THE LEBESGUE-STIELJES INTEGRAL AS APPLIED IN PROBABILITY DISTRIBUTION THEORY

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by

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ABSTRACT

Necessary definitions and theorems from real variable dealing with some properties of Lebesgue-Stieljes measures, monotone non-decreasing functions, Borel sets, functions of bounded variation and Borel measurable functions are set forth in the introduction. Chapter 2 is concerned with establishing a one to one correspondence between Lebesgue-Stieljes measures and certain equivalence classes of functions which are monotone non decreasing and continuous on the right. In Chapter 3 the Lebesgue-Stieljes Integral is defined and some of its properties are demonstrated. In Chapter 4 probability distribution function is defined and the notions in Chapters 2 and 3 are used to show that the Lebesgue-Stieljes integral of any probability distribution function can be expressed as a countable sum of positive numbers added to the Lebesgue-Stieljes integral of a continuous probability distribution function. The conclusion indicates how the Lebesgue-Stieljes integral may be used to define the probability associated with a Borel set of real numbers.

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Chapter 1

INTRODUCTION

The terminology and notation used in the thesis is defined below. Certain elementary theorems are stated without proof and proofs are indicated for a few properties of Borel sets, Lebesgue-Stieljes measures, functions of bounded variation and Borel measurable functions. These theorems and properties are used in the subsequent chapters. The proofs are included in the introduction to avoid breaking the continuity of various discussions.

DEFINITION 1.1

R is the collection of all real numbers.

DEFINITION 1.2

R* is the collection of all real numbers and $+\infty$

DEFINITION 1.3

A set is any collection of real numbers.

DEFINITION 1.4

A class is a collection of anything other than real numbers.

DEFINITION 1.5

An algebra $\mathcal{A}$ is a non empty class of subsets of R such that if A and B are in $\mathcal{A}$ so is $A \cup B$ and if A is in $\mathcal{A}$ so is $\bar{A}$.

THEOREM 1.1

An algebra $\mathcal{A}$ is closed for the taking of finite unions and intersections. R and $\emptyset$ are elements of $\mathcal{A}$.

DEFINITION 1.6

A σ-algebra $\mathcal{S}$ is an algebra where every union of a countable number of sets in $\mathcal{S}$ is again in $\mathcal{S}$.

THEOREM 1.2

A σ-algebra $\mathcal{S}$ is closed for the taking of countable intersections.

THEOREM 1.3

There exists a minimal σ-algebra which contains the class of all intervals.

Proof: Let K denote the collection of all σ-algebras that contain the class of all intervals. The class of all subsets of R is an element of K and therefore K is not empty. Let

$$\mathcal{B} = \bigcap \{ \mathcal{A} : \mathcal{A} \text{ is in } K \}$$

Then $\mathcal{B}$ is a σ-algebra and if $\mathcal{A}$ is a σ-algebra in K, $\mathcal{B}$ is a subclass of $\mathcal{A}$. Further $\mathcal{B}$ contains the class of all intervals and hence $\mathcal{B}$ is in K. $\mathcal{B}$ is therefore the minimal σ-algebra containing the class of all intervals.

DEFINITION 1.7

The class $\mathcal{B}$ is the class of Borel sets.

DEFINITION 1.8

A function on A to B mates every element of A, the domain of the function, with a unique element of B. It is not necessary that all elements of B be used.

DEFINITION 1.9

A set function, φ, is a function on a given class of sets to R* such that φ mates at least one set to an element of R.

DEFINITION 1.10

A countably additive set function, φ, is a set function such that for every $\bigcup_{i=1}^{\infty} A_i$ in the domain of φ where the A_i's are disjoint sets in the domain of φ

$$\varphi\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \varphi A_i$$

DEFINITION 1.11

A measure is a non-negative, countably additive set function defined on an algebra.

DEFINITION 1.12

A Lebesgue-Stieljes measure, μ, is a measure that mates finite numbers to finite intervals.

THEOREM 1.4

Let μ be a Lebesgue-Stieljes measure. If $B_1 \subset B_2$ and both B_1 and B_2 are in the domain of μ, then

$$\mu B_1 \leq \mu B_2.$$

Proof: Since $B_2 - B_1 = B_2 \cap \overline{B}_1$, $B_2 - B_1$ is in the domain of μ.

$$\begin{aligned} \mu B_2 &= \mu[(B_2 - B_1) \cup B_1] \\ &= \mu(B_2 - B_1) + \mu B_1 \\ &\geq \mu B_1 \end{aligned}$$

THEOREM 1.5

If μ is a Lebesgue-Stieljes measure, then

$$\mu \emptyset = 0$$

Proof:

$$\begin{aligned} \mu A &= \mu(A + \emptyset) \\ &= \mu A + \mu \emptyset. \end{aligned}$$

DEFINITION 1.13

$\mathcal{M}$ is the class of all monotone non-decreasing functions defined on R and continuous on the right.

DEFINITION 1.14

F_1 and F_2 are r-related if F_1 and F_2 are functions in $\mathcal{M}$ that differ by a constant.

THEOREM 1.6

The r-relation divides $\mathcal{M}$ into equivalence classes.

Proof: The r-relation is evidently symmetric, reflexive and transitive.

THEOREM 1.7

Every function in $\mathcal{M}$ is in one and only one equivalence class.

DEFINITION 1.15

E is the collection of all equivalence classes in $\mathcal{M}$.

DEFINITION 1.16

Let F be a function defined on R and let b be an element of R^*. Suppose F is such that $\lim_{x \to -\infty} F(x)$ exists and, in case $b = +\infty$, $\lim_{x \to +\infty} F(x)$ exists. Define $F(-\infty) = \lim_{x \to -\infty} F(x)$ and in case $b = +\infty$ define $F(b) = \lim_{x \to \infty} F(x)$. If there exists a "finite partition", $-\infty = x_0 < x_1 < \cdots < x_n = b$,

$$\sum_{i=1}^{n} | F(x_i) - F(x_{i-1}) | < k$$

for some real number k, then F is a function of bounded variation on $(-\infty, b]$. In case $b = +\infty$, F will be said to be of bounded variation on R (or simply a function of bounded variation.)

THEOREM 1.8

If F is of bounded variation on $(-\infty, b]$, then F equals the difference of two monotone non-decreasing functions on $(-\infty, b]$. The proof of this follows:

LEMMA 1.8.1

For every finite partition of $(-\infty, b]$,

$$F(b) - F(-\infty) = \sum_{i=1}^{\infty} [F(x_i) - F(x_{i-1})]$$

DEFINITION 1.16.1

The total variation of F on $(-\infty, b]$, $V_{-\infty}^{b}$, is

$$\sup \sum_{i=1}^{n} | F(x_i) - F(x_{i-1})|.$$

Evidently $V_{-\infty}^{b} \leq k$.

LEMMA 1.8.2

For every finite partition of $(-\infty, b]$

$$F(b) - F(-\infty) = \Sigma_{+} + \Sigma_{-}$$

where Σ_{+} is the sum of all the positive terms in $\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$ and Σ_{-} is the sum of the other terms.

DEFINITION 1.16.2

The positive variation of F, $P_{-\infty}^{b}$, is the supremum of Σ_{+} over all finite partitions of $(-\infty, b]$. The negative variation of F, $N_{-\infty}^{b}$, is the supremum of $-\Sigma_{-}$ for all finite partitions of $(-\infty, b]$.

LEMMA 1.8.3

$$P_{-\infty}^{b} = \frac{1}{2} [V_{-\infty}^{b} + F(b) - F(-\infty)]$$

$$N_{-\infty}^{b} = \frac{1}{2} [V_{-\infty}^{b} + F(-\infty) - F(b)]$$

Proof: Since

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \Sigma_+ + \Sigma_- = F(b) - F(-\infty)$$

and

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \Sigma_+ - \Sigma_- \leq V_{-\infty}^{b}$$

it follows that

$$\Sigma_+ \leq \tfrac{1}{2}\left[V_{-\infty}^{b} + F(b) - F(-\infty)\right]$$

and

$$-\Sigma_- \leq \tfrac{1}{2}\left[V_{-\infty}^{b} + F(-\infty) - F(b)\right].$$

On the other hand for every $\epsilon > 0$ there exists a finite partition of $(-\infty, b]$ such that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| > V_{-\infty}^{b} - \epsilon.$$

Hence for this partition a similar argument shows that

$$\Sigma_+ > \tfrac{1}{2}\left[V_{-\infty}^{b} - \epsilon + F(b) - F(-\infty)\right]$$

and

$$-\Sigma_- > \tfrac{1}{2}\left[V_{-\infty}^{b} - \epsilon - F(b) + F(-\infty)\right]$$

Thus the lemma holds.

LEMMA 1.8.4

$$V_{-\infty}^{b} = P_{-\infty}^{b} + N_{-\infty}^{b}$$

$$F(b) - F(+\infty) = P_{-\infty}^{b} - N_{-\infty}^{b}$$

Proof: These equations follow from adding and subtracting the equations of the preceding lemma.

