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THE FLOW OF A CONDUCTING GAS JET WITH ALIGNED MAGNETIC FIELD

Stanley A. Berger

PREPARED FOR:
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The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

The equations governing the flow of a conducting gas interacting with a magnetic field are coupled nonlinear partial differential equations in several dependent and independent variables. Thus, exact solutions are exceedingly rare. One class of problems which can be solved exactly involves the flow of an infinitely conducting gas when the velocity and magnetic fields are aligned everywhere. A particular boundary value problem involving such a flow is solved in this Memorandum. The results presented here should be of use to all who are working in the area of magnetohydrodynamics.

SUMMARY

When an infinitely conducting fluid flows under the influence of a magnetic field in such a way that the velocity and magnetic fields are aligned, then the governing equations may formally be reduced to those of ordinary gas dynamics. The equations are still nonlinear, but if one restricts attention to plane flows, then the hodograph technique may be employed. The equations in the hodograph plane are linear and may be solved exactly. The difficulty then remaining is to satisfy the boundary conditions. Only for restricted classes of problems can this be done. This paper treats one such class of problems: the flow of a jet of gas out of a slit in a rectangular channel. The method used is a modification of the original method of Chaplygin for nonconducting gas jets.

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THE FLOW OF A CONDUCTING GAS JET WITH ALIGNED MAGNETIC FIELD

1. INTRODUCTION

In this Memorandum we treat a particular boundary-value problem involving the flow of an infinitely-conducting gas, moving in such a manner that its velocity field is everywhere aligned with the resultant magnetic field. In such a situation the equations governing the motion can be reduced to those of ordinary gas dynamics [1-4, 6].

While this reduction is a significant one, the equations are still nonlinear and hence intractable for most boundary-value problems. If, however, we limit ourselves to plane flows, it is possible to make use of the hodograph technique. The effect of the hodograph transformation is to linearize the basic equations; the penalty paid for this simplification is that for most problems the boundary conditions in the hodograph plane are not known. These are, however, classes of problems for which this difficulty can be overcome; the outstanding example of such problems is the efflux of gas from a slit, the jet problem, first solved by Chaplygin.

Chaplygin solved the problem of jet flow from a slit in a half-plane, for which the velocity infinitely far upstream must be zero. Recently this solution has been extended to a rectangular channel with a finite, nonzero, upstream velocity. We shall be concerned with extending this solution to the magnetohydrodynamics case when the gas is infinitely-conducting and the velocity and magnetic fields are aligned. The velocity at any point will be assumed to be subcritical ($A^2 + M^2 < 1$).

The problem can be reduced to the solution of the equation for the stream function in the velocity plane under appropriate boundary conditions. The solution is obtained by separation of variables. The resulting second-order ordinary differential equation for the velocity dependence contains the hypergeometric equation as the limiting case for zero magnetic field. One of the two linearly independent solutions of the equation is analytic for velocities close to zero; the series expansion of this solution about zero velocity is discussed in the Appendix.

2. GOVERNING EQUATIONS

The steady flow of an infinitely conducting, inviscid, perfect gas in the presence of a magnetic field is governed by the following set of equations:

(1) Continuity $$\nabla\cdot(\rho\vec{q}) = 0,$$

(2) Momentum $$\rho\vec{q}\cdot\nabla\vec{q} = -\nabla p + \frac{1}{\mu}(\nabla\times\vec{B})\times\vec{B},$$

(3) Energy $$\vec{q}\cdot\nabla\left(\frac{p}{\rho^{\gamma}}\right) = 0,$$

(4), (5) Maxwell's Equations $$\left\{\begin{aligned} \nabla\cdot\vec{B} &= 0, \\ \nabla\times(\vec{q}\times\vec{B}) &= 0. \end{aligned}\right.$$

We assume $\vec{B}$ is parallel to $\vec{q}$ and write

(6) $$\vec{B} = \alpha\rho\vec{q} \ .$$

(If $\vec{B}$ and $\vec{q}$ lie in the same plane, then as a consequence of Eq. (5), $\vec{q} \times \vec{B}$ is a constant. From this it follows that if $\vec{q}$ and $\vec{B}$ are parallel anywhere, they are parallel everywhere. Thus, the condition expressed by Eq. (6) is not as restrictive as it might seem at first glance.)

Equations (1) and (4) lead to

$$(\vec{q} \cdot \nabla)\alpha = 0$$

i.e., that α is a constant along streamlines.

Equation (5) is satisfied automatically by the choice of $\vec{B}$. The momentum equation, Eq. (2), may be written as

(7) $$\nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho}\right) = \vec{q} \times (\nabla \times \vec{q}) - \frac{1}{\rho\mu} \vec{B} \times (\nabla \times \vec{B}) .$$

If we take the scalar product of this equation with $\vec{q}$ we obtain

(8) $$\frac{q^2}{2} + \int \frac{dp}{\rho} = H_o ,$$

where H_o, the total enthalpy, is constant along each streamline.

From this point on we shall assume that each of the quantities α, H_o, and $p/\rho^{\gamma} = f(S)$ is constant throughout the flow field, not just along streamlines. This would be the case if all the streamlines originated in some region where uniform conditions prevail. In the particular problem to be treated in this paper, such a situation does exist and thus the indicated assumption is not actually a restriction.

With H_0 constant throughout the flow field, Eq. (7) yields

$$\vec{q} \times (\nabla \times \vec{q}) = \frac{1}{\rho\mu} \vec{B} \times (\nabla \times \vec{B}) \quad . \tag{9}$$

If we now assume that the flow is two-dimensional (plane) and write

$$\vec{q} = u\vec{i} + v\vec{j},$$

then Eq. (9) leads to

$$\frac{\partial}{\partial x}[v(1-C\rho)] = \frac{\partial}{\partial y}[u(1-C\rho)], \tag{10}$$

where $C = \alpha^2/\mu$ = const. (throughout flow field).

For plane flow the continuity equation, Eq. (1), becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0. \tag{11}$$

The governing equations of the flow have been reduced to the three equations (8), (10), and (11). Equation (10), which is of the form of an irrotationality condition, can be identically satisfied by introducing a potential $\varphi(x,y)$ defined by

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = u(1-C\rho) \; , \\[2ex] \dfrac{\partial\varphi}{\partial y} = v(1-C\rho) \; . \end{cases} \tag{12}$$

Similarly, Eq. (11) can be satisfied by introducing the usual stream function $\Psi(x,y)$ such that

$$(13)\qquad \begin{cases} \dfrac{\partial \Psi}{\partial y} = \dfrac{\rho}{\rho_0}\, u\,, \\[2ex] \dfrac{\partial \Psi}{\partial x} = -\dfrac{\rho}{\rho_0}\, v\,. \end{cases}$$

The three governing equations may now be combined to yield a single equation in either Ψ or φ:

$$(14)\qquad \Psi_x^2\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + \Psi_y^2\Psi_{yy} - (\Psi_x^2+\Psi_y^2)\left[\frac{(1-C\rho)(M^2-1)}{M^2}\right](\Psi_{xx}+\Psi_{yy}) = 0,$$

$$(15)\qquad \varphi_x^2\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + \varphi_y^2\varphi_{yy} - (\varphi_x^2+\varphi_y^2)\left[\frac{1+C\rho(M^2-1)}{M^2}\right](\varphi_{xx}+\varphi_{yy}) = 0.$$

(See Imai [1], Hida [2], Seebass [3,4].) In the limit of zero magnetic field, $C \to 0$, these equations reduce to the corresponding equations in ordinary gas dynamics.

