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GENERALIZED UPPER BOUNDED TECHNIQUES FOR LINEAR PROGRAMMING—II

by

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ABSTRACT

A variant of the simplex method is given for solving linear programs with $M + L$ equations, L of which have the property that each variable has at most one nonzero coefficient. Special cases include transportation problems, programs with upper bounded variables, assignment and weighted distribution problems. The algorithm described uses a working basis of M rows for pivoting, pricing, and inversion which for large L can result in a substantial reduction of computation. This working basis is only $M \times M$ and is a further reduction of the size found in an earlier version, see [1]. Unfortunately, to achieve this reduction, row as well as column transformations must now be made.

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1. Introduction

As in [1] we assume that we are concerned with a linear program in which each variable has at most one non-zero coefficient in which is the last L equations to be nonnegative and the corresponding last L constant terms are positive. In section IV we indicate the necessary modifications to handle negative coefficients in those equations. By normalizing the variables and multiplying the equations by constants, we can assume without loss of generality that all nonzero coefficients and constants in the last L equations are ones, (1).

(1)

Max x_0 subject to:

$$
\begin{array}{l}
A_1^0 x_0 + A_1^1 x_1 + \dots + A_1^{n_0} x_{n_0} + A_1^{n_0+1} x_{n_0+1} + \dots + A_1^{n_1} x_{n_1} + A_1^{n_1+1} x_{n_1+1} + \dots + A_1^{n_2} x_{n_2} + \dots + A_1^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_1^N x_N = b_1 \\
A_2^0 x_0 + A_2^1 x_1 + \dots + A_2^{n_0} x_{n_0} + A_2^{n_0+1} x_{n_0+1} + \dots + A_2^{n_1} x_{n_1} + A_2^{n_1+1} x_{n_1+1} + \dots + A_2^{n_2} x_{n_2} + \dots + A_2^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_2^N x_N = b_2 \\
\vdots \\
A_M^0 x_0 + A_M^1 x_1 + \dots + A_M^{n_0} x_{n_0} + A_M^{n_0+1} x_{n_0+1} + \dots + A_M^{n_1} x_{n_1} + A_M^{n_1+1} x_{n_1+1} + \dots + A_M^{n_2} x_{n_2} + \dots + A_M^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_M^N x_N = b_M \\
x_{n_0+1} + \dots + x_{n_1} = 1 \\
x_{n_1+1} + \dots + x_{n_2} = 1 \\
\vdots \\
x_{n_{L-1}+1} + \dots + x_N = 1
\end{array}
$$

Equation 1

We review here for completeness some necessary definitions and theorems we will borrow from [1].

The $\underline{\ell^{th}}$ set of variables or columns, S_ℓ, will refer (depending on context) to those variables or columns corresponding to the columns of coefficients in (1) with 1 as their $M + \ell^{th}$ component. S_0, the 0^{th} set, is the set corresponding to columns with zeros for the $M + 1^{st}$ through $M + L^{th}$ coefficients.

We assume that the system (1) is of full rank and denote by $[\underline{A}^{j_1},\ldots,\underline{A}^{j_{M+L}}]$, a basis for the system. We always assume $\underline{A}^{j_1} = \underline{A}^0$ the coefficient of the variable to be optimized. The underscoring is to differentiate coefficient vectors with all $M + L$ components from the reduced vectors of the first M coefficients which will be denoted without the underscoring. There will be no underscoring for individual components A_i^j of these two different types of vectors even though they differ in the number of their components.

Theorem 1) At least one variable from each set S_ℓ (with the possible exception of S_0) is basic.

Theorem 2) The number of sets containing two or more basic variables is at most M.

The sets containing two or more basic variables plus S_0 are called essential sets (the term here is used in a slightly different sense than in [1]). An essential set for one basis may become an unessential one in the next.

In the next section we outline the method, in the following we formalize it as an algorithm. In the last section we indicate the modifications required to handle negative coefficients in the last L equations and finally in the appendix the method is carried out on an example.