LEMMA 1.8.5

For all x ,

$$F(x) = P_{-\infty}^{x} - [N_{-\infty}^{x} - F(-\infty)].$$

LEMMA 1.8.6

If $x < x'$, then

$$N_{-\infty}^{x} \leq N_{-\infty}^{x'} \quad \text{and} \quad P_{-\infty}^{x} \leq P_{-\infty}^{x'}$$

Proof: Obviously Σ_+ cannot be greater for $(-\infty, x]$ than for $(-\infty, x']$. Similarly $-\Sigma_-$ cannot be greater for $(-\infty, x]$ than for $(-\infty, x']$. The theorem follows from lemma 1.8.5 and lemma 1.8.6.

DEFINITION 1.17

A function g is Borel measurable if $\{x: g(x) \geq k\}$ is a Borel set for every k.

THEOREM 1.9

If g is Borel measurable, then $\{x: g(x) < k\}$, $\{x: g(x) \leq k\}$ and $\{x: g(x) > k\}$ are Borel sets for every k.

Proof: Since

$$\overline{\{x: g(x) \geq k\}} = \{x: g(x) < k\}$$

for every k and the Borel sets are closed for the taking of complements, $\{x: g(x) < k\}$ is a Borel set for every k. Since

$$\bigcap_{i=1}^{\infty} \{x: g(x) < k + \tfrac{1}{i}\} = \{x: g(x) \leq k\}$$

for every k and the Borel sets are closed for the taking of countable intersections, $\{x: g(x) \leq k\}$ is a Borel set. Finally $\{x: g(x) > k\}$

is a Borel set for all k because

$$\overline{\{x: g(x) \le k\}} = \{x: g(x) > k\}.$$

THEOREM 1.10

If g is a Borel measurable function, Kg is a Borel measurable function for every fixed real number K.

Proof: When $K = 0$, the theorem is obvious. When $K > 0$

$$\{x: Kg(x) \ge k\} = \{x: g(x) \ge \frac{k}{K}\}.$$

When $K < 0$

$$\{x: Kg(x) \ge k\} = \{x: g(x) \le \frac{k}{K}\}.$$

THEOREM 1.11

If g_1 and g_2 are Borel measurable, then $g_1 + g_2$ is Borel measurable.

Proof: If $g_1(x) + g_2(x) < k$, there exists a rational number r such that

$$g_1(x) < r < k - g_2(x).$$

Hence writing the rationals in a sequence $r_1, r_2, \cdots$,

$$\{x: g_1(x) + g_2(x) < k\} \subset \bigcup_{i=1}^{\infty} [\{x: g_1(x) < r_i\} \cap \{x: g_2(x) < k - r_i\}]$$

On the other hand, if there exists a rational number r_n such that $g_1(x) < r_n$ and $g_2(x) < k - r_n$, then $g_1(x) + g_2(x) < k$. It follows that

$$\{x: g_1(x) + g_2(x) < k\} \supset \bigcup_{i=1}^{\infty} [\{x: g_1(x) < r_i\} \cap \{x: g_2(x) < k - r_i\}].$$

Hence

$$\{x: g_1(x) + g_2(x) < k\} = \bigcup_{i=1}^{\infty} [\{x: g_1(x) < r_i\} \cap \{x: g_2(x) < k - r_i\}]$$

Taking complements

$$\{x: g_1(x)+g_2(x) \geq k\} = \bigcap_{i=1}^{\infty} [\{x: g_1(x) \geq r_i\} \cup \{x: g_2(x) \geq k - r_i\}]$$

<u>THEOREM 1.12</u>

If for every n, g_n is Borel measurable and if

$$\lim_{n \to \infty} g_n(x) = g(x)$$

then g is also Borel measurable.

Proof: Take an x in $\{x: g(x) < k\}$ and choose m large enough that $\frac{1}{m} < \frac{1}{2}[k - g(x)]$. Because of convergence there exists an N such that for every $n > N$

$$g_n(x) < g(x) + \frac{1}{m} < k - \frac{1}{m}$$

Hence x is in

$$\bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \{x: g_n(x) < k - \frac{1}{m}\}.$$

On the other hand take x in $\bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \{x: g_n(x) < k - \frac{1}{m}\}$.

Then for some m there exists an N such that for every $n > N$,

$$g_n(x) < k - \frac{1}{m}.$$

Because of convergence

$$g(x) \leq k - \frac{1}{m} < k.$$

Hence x is in

$$\{x: g(x) < k\}.$$

Then

$$\{x: g(x) < k\} = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} \{x: g_n(x) < k - \tfrac{1}{m}\}.$$

Taking complements and observing that the Borel measurability of the g_n's implies the set on the right is a Borel set, it follows that $\{x: g(x) \geq k\}$ is a Borel set and hence $g(x)$ is Borel measurable.

Chapter 2

FUNCTIONS OF $\mathcal{M}$ AND LEBESGUE-STIELJES MEASURES

It will be shown that there exists a biunique correspondence between the equivalence classes in E and all Lebesgue-Stieljes measures on $\mathcal{B}$.

THEOREM 2.1:

For every M in E there exists a unique Lebesgue-Stieljes measure, μ, such that for each F in M and for every $a < b$

$$\mu(a,b] = F(b) - F(a)$$

The proof of theorem 2.1 proceeds as follows:

DEFINITION 2.1.1

$$C_1 = \{\emptyset, (a,b], (-\infty, b], (a,\infty), R \text{ for every } a<b\}$$

LEMMA 2.1.1

C_1 is closed for the taking of finite intersections.

LEMMA 2.1.2

The complement of any set in C_1 is in C_1 or is the union of two disjoint sets in C_1.

LEMMA 2.1.3

The union of any two overlapping or abutting sets in C_1 is in C_1.

DEFINITION 2.1.2

$$\mu\emptyset = 0$$

$$\mu(a,b] = F(b) - F(a)$$

$$\mu(-\infty,b] = \lim_{x\to-\infty} \mu(x,b]$$

$$\mu(a,\infty) = \lim_{x\to\infty} \mu(a,x]$$

$$\mu R = \lim_{\substack{x\to\infty \\ y\to-\infty}} \mu(y,x]$$

LEMMA 2.1.4

Every F in a given M determines the same μ .

DEFINITION 2.1.3

$C_2 = \{A$: either A is in C_1 or $A = \bigcup_{i=1}^{n} A_i$ where the A_i's are disjoint sets in $C_1\}$

LEMMA 2.1.5

$C_1 \subset C_2$

LEMMA 2.1.6

C_2 is closed for the taking of finite unions.

Proof: First consider that if A is in C_1 and $\bigcup_{i=1}^{n} B_i$ is such that every B_i is in C_1 , $\bigcup_{i=1}^{n} B_i \cup A$ is in C_2. This follows from the distributive law for unions, Lemma 2.1.3 and the definition of C_2. Again considering the distributive law for unions, the union of any two sets in C_2 is in C_2. The lemma follows by induction.

LEMMA 2.1.7

C_2 is closed for the taking of finite intersections.

Proof: The lemma follows from the distributive law for intersections, lemma 2.1.1, the definition of C_2, and induction.

LEMMA 2.1.8

C_2 is closed for the taking of complements.

Proof: If A is in C_2 and every A_i is in C_1,

$$\bar{A} = \overline{\bigcup_{i=1}^{n} A_i}$$

$$= \bigcap_{i=1}^{n} \bar{A}_i$$

It follows from lemma 2.1.2 that every $\overline{A}_i$ is in C_2. The lemma follows from lemma 2.1.7.

LEMMA 2.1.8

C_2 is an algebra of sets.

DEFINITION 2.1.4

For every A in C_2 let

$$\mu A = \sum_{i=1}^{n} \mu A_i$$

where $\bigcup_{i=1}^{n} A_i = A$ and the A_i's are disjoint sets in C_1.

LEMMA 2.1.9

μ is uniquely defined on C_2.

Proof: If $S = \bigcup_{i=1}^{n} S_i$ where the S_i's are in C_2, then $S = \bigcup_{i=1}^{n} A_i$ where

$A_i = S_i \cap [\overline{\bigcup_{j=1}^{i-1} S_j}]$ which implies the A_i's are disjoint and in C_2.

If $\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j$, the A_i's are disjoint sets in C_1 and so are B_j's.

It follows that

$$A_i = \bigcup_{j=1}^{m} B_j \cap A_i \quad \text{and} \quad B_j = \bigcup_{i=1}^{n} A_i \cap B_j .$$

Hence

$$\mu A_i = \sum_{j=1}^{m} \mu (B_j \cap A_i) \quad \text{and} \quad \mu B_j = \sum_{i=1}^{n} \mu (A_i \cap B_j)$$

It follows that

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu (A_i \cap B_j) = \sum_{j=1}^{m} \mu B_j .$$

LEMMA 2.1.10

If A and B are in C_2 and $A \subset B$,

$\mu A \leq \mu B$.

Proof: Since $B - A = B \cap \bar{A}$ is in C_2,

$$\mu B = \mu[(B-A) \cup A] = \mu(B-A) + \mu A$$

LEMMA 2.1.11

μ is countably additive on C_2.

Proof: It is sufficient to show that if $\bigcup_{i=1}^{\infty} A_i$ is in C_2 and the A_i's are disjoint sets in C_1, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} \mu A_i$$

Consider first that if $(a,b]$ equals $\bigcup_{i=1}^{\infty} (a_i, b_i]$ where all the intervals are disjoint, then $\bigcup_{i=1}^{n} (a_i, b_i]$ is a subset of $(a,b]$. Hence for all n,

$$\mu(a,b] \geq \sum_{i=1}^{n} \mu(a_i, b_i]$$

It follows that

$$\mu(a,b] \geq \sum_{i=1}^{\infty} \mu(a_i, b_i].$$

The same inequality follows in a similar fashion for $\mu(a,\infty)$, $\mu(-\infty,b]$ and μR.