A principal difficulty, the nonlinearity of Eqs. (14) and (15), may be eliminated by interchanging independent and dependent variables, that is, by transforming from the physical plane to the hodograph plane. Proceeding in the usual manner, we obtain the following equations for Ψ in terms of the new independent variables q and θ, polar coordinates in the hodograph plane:

$$(16)\qquad q^2(1-C\rho)\Psi_{qq} + \left[\frac{(1-C\rho)^2(1+M^2) + C\rho\, M^4[3-C\rho+\gamma(C\rho-1)]}{1 - C\rho(1 - M^2)}\right] q\Psi_q$$
$$+ (1-M^2)\,[1-\rho C(1-M^2)]\Psi_{\theta\theta} = 0.$$

An equivalent equation can be written down for ω, but since this will not actually be required, we shall not set it down. Once $\Psi(q,\theta)$ is determined from Eq. (16), $\omega(q,\theta)$ may be determined from the hodograph relations, which in this case take the form

$$
(17)\quad \begin{cases} \omega_\theta = \dfrac{\rho_o}{\rho}\,\dfrac{q(1-C\rho)^2}{1-C\rho(1-M^2)}\,\Psi_q\,, \\[2ex] \omega_q = -\dfrac{\rho_o}{\rho q}\,(1-C\rho)(1-M^2)\left[\dfrac{(1-C\rho)+C\rho M^2}{1-C\rho(1-M^2)}\right]\Psi_\theta\,. \end{cases}
$$

(M^2 and ρ can be expressed in terms of q by use of Bernoulli's equation, Eq. (8).)

Once $\Psi(q,\theta)$ and $\omega(q,\theta)$ are known, the coordinates in the physical plane can be obtained from the expressions

$$
(18)\quad \begin{cases} dx = \dfrac{1}{q(1-C\rho)}\left(\cos\theta\, d\omega - \dfrac{\rho_o}{\rho}(1-C\rho)\sin\theta\, d\Psi\right), \\[2ex] dy = \dfrac{1}{q(1-C\rho)}\left(\dfrac{\rho_o}{\rho}(1-C\rho)\cos\theta\, d\Psi + \sin\theta\, d\omega\right). \end{cases}
$$

3. FORMULATION AND SOLUTION

The particular problem to be solved in this Memorandum is the outflow of an infinitely conducting gas from a slit in a rectangular vessel. This problem has been solved for the nonmagnetic case by Fal'kovich [5]. The configuration is shown in Fig. 1. The channel is assumed to be of width H, the opening of width h. Infinitely far upstream the flow is uniform with velocity v_0, and is aligned along a uniform magnetic field B_0. Since the flow velocity and magnetic field are aligned at upstream infinity, they remain aligned throughout the flow field, as indicated earlier. Also, α, H_0, and S may be taken constant throughout, since all the streamlines originate in a region of uniform conditions. Thus, the conditions that are needed for the theory presented in Sec. 2 to apply are satisfied.

Chaplygin's method for two-dimensional gas jets fails when the flow in the jet is partly subsonic and partly supersonic (see [7]). Since the hypercritical transition ($A^2 + M^2 = 1$) has the same essential character as the sonic transition, one would expect that in the present problem the restriction to be imposed is that the flow remains subcritical everywhere, that is, $A^2 + M^2 < 1$. We shall assume this to be true in what follows.

The jet is bounded by two free boundaries, C'D and CD, on which the pressure is constant. The stream function Ψ is assigned the value $\frac{1}{2}Q$ along the upper boundary of the flow, AB'C'D, and the value $-\frac{1}{2}Q$ along the lower boundary, ABCD, the total mass flow rate then being Q. The final velocity far downstream on the free boundaries is v_1.

The physical flow plane represented in Fig. 1 can be mapped onto the hodograph plane. In the hodograph plane we introduce polar coordinates θ and τ, where $\tau = q^2/q^2_{max}$. The limiting values of τ far upstream and downstream are then given by $\tau_o = v_o^2/q^2_{max}$ and $\tau_1 = v_1^2/q^2_{max}$. The hodograph representation of the flow is shown in Fig. 2, in which corresponding points are designated by the same letters. CDC' is a semicircle of radius τ_1, EAE' a semicircle of radius τ_o. The values that Ψ must take on the various boundaries in the hodograph plane are given below:

(19.1)	$\Psi = -\frac{1}{2}Q$ when $\tau = \tau_1$, $0 < \theta \leq \frac{1}{2}\pi$	DC
(19.2)	$\Psi = -\frac{1}{2}Q$ when $\tau_1 \geq \tau > 0$, $\theta = \frac{1}{2}\pi$	CB
(19.3)	$\Psi = -\frac{1}{2}Q$ when $0 \leq \tau < \tau_0$, $\theta = +0$	BA
(19.4)	$\Psi = +\frac{1}{2}Q$ when $\tau = \tau_1$, $-\frac{1}{2}\pi \leq \theta < 0$	DC'
(19.5)	$\Psi = +\frac{1}{2}Q$ when $\tau_1 \geq \tau \geq 0$, $\theta = -\frac{1}{2}\pi$	C'B'
(19.6)	$\Psi = +\frac{1}{2}Q$ when $0 \leq \tau \leq \tau_0$, $\theta = -0$	B'A

The problem now is to find a solution of the basic equation for Ψ, Eq. (16), satisfying the boundary conditions (19.1) - (19.6) in the hodograph plane. Before attempting this we must first transform Eq. (16) from independent variables q, θ to the new variables τ, θ. We also note that the Alfvén number, defined as