II. The Method

Given a feasible basis[1], we assume we have selected for each S_ℓ one basic variable x_{k_ℓ} to be the key variable. A^{k_ℓ} is said to be the key column. For S_0 the key column can be taken for convenience to be a dummy column with all zero components. We then consider the system obtained by subtracting the key columns from every other column in their respective sets (in (2) we assume for simplicity that the key variable was the first one in each set). In this modified system the value of the key variables must clearly be one. We treat these variables as we would variables at upper bound in an upper bounded variables algorithm for the simplex method and subtract their coefficients from the right hand side.

$$
(2)\qquad
\begin{array}{cccccccc|c}
y_0 \ldots y_{n_0} & y_{n_0+1} & y_{n_0+2} \cdots y_{n_1} & & \ldots & y_{n_{L-1}+1} & \ldots & y_{n_L} & \\
\hline
A^0 \ldots A^{n_0} & A^{n_0+1} & A^{n_0+2} - A^{n_0+1} \ldots A^{n_1} - A^{n_0+1} & & \ldots & A^{n_{L-1}+1} & + \ldots + & A^{n_L} - A^{n_{L-1}+1} & = b \\
 & 1 & 0 & \ldots & 0 & & & & = 1 \\
 & & & & \ddots & & & & \vdots \\
 & & & & & 1 & \ldots & 0 & = 1
\end{array}
$$

We then introduce the following notation if $A^j \in S_\ell$ we let

$$D^{k_\ell} = A^{k_\ell}$$

$$D^j = A^j - A^{k_\ell} \qquad j \neq k_\ell$$

$$(3)\qquad d = b - \sum_{\ell=0}^{L} D^{k_\ell} = b - \sum A^{k_\ell}$$

---

1 Obtaining a first feasible solution is accomplished using this method with a phase I set up as in the usual simplex method.

where the last L components of d are zero.

We assume that D^j for $j = k_\ell$ (key) to be absent from the system. The <u>working basis,</u> B, is given by $B = \{D^j \mid A^j$ is basic and not key$\}$. By virtue of theorem 1 it is clear that B has exactly M columns. We assume $B^1 = A^0$ corresponds to the coefficient of the variable to be optimized. We define the <u>derived system</u> to be

$$(4) \qquad \sum_j y_j D^j = d \quad \text{and it is easy to prove:}$$

<u>Theorem 3</u> $\quad B$ is a basis for (4).

Proof: $\quad$ Suppose $\sum_j \lambda_j B^j = 0$. Since B^j differs from $\underline{B}^j$ (the same column considered in the system depicted in (2)) by only 0 components $\sum_j \lambda_j \underline{B}^j = 0$. But this implies that the $\underline{B}^j$ plus the key columns are linearly dependent since the $\underline{B}^j$ by themselves are linearly dependent. On the other hand this set is obtained from a (nonsingular) basis by subtraction of columns from within the set which does not reduce the rank, yielding a contradiction.

By Theorem 2 there exist at most M sets with more than one basic variable. These sets and S_0 are the only sets which contain members of B; i.e., they are the <u>essential</u> (including S_0) sets.

Thus, with each feasible basis for the original system (1), we have associated a set of L key variables (and one dummy) and a basis for the derived system. We now show that we can carry out the steps of the simplex method using just the inverse B^{-1} of B, the reduced basis, and the corresponding basic solution, d, of the derived system (4).