To show the reverse inequality, first consider a and b finite. Since F is continuous on the right, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$F(a+\delta) < F(a) + \epsilon$$

Moreover for every i, there exists an $\eta_i > 0$ such that

$$F(b_i + \eta_i) < F(b_i) + \epsilon 2^{-i}.$$

Further

$$[a+\delta, b] \subset (a,b] = \bigcup_{i=1}^{\infty} (a_i, b_i] \subset \bigcup_{i=1}^{\infty} (a_i, b_i + \eta_i)$$

Hence by the Heine Borel theorem there exists an integer m such that

$$[a+\delta, b] \subset \bigcup_{i=1}^{m} (a_i, b_i+\eta_i).$$

Consequently, renaming the end points of the intervals if necessary, $a+\delta$ is in $(a_1, b_1+\eta_1)$ and for some integer k between 1 and n inclusive b is in $(a_k, b_k+\eta_k)$.

Suppose the least k is one. Then since $[a+\delta, b]$ is a subset of $(a_1, b_1+\eta_1)$, it follows that

$$F(a_1) \leq F(a+\delta) < F(a)+\epsilon$$

and

$$F(b) \leq F(b_1+\eta_1) < F(b_1) + \epsilon 2^{-1}.$$

Hence

$$F(b) - F(a) - \epsilon < F(b_1) - F(a_1) + \epsilon 2^{-1}.$$

It follows that $\mu(a,b] \leq \mu(a_1, b_1]$ which implies that

$$\mu(a,b] \leq \sum_{i=1}^{\infty} \mu(a_i, b_i].$$

Suppose the least k is greater than one. Then $b \geq b_1+\eta_1$, which implies that $b_1+\eta_1$ is in $(a,b]$. Since $b_1+\eta_1$ is not in $(a_1, b_1+\eta_1)$ there must exist an integer j greater than one such that $b_1+\eta_1$ is in $(a_j, b_j+\eta_j)$. If j is not two, let the jth interval be second and the second, the jth. Then

$$a_2 < b_1+\eta_1 < b_2+\eta_2.$$

This procedure may be repeated if necessary until the first $(a_k, b_k+\eta_k)$ where $b < b_k+\eta_k$. Then

$$[a+\delta, b] \subset \bigcup_{i=1}^{k} (a_i, b_i+\eta_i)$$

and for every integer j such that $1 < j \leq k$

$$a_j < b_{j-1} + \eta_{j-1} < b_j + \eta_j .$$

Since F is non decreasing, it follows that

$$\sum_{i=1}^{k} [F(b_i+\eta_i) - F(a_i)] = F(b_k+\eta_k) - F(a_1) + \sum_{i=1}^{k-1} [F(b_i+\eta_i) - F(a_{i+1})]$$

$$\geq F(b_k+\eta_k) - F(a_1)$$

$$> F(b) - F(a+\delta)$$

$$> F(b) - F(a) - \epsilon .$$

However

$$\sum_{i=1}^{k} F(b_i+\eta_i) < \sum_{i=1}^{k} F(b_i) + \epsilon \sum_{i=1}^{k} 2^{-i}$$

It follows that

$$\sum_{i=1}^{k} [F(b_i) - F(a_i)] > F(b) - F(a) - \epsilon \left[1 + \sum_{i=1}^{k} 2^{-i}\right]$$

Since this inequality holds for any integer greater than k,

$$\sum_{i=1}^{\infty} \mu(a_i, b_i] \geq \mu(a,b].$$

Therefore for a and b finite,

$$\mu(a,b] = \sum_{i=1}^{\infty} \mu(a_i, b_i].$$

Assume now that (a,∞) equals $\bigcup_{i=1}^{\infty} (a_i, b_i]$ where

$a = a_1 < b_1 = a_2 < \cdots$ where $\lim_{n\to\infty} b_n = \infty$. For every finite value of x greater than a, $(a,x$ C $(a_i,b_i$. It follows that there must be a b_n x.

Hence

$$\mu(a,x] \leq F(b_n) - F(a_1)$$

$$= \sum_{i=1}^{n} [F(b_i) - F(a_i)]$$

$$\leq \sum_{i=1}^{\infty} [F(b_i) -$$

$$= \sum_{i=1}^{\infty} \mu(a_i,b_i].$$

Similarly it may be shown that if $(-\infty, b]$ equals $\bigcup_{i=1}^{\infty} (a_i,b_i]$

$$\mu(-\infty,b] \leq \sum_{i=1}^{\infty} \mu(a_i,b_i]$$

and if R equals $\bigcup_{i=1}^{\infty}(a_i,b_i]$

$$\mu R \leq \sum_{i=1}^{\infty} \mu(a_i,b_i].$$

Finally every set in C_2 may be expressed as $\bigcup_{i=1}^{n} A_i$ where the A_i's are disjoint sets in C_1. If $\bigcup_{j=1}^{n} A_j$ equals $\bigcup_{i=1}^{\infty}$ $(a_i,b_i]$, it follows that

$$\bigcup_{j=1}^{n} A_j \cap (a_i,b_i] = (a_i,b_i] \text{ and } \bigcup_{i=1}^{\infty}(a_i,b_i] \cap A_j = A_j.$$

As a consquence

$$\mu(a_i,b_i] = \sum_{j=1}^{n} \mu[A_j \cap (a_i,b_i]]$$

and

$$\mu A_j = \sum_{i=1}^{\infty} \mu[(a_i,b_i] \cap A_j]$$

$$\geq \sum_{i=1}^{m} \mu[(a_i,b_i] \cap A_j]$$

Hence

$$\sum_{j=1}^{n} \mu A_j = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \mu[(a_i,b_i] \cap A_j]$$

$$\geq \sum_{j=1}^{n} \sum_{i=1}^{m} \mu[(a_i,b_i] \cap A_j]$$

$$= \sum_{i=1}^{m} \mu(a_i,b_i].$$

Letting m go to infinity gives

$$\sum_{j=1}^{n} \mu A_j = \mu[\bigcup_{i=1}^{n} A_j] = \sum_{i=1}^{\infty} \mu(a_i,b_i]$$

Hence

$$\mu[\bigcup_{i=1}^{\infty}(a_i,b_i]] = \sum_{i=1}^{\infty} \mu(a_i,b_i].$$

<u>LEMMA 2.1.12</u>

μ is a measure on C_2.

<u>DEFINITION 2.1.5</u>

For any subset, S, of R let

$$\mu^* S = \inf \sum_{i=1}^{\infty} \mu A_i$$

where every A_i is in C_2, the A_i's cover S and the infimum is with respect to all countable sequences of sets in C_2 which cover S.

LEMMA 2.1.13

μ^* is defined for all subsets of R.

LEMMA 2.1.14

If A is in C_2 ,

$$\mu^* A = \mu A.$$

Proof: Take any countable sequence of sets from C_2 which covers A. Denote the members of the sequence by $B_1, B_2, \ldots$. Then define

$$A_n = A \cap [B_n - \bigcup_{i=1}^{n-1} B_i].$$

Then A_n is in C_2, the A_n's are disjoint and $\bigcup_{i=1}^{\infty} A_i = A$. It follows from lemma 2.1.11 that

$$\mu A = \sum_{i=1}^{\infty} \mu A_i .$$

Since for all n , A_n is a subset of B_n from lemma 2.1.10,

$$\sum_{i=1}^{\infty} \mu A_i \leq \sum_{i=1}^{\infty} \mu B_i .$$

Hence

$$\mu A \leq \sum_{i=1}^{\infty} \mu B_i .$$

To complete the proof consider the sequence $A, \emptyset, \emptyset, \ldots$.

$$\mu^* A \leq \mu A + \mu \emptyset + \mu \emptyset + \cdots .$$

If $\mu^* A$ is less than μA, there will exist a sequence of sets $B_1, B_2, \ldots$ from C_2 which covers A and is such that

$$\mu^* A + \epsilon > \sum_{i=1}^{\infty} \mu B_i ,$$

where $\epsilon = \mu A - \mu^* A > 0$. This implies that

$$\mu A > \sum_{i=1}^{\infty} \mu B_i$$

which is impossible. Hence

$$\mu A = \mu^* A$$

LEMMA 2.1.15

$$\mu^* \emptyset = 0$$

LEMMA 2.1.16

If S_1 is a subset of S_2, $\mu^* S_1 \leq \mu^* S_2$.