(20) $$A = \frac{q}{(B^2/\mu\rho)^{1/2}} = \frac{\text{flow speed}}{\text{Alfvén speed}} ,$$

is related to the constant C as follows

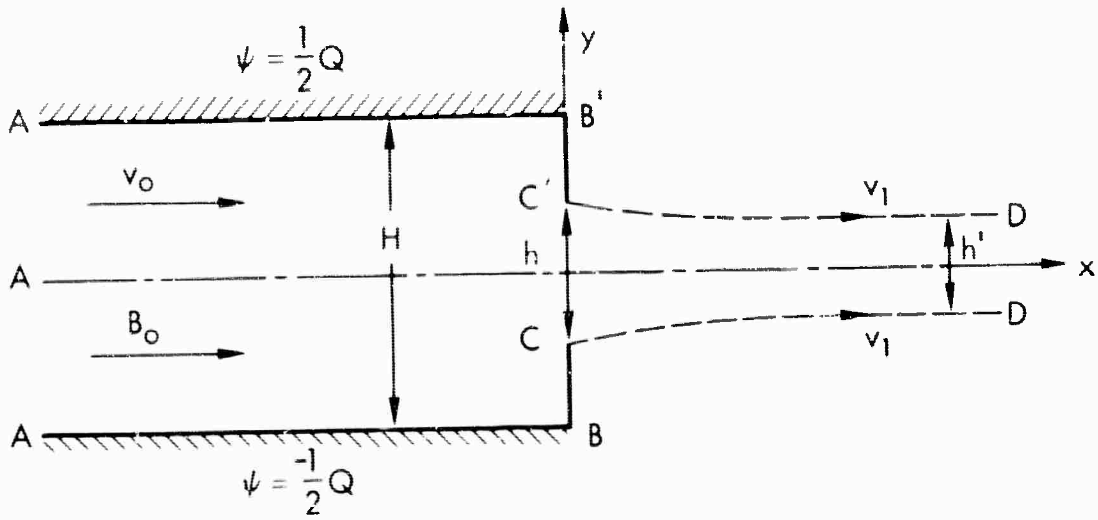

Fig. 1—Physical plane

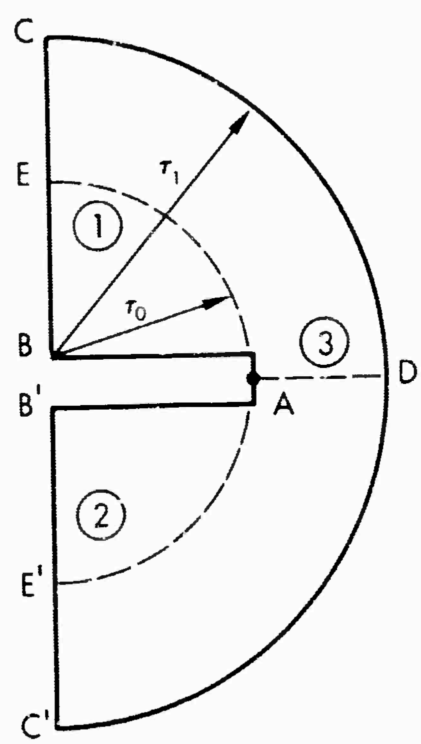

Fig. 2—Hodograph plane

(21) $$C\rho = \frac{1}{A^2} = \frac{\alpha^2\rho}{\mu} .$$

Then, introducing the stagnation Alfvén number A_0,

(22) $$A_0 = \frac{1}{\alpha}\sqrt{\frac{\mu}{\rho_o}} ,$$

we can write (assuming a perfect gas)

(23) $$\frac{A^2}{A_0^2} = \frac{\rho_o}{\rho} = (1-\tau)^{\frac{-1}{\gamma-1}}$$

so

(24) $$C\rho = \frac{1}{A_o^2}(1-\tau)^{\frac{1}{\gamma-1}} .$$

Also,

(25) $$\rho = \rho_o(1-\tau)^{\frac{1}{\gamma-1}} ,$$
$$M^2 = \frac{2}{\gamma-1}\,\frac{\tau}{1-\tau} .$$

Now making the change of variables from q, θ to τ, θ and introducing relations (24) and (25), Eq. (15) transforms into the equation

(26) $$\begin{aligned} &4\tau^2(1-\tau)\left[1 - A_0^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right]\left[1 + A_0^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\left(\frac{\gamma+1}{\gamma-1}\tau-1\right)\right]\Psi_{\tau\tau} \\ &+ 4\tau\Bigg\{\left[1 - A_0^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right]\left[(1-\tau)\left(1-A_0^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right) + \frac{\tau}{\gamma-1}\right] \\ &\qquad - 2A_0^{-2}\tau^2(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\left[\frac{\gamma-3}{(\gamma-1)^2} - \frac{1}{\gamma-1}A_0^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right]\Bigg\}\Psi_\tau \\ &\qquad + \left[1 - \frac{\gamma+1}{\gamma-1}\tau\right]\left[1 + A_0^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\left(\frac{\gamma+1}{\gamma-1}\tau-1\right)\right]^2\Psi_{\theta\theta} = 0 . \end{aligned}$$

We must now seek a solution of Eq. (26) satisfying the boundary conditions (19.1)-(19.6). We shall do this by finding separate solutions in these subareas, labelled (1), (2), (3) in Fig. 2.

For regions (1) and (2) we look for separable solutions of the form

$$(27)\quad \psi^{(1)}(\tau,\theta) = -\frac{Q}{2} + \sum_{n=1}^{\infty} a_n G_n(\tau)\sin 2n\theta,$$

$$(28)\quad \psi^{(2)}(\tau,\theta) = +\frac{Q}{2} + \sum_{n=1}^{\infty} a_n G_n(\tau)\sin 2n\theta.$$

Substituting either one of these expressions into Eq. (26), we obtain the following equation for $G_n(\tau)$:

$$(29)\quad \begin{aligned} &\tau^2(1-\tau)\left[1-A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right]\left[1+A_o^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\left(\frac{\gamma+1}{\gamma-1}\tau-1\right)\right]G_n'' \\ &+\tau\left\{\left[1-A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right]\left[1-\tau-A_o^{-2}(1-\tau)^{\frac{\gamma}{\gamma-1}}+\frac{\tau}{\gamma-1}\right]\right. \\ &\left.\qquad -2A_o^{-2}\,\tau^2(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\left[\frac{\gamma-3}{(\gamma-1)^2}\mp\frac{1}{\gamma-1}A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\right]\right\}G_n' \\ &+n^2\left[\frac{\gamma+1}{\gamma-1}\tau-1\right]\left[1+A_o^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\left(\frac{\gamma+1}{\gamma-1}\tau-1\right)\right]^2 G_n = 0. \end{aligned}$$

We shall consider this equation in greater detail later. For the present let the two linearly independent solutions of the equation be denoted by $G_n(\tau)$ and $H_n(\tau)$. It will be shown later that only one of the solutions remains bounded at $\tau = 0$; let $G_n(\tau)$ be this solution.

The expression assumed for $\psi^{(1)}(\tau,\theta)$ satisfies boundary condition (19.3) and boundary condition (19.2) on sector BE, while the expression assumed for $\psi^{(2)}(\tau,\theta)$ satisfies boundary condition (19.6) and boundary condition (19.5) on sector B'E'.

