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I. I. Shapiro

Effects of General Relativity on Interplanetary Time-Delay Measurements

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

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ABSTRACT

The predictions of general relativity regarding interplanetary time-delay measurements are explored in detail. We conclude that a fourth test of the theory is now feasible since the modifications of general relativity introduce an extra delay of about 200 μsec when radar waves are reflected from either Mercury or Venus near superior conjunction. The uses of such measurements to investigate the solar corona, a possible time dependence of the gravitational constant, and the precession of Mercury's perihelion are also discussed.

Accepted for the Air Force
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EFFECTS OF GENERAL RELATIVITY ON INTERPLANETARY TIME-DELAY MEASUREMENTS

I. INTRODUCTION

Although Einstein's theory of general relativity forms the basis of almost all cosmological arguments and has profound philosophical implications, it has been subject to very few experimental tests. The reason is not hard to find. On a laboratory scale, the deviations between the Einsteinian and, for example, the Newtonian predictions are almost always too minuscule to be detected. In fact, only three tests have been made since Einstein's theory was given its definitive form in 1916, all having been suggested in his original papers. Two relate to the interaction of matter with electromagnetic waves; the third relates to the interaction of matter with matter. The gravitational red-shift experiment, which belongs to the first category, really tests only the principle of equivalence and the Einstein prediction is obtained in other theories as well (see, for example, Nordström[1]). The most accurate such experiment was performed by Pound and Snider[2] who employed the Mössbauer effect to detect the change in frequency of gamma rays alternately "rising" and "falling" in the earth's gravitational field. Their results indicate confirmation of prediction to about 1 percent. The second test, the prediction of the bending of the path of starlight as it passes near the sun, has been subjected to repeated study during solar eclipses from 1919 to the present. Such is the difficulty of this experiment, that the results of the various observers are consistent with each other and with the predicted value to only about 25 percent.[3] Most serious reviewers are therefore of the opinion that this prediction of general relativity has not yet been verified definitively. The third test involves the prediction that the perihelion of Mercury's orbit undergoes precession of 43 seconds of arc per century, in excess of the amount calculated from Newtonian theory. The Einstein value seems to have been verified to within about 2 percent (see Clemence[4]).

A fourth test of general relativity has now been made possible by advances in radar astronomy. This test involves measuring the time delays between transmitting radar pulses toward either Venus or Mercury and detecting their echoes. These measurements must be taken at different relative orientations of the earth, the sun, and the target planet, with the most crucial ones being those near superior conjunction when the radar waves pass closest to the sun. For such configurations, as will be shown in Sec. II, predictions based on general relativity indicate that the time delays will be increased by as much as 200 μsec because of the influence of the sun's gravitational field on the speed of radio wave propagation. The increase at inferior conjunction, on the other hand, amounts to only about 10 μsec. Hence the difference, which is the significant measurable quantity, is almost as large as the maximum value of the increase.*

The actual test will entail a meticulous comparison of all the observations with the theoretical predictions. The unknown parameters (such as the initial conditions of planetary motion, and the masses and radii of the planets) will be estimated from the data using the statistical theory of parameter estimation.† The values of the parameters so determined will then be reinserted into the theory, and the resulting predictions of time delays will be compared with the observations. If the residuals, observed minus theoretical values, are smaller than or comparable to the measurement errors, then we conclude that the experiment "supports" the theory; whereas, if the residuals seem to be systematically larger than the estimated errors, then either we have overlooked some effects on our measurements or the basic theory is inadequate. We could, of course, perform the same type of analysis using the Newton instead of the Einstein theory as a basis for the comparison. In order to ascertain a priori whether the set of proposed measurements constitutes a test that can distinguish between two theories, we calculate the expected values of the measurements on the basis of both, given comparable initial conditions, to determine whether the differences exceed the anticipated measurement errors. That the radar experiments being proposed here will in fact provide a meaningful test of Einstein's theory can most readily be shown analytically by calculating the difference $\Delta\tau_r$ between the proper-time delay predicted in general relativity and the corresponding flat-space value.

II. APPROXIMATE DETERMINATION OF $\Delta\tau_r$

We wish to calculate the proper-time-delay difference $\Delta\tau_r$, given the configuration earth-sun-planet. First we calculate the delay predicted in general relativity. If the coordinates are denoted by $x^\mu (\mu = 1 \rightarrow 4)$, then the differential equations for the light rays (which travel along geodesic zero lines) are

$$\frac{d^2x^\mu}{ds^2} + \left\{ {\mu \atop \rho\ \sigma} \right\} \frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0 \quad ,$$

$$g_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \quad , \tag{2.1}$$

where the Christoffel symbols $\left\{ {\mu \atop \rho\ \sigma} \right\}$ are given in terms of the metric tensor $g_{\mu\nu} (\mu, \rho, \sigma, \nu = 1 \rightarrow 4)$ by

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* This relativistic influence on delay had been investigated previously for configurations near inferior conjunction,[5] but at that time the delay measurement errors were several orders of magnitude larger than the general relativity effect. (In addition, even if the errors were comparable to or somewhat smaller than the change introduced by relativity, no experimental test of the theory could have been made. The sensitivity of the time-delay measurements to errors in the Astronomical Unit (AU) and in the radius of Venus cannot be distinguished from the sensitivity of the delays to changes in the relativistic contribution. Even in principle, better than a 1-μsec measurement accuracy is required to separate these effects near inferior conjunction. But at that level a whole host of obscuring influences come into play, for example, topographical variations on the planetary surface. A realistic test of general relativity using only measurements made near inferior conjunction can therefore not be expected to be feasible for the foreseeable future.)

† We assume, of course, that the data are redundant, i.e., that there are far more measurements than are necessary to determine specific values for the parameters.

$$\left\{ {\mu \atop \rho\ \sigma} \right\} = \frac{1}{2} g^{\mu s}(g_{\rho s,\sigma} + g_{\sigma s,\rho} - g_{\rho\sigma,s}) \quad , \tag{2.2}$$

$$g^{\mu s} = g^{-1} \text{ cofactor } (g_{\mu s}) \quad , \tag{2.3}$$

and

$$g_{\rho\sigma,\mu} \equiv \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} \quad . \tag{2.4}$$

The contravariant metric tensor $g^{\mu s}$ is symmetric and satisfies

$$g_{\mu\nu} g^{\lambda\nu} = \delta^{\lambda}_{\mu} \quad . \tag{2.5}$$

The metric tensor $g_{\mu\nu}$ is found by solving Einstein's field equations. Schwarzschild[6] found a solution to these equations which in empty space is static, has spherical symmetry, and becomes the flat metric at infinity. Thus, a spherically symmetric distribution of matter will give rise to a gravitational field outside it which is described by Schwarzschild's solution. We therefore use this solution to represent the gravitational field of the sun.

In order to proceed further, we choose a coordinate system in which to calculate. It might be thought that our end result – the proper-time delay between pulse transmission and echo reception – would be independent of the choice of coordinate system. However, this conclusion is not valid per se because of our assumption that the positions of the earth and planet, relative to the sun, are given. Our result will be independent of the coordinate system only if we specify the planetary positions in an invariant manner. Such a description can be given in terms of the line element ds.* (The radial coordinate of each body in a given coordinate system can be determined from a prescribed value of ds; the angular variables can be treated in a similar invariant manner.) But this description would in a sense be begging the question. Our goal is really to compare the general relativity prediction with the flat-space prediction, whereas we presumably know the planetary positions only in the Newtonian framework. Determining the corresponding relativistically invariant positions is not easy. It would probably require a reanalysis of the optical data upon which the "Newtonian" positions were based. For simplicity, and since we are trying only to establish the meaningfulness of the proposed test, we shall choose the usual coordinate system (see, for example, Bergmann[7]) to represent the Schwarzschild solution and shall henceforth ignore the "coordinate-system problem," except for a comment preceding Eq. (2.17). In rectangular coordinates, we find[7]

$$g_{44} = 1 - \frac{2r_o}{r} \quad ,$$

$$g_{4s} = 0 \quad ; \quad s = 1 \rightarrow 3 \quad ,$$

$$g_{rs} = -\delta_{rs} - \frac{2r_o}{r - r_o} x_r x_s \quad ; \quad r, s = 1 \rightarrow 3 \quad , \tag{2.6}$$

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* This approach was suggested by L. Witten.

where

$$\chi_s = \frac{x^s}{r} \quad , \quad x_4 = c_o t \quad , \tag{2.7}$$

and

$$r_o = \frac{GM_s}{c_o^2} \approx 1.5 \text{ km} \quad , \tag{2.8}$$

since G is the gravitational constant, M_s the mass of the sun, and c_o the speed of light.