LEMMA 2.1.17

If S is covered by a sequence of sets, $S_1, S_2, \ldots$,

$$\mu^* S \leq \sum_{i=1}^{\infty} \mu^* S_i .$$

Proof: The statement is trivial when $\mu^* S$ is infinite. When $\mu^* S$ is finite for every S_i and every $\epsilon > 0$ there exists a sequence of sets from C_2, $A_{1i}, A_{2i}, \ldots$, which cover S_i and are such that

$$\sum_{j=1}^{\infty} \mu A_{ij} < \mu^* S_i + \epsilon 2^{-i}.$$

Hence

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu A_{ij} < \sum_{i=1}^{\infty} \mu^* S_i + \epsilon$$

Since S is covered by $S_1, S_2, \ldots$ and S_i is covered by $A_{i1}, A_{i2}, \ldots$ it follows that S is covered by $A_{11}, A_{12}, \ldots$. Hence

$$\mu^* S \leq \mu^* \left(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_{ij} \right)$$

$$\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu A_{ij}$$

$$< \sum_{i=1}^{\infty} \mu^* S_i + \epsilon$$

Consequently

$$\mu^* S \leq \sum_{i=1}^{\infty} \mu^* S_i .$$

DEFINITION 2.1.6

The class of all μ-measurable sets of real numbers, C_3, is the class of all sets of real numbers A such that

$$\mu^* S \geq \mu^* (S \cap A) + \mu^* (S - A)$$

where S is an arbitrary set of real numbers. S is called a test set.

LEMMA 2.1.18

$\emptyset$ is in C_3.

LEMMA 2.1.19

If A is in C_3, $\bar{A}$ is in C_3.

LEMMA 2.1.20

If $A_1, A_2, \cdots, A_n$ is a finite sequence of sets in C_3, $\bigcup_{i=1}^{n} A_i$ is in C_3.

Proof: Using induction, suppose A_1 and A_2 are in C_3. Then for every set S,

$$\mu^* S \geq \mu^* (S \cap A_1) + \mu^* (S - A_1).$$

Using $S - A_1$ as a test set,

$$\mu^* (S - A_1) \geq \mu^* (S - A_1 \cap A_2) + \mu^* (S - A_1 - A_2)$$

Hence

$$\mu^* S \geq \mu^* (S \cap A_1) + \mu^* (S - A_1 \cap A_2) + \mu^* (S - A_1 - A_2).$$

$$\geq \mu^*[(S\cap A_1)\cup(S-A_1\cap A_2)] + \mu^*[S-(A_1\cup A_2)]$$

$$= \mu^*[S\cap(A_1\cup A_2)] + \mu^*[S-(A_1\cup A_2)]$$

The lemma follows by induction.

LEMMA 2.1.21

If $A_1, A_2, \cdots, A_n$ is a finite sequence of disjoint sets in C_3 and S is any set of real numbers,

$$\mu^*(S\cap\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu^*(S\cap A_i).$$

Proof: Using induction again, the statement is trivial when n = 1. Making the induction hypothesis, using lemma 2.1.20 to assert that

$\bigcup_{i=1}^{n} A_i$ is in C_3 and using $S\cap\bigcup_{i=1}^{n+1} A_i$ as a test set,

$$\mu^*(S\cap\bigcup_{i=1}^{n+1} A_i) \geq \mu^*(S\cap\bigcup_{i=1}^{n+1} A_i\cap\bigcup_{i=1}^{n} A_i) + \mu^*(S\cap\bigcup_{i=1}^{n+1} A_i - \bigcup_{i=1}^{n} A_i)$$

$$\geq \mu^*(S\cap\bigcup_{i=1}^{n} A_i) + \mu^*(S\cap A_{n+1})$$

By the induction hypothesis

$$\mu^*(S\cap\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu^*(S\cap A_i).$$

It follows from lemma 2.1.17 that

$$\mu^*(S\cap\bigcup_{i=1}^{n+1} A_i) \leq \sum_{i=1}^{n+1} \mu^*(S\cap A_i).$$

Therefore

$$\mu^*(S\cap\bigcup_{i=1}^{n+1} A_i) = \sum_{i=1}^{n+1} \mu^*(S\cap A_i).$$

LEMMA 2.1.22

If $A_1, A_2, \ldots$ is a denumerable sequence of disjoint sets in C_3 and if S is an arbitrary set of real numbers,

$$\mu^*(S \cap \bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu^*(S \cap A_i)$$

Proof: Since, for every n, $\bigcup_{i=1}^{n} A_i$ is a subset of $\bigcup_{i=1}^{\infty} A_i$, it follows that

$$\mu^*(\bigcup_{i=1}^{\infty} A_i \cap S) \geq \mu^*(\bigcup_{i=1}^{n} A_i \cap S)$$

$$= \sum_{i=1}^{n} \mu^*(A_i \cap S)$$

Letting n go to infinity, it follows that

$$\mu^*(\bigcup_{i=1}^{\infty} A_i \cap S) \geq \sum_{i=1}^{\infty} \mu^*(A_i \cap S)$$

Since $A_1 \cap S, A_2 \cap S, \ldots$ cover $\bigcup_{i=1}^{\infty}(A_i \cap S)$ it follows from lemma 2.1.17 that

$$\mu^*(\bigcup_{i=1}^{\infty} A_i \cap S) \leq \sum_{i=1}^{\infty} \mu^*(A_i \cap S)$$

Thus

$$\mu^*(\bigcup_{i=1}^{\infty} A_i \cap S) = \sum_{i=1}^{\infty} \mu^*(A_i \cap S)$$

LEMMA 2.1.23

If $A_1, A_2, \ldots$ is a denumerable sequence of sets in C_3, $\bigcup_{i=1}^{\infty} A_i$ in C_3.

Proof: Taking only $A_1, A_2, \cdots, A_n$ it follows from lemma 2.1.20 that for an arbitrary set S,

$$\mu^* S \geq \mu^*(S \cap \bigcup_{i=1}^{n} A_i) + \mu^*(S - \bigcup_{i=1}^{n} A_i).$$

Moreover from lemma 2.1.20

$$\mu^*(S \cap \bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu^*(S \cap A_i)$$

and from lemma 2.1.16 and the fact that $\overline{\bigcup_{i=1}^{n} A_i} \supset \overline{\bigcup_{i=1}^{\infty} A_i}$,

$$\mu^*(S - \bigcup_{i=1}^{n} A_i) \geq \mu^*(S - \bigcup_{i=1}^{\infty} A_i)$$

Thus

$$\mu^* S \geq \sum_{i=1}^{n} \mu^*(A_i \cap S) + \mu^*(S - \bigcup_{i=1}^{\infty} A_i)$$

Letting n go to infinity

$$\mu^* S \geq \sum_{i=1}^{\infty} \mu^*(A_i \cap S) + \mu^*(S - \bigcup_{i=1}^{\infty} A_i).$$

It follows from lemma 2.1.22 that

$$\mu^* S \geq \mu^*(S \cap \bigcup_{i=1}^{\infty} A_i) + \mu^*(S - \bigcup_{i=1}^{\infty} A_i)$$

It follows that $\bigcup_{i=1}^{\infty} A_i$ is in C_3.

<u>LEMMA 2.1.24</u>

C_2 is a subset of C_3.

Proof: If A and B are two arbitrary sets in C_2, $A \cap B$ and $A - B$ are disjoint and in C_2. The union of $A \cap B$ and $A - B$ is A.

Hence

$$\mu^*(A \cap B) + \mu^*(A - B) = \mu^* A.$$

For S, an arbitrary set of real numbers, if $\mu^* S$ is infinite

$$\mu^* S = \mu^*(S \cap A) + \mu^*(S - A)$$

for all A in C_2

If μ^*S is finite, then from the definition it follows that for every $\epsilon > 0$, there exists a sequence, $A_1, A_2, \cdots$ in C_2 which covers S and is such that

$$\mu^* S + \epsilon > \sum_{i=1}^{\infty} \mu A_i$$
$$= \sum_{i=1}^{\infty} [\mu(A_i \cap A) + \mu(A_i - A)]$$

for some A in C_2. However, $S \cap A$ is a subset of $\bigcup_{i=1}^{\infty} (A_i \cap A)$ and $A_1 \cap A, A_2 \cap A, \cdots$ is a sequence of sets in C_2. Similarly S-A is a subset of $\bigcup_{i=1}^{\infty} (A_i - A)$ and $A_1 - A, A_2 - A, \cdots$ is a sequence of sets in C_2. It follows that

$$\mu^*(S \cap A) + \mu^*(S-A) \leq \sum_{i=1}^{\infty} [\mu(A_i \cap A) + \mu(A_i - A)].$$

Hence

$$\mu^* S \geq \mu^*(S \cap A) + \mu^*(S-A)$$

It follows that A is in C_3.

LEMMA 2.1.25

C_3 is a σ-algebra which contains C_2.

LEMMA 2.1.26

$\mathcal{B}$ is a subset of C_3.

Restricting the domain of μ^* to $\mathcal{B}$ it follows that:

LEMMA 2.1.27

μ^* is countably additive.

Proof: Induction may be used to show

$$\mu^* \left(\bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mu^* A_i$$

The statement is trivial for $n = 1$. Making the induction hypothesis, recalling that $\bigcup_{i=1}^{n} A_i$ is in C_3 and using $\bigcup_{i=1}^{n+1} A_i$ as a test set,

$$\mu^*\left(\bigcup_{i=1}^{n+1} A_i\right) \geq \mu^*\left(\bigcup_{i=1}^{n+1} A_i \cap \bigcup_{i=1}^{n} A_i\right) + \mu^*\left(\bigcup_{i=1}^{n+1} A_i - \bigcup_{i=1}^{n} A_i\right)$$

$$= \mu^*\left(\bigcup_{i=1}^{n} A_i\right) + \mu^* A_{n+1}.$$

The induction hypothesis, $\mu^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^* A_i$ then emplies

$$\mu^*\left(\bigcup_{i=1}^{n+1} A_i\right) \geq \sum_{i=1}^{n+1} \mu^* A_i$$

This completes the induction proof. However, since for all n, $\bigcup_{i=1}^{n} A_i$ is a subset of $\bigcup_{i=1}^{\infty} A_i$,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \mu^*\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \sum_{i=1}^{n} \mu^* A_i.$$

Since this is true for all n,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu^* A_i.$$

Since the union of the A_i's is covered by $A_1, A_2, \cdots$, from lemma 2.1.17

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^* A_i.$$

Hence

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu^* A_i.$$

LEMMA 2.1.28

μ^* restricted to $\mathcal{B}$ is unique.