In region (3), which represents the annular region CDC'E'AEC, we look for a solution of the form

$$(30) \qquad \psi^{(3)}(\tau,\theta) = -\frac{Q}{\pi}\theta + \sum_{n=1}^{\infty} [A_n G_n(\tau) + B_n H_n(\tau)]\sin 2n\theta .$$

This expression for $\psi^{(3)}$ satisfies Eq. (26) identically; it also satisfies boundary conditions (19.2) and (19.5) in sections CE and C'E', respectively. To satisfy boundary condition (19.1) and boundary condition (19.4) on arc CDC' (on which $\tau = \tau_1$) we require

$$-\frac{1}{2}Q = -\frac{Q}{\pi}\theta + \sum_{n=1}^{\infty} [A_n G_n(\tau_1) + B_n H_n(\tau_1)] \sin 2n\theta, \quad 0 < \theta \leq \frac{1}{2}\pi ,$$

$$\frac{1}{2}Q = -\frac{Q}{\pi}\theta + \sum_{n=1}^{\infty} [A_n G_n(\tau_1) + B_n H_n(\tau_1)] \sin 2n\theta, \quad -\frac{1}{2}\pi \leq \theta < 0 ,$$

or equivalently,

$$(31) \qquad \frac{Q}{\pi}\theta \mp \frac{1}{2}Q = \sum_{n=1}^{\infty} [A_n G_n(\tau_1) + B_n H_n(\tau_1)] \sin 2n\theta,$$

where the upper sign is taken when $\theta > 0$, and the lower one when $\theta < 0$.

Since the left-hand side of Eq. (31) is an antisymmetric function of θ in the interval $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$, it can be expanded as a Fourier sine series

$$(32)\quad \frac{Q}{\pi}\theta \mp \frac{1}{2}Q = -\frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\theta,\ -\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi\ .$$

Equating the right-hand sides of Eqs. (31) and (32) we obtain

$$(33)\qquad A_nG_n(\tau_1) + B_nH_n(\tau_1) = -\frac{Q}{\pi n}\ .$$

This completes the satisfaction of boundary conditions (19.1)-(19.6).

There are additional boundary conditions which must be satisfied. These result from the requirement that the expression for $\Psi^{(3)}(\tau,\theta)$ be the analytic continuation of $\Psi^{(1)}$ and $\Psi^{(2)}$ into region (3). This leads to the conditions

$$(34)\quad \begin{aligned} &\Psi^{(3)}(\tau_o,\theta) = \Psi^{(1)}(\tau_o,\theta),\ \frac{\partial\Psi^{(3)}}{\partial\tau}(\tau_o,\theta) = \frac{\partial\Psi^{(1)}}{\partial\tau}(\tau_o,\theta),\ 0 \leq \theta \leq \frac{1}{2}\pi,\\ &\Psi^{(3)}(\tau_o,\theta) = \Psi^{(2)}(\tau_o,\theta),\ \frac{\partial\Psi^{(3)}}{\partial\tau}(\tau_o,\theta) = \frac{\partial\Psi^{(2)}}{\partial\tau}(\tau_o,\theta),\ -\frac{1}{2}\pi \leq \theta \leq 0.\end{aligned}$$

Substituting the expressions for $\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}$ into Eqs. (34) we obtain

$$(35)\quad \begin{cases} -\dfrac{Q}{\pi}\theta + \sum\limits_{n=1}^{\infty}[A_nG_n(\tau_o) + B_nH_n(\tau_o)]\ \sin 2n\theta \\ \qquad = -\dfrac{Q}{2} + \sum\limits_{n=1}^{\infty} a_nG_n(\tau_o)\sin 2n\theta, \qquad 0 \le \theta \le \frac{1}{2}\pi, \\ \sum\limits_{n=1}^{\infty}[A_nG_n'(\tau_o) + B_nH_n'(\tau_o)]\ \sin 2n\theta \\ \qquad = \sum\limits_{n=1}^{\infty} a_nG_n'(\tau_o)\ \sin 2n\theta, \qquad 0 \le \theta \le \frac{1}{2}\pi, \end{cases}$$

$$(36)\quad \begin{cases} -\dfrac{Q}{\pi}\theta + \sum\limits_{n=1}^{\infty}[A_nG_n(\tau_o) + B_nH_n(\tau_o)]\ \sin 2n\theta \\ \qquad = \dfrac{Q}{2} + \sum\limits_{n=1}^{\infty} a_nG_n(\tau_o)\ \sin 2n\theta, \qquad -\frac{1}{2}\pi \le \theta \le 0, \\ \sum\limits_{n=1}^{\infty}[A_nG_n'(\tau_o) + B_nH_n'(\tau_o)]\ \sin 2n\theta \\ \qquad = \sum\limits_{n=1}^{\infty} a_nG_n'(\tau_o)\ \sin 2n\theta, \qquad -\frac{1}{2}\pi \le \theta \le 0 \quad . \end{cases}$$

Combining the first equations from Eqs. (35) and (36), we can write

$$(37)\quad \frac{Q}{\pi}\theta \mp \frac{1}{2}Q = \sum_{n=1}^{\infty}[(A_n-a_n)G_n(\tau_o) + B_nH_n(\tau_o)]\ \sin 2n\theta \ ,$$

where the upper sign is taken when $\theta > 0$, the lower one when $\theta < 0$. As before, the left-hand side can be written as a Fourier sine series (Eq. (32)); Eq. (37) then reduces to

$$(38)\quad (A_n - a_n)\, G_n(\tau_o) + B_n H_n(\tau_o) = -\frac{Q}{\pi n}\,.$$

The second equations of Eqs. (35) and (36) yield the single relation

$$(39)\quad (A_n - a_n) G_n'(\tau_o) + B_n H_n'(\tau_o) = 0.$$

Equations (33), (38), and (39) determine the three sets of unknown coefficients: a_n, A_n, and B_n. Solving these equations, we find

$$(40)\quad \begin{cases} a_n = -\dfrac{Q}{\pi n}\dfrac{1}{G_n(\tau_1)}\left[1 + \dfrac{G_n'(\tau_o)H_n(\tau_1) - G_n(\tau_1)H_n'(\tau_o)}{W_n(\tau_o)}\right], \\[2ex] A_n = \dfrac{-Q}{\pi n}\dfrac{1}{G_n(\tau_1)}\left[1 + \dfrac{G_n'(\tau_o)H_n(\tau_1)}{W_n(\tau_o)}\right], \\[2ex] B_n = \dfrac{Q}{\pi n}\dfrac{G_n'(\tau_o)}{W_n(\tau_o)}\,, \end{cases}$$

where $W_n(\tau_o)$ is the Wronskion of the solutions $G_n(\tau), H_n(\tau)$ evaluated at τ_o, that is

(41) $$W_n(\tau_o) = G_n(\tau_o)H_n'(\tau_o) - G_n'(\tau_o)H_n(\tau_o) \ .$$

If we now substitute these results into the expressions for $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$, the solution takes the form

(42) $$\psi^{(1)}(\tau,\theta) = -\frac{Q}{2} - \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{nG_n(\tau_1)} \left[1 + \frac{G_n'(\tau_o)H_n(\tau_1) - G_n(\tau_1)H_n'(\tau_o)}{W_n(\tau_o)}\right] G_n(\tau)\ \sin 2n\theta,$$