The first step is to obtain a set of prices for (1). Let us denote by $\pi = (\pi_1, \ldots, \pi_M)$ the prices on the first M equations and $\mu = (\mu_1, \ldots, \mu_L)$ the prices on the last L. These prices are determined uniquely by the condition that

$$(\pi , \mu)\underline{A}^0 = (\pi, \mu)\underline{A}^{j_1} = 1$$

$$(\pi , \mu)\underline{A}^{j_i} = 0 \qquad i=2,\ldots,M+L$$

Let $\hat{\pi} = (B^{-1})_1$, the first row of the inverse of the working basis B. It has the property that

$$\hat{\pi}B^1 = \hat{\pi}A^0 = 1$$

$$\hat{\pi}B^j = 0 \qquad j=2,\ldots,M$$

i.e., $\hat{\pi}$ is a set of prices for (4). To extend $\hat{\pi}$ to a set of prices for (2) is trivial we simply set

$$\text{(5)} \qquad \hat{\mu}_\ell = - \hat{\pi}A^{k_\ell} \qquad \ell=1,\ldots,L .$$

Now consider for basic columns A^{j_i}

$$(\hat{\pi} , \hat{\mu})\underline{A}^{j_i} = (\hat{\pi}, \hat{\mu})\underline{A}^{k_\ell} = 0 \text{ if } A^{j_i} \text{ is key (5)}$$

or

$$\begin{aligned}(\hat{\pi} , \hat{\mu})\underline{A}^{j_i} &= (\hat{\pi} , \hat{\mu})(\underline{B}^i + \underline{A}^k) \text{ for some } k \\ &= (\hat{\pi} , \hat{\mu})\underline{B}^i + (\hat{\pi} , \hat{\mu})\underline{A}^k \\ &= 0 + 0 \text{ if } A^{j_i} \text{ is not key .}\end{aligned}$$

Thus $(\hat{\pi} , \hat{\mu})$ is a set of prices for the original system (1).

Using these prices we can "price out" the columns of (1) to find one to enter into the basis. Using the usual simplex criterion, the incoming column A^s would be chosen by

$$\Delta_s = (\pi , \mu)\underline{A}^s = \min_j (\pi , \mu)\underline{A}^j = \min_j \Delta_j$$

where

$$\Delta_j = \sum \pi_i A_i^j + \mu_\ell \quad \text{for } A^j \in S_\ell .$$

If $\Delta_s \geqq 0$, we have an optimal basic feasible solution and we're done; otherwise we bring $\underline{A}^s$ into the basis. Assume $A^s \in S_\sigma$. To do this, we must express $\underline{A}^s$ and $\underline{b}$ in terms of the current basis for (1). If we let

$$\bar{D}^s = B^{-1}D^s = B^{-1}(A^s - A^{k_\sigma})$$

then

$$(6) \qquad (A^s - A^{k_\sigma}) = \sum_{i=1}^{M} \bar{D}_i^s B^i = \sum \bar{D}_i^s (A^{\eta_i} - A^{\nu_i})$$

where η_i indicates the column number in (2) corresponding to the i^{th} column of the working basis and ν_i denotes the column number of the corresponding key variable.

We denote the representation of A^s in terms of the current basis by $\bar{A}_i^s$, that is:

$$\underline{A}^s = \sum_{i=1}^{M+L} \bar{A}_i^s \underline{A}^{j_i} .$$

From (6) we see

$$
(7) \qquad \bar{A}_j^s = \begin{cases} 1 - \sum\limits_{\nu_t = k_\sigma} \bar{D}_t^s & \text{if } A^{j_i} = A^{k_\sigma} \\ \bar{D}_t^s & \text{if } A^{j_i} = A^{\eta_t} \text{ for some } t \\ - \sum\limits_{\nu = j_i} \bar{D}_\nu^s & \text{if } A^{j_i} = A^{\nu_t} \text{ for some } t \\ 0 & \text{otherwise} \end{cases}
$$

The current values for the variables in the basis $\bar{b}_i$ are given either by updating the values of the previous iteration in the usual way or recomputed in a similar way to $\bar{A}_j^s$ above. That is