Proof: First consider some B in $\mathcal{B}$ such that μ^*B is finite. It is necessary to show that if μ_1 is a measure on $\mathcal{B}$ such that $\mu_1 A$ equals $\mu^* A$ equals μA for every A in C_2, then

$$\mu_1 B = \mu^* B.$$

To show this equality consider that for every B in

$$\mu^* B = \inf \sum_{i=1}^{\infty} \mu A_i.$$

Hence for every $\epsilon > 0$, there exists a sequence $A_1, A_2, \cdots$ in C_2 which covers B and is such that

$$\mu^* B + \epsilon > \sum_{i=1}^{\infty} \mu A_i$$

Assuming the A_i's are disjoint it follows that

$$\begin{aligned}\sum_{i=1}^{\infty} \mu A_i &= \sum_{i=1}^{\infty} \mu_1 A_i \\ &= \mu_1\left(\bigcup_{i=1}^{\infty} A_i\right) \\ &\geq \mu_1 B.\end{aligned}$$

Hence

$$\mu^* B \geq \mu_1 B.$$

To show the reverse inequality, consider that since B is in C_3,

$$\begin{aligned}\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) &\geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i \cap B\right) + \mu^*\left(\bigcup_{i=1}^{\infty} A_i - B\right). \\ &= \mu^* B + \mu^*\left(\bigcup_{i=1}^{\infty} A_i - B\right).\end{aligned}$$

Since the A_i's are disjoint

$$\begin{aligned}\epsilon &> \sum_{i=1}^{\infty} \mu^* A_i - \mu^* B \\ &= \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) - \mu^* B \\ &= \mu^*\left(\bigcup_{i=1}^{\infty} A_i - B\right)\end{aligned}$$

But for every $\epsilon > 0$, there exists a sequence of sets $B_1, B_2, \cdots$ in C_2 that cover $\bigcup_{i=1}^{\infty} A_i - B$ and are such that

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i - B\right) < \sum_{i=1}^{\infty} \mu B_i .$$

Taking the A_i's to be disjoint, it follows that

$$\begin{aligned}\sum_{i=1}^{\infty} \mu B_i &= \sum_{i=1}^{\infty} \mu_1 B_i \\ &\geq \mu_1\left(\bigcup_{i=1}^{\infty} A_i - B\right)\end{aligned}$$

Hence

$$\begin{aligned}\mu_1\left(\bigcup_{i=1}^{\infty} A_i - B\right) &< \epsilon + \mu^*\left(\bigcup_{i=1}^{\infty} A_i - B\right) \\ &< \epsilon + \epsilon\end{aligned}$$

Furthermore

$$\begin{aligned}\mu^* B &\leq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \\ &= \sum_{i=1}^{\infty} \mu^* A_i\end{aligned}$$

$$= \sum_{i=1}^{\infty} \mu_1 A_i$$

$$= \mu_1 \left(\bigcup_{i=1}^{\infty} A_i \right)$$

$$= \mu_1 \left(\bigcup_{i=1}^{\infty} A_i - B \right) + \mu_1 B$$

$$< 2\epsilon + \mu_1 B.$$

It follows that

$$\mu^* B \leq \mu_1 B.$$

Hence

$$\mu_1 B = \mu^* B$$

Assuming $\mu^* B$ is infinite, express R as the infinite union of bounded disjoint intervals. Then $R = \bigcup_{i=1}^{\infty} (a_i, b_i]$. Further

$B = B \cap \bigcup_{i=1}^{\infty} (a_i, b_i]$ and

$$\mu^* B = \sum_{i=1}^{\infty} \mu^* (B \cap (a_i, b_i])$$

$$= \sum_{i=1}^{\infty} \mu_1 (B \cap (a_i, b_i])$$

$$= \mu_1 B.$$

<u>THEOREM 2.2</u>:

If is a Lebesgue-Stieljes measure on $\mathcal{B}$, then there exists a unique equivalence class M in E such that for every F in M and every $a < b$

$$\mu(a, b] = F(b) - F(a).$$

DEFINITION 2.2.1

$$F_o(x) = \begin{cases} \mu(o,x] & \text{for } x > o \\ 0 & \text{for } x = o \\ \mu(x,o] & \text{for } x < o \end{cases}$$

The proof of this theorem comes from the following lemmas;

LEMMA 2.2.1

For every $a < b$,

$$\mu(a,b] = F_o(b) - F_o(a)$$

Proof: There are five cases:

Case 1: If $o < a$,

$$F_o(a) = \mu(o,a] \text{ and } F_o(b) = \mu(o,b]$$

However

$$\mu(o,b] = \mu(o,a] + \mu(a,b]$$

It follows that

$$F_o(b) - F_o(a) = \mu(a,b].$$

Case 2: If $o = a$,

$$F_o(a) = 0 \text{ and } F_o(b) = \mu(0,b] = \mu(a,b]$$

Clearly

$$F_o(b) - F_o(a) = \mu(a,b].$$

The other three cases, when $a < 0 < b$, $b = 0$ and $a < b < o$, follow in a similar fashion.

LEMMA 2.2.2

F_o is a monotone, non decreasing function defined for every x in R.

Proof: For every $x > a$

$$F_o(x) - F_o(a) = \mu(a,x].$$

Since $\mu(a,x] \geq 0$, it follows that for every $x > a$

$$F_0(x) \geq F_0(a)$$

Since a is chosen arbitrarily, F_0 is a monotone increasing function.

Clearly F_0 is defined for all x in R.

LEMMA 2.2.3

F_0 is continuous from the right at every point of R.

Proof: Select an arbitrary real number, a. Then

$$(a, a+1] = \bigcup_{i=1}^{\infty} (a + \tfrac{1}{i+1}, a + \tfrac{1}{i}]$$

and

$$\mu(a, a+1] = \sum_{i=1}^{\infty} \mu(a + \tfrac{1}{i+1}, a + \tfrac{1}{i}]$$

The sequence of partial sums represented by this infinite series is monotone increasing and bounded by $\mu(a, a+1]$. Hence for every $\epsilon > 0$ there exists an N such that for every $n > N$

$$\sum_{i=n}^{\infty} \mu(a + \tfrac{1}{i+1}, a + \tfrac{1}{i}] < \epsilon.$$

However since

$$\mu(a, a + \tfrac{1}{n}] = \sum_{i=n}^{\infty} \mu(a + \tfrac{1}{i+1}, a + \tfrac{1}{i}],$$

it follows that

$$F_0(a + \tfrac{1}{n}) - F_0(a) < \epsilon$$

Since F_0 is monotone non-decreasing

$$F_0(x) - F_0(a) < \epsilon$$

for every x in $(a, a+\frac{1}{n})$. Therefore F_0 is continuous on the right at a where a is an arbitrary real number.

LEMMA 2.2.4

F_o is in $\mathcal{M}$.

DEFINITION 2.2.2

Let M be the equivalence class in $\mathcal{M}$ which contains F_o.

LEMMA 2.2.5

For every F in M and every $a < b$,

$$\mu(a,b] = F(b) - F(a).$$

Proof: Since F is in M, there exists a real number C such that for every x in R,

$$F(x) = F_o(x) + C.$$

Thus

$$\begin{aligned}\mu(a,b] &= F_o(b) - F_o(a) \\ &= F_o(b) + C - F_o(a) - C \\ &= F(b) - F(a).\end{aligned}$$

This completes the proof of the theorem. The following is noted however:

LEMMA 2.2.6

$$\lim_{x \to a} \mu(x,a) = 0.$$

Proof: The proof is similar to the proof of lemma 2.2.3.

Chapter 3

PART I: THE DEFINITION OF THE LEBESGUE-STIELJES INTEGRAL

The Lebesgue-Stieljes integral of a bounded point function g with respect to a Lebesgue-Stieljes measure μ, or with respect to any function F in the equivalence class of $\mathcal{M}$ that corresponds to μ over a Borel set B such that μB is finite will be defined. The definition will be extended to functions g that are not bounded on B, to Lebesgue-Stieljes measures μ such that μB is infinite, to functions F that are monotone non-decreasing on R but not continuous on the right and finally to functions F of bounded variation on B. Some preliminary definitions are necessary:

<u>DEFINITION 3.1.1</u>

For a given Borel set B, D_n is defined to be a collection of n disjoint Borel sets $B_1, B_2, \cdots, B_n$ such that

$$\bigcup_{i=1}^{n} B_i = B.$$

<u>DEFINITION 3.1.2</u>

The upper Darboux sum of a bounded function g with respect to a Lebesgue-Stieljes measure μ and a given D_n on a Borel set B of finite μ-measure is

$$\sum_{i=1}^{n} M_i \, \mu B_i$$

where $B_1, B_2, \cdots, B_n$ are the elements of D_n and M_i is the supremum of g on B_i.