(43) $$\psi^{(2)}(\tau,\theta) = \frac{Q}{2} - \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{nG_n(\tau_1)} \left[1 + \frac{G_n'(\tau_o)H_n(\tau_1) - G_n(\tau_1)H_n'(\tau_o)}{W_n(\tau_o)}\right] G_n(\tau)\ \sin 2n\theta,$$

(44) $$\psi^{(3)}(\tau,\theta) = -\frac{Q}{\pi}\theta - \frac{Q}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{nG_n(\tau_1)}\left\{1 + \frac{G_n'(\tau_o)H_n(\tau_1)}{W_n(\tau_o)}\right\} G_n(\tau) - \left\{\frac{G_n'(\tau_o)}{n\ W_n(\tau_o)}\right\} H_n(\tau)\right] \sin 2n\theta \ .$$

Equation (44) may also be written

(45) $$\frac{\pi}{Q}\ \psi^{(3)}(\tau,\theta) = -\theta - \sum_{n=1}^{\infty}\frac{X_n(\tau)}{n}\ \sin 2n\theta,$$

where

$$(46)\quad X_n(\tau) = \frac{G_n(\tau)}{G_n(\tau_1)} - \frac{1}{W_n(\tau_o)} \frac{H_n(\tau)G_n(\tau_1) - H_n(\tau_1)G_n(\tau)}{G_n(\tau_1)} \quad G_n'(\tau_o) \ .$$

Now that the solution in the hodograph plane has been obtained, it is necessary to transform back to the physical plane. If we introduce the hodograph equations, Eqs. (17), into the transformation relations, Eqs. (18), we obtain

$$dx = \frac{1}{q}\left\{\left[-\frac{\rho_o}{\rho}\sin\theta\ \Psi_\theta + \frac{C_\rho\theta}{1-A^{-2}}\frac{\rho_o}{\rho}q(1-C\rho)^2\right.\right.$$

$$\left.\left[1 - C\rho(1 + \frac{q}{\rho}\frac{d\rho}{dq})\right]^{-1}\Psi_q\right]d\theta$$

$$\left. + \left[-\frac{\rho_o}{\rho}\sin\theta\ \Psi_q + \frac{\cos\theta}{1-A^{-2}}\left[q(1-C\rho)\frac{d}{dq}\left(\frac{\rho_o}{\rho q}\right)\Psi_\theta\right]\right]dq\right\},$$

$$dy = \frac{1}{q}\left\{\left[\frac{\sin\theta}{1-A^{-2}}q(1-C\rho)\frac{d}{dq}\left(\frac{\rho_o}{\rho q}\right)\Psi_\theta + \frac{\rho_o}{\rho}\cos\theta\ \Psi_q\right]dq\right.$$

$$\left. + \left[\frac{\sin\theta}{1-A^{-2}}\frac{\rho_o}{\rho}q(1-C\rho)^2\left[1 - C\rho(1 + \frac{q}{\rho}\frac{d\rho}{dq})\right]^{-1}\Psi_q + \frac{\rho_o}{\rho}\cos\theta\Psi_\theta\right]d\theta\right\}.$$

Integrating these expressions results in

$$x = \frac{\rho_o}{\rho}\frac{1}{q}\left\{-\int^\theta \Psi_\theta \sin\theta d\theta + \frac{q(1-C\rho)^2}{1-A^{-2}}\right.$$

$$\left.\left[1 - C\rho(1 + \frac{q}{\rho}\frac{d\rho}{dq})\right]^{-1}\int^\theta \Psi_q \cos\theta d\theta\right\} + x_o(q),$$

$$y = \frac{1}{q}\left\{ \frac{\rho_o}{(1-A^{-2})}\, q(1-C\rho)^2 \left[1-C\rho\left(1+\frac{q}{\rho}\frac{d\rho}{dq}\right)\right]^{-1} \int^{\theta} \Psi_q \sin\theta d\theta + \frac{\rho_o}{\rho} \int^{\theta} \Psi_\theta \cos\theta d\theta \right\}$$

$$+ y_o(q) \; .$$

Making the change to the variable τ and introducing the relations Eqs. (21)-(25), we finally obtain the following for the coordinates in the physical plane:

$$(47) \quad x = \frac{(1-\tau)^{-\beta}}{q} \left[-\int^{\theta} \Psi_\theta \sin\theta d\theta + 2\tau[1 - A_o^{-2}(1-\tau)^{\beta}] \right.$$

$$[1-A_o^{-2}(1-\tau)^{\beta}\{1-2\beta\tau(1-\tau)^{-1}\}]^{-1} \times$$

$$\left. \int^{\theta} \Psi_\tau \cos\theta d\theta \right] + x_o(\tau) ,$$

$$(48) \quad y = \frac{1}{q}(1-\tau)^{-\beta} \left[2\tau[1-A_o^{-2}(1-\tau)^{\beta}]\; [1-A_o^{-2}(1-\tau)^{\beta} \right.$$

$$\{1-2\beta\tau(1-\tau)^{-1}\}]^{-1} \int^{\theta} \Psi_\tau \sin\theta d\theta$$

$$\left. + \int^{\theta} \Psi_\theta \cos\theta d\theta \right] + y_o(\tau) ,$$

where for convenience we have introduced $\beta = 1/(\gamma-1)$.

The y-coordinate in region (3) can be found by substituting the expression for $\Psi^{(3)}$, Eq. (45), into Eq. (48).

We obtain

$$(49) \quad y^{(3)} = -\frac{Q(1-\tau)^{-\beta}}{\pi q}\left\{\tau R(\tau)\sum_{n=1}^{\infty}\frac{X_n'(\tau)}{n}\left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right]\right.$$

$$\left. + \sin\theta + \sum_{n=1}^{\infty} X_n(\tau)\left[\frac{\sin(2n+1)\theta}{2n+1} + \frac{\sin(2n-1)\theta}{2n-1}\right]\right\},$$

where

$$(50) \quad R(\tau) = [1-A_0^{-2}(1-\tau)^{\beta}][1-A_0^{-2}(1-\tau)^{\beta}\{1-2\beta\tau(1-\tau)^{-1}\}]^{-1} .$$

(Since $y = 0$ when $\theta = 0$, $y_o(\tau)$ has been set equal to zero in Eq. (49).)