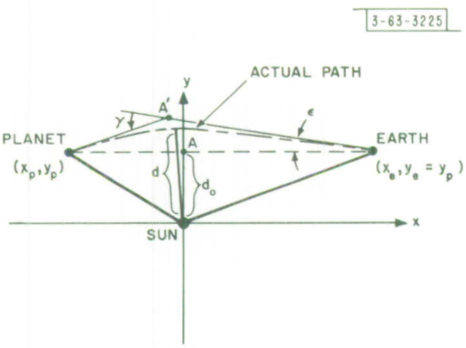

Fig. 1. Geometry of path of radar pulse: rectangular coordinates.

Using Eqs. (2.1) and (2.6), we can find the round-trip coordinate-time delay between pulse transmission and echo reception. An approximate result can be obtained almost immediately by assuming that the "spatial" path is a straight line (see Fig. 1) and choosing the coordinate axes so that this path is parallel to the x-axis. We also assume, for simplicity, that the earth remains stationary between transmission and echo detection. Although affecting the time delay slightly, this assumption has a negligible effect on $\Delta\tau_r$. (However, see Sec. III.)

The straight-line path assumption has two effects on the calculated time delay:

(a) The decrease in path length (see Fig. 1) causes the actual delay to be underestimated.

(b) The closer approach to the sun of the straight-line path results in the light ray passing through a higher gravitational potential, and suffering therefore a greater decrease in speed and a consequent increase in delay.

We can estimate quantitatively the contribution of each. For the first effect listed, we compare the length of the straight-line path with the two straight-line segments, one tangent to the actual light-ray path at the earth's position, the other tangent to the actual path at the planet's position, as shown in Fig. 1. The mutual inclination of these two segments, which represents the total angular deflection of the ray, we call γ. As was originally shown by Einstein, even for d equal to one solar radius ($d = R_s$), the angle γ is no more than 1.75 seconds of arc, i.e., about 8.5×10^{-6} radians. Assuming, again for simplicity, that $|x_p| = |x_e|$, it is easy to show that the actual path length L' satisfies the inequality

$$L < L' < \frac{L}{\cos\epsilon} \approx L\left(1 + \frac{\epsilon^2}{2}\right) \approx L\left(1 + \frac{\gamma^2}{8}\right) \quad , \tag{2.9}$$

where L is the length (expressed in light time) of the straight path, and ϵ is defined in the figure. Hence,

$$\Delta L \equiv L' - L < L\,\frac{\gamma^2}{8} \lesssim 9 \times 10^{-12}\,L \quad . \tag{2.10}$$

Even for $L = 10^3$ sec, we find ΔL to be only about 9 nsec which is completely negligible considering presently achievable accuracies.

To estimate the contribution of the second effect, we use a crude approximation to the general relativistic expression for the speed of a light wave in the presence of a gravitational potential Φ:

$$c \approx c_o\left(1 + \frac{2\Phi}{c_o^2}\right) \quad , \tag{2.11}$$

where c_o is its speed in the absence of a gravitational field. The difference Δc between the speed of a light wave at the point A of closest approach to the sun along the straight-line path and the speed at the intersection A' of the two line segments that are tangent to the actual path at earth and planet, respectively, is then given by

$$\Delta c = c(A) - c(A') = \frac{2}{c_o}\,[\Phi(A) - \Phi(A')]$$

$$= \frac{2GM_s}{c_o}\left[\frac{1}{d_o} - \frac{1}{(d_o + \Delta d)}\right] \approx \frac{2GM_s}{c_o}\,\frac{\Delta d}{d_o^2} \quad , \tag{2.12}$$

where

$$\Delta d = \frac{L}{4}\,\sin\epsilon \approx L\,\frac{\gamma}{8} \quad . \tag{2.13}$$

Equation (2.12) represents an upper bound on the maximum difference in speed along the two paths. Hence, a gross upper bound on the effect ΔL on time delay of assuming a straight-line path is given by

$$\Delta L < L\,\frac{\Delta c}{c_o} \approx r_o\left(\frac{L}{d_o}\right)^2\left(\frac{\gamma}{4}\right) \quad . \tag{2.14}$$

For $L = 10^3$ sec, we find $\Delta L < 2 \times 10^{-6}$ sec even when $d_o = R_s \approx 2.3$ sec. Hence, this effect of the approximation is also negligible.

This type of result is actually a general one for refraction phenomena where the change in index is small: The effect of the change in path on the time delay is of higher order in the change of index than is the effect of the change in speed.

Having justified our straight-line path assumption, we now calculate the coordinate-time delay t explicitly. Since we have determined the path, albeit by assumption, we need only the second of Eqs. (2.1) to determine t. Considering the rectilinear nature of the path and its direction parallel to the x-axis, we find from Eqs. (2.6) and (2.7) that

$$g_{44}c_o^2dt^2 + g_{11}dx^2 = 0 \quad ,$$

i.e., that

$$c = \frac{dx}{dt} = c_o\left(-\frac{g_{44}}{g_{11}}\right)^{1/2} = c_o\left[\frac{1-\frac{2r_o}{r}}{1+\frac{2r_o}{r-r_o}\frac{x^2}{r^2}}\right]^{1/2} \approx c_o\left[1-\frac{r_o}{r}\left(1+\frac{x^2}{r^2}\right)\right] \quad . \tag{2.15}$$

Since t increases monotonically over the round trip, whereas x first decreases monotonically and then increases monotonically, we find that the round-trip delay t_r is given by

$$t_r = -\int_{x_e}^{x_p}\frac{dx}{c} + \int_{x_p}^{x_e}\frac{dx}{c} = 2\int_{x_p}^{x_e}\frac{dx}{c} \approx \frac{2}{c_o}\int_{x_p}^{x_e}\left[1+\frac{r_o}{r}\left(1+\frac{x^2}{r^2}\right)\right]dx \quad . \tag{2.16}$$

Using Dwight[8] leads to*

$$t_r = \frac{2}{c_o}(x_e - x_p) + \frac{2r_o}{c_o}\left[2\log_e\left(\frac{x_e + r_e}{x_p + r_p}\right) - \left(\frac{x_e}{r_e} - \frac{x_p}{r_p}\right)\right] \quad , \tag{2.17}$$

where in consideration of our straight-line path,

$$r_{e,p}^2 = x_{e,p}^2 + d_o^2 \quad . \tag{2.18}$$

What we seek, of course, is not the coordinate-time delay, but the proper-time delay τ_r which is measured by the earth observer. As is well known,[7]

$$\tau_r = \frac{1}{c_o}\int_{s_1}^{s_2} ds \quad , \tag{2.19}$$

where s_1 is the (four-dimensional) position of the earth at transmission, and s_2 is the corresponding position at echo reception. Since, for the earth, the geodesic is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad , \tag{2.20}$$

and since we assumed that the earth remains stationary between pulse transmission and echo reception, we find

$$ds^2 = g_{44}c_o^2 dt^2 \quad , \tag{2.21}$$

and hence,†

$$\tau_r = \int_{t_1}^{t_2} g_{44}^{1/2}dt \approx (t_2 - t_1)\left(1 - \frac{r_o}{r_e}\right) \quad . \tag{2.22}$$

But $(t_2 - t_1)$ is just the left side of Eq. (2.16), and we obtain, finally,

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* If a harmonic coordinate system were employed (see Fock[9]), Eq. (2.17) would not contain the nonlogarithmic term inside the brace.[10] However, in the harmonic coordinate system the numerical values for the planetary positions are related to the relevant invariant quantities in a different way than in our coordinate system. As we explained earlier, these differences are of no intrinsic importance.