Let $\bar{d} = \bar{d}_1, \ldots, \bar{d}_M$ be given by

$$
(8) \qquad \bar{d} = B^{-1}(b - \sum A^{k_\ell}) = B^{-1} d
$$

then

$$
(b - \sum A^{k_\ell}) = \sum d_i B^i = \sum d_i (A^{\eta_i} - A^{\nu_i})
$$

and as in (7) the $\bar{b}_i$ are given by

$$
(9) \qquad \bar{b}_i = \begin{cases} 1 - \sum\limits_{\nu_t = j_i} d_t & \text{if } A^{j_i} \text{ is key} \\ d_t & \text{if } A^{j_i} = A^{\eta_t} \text{ for some } t \\ 0 & \text{otherwise} \end{cases}
$$

Finding the variable to leave the basis is accomplished in exactly the same way as in the ordinary simplex method. Let

(10) $$\theta \triangleq \frac{\bar{b}_r}{\bar{A}_r^s} \triangleq \min_{\bar{A}_i^s > 0} \frac{\bar{b}_i}{\bar{A}_i^s} \quad , \; i = 1,\ldots,M + L .$$

where we of course require that $\bar{A}_r^s > 0$. Let us assume $A^{j_r} \in S_\rho$. Three cases can occur in the updating process:

(a) If S_σ is not essential and $A^{j_r} \in S_\sigma$; i.e., the outgoing variable is the key variable in S_σ then B remains unchanged, and A^s simply replaces A^{j_r} as the key variable in S_σ . This requires the updating of $\bar{d}$ which is accomplished as follows:

$$\bar{d} := B^{-1}\left(b - \sum_{\ell \neq \sigma} A^{k_\ell} - A^{k_\sigma} + A^{j_r} - A^s \right)^{1/}$$

$$= B^{-1}\left(b - \sum_{\ell \neq \sigma} A^{k_\ell} - A^{k_\sigma} \right) - B^{-1}(A^s - A^{j_r})$$

$$= \bar{d} - B^{-1}(A^s - A^{j_r})$$

$$= \bar{d} - \bar{D}^s$$

Observing that $A^{j_r} = A^{k_\sigma}$, this is easy to compute since we already have $\bar{d}$ and the second term was generated in determining the $\bar{A}_i^s$.

(b) If A^{j_r} is not a key variable, then we update B^{-1} simply by pivoting on the column $\bar{D}^s$ on the row which $A^{j_r} - A^{k_\rho}$ occupies

---

1/ Where the symbol ": = " does not indicate equality but rather that the expression on the right replaces or (updates) the variable on the left.

in the working basis. In symbols $B^{-1} := PB^{-1}$ where P is the matrix which performs the pivot. $\bar{d}$ is updated by applying this pivot to the old $\bar{d}$.

(c) If $A^{j_r} \in S_\rho$ is a key variable in a different set than the one containing $A^s \in S_\sigma$, we must first change the key variable in S_ρ . To change the inverse of the pseudo basis B , we consider all columns of B of the form $A^j - A^{j_r}$ there must be at least one such since after A^s enters the basis S_ρ must contain a basic variable (Theorem 1). One of these columns, call it A^k , is to become the key variable. To get the new pseudo-basis $\bar{B}$ from the old one B we wish to multiply the column (in B) $A^k - A^{j_r}$ by -1 to obtain $A^{j_r} - A^k$ and we wish to subtract $A^k - A^{j_r}$ from every other column of the form $A^j - A^{j_r}$ for $j \neq k$ to obtain $A^j - A^k$. That is

$B := BT$ where T is of the form

(12)
$$T = \begin{bmatrix} 1 & & & & & & & & & \\ & \cdot & & & & & & & & \\ & & 1 & & & & & & & \\ 0 \ldots 0 & -1 \ldots & -1 & -1 & -1 \ldots & -1 & 0 \ldots 0 \\ & & & & 1 & & & & \\ & & & & & \cdot & & & \\ & & & & & & \cdot & & \\ & & & & & & & 1 \end{bmatrix}$$

Where the -1's occur in the columns corresponding to $A^j \in S_\rho$, and the row corresponds to the new key variable, A^k .

$$B^{-1} := T^{-1}B^{-1}$$

and it is easily verified that $T^{-1} = T$

$$T^{-1} = T$$

since applying the process twice replaces A^{j_r} as the key variable. The values for $\bar{d}$ are updated by applying T^{-1}. Now with the new key variable in S_ρ we simply apply the process outlined in b .