Along the outer boundaries of the jet CD and C'D the velocity is constant, equal to τ_1. Setting $\tau = \tau_1$ in Eq. (49), we obtain

$$(51) \quad y^{(3)}(\tau_1,\theta) = -\frac{Q(1-\tau_1)^{-\beta}}{\pi q_1}\left\{\tau_1 R(\tau_1)\sum_{n=1}^{\infty}\frac{X_n'(\tau_1)}{n}\right.$$

$$\left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right]$$

$$\left. + \sin\theta + \sum_{n=1}^{\infty} X_n(\tau_1)\left[\frac{\sin(2n+1)\theta}{2n+1} + \frac{\sin(2n-1)\theta}{2n-1}\right]\right\}.$$

From the expression for $X_n(\tau)$, Eq. (46), we see that $X_n(\tau_1) = 1$. The last sum in the above equation can then be written

$$\sum_{n=1}^{\infty} \left[\frac{\sin(2n+1)\theta}{2n+1} + \frac{\sin(2n-1)\theta}{2n-1}\right] = \sin\theta + 2\sum_{n=1}^{\infty} \frac{\sin(2n+1)\theta}{2n+1} \quad .$$

The right-hand side is the Fourier sine series expansion of the function $-\sin\theta \pm \frac{\pi}{2}$ (with + for $\theta > 0$, − for $\theta < 0$). Hence we can write

$$(52) \quad \sum_{n=1}^{\infty} \left[\frac{\sin(2n+1)\theta}{2n+1} + \frac{\sin(2n-1)\theta}{2n-1}\right] = -\sin\theta \pm \frac{\pi}{2}, \quad \left(\begin{array}{l} + \text{ for } \theta > 0, \\ - \text{ for } \theta < 0 \end{array}\right) .$$

Substituting this expression in Eq. (51) yields

$$(53) \quad y^{(3)}(\tau_1,\theta) = -\frac{Q(1-\tau_1)^{-\beta}}{\pi q_1} \left\{ \tau_1 R(\tau_1) \sum_{n=1}^{\infty} \frac{X_n'(\tau_1)}{n} \left[\frac{\sin(2n-1)\theta}{2n-1} - \frac{\sin(2n+1)\theta}{2n+1}\right] \pm \frac{\pi}{2} \right\}, \quad \theta \gtrless 0 .$$

When $\theta = -\frac{\pi}{2}$, $y^{(3)}$ is equal to half the width of the opening of the jet, $\frac{h}{2}$, and thus

$$y^{(3)}(\tau_1, -\tfrac{\pi}{2}) = \frac{h}{2} = \frac{Q(1-\tau_1)^{-\beta}}{\pi q_1}\left\{\tau_1 R(\tau_1)\sum_{n=1}^{\infty}\frac{X_n'(\tau_1)}{n}\left[\frac{\sin\frac{2n-1)\pi}{2}}{2n-1} - \frac{\sin\frac{(2n+1)\pi}{2}}{2n+1}\right] + \frac{\pi}{2}\right\},$$

or,

$$h = \frac{2Q(1-\tau_1)^{-\beta}}{\pi q_1}\left\{-4\tau_1 R(\tau_1)\sum_{n=1}^{\infty}\frac{(-1)^n X_n'(\tau_1)}{4n^2-1} + \frac{\pi}{2}\right\}. \tag{54}$$

By continuity of mass, we have

$$Q = \text{mass flow} = \frac{\rho_1}{\rho_o} q_1 h' , \tag{55}$$

where h' is the width of the jet at $x = +\infty$. Using Eq. (25), we may rewrite (55) as

$$Q = q_1(1-\tau_1)^{\beta}\, h' .$$

Using this expression to eliminate Q in Eq. (54), we obtain

$$\frac{1}{k} = \frac{h}{h'} = 1 - \frac{8}{\pi}\,\tau_1 R(\tau_1)\sum_{n=1}^{\infty}\frac{(-1)^n X_n'(\tau_1)}{4n^2-1},$$

where k is the coefficient of contraction of the jet.

Differentiating Eq. (46), we find

$$X_n'(\tau_1) = \frac{G_n'(\tau_1)}{G_n(\tau_1)} - \frac{1}{W_n(\tau_o)}\,\frac{H_n'(\tau_1)G_n(\tau_1)-H_n(\tau_1)G_n'(\tau_1)}{G_n(\tau_1)}\,G_n'(\tau_o),$$

or

$$(56)\quad X_n'(\tau_1) = \frac{G_n'(\tau_1)}{G_n(\tau_1)} - \frac{W_n(\tau_1)}{W_n(\tau_o)}\frac{G_n'(\tau_o)}{G_n(\tau_1)} \ .$$

Introducing this into the equation for 1/k above, we obtain

$$(57)\quad \frac{1}{k} = 1 - \frac{8}{\pi}\tau_1 R(\tau_1) \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}\left[\frac{G_n'(\tau_1)}{G_n(\tau_1)} - \frac{W_n(\tau_1)}{W_n(\tau_o)}\frac{G_n'(\tau_o)}{G_n(\tau_1)}\right] \ .$$

For an infinitely broad vessel ($H \to \infty$), τ_o must be zero and the expression above becomes (denoting the value of k in this case by k_∞).

$$\frac{1}{k_\infty} = 1 - \frac{8}{\pi}\tau_1 R(\tau_1) \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}\left[\frac{G_n'(\tau_1)}{G_n(\tau_1)} - \frac{W_n(\tau_1)}{W_n(0)}\frac{G_n'(0)}{G_n(\tau_1)}\right] \ .$$

This should be compared with the Chaplygin formula for the nonmagnetic case

$$\left(\frac{1}{k_\infty}\right)_{\text{nonmagnetic}} = 1 - \frac{8}{\pi}\tau_1 \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}\frac{Z_n'(\tau_1)}{Z_n(\tau_1)} \ ,$$

where

$$Z_n(\tau) = \tau^n F(a_n, b_n, 2n+1; \tau) \ .$$

APPENDIX

The equation obtained after separating variables was shown to be Eq. (29), that is,

$$\tau^2(1-\tau)\ [1-A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}]\ [1 + A_o^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}(\frac{\gamma+1}{\gamma-1}\ \tau-1)]G_n''(\tau)$$

$$+ \tau \Big\{ [1 - A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}]\ [1-\tau-A_o^{-2}(1-\tau)^{\frac{\gamma}{\gamma-1}} + \frac{\tau}{\gamma-1}]$$

(A-1)

$$- 2A_o^{-2}\tau^2(1-\tau)^{\frac{2-\gamma}{\gamma-1}}[\frac{\gamma-3}{(\gamma-1)^2} - \frac{1}{\gamma-1}\ A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}] \Big\}\ G_n'(\tau)$$

$$+ n^2\ [\frac{\gamma+1}{\gamma-1}\ \tau-1]\ [1 + A_o^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}(\frac{\gamma+1}{\gamma-1}\ \tau-1)]^2\ G_n(\tau) = 0.$$

The solution we require is the one which remains bounded when $\tau = 0$. In this Appendix we shall present a brief discussion of Eq. (A-1). Complete details will be given in a subsequent paper. First, we note that when the magnetic field vanishes, i.e., $A_o^{-2} = 0$, the equation reduces to

(A-2) $$\tau^2(1-\tau)\overline{G}_n'' + \tau[1 + (\beta-1)\tau]\overline{G}_n' + n^2[(1 + 2\beta)\tau-1]\overline{G}_n = 0.$$

This is the usual equation which arises in the study of the hodograph technique [7]. The solution of this equation which remains bounded at $\tau = 0$ is given by

$$\text{(A-3)} \qquad \bar{G}_n(\tau) = \tau^n F(a_n, b_n, 2n+1; \tau),$$

where a_n, b_n are determined from

$$\text{(A-4)} \qquad a_n + b_n = 2n - \beta,$$

$$a_n b_n = -\beta n(2n+1),$$

and $F(a_n, b_n, 2n+1; \tau)$ is the hypergeometric function, whose power series,

$$\text{(A-5)} \quad F(a_n, b_n, 2n+1; \tau) = \frac{\Gamma(2n+1)}{\Gamma(a_n)\Gamma(b_n)} \sum_{m=0}^{\infty} \frac{\Gamma(a_n+m)\Gamma(b_n+m)}{\Gamma(2n+1+m)} \frac{\tau^m}{m!},$$

is uniformly convergent in the domain $|\tau| < 1$.