† The contribution of the earth's orbital motion to the ratio $\tau/(t_2 - t_1)$ is actually $r_o/(2r_e)$. But this contribution is the same as the corresponding contribution to the flat-space delay. Since we are here concerned only with the difference in these delays, neglect of the earth's motion is not serious.

$$\tau_r \approx \frac{2}{c_o}(x_e - x_p) + \frac{2r_o}{c_o}\left\{2\log_e \frac{x_e + (x_e^2 + d_o^2)^{1/2}}{x_p + (x_e^2 + d_o^2)^{1/2}} - \left[\frac{2x_2 - x_p}{(x_e^2 + d_o^2)^{1/2}} - \frac{x_p}{(x_p^2 + d_o^2)^{1/2}}\right]\right\} \quad . \tag{2.23}$$

The calculation of the corresponding flat-space delay τ_{fs} is trivial; the result is simply

$$\tau_{fs} = \frac{2}{c_o}(x_e - x_p) \quad , \tag{2.24}$$

and hence the difference $\Delta\tau_r$ is given by

$$\Delta\tau_r \approx \frac{2r_o}{c_o}\left\{2\log_e \frac{x_e + (x_e^2 + d_o^2)^{1/2}}{x_p + (x_p^2 + d_o^2)^{1/2}} - \left[\frac{2x_e - x_p}{(x_e^2 + d_o^2)^{1/2}} - \frac{x_p}{(x_p^2 + d_o^2)^{1/2}}\right]\right\} \quad . \tag{2.25}$$

This expression simplifies considerably in several cases. Near superior conjunction, for example, we find*

$$\Delta\tau_r \approx \frac{4r_o}{c_o}\left[\log_e \left|\frac{4x_e x_p}{d_o^2}\right| - \left(\frac{3x_e - x_p}{2x_e}\right)\right] \quad ; \quad d_o \ll x_e, |x_p| \quad , \tag{2.26}$$

and near inferior conjunction,

$$\Delta\tau_r \approx \frac{4r_o}{c_o}\left[\log_e \left(\frac{x_e}{x_p}\right) - \left(\frac{x_e - x_p}{2x_e}\right)\right] \quad ; \quad d_o \ll x_e, x_p \quad . \tag{2.27}$$

At elongation, Eq. (2.25) reduces to

$$\Delta\tau_r \approx \frac{4r_o}{c_o}\left[\log_e \left(\frac{2x_e}{d_o}\right) - 1\right] \quad ; \quad x_p \approx 0 \quad ; \quad d_o^2 \ll x_e^2 \quad . \tag{2.28}$$

This last form is valid only for Mercury since, for Venus, $x_e \approx d_o$ at elongation.

Although g_{44} and g_{rs} contribute equally to the dominant logarithmic term in the expression for $\Delta\tau_r$, their relative contributions to the second term vary with the path of the radar pulse. The maximum magnitude of the difference between these contributions is, however, too small to be reliably detected experimentally in the near future.

In Fig. 2, we show the values of $\Delta\tau_r$ as a function of the angle between the earth-sun line and the earth-Mercury line. This function, of course, is double-valued, one branch corresponding to the planet's being on the same "side" of the sun as the earth, the other branch to the planet's being on the opposite side. Both branches are shown. Figure 3 presents the same results for Venus. In constructing these figures the orbital eccentricities were neglected, and the round-trip delays corresponding to several of the orientations were included.

III. RIGOROUS DETERMINATION OF $\Delta\tau_r$

We now proceed to a rigorous derivation of Eq. (2.25). That is, here we shall make a formal expansion of the solution to the equations of motion in powers of r_o. Using Eqs. (2.1) and (2.6),

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* Note the identity: $-\log_e [-x + (x^2 + d^2)^{1/2}] = \log_e [x + (x^2 + d^2)^{1/2}] - \log_e d^2$.

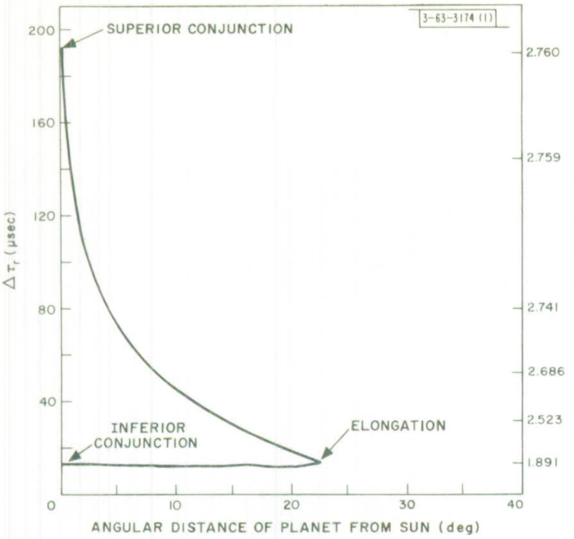
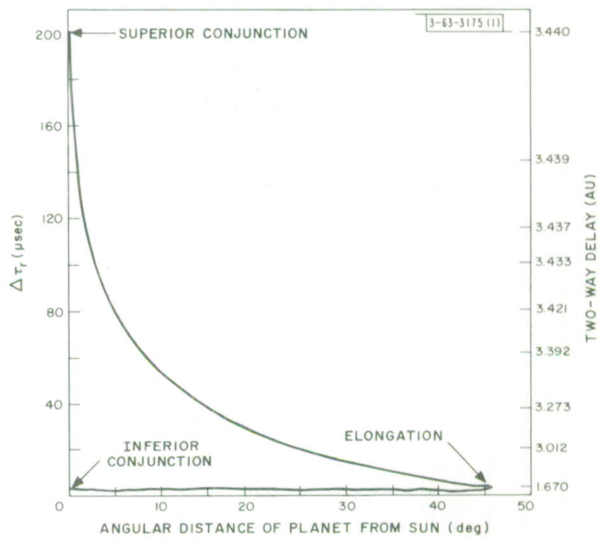


Fig. 2. Effect of general relativity on earth-Mercury time delays.

Fig. 3. Effect of general relativity on earth-Venus time delays.



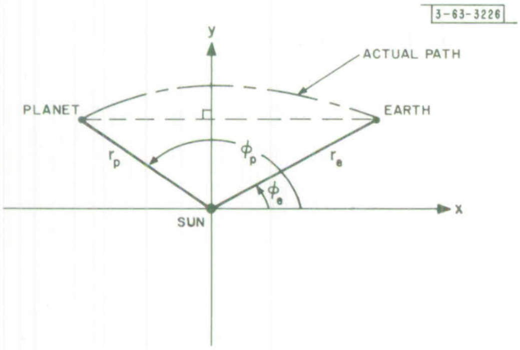


Fig. 4. Geometry of path of radar pulse: polar coordinates.

and making the transformation from rectangular to polar coordinates, we find[11] the equations of motion of a light ray to be

$$e^{\mu}\left(\frac{dt}{ds}\right)^2 - e^{-\mu}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\varphi}{ds}\right)^2 = 0 \quad ,$$

$$e^{\mu}\frac{dt}{ds} = k \quad ,$$

$$r^2\frac{d\varphi}{ds} = h \quad , \tag{3.1}$$
*

where r and φ are polar coordinates in the plane of motion of the light ray, and t is the time coordinate. The quantity e^{μ} stands for

$$e^{\mu} = 1 - \frac{2r_o}{r} \quad , \tag{3.2}$$

whereas k and h are constants of integration with s being a parameter.