With our updated B , y and key variables we are now ready to make another iteration. If the inverse of the pseudo-basis is expressed in product form we have

$$B^{-1} = \prod^{t} T^{t}$$

where each T^t is either of the form (12) or (13), the latter resulting from a pivot on the r^{th} element of the column $(\alpha_1,\ldots,\alpha_m)^T$.

$$(13) \qquad P = \begin{bmatrix} 1 & & & -\alpha_1/\alpha_r & & & \\ & \ddots & & \vdots & & & \\ & & 1 & -\alpha_{r-1}/\alpha_r & & & \\ & & & 1/\alpha_r & & & \\ & & & \vdots & 1 & & \\ & & & & & \ddots & \\ & & & -\alpha_m/\alpha_r & & & 1 \end{bmatrix}$$

As W. Orchard-Hays has remarked to one of the authors we can if we wish, express each transformation of the form (12) as a sequence of transformation of the form (13). Suppose we wish to express a matrix T in terms of transformations of the form (13), where -1's appear in columns h_0, $h_1,\ldots,h_i$ and suppose that the -1 in column h_0 lies on the diagonal; in other words, all the -1's are in the h_0^{th} row. Let $P(r,s)$ denote the pivot matrix (13) with $\alpha_j = 0$

$j \neq r$, or s, $\alpha_r = 1$ and $\alpha_s = -1$. When multiplying on the left this matrix has the effect of subtracting the r^{th} row from the s^{th} row. Finally let $P(r,r)$ be the matrix with all plus ones on the diagonal except in the r^{th} diagonal element which is -1. Every other element is zero. When multiplying on the right, this has the effect of multiplying the r^{th} column by -1. It is then easy to see that