When $n = 1$, Eq. (A-1) may be solved exactly (see [4]). For $n \neq 1$, the primary difficulty in solving the equation resides in the fact that, for general γ, the coefficients involve nonintegral powers of τ. Note, however, that if

$$\text{(A-6)} \qquad \frac{1}{\gamma-1} = k, \text{ or } \gamma = 1 + \frac{1}{k},$$

where k is a positive integer, then the coefficients are polynomials in τ. In this case the equation becomes

$$
\begin{aligned}
&\tau^2(1-\tau)[1-A_o^{-2}(1-\tau)^k][1+A_o^{-2}(1-\tau)^{k-1}((2k+1)\tau-1)]G_n'' \\
&+\tau\{[1-A_o^{-2}(1-\tau)^k][1-\tau-A_o^{-2}(1-\tau)^{k+1}+k\tau] \\
&\quad - 2A_o^{-2}\tau^2(1-\tau)^{k-1}[(k-2k^2)-kA_o^{-2}(1-\tau)^k]\}G_n' \\
&+n^2[(2k+1)\tau-1][1+A_o^{-2}(1-\tau)^{k-1}((2k+1)\tau-1)]^2\ G_n = 0.
\end{aligned}
\tag{A-7}
$$

The finite singularities of this equation are given by

$$
\begin{aligned}
&\tau = 0, \\
&\tau = 1, \\
&\tau = 1 - A_o^{2/k}, \\
&(1-\tau)^{k-1}[1-(2k+1)\tau] = A_o^2.
\end{aligned}
\tag{A-8}
$$

Since k is a positive integer, the last expression has exactly k roots. Thus Eq. (A-7) has (3+k) finite singularities. In the special case $k = 1$ ($\gamma = 2$) the four finite singularities are

$$
\begin{aligned}
&\tau = 0, \\
&\tau = 1, \\
&\tau = 1 - A_o^2, \\
&\tau = \tfrac{1}{3}(1-A_o^2).
\end{aligned}
\tag{A-9}
$$

One can show that $\tau = \infty$ is also a singular point.

Although $\tau = 0$ is a singular point of the general equation, Eq. (29) or (A-1), this equation, like the hypergeometric equation to which it reduces when $A_o^{-2} = 0$, has one regular solution near $\tau = 0$. It is this solution which has been called $G_n(\tau)$. To find this regular solution we write $G_n(\tau)$ as

$$G_n(\tau) = \sum^{\infty} a_{nk}\tau^k,$$

and substitute in Eq. (A-1). At the same time the coefficients of the equation are also expanded in series in τ. The following recurrence relation for the coefficients a_{nk} is then obtained:

$$(1-A_o^{-2})^2(k^2-n^2)a_{nk} - [(k-1)(k-2) + \frac{1}{\gamma-1}\{\gamma-2 + 3A_o^{-2} - (\gamma+1)A_o^{-4}\}(k-1)$$

(A-10)
$$-n^2\{\frac{\gamma+1}{\gamma-1} + \frac{1}{\gamma-1}A_o^{-2}[-6\gamma + A_o^{-2}(5\gamma-1)]\}]\,a_{n,k-1} + n^2\frac{A_o^{-2}}{(\gamma-1)^2}$$

$$[8\gamma^2 - 7\gamma - A_o^{-2}(12\gamma^2 - 11\gamma + 1)]a_{n,k-2},$$

$$+ A_o^{-2}\left[\sum_{\substack{\ell=1\\ s=k-1\\ \ell+s=k}}^{\substack{s=0\\ \ell=k}} C_\ell^1\, s(s-1)a_{ns} - \sum_{\substack{\ell=2\\ s=k-2\\ \ell+s=k}}^{\substack{s=0\\ \ell=k}} C_\ell^2\, s a_{ns} + \right.$$

$$\left. n^2\sum_{\substack{\ell=3\\ s=k-3\\ \ell+s=k}}^{\substack{s=0\\ \ell=k}} C_\ell^3\, a_{ns}\right] = 0, \qquad k \geq 0,$$

where

$$\text{(A-11-a)}\quad c_\ell^1 = \frac{\gamma+1}{\gamma-1} C\binom{\frac{1}{\gamma-1}}{\frac{1}{\gamma-1}-\ell+1} - \frac{\gamma+1}{\gamma-1} A_o^{-2}\, C\binom{\frac{2}{\gamma-1}}{\frac{2}{\gamma-1}-\ell+1}$$

$$-C\binom{\frac{1}{\gamma-1}}{\frac{1}{\gamma-1}-\ell} - C\binom{\frac{\gamma}{\gamma-1}}{\frac{\gamma}{\gamma-1}-\ell} + A_o^{-2}\, C\binom{\frac{2}{\gamma-1}}{\frac{2}{\gamma-1}-\ell},$$

$$\text{(A-11-b)}\quad c_\ell^2 = C\binom{\frac{1}{\gamma-1}}{\frac{1}{\gamma-1}-\ell} + C\binom{\frac{\gamma}{\gamma-1}}{\frac{\gamma}{\gamma-1}-\ell} - A_o^{-2}\, C\binom{\frac{\gamma+1}{\gamma-1}}{\frac{\gamma+1}{\gamma-1}-\ell}$$

$$+ \frac{2-\gamma}{\gamma-1} C\binom{\frac{1}{\gamma-1}}{\frac{1}{\gamma-1}-\ell+1} + 2\,\frac{\gamma-3}{(\gamma-1)^2} C\binom{\frac{2-\gamma}{\gamma-1}}{\frac{2-\gamma}{\gamma-1}-\ell+2}$$

$$+ \frac{2A_o^{-2}}{1-\gamma}\, C\binom{\frac{3-\gamma}{\gamma-1}}{\frac{3-\gamma}{\gamma-1}-\ell+2},$$

$$\text{(A-11-c)}\quad c_\ell^3 = 2\left(\frac{\gamma+1}{\gamma-1}\right)^2 C\binom{\frac{2-\gamma}{\gamma-1}}{\frac{2-\gamma}{\gamma-1}-q+2} - 3A_o^{-2}\left(\frac{\gamma+1}{\gamma-1}\right)^2 C\binom{2\left(\frac{2-\gamma}{\gamma-1}\right)}{2\left(\frac{2-\gamma}{\gamma-1}\right)-q+2}$$