Our goal can be accomplished as follows: We assume as before that the positions of the earth and target planet are given and that the earth remains stationary between transmission and detection of the radar pulse, and we calculate the round-trip time delay twice, once using the values for $g_{\mu\nu}$ given in Eq. (2.6), and once letting $g_{\mu\nu} = \pm\delta_{\mu\nu}$ with the plus sign holding for the temporal component and the minus sign for the spatial components. Each of these values for the difference in coordinate time is then separately converted to the appropriate proper-time delay. The two values are then subtracted, yielding $\Delta\tau_r$. One aspect of the subtraction operation deserves special mention: The distance of closest approach is different in the two calculations and the relation between them must be established to the proper accuracy.

Let us do the flat-space calculation first. We refer to Fig. 4 for the appropriate geometry. For convenience, we orient the x-y axes such that, for the flat-space path, the point on the path closest to the sun lies on the y-axis. (This orientation, as before, introduces certain algebraic simplifications in the calculations.) We denote this distance of closest approach by d_o in the flat-space case, and by d in the curved-space calculation.

For the flat-space case, of course, we could deduce the result by inspection. However, to illustrate the general method we proceed as follows: Here Eqs. (3.1) reduce to

$$\left(\frac{dt}{ds}\right)^2 - \left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\varphi}{ds}\right)^2 = 0 \quad ,$$

$$\frac{dt}{ds} = k \quad ,$$

$$r^2\frac{d\varphi}{ds} = h \quad . \tag{3.3}$$

Combining the second and third yields

$$\frac{dt}{d\varphi} = r^2\frac{k}{h} \quad , \tag{3.4}$$

whereas multiplying the first by $(ds/d\varphi)^2$ leads to

$$\left(\frac{dt}{d\varphi}\right)^2 - \left(\frac{dr}{d\varphi}\right)^2 - r^2 = 0 \quad . \tag{3.5}$$

---

* Throughout we use units in which c_o is unity.

Substituting (3.4) into (3.5) yields

$$r^4 \frac{k^2}{h^2} - \left(\frac{dr}{d\varphi}\right)^2 - r^2 = 0 \quad ,$$

i.e.,

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{k^2}{h^2} r^4 - r^2 \quad . \tag{3.6}$$

Defining

$$R \equiv \frac{h}{k} \quad , \tag{3.7}$$

we obtain

$$d\varphi = \pm \frac{dr}{\left(\frac{r^4}{R^2} - r^2\right)^{1/2}} = \pm \frac{R dr}{r(r^2 - R^2)^{1/2}} \quad . \tag{3.8}$$

Since φ is monotonically increasing for the first half of the round trip and monotonically decreasing for the second half, whereas r first decreases to d_o, then increases to r_p, etc., we see that the minus sign applies for the first part of the one-way trip and the plus sign for the second, etc. We will calculate explicitly only the one-way delay: The two paths are identical so that the total delay t_{fs} is just twice the one-way result. Integrating (3.8) yields[8]

$$\varphi = \pm \cos^{-1}\left(\frac{R}{r}\right) + \varphi_k \quad ; \quad 0 \leqslant \cos^{-1}\left(\frac{R}{r}\right) \leqslant \frac{\pi}{2} \quad , \tag{3.9}$$

where the minus sign holds for $0 \leqslant \varphi \leqslant \pi/2$ and the plus sign for $\pi/2 < \varphi \leqslant \pi$. We can determine R and φ_k in terms of the boundary conditions:

$$\varphi_e = -\cos^{-1}\left(\frac{R}{r_e}\right) + \varphi_k \quad ,$$

$$\varphi_p = \cos^{-1}\left(\frac{R}{r_p}\right) + \varphi_k \quad . \tag{3.10}$$

These equations can be solved in a straightforward manner to obtain R and φ_k. By subtracting the first from the second and taking the cosine of both sides, we obtain

$$\cos(\varphi_p - \varphi_e) = \frac{1}{r_e r_p} \left[R^2 - (r_e^2 - R^2)^{1/2} (r_p^2 - R^2)^{1/2}\right] \quad .$$

Eliminating the radicals, we find

$$[r_e r_p \cos(\varphi_p - \varphi_e) - R^2]^2 = (r_e^2 - R^2)(r_p^2 - R^2) \quad .$$

Hence,

$$R = \frac{r_e r_p \sin(\varphi_p - \varphi_e)}{[r_e^2 + r_p^2 - 2 r_e r_p \cos(\varphi_p - \varphi_e)]^{1/2}} \quad . \tag{3.11}$$

The value of φ_k then follows directly from Eqs. (3.10). The denominator in (3.11) is simply

$$r_e \cos\varphi_e + r_p \cos(\pi - \varphi_p) = r_e \cos\varphi_e - r_p \cos\varphi_p \quad ,$$

whereas the numerator is

$$d_o(r_e \cos\varphi_e - r_p \cos\varphi_p) \quad ,$$

since

$$r_e \sin\varphi_e = r_p \sin\varphi_p = d_o \quad .$$

Hence,

$$R = d_o \quad , \tag{3.12}$$

and inserting this result in Eq. (3.10), we find

$$\varphi_k = \frac{\pi}{2} \quad , \tag{3.13}$$

since $\sin^{-1} x + \cos^{-1} x = \pi/2$, when $x < \pi/2$; and $\sin^{-1} x - \cos^{-1} x = \pi/2$, when $\pi/2 < x \leqslant \pi$. These results also follow from the fact that

$$\left.\frac{dr}{d\varphi}\right|_{r=d_o} = 0 \quad .$$

Thus,

$$\left.\frac{dr}{d\varphi}\right|_{r=d_o} = \frac{d_o(d_o^2 - R^2)^{1/2}}{R} = 0 \Longrightarrow R = d_o \quad .$$

Since $\varphi = \pi/2$ when $r = d_o$, by our choice of axes, we have

$$\frac{\pi}{2} = -\cos^{-1}(1) + \varphi_k \Longrightarrow \varphi_k = \frac{\pi}{2} \quad .$$

To obtain the time delay t_{fs} explicitly, we use Eqs. (3.4) and (3.5):

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{d\varphi}{dt}\right)^2 \left(\frac{dr}{d\varphi}\right)^2 = \frac{R^2}{r^4}\left(\frac{r^4}{R^2} - r^2\right) = 1 - \frac{R^2}{r^2} \quad . \tag{3.14}$$

Using Eq. (3.12), we find

$$dt = \pm \frac{rdr}{(r^2 - d_o^2)^{1/2}} \quad . \tag{3.15}$$

Again, we find t increases monotonically, whereas r decreases monotonically from r_e to d_o and then increases monotonically to r_p. The plus sign therefore holds for the second part and the minus sign for the first. Remembering the factor of two needed to convert to the round-trip value, we find[8]

$$t_{fs} = 2\int_{d_o}^{r_e} \frac{rdr}{(r^2 - d_o^2)^{1/2}} + 2\int_{d_o}^{r_p} \frac{rdr}{(r^2 - d_o^2)^{1/2}} = 2\,[(r_e^2 - d_o^2)^{1/2} - (r_p^2 - d_o^2)^{1/2}] \quad , \tag{3.16}$$

But, as mentioned above,

$$d_o = r_e \sin\varphi_e = r_p \sin\varphi_p \quad , \tag{3.17}$$

and hence, reinserting c_o explicitly, we find

$$t_{fs} = \frac{2}{c_o}(x_e - x_p) \quad , \tag{3.18}$$

since x_p is negative for the configuration of Fig. 4.

