FOUR-COLOR REDUCIBILITY OF PLANAR GRAPHS CONTAINING SUBGRAPHS WITH FOUR-POINT BOUNDARIES

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It is well known that a planar graph containing a proper subgraph with only three points in the boundary between the subgraph and its complement is four-color reducible—that is, if the subgraph and the complement plus the boundary are each colorable with four colors, then the original graph is four-colorable. A proof of reducibility has been lacking for the case of subgraphs whose boundaries contain four points. In the case of the three-point boundary, if the graph is fully triangulated, the three boundary points form a triangle, and they must be assigned different colors: hence a common coloring can be obtained by relabeling one of the two components so that they match on the three points. If the boundary contains four points, however, in the fully triangulated case the boundary points will form a square. The two components can be completed by connecting opposite corners of the square, for example, z, y in Fig 1, in which case z and y must be assigned different colors, but there is no way of determining whether the two remaining corners will have the same or different colors.

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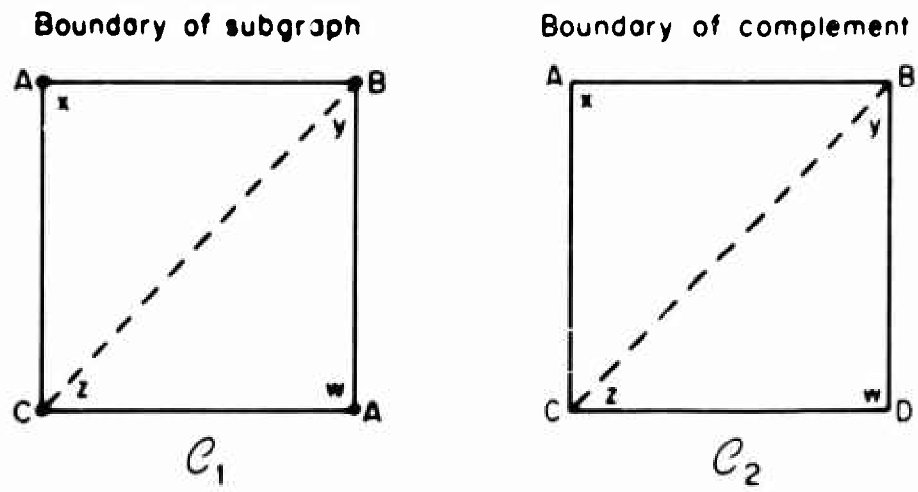

Fig. 1

In the following it will be shown, under an inductive hypothesis that graphs smaller than the original graph can be four-colored, that each of the components will have several possible colorings, and there will always be a pair of colorings for the components that will match.

Under the inductive hypothesis, out of the four possible colorings of the boundary squares for subgraphs and complements completed as in Fig. 1, the only two cases that will not match after relabeling are the ones where x and w have the same colors in one component and different colors in the other. Assume that the situation is as shown in Fig. 1. Now, in $\mathcal{C}_2$ the color of w can be changed to A without affecting the color of x unless there is a chain of points within the complement connecting x and w alternatively colored A, D, A, ... A, D, as shown in Fig. 2. For the color of w can be changed to A, the color of all points connected to w and colored A can be changed to D, and so on. In the case of finite graphs, this process will terminate, and, in the absence of an A - D chain between x and w, the color of x will remain unchanged. Hence, in the absence of such a chain, $\mathcal{C}_2$ can be recolored to a $\mathcal{C}_2'$, which matches $\mathcal{C}_1$.

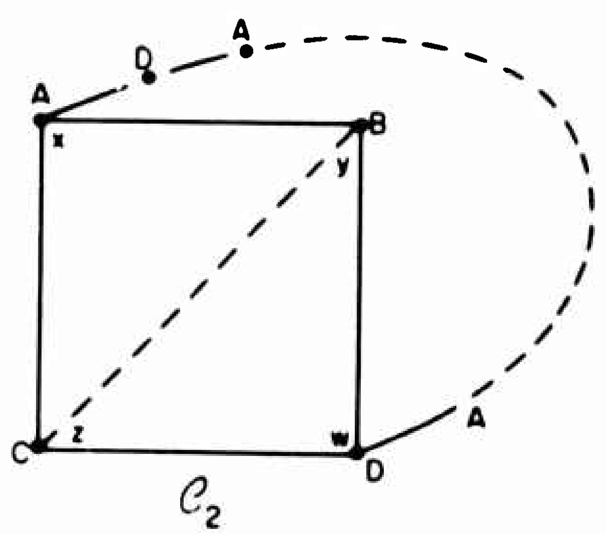

Fig. 2

In the case where there is an A - D chain between x and w, then y and z are isolated in the sense that if the added connection between y and z is removed, there cannot be a B - C chain between y and z, for it would have to intersect the A - D chain between x and w, and none of the points on that chain are colored either B or C. Hence when an A - D chain between x and w exists, $\mathcal{C}_2$ can be recolored to $\mathcal{C}''_2$ so that y and z have the same color. This does not produce a match with $\mathcal{C}_1$.

In this case, consider the situation with the added connection between x and w, rather than between y and z, for the subgraph. (See Fig. 3.) Under the inductive

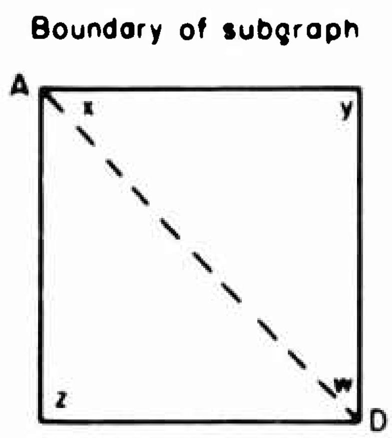

Fig. 3

hypothesis, there is a coloring for the completed subgraph, in which x and w have different colors, but the colors of y and z are not determined. There are two possibilities: $\mathcal{C}'_1$ where y and z have different colors; and $\mathcal{C}''_1$, where y and z have the same color. $\mathcal{C}'_1$ matches $\mathcal{C}_2$, and $\mathcal{C}''_1$ matches $\mathcal{C}''_2$.

A similar argument will, of course, hold for the converse case where the original colorings have different colors for x and w in the subgraph and the same color for x and w in the complement. Thus, in all cases there is a pair of compatible colorings.