$$+ A_o^{-2}\left(\frac{\gamma+1}{\gamma-1}\right)^3 C\binom{2\left(\frac{2-\gamma}{\gamma-1}\right)}{2\left(\frac{2-\gamma}{\gamma-1}\right)-q+3}$$

$$+ 2\, C\binom{\frac{2-\gamma}{\gamma-1}}{\frac{2-\gamma}{\gamma-1}-q} - A_o^{-2}\, C\binom{2\left(\frac{2-\gamma}{\gamma-1}\right)}{2\left(\frac{2-\gamma}{\gamma-1}\right)-q}$$

$$- 4\,\frac{\gamma+1}{\gamma-1} C\binom{\frac{2-\gamma}{\gamma-1}}{\frac{2-\gamma}{\gamma-1}-q+1}$$

$$+ 3\, A_o^{-2}\,\frac{\gamma+1}{\gamma-1} C\binom{2\left(\frac{2-\gamma}{\gamma-1}\right)}{2\left(\frac{2-\gamma}{\gamma-1}\right)-q+1},$$

and $C\binom{n}{m}$ is defined in the usual way

$$C\binom{n}{m} = \frac{n!}{(n-m)!\,m!} \quad .$$

According to (A-10) each a_{nk} is given in terms of all previous a_{nk}, whereas in the nonmagnetic case the recurrence relation involves a_{nk} and $a_{n,k-1}$ only. Thus, it seems impossible to write down the various a_{nk} explicitly. However, we do note that the first nonzero coefficient occurs when $k = n$. It is then more convenient to write our solution as follows

$$\text{(A-12)} \qquad G_n(\tau) = \sum_{k=0}^{\infty} a_{nk}\,\tau^k = \sum_{k=n}^{\infty} a_{nk}\,\tau^k = \tau^n \sum_{k=n}^{\infty} a_{nk}\,\tau^{k-n}$$

$$= \tau^n \sum_{p=0}^{\infty} a_{n,p+n}\tau^p = \tau^n \sum_{p=0}^{\infty} \bar{a}_{np}\,\tau^p \quad .$$

The recurrence relation for $\bar{a}_{np}$, obtained directly from (A-10), is found to be

$$(1-A_o^{-2})^2 p(p+2n)\bar{a}_{np} - \Big[[(p+n-1)(p+n-2)+ \frac{1}{\gamma-1}\{\gamma-2 + 3A_o^{-2}-(\gamma+1)A_o^{-4}\}(p+n-1)$$

$$- n^2\{\frac{\gamma+1}{\gamma-1} + \frac{1}{\gamma-1} A_o^{-2}(-6\gamma+A_o^{-2}(5\gamma-1))\}\Big]\bar{a}_{n,p-1} + n^2 \frac{A_o^{-2}}{(\gamma-1)^2}$$

$$[8\gamma^2-7\gamma-A_o^{-2}(12\gamma^2-11\gamma+1)]\bar{a}_{n,p-2}$$

$$+ A_o^{-2}\left[\sum_{\substack{\ell=1\\ s=p+n-1\\ s+\ell=p+n}}^{\substack{s=0\\ \ell=p+n}} C_\ell^1\, s(s-1)\bar{a}_{n,s-n} - \right. \tag{A-13}$$

$$\left. \sum_{\substack{\ell=2\\ s=p+n-2\\ s+\ell=p+n}}^{\substack{s=0\\ \ell=p+n}} C_\ell^2\, s\, \bar{a}_{n,s-n} + n^2 \sum_{\substack{\ell=3\\ s=p+n-3\\ s+\ell=p+n}}^{\substack{s=0\\ \ell=p+n}} C_\ell^3\, \bar{a}_{n,s-n}\right] = 0 .$$

The first few terms in the series for $G_n(\tau)$ may now be calculated from (A-13). We find

$$G_n(\tau) = \bar{a}_{no}\, \tau^n \Bigg\{ 1 + \frac{1}{(1-A_o^{-2})^2(1+2n)}\, [n(n-1)+Bn-n^2D+n(n-1)A_o^{-4}]\tau$$

$$+ \frac{1}{(1-A_o^{-2})^2 2^2(1+n)}\, \Bigg\{[n(n+1)+B(n+1)-n^2D+A_o^{-4}n(n+1)]$$

$$\frac{[n(n-1)+Bn-n^2D+n(n-1)A_o^{-4}]}{(1-A_o^{-2})^2(1+2n)} \tag{A-14}$$

$$- \left[n^2E + \frac{n(n-1)A_o^{-2}}{(\gamma-1)^2}\{\gamma+A_o^{-2}(1-3\gamma)\} - \frac{A_o^{-2}n}{(\gamma-1)^2}\{\gamma-3+A_o^{-2}(1-3\gamma)\}\right]\Bigg\}\tau^2$$

$$+ \ldots \Bigg\},$$

where

$$B = \frac{1}{\gamma-1}\,[\gamma-2 + 3\,A_o^{-2} - (\gamma+1)A_o^{-4}]\,,$$

(A-15) $$D = \frac{\gamma+1}{\gamma-1} + \frac{1}{\gamma-1}\,A_o^{-2}(-6\gamma + A_o^{-2}(5\gamma-1))\,,$$

$$E = \frac{A_o^{-2}}{(\gamma-1)^2}\,[8\gamma^2 - 7\gamma - A_o^{-2}(12\gamma^2 - 11\gamma + 1)]\,.$$

If A_o^{-2} is set equal t zero in (A-14) the series reduces to

(A-16) $$G_n(\tau) = \bar{a}_{no}\,\tau^n\,F(a_n, b_n, 2n+1;\tau)\,,$$

where a_n and b_n are the same quantities defined in (A-4).

In addition to $G_n(\tau)$, it is necessary that we know $W_n(\tau)$, the Wronskian of the two linearly independent solutions of (A-1). This is given by

(A-17) $$W_n(\tau) = W_0\left(\exp - \int_{\tau_o}^{\tau} \frac{b(\tau)}{a(\tau)}\,d\tau\right),$$

where

$$a(\tau) = \tau(1-\tau)[1-A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}]\;[1 + A_o^{-2}(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\,(\tfrac{\gamma+1}{\gamma-1}\,\tau-1)]\,,$$

$$b(\tau) = [1-A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}]\;[1-\tau-\bar{A}_o^{2}(1-\tau)^{\frac{\gamma}{\gamma-1}} + \tfrac{\tau}{\gamma-1}]$$

$$- 2A_o^{-2}\,\tau^2(1-\tau)^{\frac{2-\gamma}{\gamma-1}}\,[\frac{\gamma-3}{(\gamma-1)^2} - \frac{1}{\gamma-1}\,A_o^{-2}(1-\tau)^{\frac{1}{\gamma-1}}\,.$$

Equations (A-14) and (A-17) are sufficient to enable one to calculate k, the coefficient of contraction of the jet, using Eq. (57).

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