$$T = P(h_0, h_k)P(h_0, h_{k-1})\ldots P(h_0, h_1)P(h_0, h_0)$$

FLOW DIAGRAM OF THE GENERALIZED UPPER-BOUND ALGORITHM

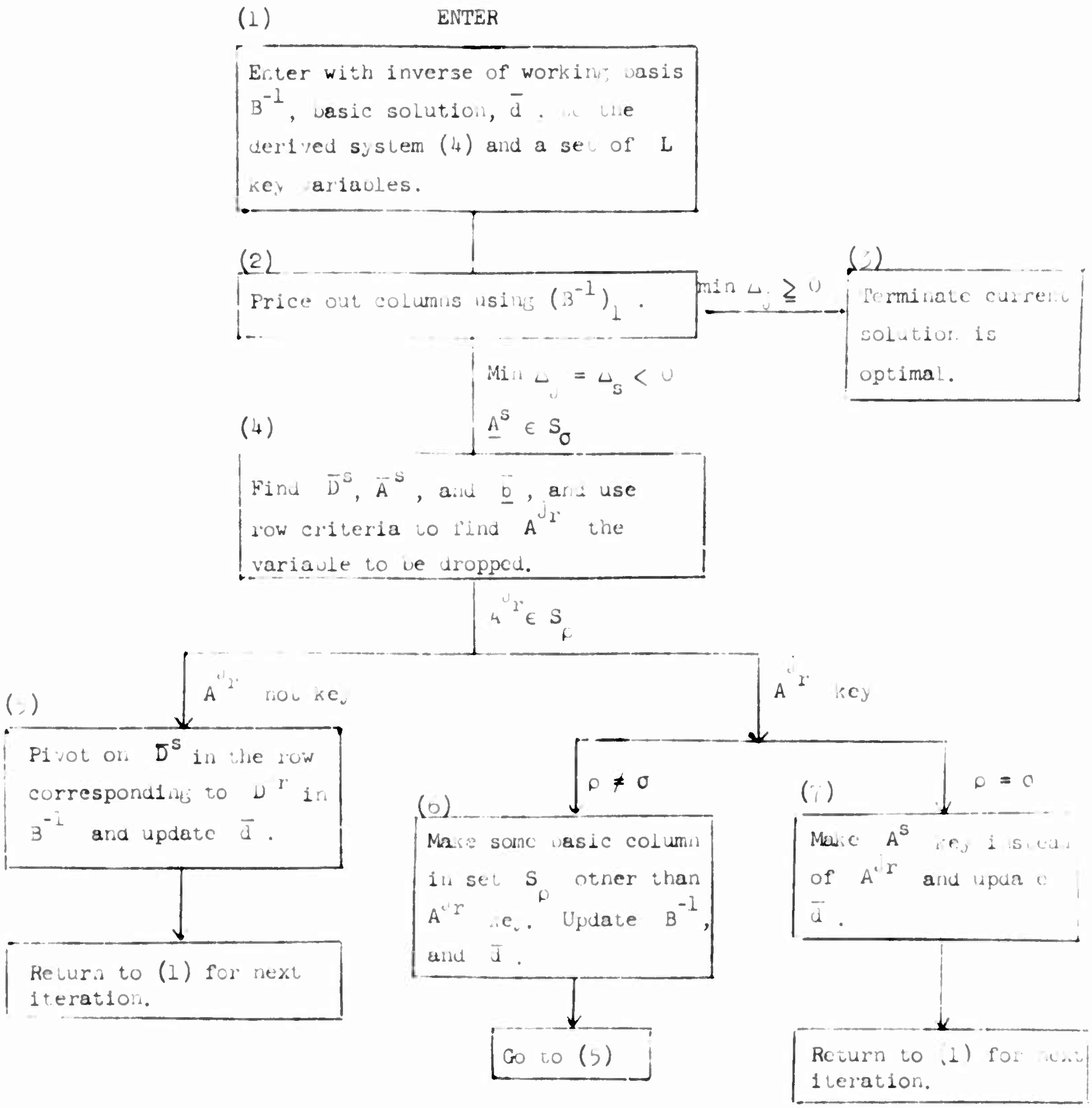

FIGURE 1

III. Description of Algorithm

Referring to figure 1, the algorithm takes place in the following steps:

(1) We assume we enter the algorithm with the inverse B^{-1} of the working basis, the value $\bar{d}$ of the appropriate basic solution of the derived system (4) and the set of key variables. To get this initial solution, the usual phase I procedure can be carried out in the obvious way.

(2) Let $\pi_i = (B^{-1})^i_1$ for $i = 1,\ldots,M$ and for each set $S_\ell (\ell \neq 0)$ let $\mu_\ell = -\sum_{i=1}^{M} \pi_i A_i^{k_\ell}$, where A^{k_ℓ} is the key column in S_ℓ. Let

$$\Delta_j = \sum_{i=1}^{M} \pi_i A_i^j + \mu_\ell \quad \text{for} \quad A^j \in S_\ell .$$

Let $\Delta_s = \min \Delta_j$ and suppose $A^s \in S_\sigma$. If $\Delta_s \geq 0$, we go to Step (3); otherwise skip to (4).

(3) Terminate we have an optimal solution.

(4) Find $\bar{D}^s = B^{-1} \quad D^s = B^{-1}(A^s - A^{k_\sigma})$, $\bar{A}^s$ by means of Equation (7) and $\bar{b}$ by means of Equation (9). Use the usual simplex decision rule Equation (10) to find the variable to be dropped A^{j_r} and suppose $A^{j_r} \in S_\sigma$. If A^{j_r} is key, go to step (5); if not and $\rho \neq \sigma$, go to (6); if $\rho = \sigma$, go to step (7).

(5) We pivot with respect to $\bar{D}^s$ in the row corresponding to D^{j_r} in B^{-1} and update $\bar{d}$ by applying the pivot transform to it. We then return to step (1) for another iteration.