<u>DEFINITION 3.1.3</u>

The lower Darboux sum of a bounded function g with respect to a Lebesgue-Stieljes measure μ and with respect to a given D_n over a Borel set B of finite μ-measure is

$$\sum_{i=1}^{n} m_i \mu B_i$$

where $B_1, B_2, \cdots, B_n$ are the elements of D and M is the infimum of g on B_i.

DEFINITION 3.1.4

The upper integral of a bounded function g with respect to a Lebesgue-Stieljes measure μ over a Borel set B of finite μ-measure is

$$\inf \sum_{i=1}^{n} M_i \mu B_i$$

where the infimum is taken with respect to all D_n's for all values of n. The upper integral is denoted by

$$\overline{\int}_B g\,d\mu \quad \text{or} \quad \overline{\int}_B g\,dF$$

where F is any function in the equivalence class corresponding to μ.

DEFINITION 3.1.5

The lower integral of a bounded function g with respect to a Lebesgue-Stieljes measure μ over a Borel set B of finite μ-measure is

$$\sup \sum_{i=1}^{n} m_i \mu B_i$$

where the supremum is taken with respect to all D_n's for all values of n. The lower integral is denoted by

$$\underline{\int}_B g\,d\mu \quad \text{or} \quad \underline{\int}_B g\,dF$$

where F is defined as in definition 3.1.4.

DEFINITION 3.1

A bounded function g is Lebesgue-Stieljes integrable with respect to the Lebesgue-Stieljes measure μ over a Borel set B of finite μ-measure if the upper and lower integrals are equal and finite. The common value of the upper and lower integrals is called the Lebesgue-Stieljes integral and is denoted by

$$\int_B g\,d\mu \quad \text{or} \quad \int_B g\,dF$$

<u>THEOREM 3.1</u>:

A necessary and sufficient condition for the existence of $\int_B g\,d\mu$ is that for every $\epsilon > 0$, there exists a D_n such that

$$\sum_{i=1}^{n} (M_i - m_i)\,\mu B_i < \epsilon$$

Proof: Clearly each upper Darboux sum is greater than or equal to each lower Darboux sum. It follows that

$$\overline{\int}_B g\,d\mu \geq \underline{\int}_B g\,d\mu$$

For every $\epsilon > 0$ there exists a D_n' and a D_m'' such that

$$\sum_{i=1}^{n} M_i\,\mu B_i' - \overline{\int}_B g\,d\mu < \epsilon/2$$

and

$$\underline{\int}_B g\,d\mu - \sum_{i=1}^{m} m_i\,\mu B_i'' < \epsilon/2$$

These inequalities continue to hold when D_n' and D_m'' are replaced by $D_\ell = \{ B_1' \cap B_1'', B_1' \cap B_2'', \cdots, B_n' \cap B_m'' \}$. If $\int_B g\,d\mu$ exists, the upper and lower integrals are equal by definition. It follows that the stated condition is necessary. Conversely, if the condition is satisfied

$$\overline{\int}_B g\,d\mu - \underline{\int}_B g\,d\mu < \epsilon$$

Since the inequalities hold for every $\epsilon > 0$ the upper and lower integrals must be equal.

THEOREM 3.2:

If μB is finite and g is Borel measurable and bounded on B, the $\int_B g \, d\mu$ exists.

Proof: Take $\frac{\epsilon}{1+\mu B} > 0$ and any finite number of points

$y_0, y_1, \cdots, y_n$ such that

$$y_0 = m = \inf f(x) \quad \text{for all} \quad x \text{ in } B,$$

$$y_n = M = \sup f(x) \quad \text{for all} \quad x \text{ in } B,$$

$$m = y_0 < y_1 < \cdots < y_n = M,$$

and

$$y_i - y_{i-1} < \frac{\epsilon}{1+\mu B} \quad \text{for } i = 1, 2, \cdots, n.$$

Let

$$B_i = \{x : x \text{ is in } B \text{ and } y_{i-1} < g(x) \le y_i\}$$
$$= \{x : x \text{ is in } B \text{ and } g(x) \le y_i\} \cap \{x : x \text{ is in } B \text{ and } g(x) > y_{i-1}\}$$

Since g is Borel measurable, B_i is the intersection of two Borel sets and therefore a Borel set. It follows that μB_i is defined. If M_i is the supremum of $g(x)$ for x in B_i and m_i is the infimum of $g(x)$ for x in B_i, it follows that

$$y_{i-1} < m_i \le g(x) \le M_i \le y_i.$$

It follows that

$$M_i - m_i < \frac{\epsilon}{1+\mu B}$$

Therefore

$$0 \leq \sum_{i=1}^{n} (M_i - m_i)\,\mu B_i$$

$$< \frac{\epsilon}{1+\mu B} \sum_{i=1}^{n} \mu B_i = \epsilon \frac{\mu B}{1+\mu B} < \epsilon$$

It follows from theorem 3.1 that the Lebesgue-Stieljes intergral exists.

DEFINITION 3.2

If g is not bounded on B, define

$$\int_B g\,d\mu = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_B g_{a,b}\,d\mu$$

where

$$g_{a,b}(x) = \begin{cases} a & \text{for } g(x) \quad a \\ g(x) & \text{for } a \quad g(x) \quad b \\ b & \text{for } b \quad g(x) \end{cases}$$

provided the above limit exists.

DEFINITION 3.3

If μ B is infinite, define

$$\int_B g\,d\mu = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_{B_{a,b}} g\,d\mu$$

where

$$B_{a,b} = B \cap (a,b)$$

provided the limit exists.

DEFINITION 3.5

If F is monotone non-decreasing but not continuous on the right, define

$$\int_B g\,dF = \int_B g\,dF^*$$

where

$$F^*(x) = \lim_{x' \to x} F(x')$$

for all x in B provided the integral with respect to F^* exists.

DEFINITION 3.6

If F is of bounded variation on R, define

$$\int_B g\,dF = \int_B g\,dF_1 - \int_B g\,dF_2$$

where

$$F = F_1 - F_2$$

and F_1 and F_2 are monotone non-decreasing provided the integrals with respect to F_1 and F_2 exist.

PART II: PROPERTIES OF THE LEBESGUE-STIELJES INTEGRAL

Properties will be derived for the Lebesgue-Stieljes integral of a bounded Borel measurable function g with respect to a Lebesgue-Stieljes measure μ over a Borel set B of finite μ measure. These properties will be useful in Chapter 4.

THEOREM 3.3

If $g_1, \cdots, g_n$ is a finite collection of bounded Borel measurable functions,

$$\int_B \left(\sum_{i=1}^{n} g_i\right) d\mu = \sum_{i=1}^{n} \int_B g_i \, d\mu$$

Proof: Since g_1 and g_2 are bounded and Borel measurable, so is $g_1 + g_2$. It follows that

$$\int_B (g_1+g_2)\, d\mu = \sup \sum_{i=1}^{n} m_{oi}\, \mu B_i$$

$$= \inf \sum_{i=1}^{n} M_{oi}\, \mu B_i$$

where m_{oi} is the infimum of $g_1(x) + g_2(x)$ for all x in B_i and M_{oi} is the supremum of $g_1(x) + g_2(x)$ for all x in B_i. Furthermore

$$m_{1i} + m_{2i} \leq m_{oi}$$

where m_{1i} is the infimum of $g_1(x)$ and m_{2i} is the infimum of $g_2(x)$ for all x in B_i; moreover

$$M_{1i} + M_{2i} \geq M_{oi}$$

where M_{1i} and M_{2i} are defined in the obvious way. It follows that

$$\sum_{i=1}^{n} m_{oi}\, \mu B_i \geq \sum_{i=1}^{n} (m_{1i} + m_{2i})\, \mu B_i$$

and

$$\sum_{i=1}^{n} M_{0i}\,\mu B_i \le \sum_{i=1}^{n} (M_{1i}+M_{2i})\,\mu B_i$$

Since g_1 and g_2 are integrable, it follows that

$$\int_B g_1 d\mu + \int_B g_2 d\mu = \sup \sum_{i=1}^{n} m_{1i}\,\mu B_i + \sup \sum_{i=1}^{n} m_{2i}\,\mu B_i$$

$$= \sup \sum_{i=1}^{n} (m_{1i}+m_{2i})\,\mu B_i$$

$$= \inf \sum_{i=1}^{n} (M_{1i}+M_{2i})\,\mu B_i$$

But since

$$m_{1i}+m_{2i} \le m_{0i} \le M_{0i} \le M_{1i}+M_{2i}\,,$$

it follows that

$$\int_B g_1\, d\mu + \int_B g_2 d\mu = \int_B (g_1+g_2)\, d\mu$$

The conclusion follows by induction.