Now we proceed analogously for the curved-space case. Combining the second and third members of Eqs. (3.1), we find the analog to Eq. (3.4):

$$\frac{dt}{d\varphi} = e^{-\mu} r^2 \frac{k}{h} \quad . \tag{3.19}$$

Multiplying the first by $(ds/d\varphi)^2$, as before, leads to

$$e^{\mu}\left(\frac{dt}{d\varphi}\right)^2 - e^{-\mu}\left(\frac{dr}{d\varphi}\right)^2 - r^2 = 0 \quad . \tag{3.20}$$

Substituting (3.19) into (3.20) yields

$$e^{-\mu} r^4 \frac{k^2}{h^2} - e^{-\mu}\left(\frac{dr}{d\varphi}\right)^2 - r^2 = 0 \quad ,$$

i.e.,

$$\left(\frac{dr}{d\varphi}\right)^2 = e^{\mu}\left(e^{-\mu} r^4 \frac{k^2}{h^2} - r^2\right)$$

$$= r^4 \frac{k^2}{h^2} - r^2 e^{\mu} \quad . \tag{3.21}$$

Again defining R as in Eq. (3.7), we obtain

$$d\varphi = \pm \frac{Rdr}{r(r^2 - R^2 e^{\mu})^{1/2}} \quad . \tag{3.22}$$

As before, we note that for the configuration drawn in Fig. 4, φ increases as r first decreases from r_e to d and then increases from d to r_p. Then φ decreases as r first decreases from r_p to d and then increases from d to r_e. Since the round-trip is symmetric, we will consider only the one-way case and multiply the result by two. We have, therefore,

$$\varphi = -\int \frac{Rdr}{r(r^2 - R^2 e^{\mu})^{1/2}} \quad ; \quad r_e \geqslant r \geqslant d \quad ; \quad \varphi_e \leqslant \varphi \leqslant \varphi(d)$$

$$= \int \frac{Rdr}{r(r^2 - R^2 e^{\mu})^{1/2}} \quad ; \quad d \leqslant r \leqslant r_p \quad ; \quad \varphi(d) \leqslant \varphi \leqslant \varphi_p \quad . \tag{3.23}$$

Let us first consider the case $r_e \geqslant r \geqslant d$. By changing the integration from r decreasing to r increasing, we have

$$\varphi = \int \frac{R\,dr}{r(r^2 - R^2 e^{\mu})^{1/2}} \quad .$$

But

$$e^{\mu} = 1 - \frac{2r_o}{r} \quad ,$$

and since $r = d$ at $(dr/d\varphi) = 0$, we find from Eq. (3.21) that

$$\frac{d^2}{R^2} - \left(1 - \frac{2r_o}{r}\right) = 0 \quad ,$$

i.e., that

$$R^2 = \frac{d^2}{1 - \frac{2r_o}{d}} \quad ,$$

or

$$R \approx \frac{d}{\left(1 - \frac{2r_o}{d_o}\right)^{1/2}} \approx d\left(1 + \frac{r_o}{d_o}\right) \quad . \tag{3.24}$$

We determine d from the initial conditions, after integrating Eq. (3.23). For this latter purpose, we substitute Eqs. (3.2) and (3.24) into (3.23):

$$\varphi = \frac{d}{\left(1 - \frac{2r_o}{d}\right)^{1/2}} \int \frac{dr}{r\left[r^2 - \frac{d^2}{\left(1 - \frac{2r_o}{d}\right)}\left(1 - \frac{2r_o}{r}\right)\right]^{1/2}}$$

$$= d^{3/2} \int \frac{dr}{r^{1/2}\,[r^3 d - d^3 r - 2r_o(r^3 - d^3)]^{1/2}} \tag{3.25}$$

But

$$r^3 - d^3 = (r - d)\,(r^2 + rd + d^2) \quad . \tag{3.26}$$

Hence,

$$\varphi = d^{3/2} \int \frac{dr}{[r(r-d)]^{1/2}\,[rd(r+d) - 2r_o(r^2 + rd + d^2)]^{1/2}} \quad .$$

Since $r_o << d$, we expand:

$$\varphi \approx d \int \frac{dr}{r(r^2 - d^2)^{1/2}} \left[1 + \frac{r_o}{d} + \frac{r_o d}{r(r+d)}\right] \tag{3.27}$$

and also find[8]

$$\int \frac{dr}{r(r^2 - d^2)^{1/2}} = \frac{1}{d}\cos^{-1}\left|\frac{d}{r}\right| \quad ; \quad 0 \leqslant \cos^{-1}\left|\frac{d}{r}\right| \leqslant \frac{\pi}{2} \quad . \tag{3.28}$$

To evaluate the second integral, we note that

$$\frac{1}{r^2(r+d)} = \frac{1}{dr^2} - \frac{1}{d^2 r} + \frac{1}{d^2(r+d)} \quad . \tag{3.29}$$

Hence,

$$\varphi \approx \left(1 + \frac{r_o}{d}\right) \cos^{-1} |\frac{d}{r}| + r_o d^2 \int \frac{dr}{(r^2 - d^2)^{1/2}} \left[\frac{1}{dr^2} - \frac{1}{d^2 r} + \frac{1}{d^2(r+d)}\right] \quad . \tag{3.30}$$

Using Dwight[8] and Gröbner and Hofreiter,[12] we find

$$\varphi = \cos^{-1} |\frac{d}{r}| + \frac{r_o}{d} (r^2 - d^2)^{1/2} [\frac{2r + d}{r(r+d)}] + \varphi_k \quad . \tag{3.31}$$

We could determine the constants of integration, d and φ_k, in terms of the boundary conditions. Equivalently, let φ_d be the angle at which r = d. Then, we have

$$\varphi_d - \varphi_e = \cos^{-1} |\frac{d}{r_e}| + \frac{r_o}{d} (r_e^2 - d^2)^{1/2} \left[\frac{2r_e + d}{r_e(r_e + d)}\right] \quad ,$$

$$\varphi_p - \varphi_d = \cos^{-1} |\frac{d}{r_p}| + \frac{r_o}{d} (r_p^2 - d^2)^{1/2} \left[\frac{2r_p + d}{r_p(r_p + d)}\right] \quad . \tag{3.32}$$

Since we are solving only to first order in r_o, we can replace d by d_o in the coefficients of r_o. Adding the two members of Eq. (3.32), we find

$$\cos\left(\cos^{-1} |\frac{d}{r_p}| + \cos^{-1} |\frac{d}{r_e}|\right) = \frac{1}{r_e r_p} [d^2 - (r_p^2 - d_o^2)^{1/2} (r_e^2 - d_o^2)^{1/2}]$$

$$= \cos(\varphi_p - \varphi_e) + \frac{r_o}{d_o} f \sin(\varphi_p - \varphi_e) \quad , \tag{3.33}$$

where

$$f \equiv \frac{(r_p^2 - d_o^2)^{1/2} (2r_p + d_o)}{r_p(r_p + d_o)} + \frac{(r_e^2 - d_o^2)^{1/2} (2r_e + d_o)}{r_e(r_e + d_o)} \quad . \tag{3.34}$$

From Eq. (3.33) it follows that

$$d = \frac{r_e r_p \left\{1 - \left[\cos(\varphi_p - \varphi_e) + \frac{r_o}{d_o} f \sin(\varphi_p - \varphi_e)\right]^2\right\}^{1/2}}{\left\{r_e^2 + r_p^2 - 2r_e r_p \left[\cos(\varphi_p - \varphi_e) + \frac{r_o}{d_o} f \sin(\varphi_p - \varphi_e)\right]\right\}^{1/2}} \quad , \tag{3.35}$$

which is the analog of Eq. (3.11). Expanding the right side of (3.35) leads to

$$d \approx \frac{r_e r_p \sin(\varphi_p - \varphi_e)}{[r_e^2 + r_p^2 - 2r_e r_p \cos(\varphi_p - \varphi_e)]^{1/2}} \left\{1 - \frac{r_o f}{d_o}\right.$$

$$\left.\times \left[\operatorname{ctn}(\varphi_p - \varphi_e) - \frac{r_e r_p \sin(\varphi_p - \varphi_e)}{r_e^2 + r_p^2 - 2r_e r_p \cos(\varphi_p - \varphi_e)}\right]\right\} \quad . \tag{3.36}$$

Using Eqs. (3.11) and (3.12), we find

$$d \approx d_o\left(1 + \frac{r_o}{d_o} f \left\{\frac{d_o}{[r_e^2 + r_p^2 - 2r_e r_p \cos(\varphi_p - \varphi_e)]^{1/2}} - \operatorname{ctn}(\varphi_p - \varphi_e)\right\}\right) \quad . \tag{3.37}$$