(6) Make some basic column say A^k , $k \neq j_r$ in set S_ρ key instead of A^{j_r}. Update B^{-1} by applying a column transformation of the form (12) and update B^{-1} by $B^{-1} := T^{-1}B^{-1}$. $\bar{d}$ is updated by $\bar{d} := T^{-1}\bar{d}$. We then can go to step (5).

(7) Make A^s key instead of A^{j_r} and update $\bar{d}$ by

$$\bar{d} := \bar{d} - \bar{D}^s .$$

Return to step (1).

IV. Negative Coefficients:

When negative coefficients appear in the last $M + L$ equations, the algorithm is changed in a quite obvious way. We can assume without loss of generality that the coefficients in the last L equations are +1 or -1 and the last L right hand side components are +1. Theorems 1, 2, and 3 still hold, and we can require that each key column have a <u>+1</u> in the last L equations since clearly each set must have such a column which is basic. In the pricing process if the column A^j to be priced has a negative coefficient in the last L components, the appropriate μ is subtracted rather than added to πA^j . To form the difference columns D^j the key column is added to columns with a - 1 rather than subtracted and appropriate modifications in equations (7) and (9) must be made to reflect this. Other than these slight modifications, the algorithm proceeds exactly as before.

Example: Consider the following example (Figure 2) with $M = 3$. We seek to maximize x_0.

S_0	S_1			S_2	S_3	S_4		S_5		
$\underline{A}^0$	$\underline{A}^1$	$\underline{A}^2$	$\underline{A}^3$	$\underline{A}^4$	$\underline{A}^5$	$\underline{A}^6$	$\underline{A}^7$	$\underline{A}^8$	$\underline{A}^9$	$\underline{b}$
1	0	2	0	3	4	5	1	-1	-12	15
1	1	-1	0	2	1	4	2	-3	6	7
0	0	0	1	0	0	0	0	0	0	0
	1	1	1							1
				1						1
					1					1
						1	1			1
								1	1	1
$\bar{\underline{b}}^T = (3$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	1		1	)	
	x			x	x	x		x		

Figure 2

The initial basis is $A^0A^1A^2A^3A^4A^5A^6A^8$ and $A^1A^4A^5A^6A^8$ are key. The working basis $B = \{A^0 - 0,\ A^2 - A^1,\ A^3 - A^1\}$; hence

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which has an inverse:

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

With the aid of (5) we find the prices $[\pi, \mu] = [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{9}{2}, 2]$.

We then price out and find $A^7 \epsilon S_4$ wins.

$$\bar{A}^7 - \bar{A}^6 = B^{-1}[A^7 - A^6] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -\frac{1}{2} \\ 0 \end{bmatrix},$$

i.e., $A^7 - A^6 = -3A^0 - \frac{1}{2}(A^2 - A^1)$

or $\underline{A}^7 = -3\underline{A}^0 + \frac{1}{2}\underline{A}^1 - \frac{1}{2}\underline{A}^2 + \underline{A}^6$

giving a representation of $\underline{A}^7$ in terms of the full basis.

$$\bar{\underline{A}}^7 = (-3, \tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, 1, 0)^T$$

We obtain the values of the variables by considering

$$B^{-1}[b - \sum A^{k\ell}] = B^{-1} \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

This means

$$\underline{b} - \sum \underline{A}^{k\ell} = 3\underline{A}^0 - \tfrac{1}{2}\underline{A}^1 + \tfrac{1}{2}\underline{A}^2$$

or

$$\underline{b} = 3\underline{A}^0 + \tfrac{1}{2}\underline{A}^1 + \tfrac{1}{2}\underline{A}^2 + \underline{A}^4 + \underline{A}^5 + \underline{A}^6 + \underline{A}^8$$

hence

$$\bar{\underline{b}} = [3, \tfrac{1}{2}, \tfrac{1}{2}, 0, 1, 1, 1, 1]^T.$$

We now determine the variable going out of the basis by

$$\theta = \min_{\bar{A}_i^7 > 0} \frac{\bar{b}_i}{\bar{A}_i^7} \triangleq \frac{\bar{b}_r}{\bar{A}_r^7} = 1$$

and r could be 2 or 7. Taking it to be 7, we see that since the set S_4 is inessential and $\theta = 1$ we just replace A^6 by A^7 as a key variable and B remains unchanged. The new multipliers are

$$[\bar{\pi}, \mu] = [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; -\tfrac{1}{2}, -\tfrac{5}{2}, -\tfrac{5}{2}, -\tfrac{3}{2}, 2]$$

and this time $\underline{A}^9 \in S_5$ prices out optimally.