THEOREM 3.4

If B_1 and B_2 are disjoint Borel sets of finite μ-measure whose union is B , then

$$\int_B g\, d\mu = \int_{B_1} g\, d\mu + \int_{B_2} g\, d\mu .$$

Proof: Define

$$g_1(x) = \begin{cases} g(x) & \text{for } x \text{ in } B_1 \\ 0 & \text{for } x \text{ in } B_2 \end{cases}$$

and

$$g_2(x) = \begin{cases} g(x) & \text{for } x \text{ in } B_2 \\ 0 & \text{for } x \text{ in } B_1 \end{cases}$$

Then

$$\int_B g\,d\mu = \int_B (g_1+g_2)\,d\mu$$
$$= \int_{B_1} g_1\,d\mu + \int_{B_2} g_2\,d\mu$$
$$= \int_{B_1} g\,d\mu + \int_{B_2} g\,d\mu.$$

THEOREM 3.5

$$m\mu B \leq \int_B g\,d\mu \leq M\mu B$$

where m is the infimum and M, the supremum of g(x) for all x in B.

Proof: Letting $B_1, B_2, \cdots, B_n$ be a sequence of disjoint Borel sets whose union is B and letting M_i be the supremum of g(x) for all x in B_i, it follows that $M_i \leq M$ for all i. Thus

$$\sum_{i=1}^{n} M_i \mu B_i \leq \sum_{i=1}^{n} M \mu B_i$$
$$= M\mu B.$$

Similarly

$$\sum_{i=1}^{n} m_i \mu B_i \leq m\mu B.$$

But when the Lebesgue-Stieljes intergral exists

$$\inf \sum_{i=1}^{n} M_i \mu B_i = \sup \sum_{i=1}^{n} m_i \mu B_i.$$

It follows that

$$m\mu B \leq \int_B g\,d\mu \leq M\mu B.$$

COROLLARY 3.5.1

If $\mu B = 0$, $\int_B g\,d\mu = 0$

THEOREM 3.6

$$\left|\int_B g\,d\mu\right| \le \int_B |g|\,d\mu.$$

Proof: Clearly

$$|g(x)| + g(x) \ge 0 \quad \text{and} \quad |g(x)| - g(x) \ge 0$$

Letting m_1 be the infimum of $|g(x)| + g(x)$ for all x in B and m_2, the infimum of $|g(x)| - g(x)$, it follows that

$$\begin{aligned} 0 &\le m_1 \mu B \\ &\le \int_B (|g| + g)\,d\mu \\ &= \int_B |g|\,d\mu + \int_B g\,d\mu. \end{aligned}$$

Hence

$$-\int_B g\,d\mu \le \int_B |g|\,d\mu$$

Similarly

$$\begin{aligned} 0 &\le m_2 \mu B \\ &\le \int_B (|g| - g)\,d\mu \\ &= \int_B |g|\,d\mu + \int_B (-g)\,d\mu. \end{aligned}$$

Since it follows directly from the definition that a constant may be factored across the integral sign,

$$\int_B g\,d\mu \le \int_B |g|\,d\mu.$$

Hence

$$\left|\int_B g\,d\mu\right| \le \int_B |g|\,d\mu.$$

THEOREM 3.7

Suppose μ is a Lebesgue-Stieljes measure, B is some Borel set of finite μ-measure and $g_1, g_2, \cdots$ is a sequence of Borel measurable functions defined on B and such that for every n and for every x in B there exists a real number K such that

$$|g_n(x)| < K$$

Suppose moreover that

$$\lim_{n \to \infty} g_n(x) = g(x)$$

almost everywhere i.e. for all x in $B-B_0$ where $\mu B_0 = 0$. Finally suppose that g is bounded on B. It follows that

$$\lim_{n \to \infty} \int_B g_n d\mu = \int_B \lim_{n \to \infty} g_n d\mu$$

Proof: Since g is bounded on B and $\mu B_0 = 0$, it is obvious that

$$\int_{B_0} g d\mu = 0$$

Letting $B-B_0 = B^*$, g is the limit of a sequence of Borel measurable functions on B^* and hence g is Borel measurable. Moreover $|g(x)| \leq K$ for all x in B^*. It follows that g(x) is integrable over B^* and hence the following integrals exist and

$$\int_{B^*} g d\mu = \int_{B^*} g d\mu + \int_{B_0} g d\mu$$

$$= \int_B g d\mu.$$

For every $\epsilon > 0$ a non-decreasing sequence of subsets of B^* may be defined as follows:

$$B_1 = \{x : |g_n(x) - g(x)| < \epsilon' \text{ for } n = 1, 2, \cdots\}$$

$$B_2 = \{x : |g_n(x) - g(x)| < \epsilon' \text{ for } n = 2, 3, \cdots\}$$

$$\cdots$$

$$B_i = \{x : |g_n(x) - g(x)| < \epsilon' \text{ for } n = i, i+1, \cdots\}$$

$$, \cdots$$

where $\epsilon' = \dfrac{\epsilon}{2(1+\mu B)}$

Since B_i is a subset of B^* for every i

$$\bigcup_{i=1}^{\infty} B_i \subset B^*.$$

Furthermore x in B^* implies $\lim_{n \to \infty} g_n(x)$ equals $g(x)$. This means that for every $\epsilon > 0$ there exists an m such that for every $n > m$, x is in B_n.

In symbols

$$B^* \subset \bigcup_{i=1}^{\infty} B_i$$

It follows that

$$B^* = \bigcup_{i=1}^{\infty} B_i$$

Since $B_1 \subset B_2 \subset \cdots$,

$$\bigcup_{i=1}^{\infty} B_i = B_1 \cup \bigcup_{i=1}^{\infty} (B_{i+1} - B_i).$$

Thus since the sets on the right are disjoint Borel sets

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu B_1 + \sum_{i=1}^{\infty} \mu(B_{i+1} - B_i)$$

$$= \lim_{n\to\infty} (\mu B_1 + \mu B_2 - \mu B_1 + \cdots + \mu B_n - \mu B_{n-1})$$

$$= \lim_{n\to\infty} \mu B_n$$

This means that there exists an m such that for every $n > m$

$$|\mu B - \mu B_n| < \frac{\epsilon}{4K}.$$

Since it may be easily shown that $|g_n - g|$ is a bounded, Borel measurable function, it follows that $|g_n - g|$ is integrable and hence for every $n > m$

$$\int_B |g_n - g|\, d\mu = \int_{B_n} |g_n - g|\, d\mu + \int_{B-B_n} |g - g_n|\, d\mu$$

$$< \epsilon' \mu B_n + 2K\, \mu(B - B_n)$$

$$< \frac{\epsilon \mu B}{2(1+\mu B)} + 2K \frac{\epsilon}{4K} < \epsilon.$$

Since

$$\left|\int_B (g_n - g)\, d\mu\right| \le \int_B |g_n - g|\, d\mu,$$

it follows that for every $\epsilon > 0$, there exists an m such that for every $n > m$

$$\left| \int_B g_n d\mu - \int_B g d\mu \right| < \epsilon$$

THEOREM 3.8

If $\sum_{i=1}^{\infty} f_i(x) = g(x)$ almost everywhere on some Borel set B of infinite μ-measure, if g is bounded on B and if there exists a real number K such that for all x in B and for all n, $\left| \sum_{i=1}^{n} f_i(x) \right| < K$, then

$$\sum_{i=1}^{\infty} \int_B f_i \, d\mu = \int_B \left(\sum_{i=1}^{\infty} f_i \right) d\mu.$$

Proof: This is an immediate consequence of theorem 3.7 considering the sequence of partial sums.

THEOREM 3.9

If $B_1, B_2, \cdots$ is a sequence of disjoint Borel sets whose union is B,

$$\int_B g \, d\mu = \sum_{i=1}^{\infty} \int_{B_i} g \, d\mu.$$

Proof: Define $e_i(x)$ to be 1 when x is in B_i and zero otherwise. Then for all x in B,

$$g(x) = \sum_{i=1}^{\infty} e_i(x) \, g(x).$$

Hence

$$\begin{aligned} \int_B g \, d\mu &= \int_B \left(\sum_{i=1}^{\infty} e_i g \right) d\mu \\ &= \sum_{i=1}^{\infty} \int_B e_i g \, d\mu \\ &= \sum_{i=1}^{\infty} \int_{B_i} g \, d\mu. \end{aligned}$$

Chapter 4

PROBABILITY INTEGRALS

In this section probability measure and probability distribution function will be defined. Then it will be shown how the Lebesgue-Stieljes integral of a bounded Borel measurable function g with respect to a probability measure may be expressed as a countable sum of positive numbers plus the Lebesgue-Stieljes integral of g with respect to a function that is everywhere continuous.

DEFINITION 4.1

If F is monotone non-decreasing, defined on R and continuous from the right, i.e. if F is an element of $\mathcal{M}$ and if $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$, then F is a probability distribution function.

DEFINITION 4.2

If μ is the unique measure determined by a probability distribution function, μ is a probability measure and for any Borel set B, μB will be denoted by P(B). P(B) is the "probability" that x is in B.

THEOREM 4.1

If P is a probability measure,

Moreover for all Borel sets B,

THEOREM 4.2

If μ is a Lebesgue-Stieljes measure, $\mu\{a\}$ is greater than zero if and only if a is a point of discontinuity for every function F in the equivalence class corresponding to μ .

