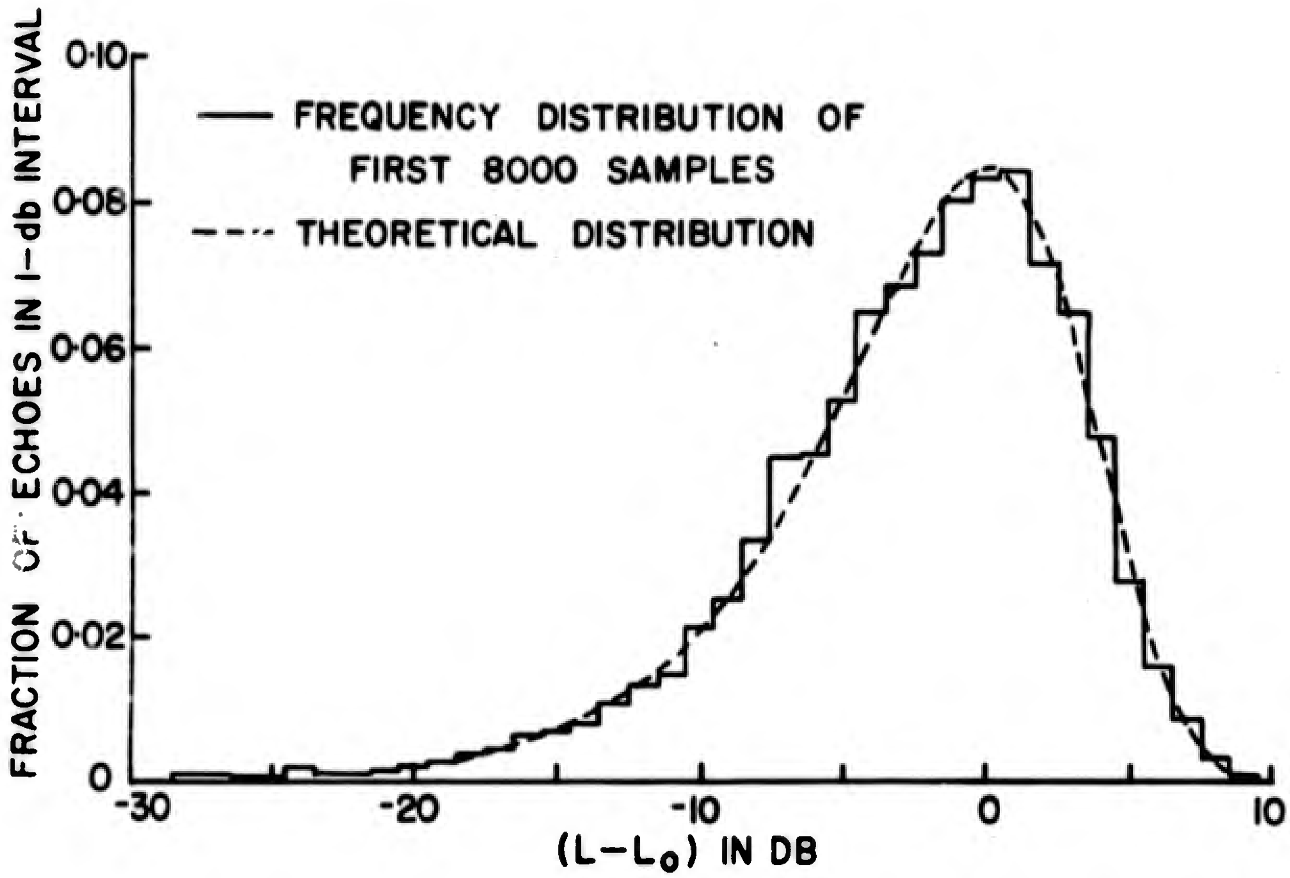

Fig. A-1: Histogram of the first 8000 simulated echoes generated by the Monte Carlo routine. The theoretical probability distribution is shown for comparison.

generated by this technique is given in Table A-2.

TABLE A-2: SAMPLE SEQUENCE OF ECHO INTENSITY LEVELS GENERATED BY THE MONTE CARLO TECHNIQUE.

Values of L in db (L_o = 0 db)

-5	-1	-1	-7	-2	-9	1	0	-4	-2	-7	-8
3	2	-9	-5	-2	-2	-5	2	1	-3	-3	-4
-3	2	0	2	-20	-2	1	-8	-1	0	0	-3
-2	4	-4	1	-1	-10	6	-2	-1	-21	-5	-1
0	-7	2	1	-6	6	-2	4	-4	-12	-8	-3

The real test of the technique is, of course, the frequency distribution of a large number of simulated echoes. Such a frequency distribution is shown in Fig. A-1, for the smallest number of samples (8000) used in the numerical simulations discussed in the body of the report. The "ideal" distribution, corresponding to equation (2-5) and plotted from Table A-1, is also shown for comparison. The agreement is obviously good.

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