$$B^{-1}[A^9 - A^8] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -11 \\ 9 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ 0 \end{bmatrix},$$

that is, $\underline{A}^9 = -\underline{A}^0 + 5\underline{A}^1 - 5\underline{A}^2 + \underline{A}^8$ and

$$\underline{A}^9 = [-1, 5, -5, 0, 0, 0, 1, 0].$$

$$B^{-1}[\underline{b} - \sum^k \underline{A}^{\ell}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$b = 6A^0 + A^2 - A^1 + A^1 + A^4 + A^5 + A^7 + A^8$$

$$= 6A^0 + A^2 + A^4 + A^5 + A^7 + A^8$$

$$\underline{b} = [\ 6,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 1\ ]$$

$$\theta = \min_{\bar{\bar{A}}_i^9 > 0} \frac{\bar{\bar{b}}_i}{\bar{\bar{A}}_i^9} = \frac{\bar{\bar{b}}_2}{\bar{\bar{A}}_2^9} = \frac{0}{6}; \quad r = 2, \quad j_r = 1$$

therefore, we want to drop A^1, which however is key. So first we must replace A^1 by A^2 as a key variable. To do this we take our current working basis

$$B = [A^0 - 0\ ,\ A^2 - A^1\ ,\ A^3 - A^1]$$

and postmultiply it by

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which has the effect of subtracting the second column from the third and reversing the sign of the second column.

$$B' = BT = [A^0 - 0\ ,\ A^1 - A^2\ ,\ A^3 - A^2]$$

$$(B')^{-1} = T^{-1}B^{-1}$$

where

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$(B')^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

We then pivot in the vector column

$$(B')^{-1}(A^9 - A^8) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -11 \\ 9 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$$

on the second component.

This gives us a new inverse basis

$$B^{-1} = PB'^{-1}$$

$$= \begin{bmatrix} 1 & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{20} & \frac{11}{20} & \frac{7}{20} \\ \frac{1}{20} & \frac{1}{20} & -\frac{3}{20} \\ 0 & 0 & 1 \end{bmatrix}$$

The new prices are

$$\left(\frac{9}{20}, \frac{11}{20}, \frac{7}{20}; -\frac{7}{20}, -\frac{47}{20}, -\frac{31}{20}, \frac{42}{20}\right)$$

and upon pricing out we find that all columns price out nonnegatively and the optimal solution is given by

$$\bar{B}^{-1}[b - \sum A^{k}\ell] = \begin{bmatrix} \frac{9}{20} & \frac{11}{20} & \frac{7}{20} \\ \frac{1}{20} & \frac{1}{20} & -\frac{3}{20} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

and

$$b - A^2 - A^4 - A^5 - A^7 - A^8 = 6A^0$$

or

$$\underline{\bar{\bar{b}}} = [6, 1, 0, 1, 1, 1, 1, 0]$$

the values of the basic variables. Another way to compute $\bar{\bar{b}}$ is, of course, by $\bar{\bar{b}} = \bar{b} - \theta\bar{\bar{A}}^{9}$, the usual formula for updating the values of the basic variables is the simplex method.

REFERENCE

[1] Dantzig, George B., and R.M. Van Slyke, "Generalized Upper Bounding Techniques - I," ORC 64-17 (RR), Operations Research Center, University of California, also to appear in the Proceedings of the IBM Symposium on Combinatorial Methods (March 1964).