Proof: First assuming that a is a point of discontinuity, it follows from the fact that every F is monotone non-decreasing and continuous on the right that

$$\lim_{x \to a-} F(x) < \lim_{x \to a+} F(x) = F(a).$$

But

$$\lim_{x \to a-} \mu(x,a] = \lim_{x \to a-} \mu(x,a) + \mu\{a\}$$
$$= \mu\{a\}$$

Further

$$\lim_{x \to a-} \mu(x,a] = \lim (F(a) - F(x)) > 0$$

It follows that $\mu\{a\} > 0$

On the other hand, assuming that $\mu\{a\} > 0$

$$\lim_{x \to a-} (F(a) - F(x)) = P\{a\} > 0$$

Hence F is discontinuous at a. This clearly holds for all F in the equivalence class corresponding to μ.

COROLLARY 4.2.1

$\mu\{a\} = 0$ if and only if a is a point of continuity of F for all F in the equivalence class corresponding to μ.

THEOREM 4.3

For all functions F in $\mathcal{M}$ there are at most a countable number of discontinuities.

Proof: Suppose a is a point of discontinuity for F. Then

$$F(a) = \lim_{x \to a+} F(x) > \lim_{x \to a-} F(x).$$

Mate a with a rational number r such that

$$\lim_{x \to a-} F(x) < r < \lim_{x \to a+} F(x).$$

Since F is monotone non-decreasing, each distinct point of discontinuity corresponds to a distinct rational number. Since the rationals are denumerable, the points of discontinuity are countable.

THEOREM 4.4

If F is a probability distribution function

$$F = f + S$$

where f is continuous on R.

Proof: Let $\bar{x}$ be the points of discontinuity for F. $\bar{x}$ is a Borel set since $\bar{x}$ is a countable union of distinct points and each point is a Borel set. Moreover $R-\bar{x}$ is the points of continuity for F and is also a Borel set. Suppose P is the probability measure that corresponds to F. Then for all Borel sets B,

$$P(B) = P(B \cap \bar{X}) + P(B - \bar{X})$$

Define

$$\mu_1 B = P(B - \bar{X}) \quad \text{and} \quad \mu_2 B = P(B \cap \bar{X})$$

Then μ_1 and μ_2 are bounded Lebestue-Stieljes measures. For all x define

$$f(x) = \mu_1(-\infty, x] \quad \text{and} \quad S(x) = \mu_2(-\infty, x].$$

Then f is in the equivalence class corresponding to μ_1 and S is in the equivalence class corresponding to μ_2.

Then for all x ,

$$f(x) + S(x) = \mu_1(-\infty, x] + \mu_2(-\infty, x]$$

$$= P[(-\infty, x] - \underline{\overline{x}}] + P[(-\infty, x] \cap \underline{\overline{x}}]$$

$$= P(-\infty, x]$$

$$= F(x)$$

If x is in $\underline{\overline{x}}$,

$$\mu_1\{x\} = P[\{x\} - \underline{\overline{x}}] = P\phi = 0$$

If x is in R-$\underline{\overline{x}}$

$$\mu_1\{x\} = P[\{x\} - \underline{\overline{x}}] = P\{x\} = 0$$

Hence $\mu_1\{x\}$ is zero for all x in R. Since f is in the equivalence class corresponding to μ_1 it follows that f is continuous for all x.

THEOREM 4.5

If x_1 and x_2 are two points in $\underline{\overline{x}}$ and no points of $\underline{\overline{x}}$ are in (x_1, x_2),

Then for every x in $[x_1, x_2)$, $S(x) = S(x_1)$.

Proof: For every x in $[x_1, x_2)$,

$$P[(-\infty, x] \cap \underline{\overline{x}}] = P[(-\infty, x_1] \cap \underline{\overline{x}}] + P[(x_1, x) \cap \underline{\overline{x}}]$$

$$= P[(-\infty, x_1] \cap \underline{\overline{x}}]$$

Hence

$$P[(-\infty, x] \cap \underline{\overline{x}}] = \mu_2(-\infty, x]$$

$$= S(x) = S(x_1).$$

THEOREM 4.6

S is continuous at all x in R-$\underline{\overline{x}}$ and discontinuous at x in $\underline{\overline{x}}$.

Proof: For x in $R-\underline{\bar{x}}$,

$$\mu_2\{x\} = P[\{x\} \cap \underline{\bar{x}}] = P\emptyset = 0$$

Hence x is a point of continuity for S.

For x in $\underline{\bar{x}}$

$$\mu_2\{x\} = P[\{x\} \cap \underline{\bar{x}}] = P\{x\} > 0$$

Hence x is a point of discontinuity for S.

DEFINITION 4.3

A function having the properties attributed to S in theorems 4.5 and 4.6 will be called a generalized step function.

THEOREM 4.7

$$\int_B g\,dF = \sum_{x_i \text{ in } B} g(x_i)\mu_2\{x_i\} + \int_{B-\underline{\bar{x}}} g\,d\mu_1$$

where g is bounded and Borel measurable on the Borel set B, F is a probability distribution function and $\underline{\bar{x}}$, μ_1 and μ_2 are as defined in theorem 4.4.

Proof: By definition

$$\begin{aligned} \int_B g\,dF &= \inf \sum_{i=1}^{n} M_i P(B_i) \\ &= \inf \sum_{i=1}^{n} M_i[\mu_1 B_i + \mu_2 B_i] \\ &= \inf \sum_{i=1}^{n} M_i \mu_1 B_i + \inf \sum_{i=1}^{n} M_i \mu_2 B_i \end{aligned}$$

$$= \int_B g\,d\mu_2 + \int_B g\,d\mu_1$$

$$= \int_{B\cap\bar{X}} g\,d\mu_2 + \int_B g\,d\mu_1$$

From theorem 3.9

$$\int_B g\,dF = \sum_{x_i \text{ in } B\cap\bar{X}} \left[\int_{\{x_i\}} g\,d\mu_2\right] + \int_B g\,d\mu_1$$

$$= \sum_{x_i \text{ in } B\cap\bar{X}} g(x_i)\mu_2\{x_i\} + \int_B g\,d\mu_1$$

Chapter 5

CONCLUSION

A bounded Borel measurable function which gives an important special case of the general formulas in the preceding chapters is $g(x) = 1$ for all x. If $g(x) = 1$, for any Borel set, B, and any probability distribution function, F, the probability that x is in B is given by

$$P(B) = \int_B dF$$

Suppose F is continuous everywhere. It may be shown that F has a derivative at every point with the possible exception of a set of Lebesgue measure zero [2]. If the derivative of F, F', exists everywhere, it is called the probability density function. Furthermore it may be shown that

$$\int_B F' dx = \int_B dF = P(B).$$

In particular if B is the interval from a to b

$$P(B) = \int_a^b F' dx = F(b) - F(a)$$

This is true regardless of whether the interval is (a,b), (a,b], [a,b) or [a,b]. If B is a single point P(B) is clearly zero.

Suppose F is a generalized step function, i.e. $F(x) = S(x)$. The set of points at which F is discontinuous $\underline{X}$ is either a finite or denumerable set. The function p is called the probability density function for F where

$$p(x_i) = F(x_i) - \lim_{x \to x_i -} F(x_i) \quad \text{for } x_i \text{ in } \underline{X}$$

and

$$p(x) = 0 \quad \text{for } x \text{ not in } \underline{X}.$$

The probability that x is in a Borel set B is

$$\int_B dF = \sum_{x_i \text{ in } B} p(x_i)$$

The set $\bar{X}$ may be such that every point of $\bar{X}$ is in an interval containing no other points of $\bar{X}$. In this case $\bar{X}$ is said to be discrete. The discrete case includes the case where $\bar{X}$ has a finite number of points in every finite interval. In this case F is a step function in the ordinary sense [3]. It may also happen that $\bar{X}$ is discrete but has a denumerable number of points in some finite interval. For example let

$$\bar{X} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots\}$$

and define

$$F(x) = \begin{cases} 0 & \text{for } x \leq 1. \\ \frac{1}{2^n} & \text{for } x \text{ in } [\frac{1}{n+1}, \frac{1}{n}),\ n = 1, 2, \cdots. \\ 1 & \text{for } x \text{ in } [1, \infty). \end{cases}$$

Also the set $\bar{X}$ may be such that there exists a denumerable set of x_i's in every interval. In this case $\bar{X}$ is said to be everywhere dense. For example let $\bar{X}$ be the set of all rational numbers, $r_1, r_2, \cdots$. Define

$$p(r_n) = \frac{1}{2^n}.$$

and let

$$F(x) = \sum_{r_i \leq x} p(r_i)$$

Finally F may of course be the sum of a non zero continuous function and a non zero generalized step function.

The two cases usually discussed in elementary probability courses are where F is everywhere differentiable (and hence continuous), and where F is an ordinary step function.

BIBLIOGRAPHY

1. Cramér, H. Mathematical methods of statistics. Princeton University Press, 1957.

2. Hobson, E. W. The theory of functions of a real variable and the theory of Fourier's series, v. 1. Dover Publications, Inc., 1957.

3. James, G. and R. C. James. Mathematics dictionary. D. Van Nostrand Company, Inc., 1959.

4. Loève, M. Probability theory. D. Van Nostrand Company, Inc., 1955.

5. McShane, E. J. and T. A. Botts. Real analysis. D. Van Nostrand Company, Inc., 1959.

6. Munroe, M. E. Introduction to measure and integration. Addison-Wesley Publishing Company, Inc., 1953.

7. Royden, H. L. Real analysis. The Macmillan Company, 1963.