But from the law of cosines, we have

$$[r_e^2 + r_p^2 - 2r_e r_p \cos(\varphi_p - \varphi_e)]^{1/2} = x_e - x_p \quad . \tag{3.38}$$

It also follows from the above that

$$\cos(\varphi_p - \varphi_e) = \frac{d_o^2 + x_e x_p}{r_e r_p} \quad ,$$

and that

$$\sin(\varphi_p - \varphi_e) = \frac{d_o(x_e - x_p)}{r_e r_p} \quad .$$

Hence,

$$\frac{d_o}{x_e - x_p} - \frac{d_o^2 + x_e x_p}{r_e r_p}\left[\frac{r_e r_p}{d_o(x_e - x_p)}\right] = -\frac{x_e x_p}{(x_e - x_p)} \quad ,$$

which implies that

$$d \approx d_o\left[1 - \frac{r_o}{d_o^2}\left(\frac{x_e x_p}{x_e - x_p}\right) f\right] \quad . \tag{3.39}$$

Thus, $d > d_o$ since $f > 0$ and $x_p < 0$. But we expect this (see Fig. 4), and thus at least our analysis is not obviously in error. To find φ_d, we can use either of Eqs. (3.32). Let us take the first:

$$\varphi_d = \varphi_e + \cos^{-1}\left[\frac{d_o}{r_e} - \frac{r_o}{d_o r_e}\left(\frac{x_e x_p}{x_e - x_p}\right) f\right] + \frac{r_o}{d_o}(r_e^2 - d_o^2)^{1/2}\left[\frac{2r_e + d_o}{r_e(r_e + d_o)}\right] \quad .$$

We note that if

$$\cos^{-1} A = B \quad ,$$

and

$$\cos^{-1}(A + \epsilon) = B + \delta \quad ,$$

then for $\epsilon, \delta << 1$,

$$A + \epsilon = \cos(B + \delta) \approx \cos B - \delta \sin B \quad ,$$

and

$$\delta = \frac{\cos B - (A + \epsilon)}{\sin B} = -\frac{\epsilon}{\sin B} \quad .$$

In our case,

$$B = \cos^{-1}\left(\frac{d_o}{r_e}\right) \quad .$$

Hence,

$$\cos B = \frac{d_o}{r_e} \quad ; \quad \sin B = \frac{x_e}{r_e} \quad ,$$

and

$$\cos^{-1}\left[\frac{d_o}{r_e} - \frac{r_o}{d_o r_e}\left(\frac{x_e x_p}{x_e - x_p}\right) f\right] = \cos^{-1}\left(\frac{d_o}{r_e}\right) + \frac{r_o}{d_o r_e}\left[\frac{x_e x_p}{(x_e - x_p)}\right] f\left(\frac{r_e}{x_e}\right) \quad .$$

But, from Eqs. (3.10), (3.12), and (3.13), we find

$$\cos^{-1}\left(\frac{d_o}{r_e}\right) = \frac{\pi}{2} - \varphi_e \quad ,$$

and therefore,

$$\varphi_d = \frac{\pi}{2} + \frac{r_o}{d_o}\left(\frac{x_p f}{x_e - x_p}\right) + \frac{r_o}{d_o} x_e \left[\frac{2r_e + d_o}{r_e(r_e + d_o)}\right] \quad .$$

Using the expression for f from Eq. (3.34), we get

$$f = -\frac{x_p(2r_p + d_o)}{r_p(r_p + d_o)} + \frac{x_e(2r_e + d_o)}{r_e(r_e + d_o)} \quad ,$$

and

$$\varphi_d = \frac{\pi}{2} - \frac{r_o}{d_o}(x_e - x_p)^{-1}\left[\frac{x_p^2(2r_p + d_o)}{r_p(r_p + d_o)} - \frac{x_e^2(2r_e + d_o)}{r_e(r_e + d_o)}\right] \quad . \tag{3.40}$$

Is this result reasonable? If $r_e = r_p$, then $x_e = -x_p$ by our original assumption, and therefore $\varphi_d = \pi/2$ as we would expect from symmetry arguments. If $|x_p| << x_e$, we find $\varphi_d > \pi/2$. This deduction is also in accordance with expectations (or so it seems on cursory inspection!). At least (3.40) has the expected symmetry properties under reflection through the y-axis.

To obtain the time delay, we proceed as before. From Eqs. (3.19) and (3.21) we find

$$\left(\frac{d\varphi}{dt}\right)^2 \left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{dr}{dt}\right)^2 = e^{2\mu} \frac{R^2}{r^4}\left(\frac{r^4}{R^2} - r^2 e^{\mu}\right) \quad ,$$

and

$$\frac{dr}{dt} = \pm e^{\mu}\left(1 - \frac{R^2}{r^2} e^{\mu}\right)^{1/2} \quad . \tag{3.41}$$

Thus, the coordinate delay from $r = r_e$ to $r = d$, is

$$t_{ed} = \int_d^{r_e} \frac{e^{-\mu} dr}{\left(1 - \frac{R^2}{r^2} e^{\mu}\right)^{1/2}} \quad , \tag{3.42}$$

and the corresponding delay t_{dp} from $r = d$ to $r = r_p$ is

$$t_{dp} = \int_d^{r_p} \frac{e^{-\mu} dr}{\left(1 - \frac{R^2}{r^2} e^{\mu}\right)^{1/2}} \quad . \tag{3.43}$$

To evaluate the indefinite integral to first order in r_o, we find, using Eqs. (3.2) and (3.24), that

$$\frac{e^{-\mu}}{\left(1 - \frac{R^2}{r^2} e^{\mu}\right)^{1/2}} \approx (r^2 - d^2)^{-1/2} \left[r + r_o\left(2 + \frac{d_o}{r + d_o}\right)\right] \quad .$$

Hence,

$$t_{ed} = \int_d^{r_e} \frac{r dr}{(r^2 - d^2)^{1/2}} + r_o \left[\int_{d_o}^{r_e} \frac{2dr}{(r^2 - d_o^2)^{1/2}} + \int_{d_o}^{r_e} \frac{d_o dr}{(r + d_o)(r^2 - d_o^2)^{1/2}}\right] \quad . \tag{3.44}$$

Integrating yields[8,12]

$$t_{ed} = (r_e^2 - d^2)^{1/2} + r_o \left[2 \log_e \frac{r_e + (r_e^2 - d_o^2)^{1/2}}{d_o} + \frac{(r_e^2 - d_o^2)^{1/2}}{(r_e + d_o)}\right] \quad .$$

Thus, the total two-way delay t_r is given by

$$\begin{aligned} t_r \equiv 2(t_{ed} + t_{dp}) = \frac{2}{c_o} & \left[(r_e^2 - d^2)^{1/2} + (r_p^2 - d^2)^{1/2}\right] \\ & + \frac{4r_o}{c_o} \left\{\log_e \left[\frac{r_e + (r_e^2 - d_o^2)^{1/2}}{d_o}\right] \left[\frac{r_p + (r_p^2 - d_o^2)^{1/2}}{d_o}\right]\right. \\ & \left. + \frac{1}{2} \left[\frac{(r_e^2 - d_o^2)^{1/2}}{(r_e + d_o)} + \frac{(r_p^2 - d_o^2)^{1/2}}{(r_p + d_o)}\right]\right\} \quad , \end{aligned} \tag{3.45}$$

where we have reinserted c_o explicitly. In view of Eq. (3.39), we can write:

$$d^2 \approx d_o^2 - 2r_o\left(\frac{x_e x_p}{x_e - x_p}\right) f \quad ,$$

which leads to

$$(r_e^2 - d^2)^{1/2} \approx x_e \left[1 + r_o \frac{x_p}{x_e(x_e - x_p)} f\right] \quad . \tag{3.46}$$

Similarly,

$$(r_p^2 - d^2)^{1/2} \approx -x_p \left[1 + r_o \frac{x_e}{x_p(x_e - x_p)} f\right] \quad . \tag{3.47}$$

We can therefore rewrite Eq. (3.45) as

$$t_r = \frac{2}{c_o} (x_e - x_p) + \frac{4r_o}{c_o} \left[\log_e\left(\frac{r_e + x_e}{d_o}\right)\left(\frac{r_p - x_p}{d_o}\right) - \frac{1}{2}\left(\frac{x_e}{r_e} - \frac{x_p}{r_p}\right)\right] \quad , \tag{3.48}$$

which is exact up to terms of order $(r_o/c_o)^2$.

Both Eqs. (3.18) and (3.45) represent the time delays in coordinate time. However, as before, we seek the proper time measured by the earth observer. We therefore use Eq. (2.19) to convert each of these results to proper time. Consider Eq. (3.18) first. Here Eq. (2.20) becomes

$$ds^2 = c_o^2 dt^2 \quad , \tag{3.49}$$

since we have assumed that the earth remains fixed which implies that $(dr)^2 = (d\varphi)^2 = 0$. For the nonflat metric, we have

$$ds^2 = c_o^2 \left(1 - \frac{2r_o}{r_e}\right) dt^2 \quad , \tag{3.50}$$

since the temporal part of the metric depends on the gravitational field. Thus

$$\tau_r = \int_o^{t_r} \left(1 - \frac{2r_o}{r_e}\right)^{1/2} dt \approx t_r \left(1 - \frac{r_o}{r_e}\right) \quad , \tag{3.51}$$

and to first order in r_o, we find

$$\tau_r = \frac{2}{c_o}(x_e - x_p) + \frac{4r_o}{c_o}\left[\log_e\left(\frac{r_e + x_e}{d_o}\right)\left(\frac{r_p - x_p}{d_o}\right) - \frac{1}{2}\left(\frac{x_e}{r_e} - \frac{x_p}{r_p} + \frac{x_e - x_p}{r_e}\right)\right] \quad , \tag{3.52}$$

whereas

$$\tau_{fs} = \frac{2}{c_o}(x_e - x_p) \quad . \tag{3.53}$$

Since

$$\log\left(\frac{r - x}{d^2}\right) = \log\left(\frac{1}{r + x}\right) \quad ,$$

we can write

$$\Delta\tau_r = \tau_r - \tau_{fs} \approx \frac{2r_o}{c_o}\left[2 \log_e\left(\frac{r_e + x_e}{r_p + x_p}\right) - \left(\frac{2x_e - x_p}{r_e} - \frac{x_p}{r_p}\right)\right] \quad . \tag{3.54}$$

The above formula was derived on the assumption that the planet and earth are on opposite sides of the y-axis. Suppose both are on the same side; then the left side of the second equation of (3.32) should read $\varphi_d - \varphi_p$. Hence

$$\cos\left(\cos^{-1}\left|\frac{d}{r_e}\right| - \cos^{-1}\left|\frac{d}{r_p}\right|\right) = \frac{1}{r_e r_p}\left[d^2 + (r_p^2 - d^2)^{1/2}(r_e^2 - d^2)^{1/2}\right]$$

$$= \cos(\varphi_p - \varphi_e) + \frac{r_o}{d_o} f' \sin(\varphi_p - \varphi_e) \quad , \tag{3.55}$$

where

$$f' = \frac{(r_e^2 - d_o^2)^{1/2}(2r_e + d_o)}{r_e(r_e + d_o)} - \frac{(r_p^2 - d_o^2)^{1/2}(2r_p + d_o)}{r_p(r_p + d_o)} \quad . \tag{3.56}$$

As previously, we find

$$d' \approx d_o\left[1 - \frac{r_o}{d_o^2}\left(\frac{x_e x_p}{x_e - x_p}\right) f'\right] \quad . \tag{3.57}$$

We also obtain the analog of Eq. (3.45) as:

$$t_r' = 2(t_{ed} - t_{dp}) = \frac{2}{c_o}\left[(r_e^2 - d^2)^{1/2} - (r_p^2 - d^2)^{1/2}\right] + \frac{4r_o}{c_o}$$

$$\times \left\{ \log_e \frac{r_e + (r_e^2 - d_o^2)^{1/2}}{r_p + (r_p^2 - d_o^2)^{1/2}} + \frac{1}{2}\left[\frac{(r_e^2 - d_o^2)^{1/2}}{(r_e + d_o)} - \frac{(r_p^2 - d_o^2)^{1/2}}{(r_p + d_o)}\right]\right\} \quad . \tag{3.58}$$

But, in analogy with Eqs. (3.46) and (3.47), we find

$$(r_e^2 - d^2)^{1/2} - (r_p^2 - d^2)^{1/2} \approx x_e - x_p - r_o f' \quad .$$

Hence,

$$t_r' = \frac{2}{c_o}(x_e - x_p) + \frac{4r_o}{c_o}\left[\log_e\left(\frac{r_e + x_e}{r_p + x_p}\right) - \frac{1}{2}\left(\frac{x_e}{r_e} - \frac{x_p}{r_p}\right)\right] \quad ,$$

and finally,

$$\Delta\tau_r' = \frac{4r_o}{c_o}\left[\log_e\left(\frac{r_e + x_e}{r_p + x_p}\right) - \frac{1}{2}\left(\frac{2x_e - x_p}{r_e} - \frac{x_p}{r_p}\right)\right] \quad . \tag{3.59}$$

Comparing (3.59) and (3.54), we find that, in general,

$$\Delta\tau_r = \frac{4r_o}{c_o}\left[\log_e\left(\frac{r_e + x_e}{r_p + x_p}\right) - \frac{1}{2}\left(\frac{2x_e - x_p}{r_e} - \frac{x_p}{r_p}\right)\right] + O\left(\frac{r_o^2}{c_o^2}\right) \quad . \tag{3.60}$$

Considering Eq. (2.18), we see that the result (3.60) is in precise accord with Eq. (2.25).

At this point we should mention that in an interplanetary radar experiment, in addition to the time delay, the Doppler shift of the radar wave is also measured; but, although the effect on time delay of the change in c is cumulative, the corresponding general relativistic effect on Doppler cancels out over the round trip.* In our development thus far we have assumed that the earth remains stationary. Whereas such an approximation has no significant effect on $\Delta\tau_r$, the change Δf_r in Doppler attributable to $\Delta\tau_r$ comes about solely because of the time variation of $\Delta\tau_r$ introduced by the relative motions of earth and planet. Since, to first order in v/c, the Doppler shift Δf is related to the transmitted frequency f and to the time delay τ by

$$\Delta f = -f\dot{\tau} \quad , \tag{3.61}$$

and since $\Delta\tau_r$ is most sensitive to changes in d_o, we can approximate Δf_r by

$$\Delta f_r \approx -f\left(\frac{\partial \Delta\tau_r}{\partial d_o}\right)\left(\frac{\partial d_o}{\partial t}\right) \quad , \tag{3.62}$$

where by straightforward differentiation of Eqs. (3.60) and (3.11), we find

$$\frac{\partial \Delta\tau_r}{\partial d_o} = \frac{4r_o}{c_o} d_o\left[\frac{1}{r_e(r_e + x_e)} - \frac{1}{r_p(r_p + x_p)} + \frac{1}{2}\left(\frac{2x_e - x_p}{r_e^3} - \frac{x_p}{r_p^3}\right)\right] \quad , \tag{3.63}$$

* If a suitably calibrated frequency-measuring device were stationed on one of the inner planets or in orbit in their vicinity, then the general relativistic effect on frequency, the so-called gravitational red shift, would indeed be detectable, amounting to a change in frequency at X-band of about 20 cps at the orbit of Venus. Of course, when the receiver is inside earth's orbit, the change is a violet shift.

and

$$\frac{\partial d_o}{\partial t} = r_e r_p(\dot{\varphi}_p - \dot{\varphi}_e)\left\{\frac{\cos(\varphi_p - \varphi_e)}{[r_e^2 + r_p^2 - 2r_e r_p \cos(\varphi_p - \varphi_e)]^{1/2}}\right.$$

$$\left. - \frac{r_e r_p \sin^2(\varphi_p - \varphi_e)}{[r_e^2 + r_p^2 - 2r_e r_p \cos(\varphi_p - \varphi_e)]^{3/2}}\right\} \quad . \qquad (3.64)$$

Near superior conjunction, this approximation to Δf_r reduces to

$$\Delta f_r \approx \pm f \frac{8r_o}{c_o}\left|\frac{x_e x_p(\dot{\varphi}_p - \dot{\varphi}_e)}{d_o(x_e - x_p)}\right| \quad , \qquad (3.65)$$

where the minus sign is to be used for the pre-superior-conjunction configuration and the plus sign for the post-conjunction configuration. Even at X-band frequencies and with grazing incidence ($d_o \approx R_s$), Eq. (3.65) yields only a 3.6-cps effect. Furthermore, since the planetary orbits are not coplanar, the magnitude of $\partial d_o/\partial t$ will in general be less than indicated by Eq. (3.64). At $d_o = 3R_s$, the closest approach possible with the Haystack antenna (see Sec. IV), Δf_r would fall to about 1 cps or less. Aside from the questions of frequency stability and of other influences on the Doppler shift,[5] it is doubtful that the center frequency of the Doppler-broadened echo from a rotating planet could be located with sufficient accuracy to detect Δf_r reliably.

IV. OTHER INFLUENCES ON TIME DELAYS

Are the relativistic effects on interplanetary time delays likely to be obscured by others? The most important candidates in this latter category are the imprecise knowledge of planetary orbits and the presence of interplanetary plasma. A moment's reflection suffices to show that the orbits of the earth and the target planet can be determined with more than the required precision from optical and time-delay measurements distributed around the orbits of both planets. For example, time-delay observations of Mercury from earth could be made at all positions of Mercury along its orbit with the radar wave never passing near the sun. Observations along one half could be obtained from elongation to elongation through inferior conjunction, and along the other half during a corresponding elongation-inferior conjunction-elongation period, but with the earth on the opposite side of its orbit. In addition, since the received power varies with the inverse fourth power of the interplanetary range, measurements made near inferior conjunction will undoubtedly be more accurate than the corresponding ones made near superior conjunction. Hence, the precision of orbital determination should be at least as high as the accuracy with which one can make the crucial time-delay measurements near superior conjunction. More generally, we can see from this analysis that the sensitivity of the time delays to changes in $\Delta\tau_r$ will be different from the corresponding sensitivities to changes in the initial conditions of the orbits and in the planetary masses and radii. A parameter characterizing $\Delta\tau_r$ could therefore be estimated from the data simultaneously with the other relevant ones, without incurring any appreciable accuracy penalty from inseparability of effects. The topographical variations on the target planets are probably small enough so that even the most accurate measurements will not be significantly degraded thereby.

The effect $\Delta\tau_m$ of the interplanetary medium on the two-way time delay can be represented by[5]

$$\Delta\tau_m \approx \frac{8.2 \times 10^7}{f^2 c_o} \int_{x_p}^{x_e} N(\ell)\, d\ell \ \text{sec} \quad , \tag{4.1}$$

where N is expressed in electron/cm^3, f in cps, c in cm/sec, and ℓ in cm. Equation (4.1) is valid only if f lies sufficiently above the plasma frequency of the medium; this condition will hold for all our considerations. Using recently compiled results on the solar corona (see Fig. 11* in Erickson[13]), we find that during a "quiet-sun" period, N(r) can be represented reasonably well by

$$N(r) = 5 \times 10^5 \left(\frac{R_s}{r}\right)^2 \ e\ell/cm^3 \quad ; \quad r^2 = \ell^2 + d^2 \tag{4.2}$$

from $r = 4R_s$ to $r = 20R_s$. Inside this range the actual N increases more rapidly with decreasing r, whereas outside it decreases more rapidly with increasing r. For a period of maximum solar activity, Fig. 11 of Erickson[13] shows that N is probably about a factor of five higher in the radial range represented by Eq. (4.2). Substituting Eq. (4.2) into (4.1), yields[8]

$$\Delta\tau_m \approx \frac{6.5 \times 10^{24}}{f^2 d} \left[\tan^{-1}\left(\frac{x_e}{d}\right) - \tan^{-1}\left(\frac{x_p}{d}\right)\right] \text{sec} \quad . \tag{4.3}$$

Here d, x_e, and x_p are expressed in centimeters and are defined as in Fig. 1. For $d \ll x_e, |x_p|$ and with $x_p < 0$, we find

$$\Delta\tau_m \approx \frac{6.5 \times 10^{24}}{f^2 d} \pi \quad . \tag{4.4}$$

For the Arecibo Ionospheric Observatory's frequency of 430 Mcps, the lowest at which interplanetary time-delay measurements are currently being made, Eq. (4.4) yields $\Delta\tau_m \approx 4 \times 10^{-4}$ sec for observations of Mercury near superior conjunction with $d \approx 4R_s$. (This latter value corresponds to an angular distance from the sun of 1°, the smallest at which Arecibo measurements can be made.) In this case, $\Delta\tau_r$ would equal about 1.4×10^{-4} sec, as can be seen from Fig. 2, and would most likely be masked by the uncertainty in $\Delta\tau_m$. Although $\Delta\tau_m$ varies inversely with d, whereas the corresponding dependence in $\Delta\tau_r$ is logarithmic, the difference $\Delta\tau_r - \Delta\tau_m$ is nowhere large enough and positive for a really reliable and accurate result to be obtained solely from Arecibo data. Since, for sufficiently high f, $\Delta\tau_m$ varies as the inverse square of the radar frequency, this plasma effect will be reduced by a factor of almost 400 (and will therefore be unimportant) for measurements made at the 8350-Mcps frequency of the newly constructed, but not yet fully instrumented, Haystack radar facility. The Jet Propulsion Laboratory Goldstone radar, operated at a frequency of 2388 Mcps, could probably also make these time-delay measurements in a "quiet-sun" period without undue interference from the solar corona. However, during a maximum of solar activity, it is doubtful whether measurements at 2388 Mcps could be used

* Note the caption on this figure: "A complication of data.... " It unwittingly summarizes the experimental situation quite well.

alone to perform an accurate test of general relativity: The plasma would cause an increase in time delay at $d = 4R_s$ of about 0.7×10^{-4} sec and at $d = 20R_s$ of about 0.15×10^{-4} sec, with an uncertainty of perhaps a factor of two or three.* But in the latter case, $\Delta\tau_r$ is only 0.7×10^{-4} sec. In any event, simultaneous equivalently accurate measurements at two well-spaced frequencies will allow the plasma effect to be deduced and subtracted since $\Delta\tau_m$ is frequency dependent and $\Delta\tau_r$ is not.†

Other possibly relevant effects on the delays are easily disposed of. A previous study[5] has shown that the earth's and planet's atmospheres and ionospheres will not significantly affect time delays, even for f = 430 Mcps. The effect of the earth's gravity and motion on the laboratory clock is unimportant for this experiment, since the clock rate remains constant over a year to within about one part in 10^{10}. The gravitational effects of the earth, moon, and target planet on the delays are much smaller than those of the sun, but in any case the former (excepting the moon) could be neglected since these contributions will be almost identical in each measurement and consequently indistinguishable from a small decrease in the planet's radius. Similarly, any bias introduced into the time delay by the radar system will not affect this experiment, provided only that such bias is independent of the relative orientations of the earth, sun, and planet. Any lack of precision in the determination of c in terms of terrestrial units (such as in km/sec) is clearly irrelevant to our experiment since time delays only are of concern.

In making the time-delay measurements, we must consider the radio interference introduced by the sun. For the Haystack facility, the antenna beamwidth is sufficiently narrow and the near sidelobes are of sufficiently low gain that the beam can be directed within a half-degree of the solar limb without the radio emanations introducing a significant increase in the over-all system noise temperature. For Arecibo, the closest possible approach is about 1°, partly because of the larger radio diameter of the sun at 430 Mcps and partly because of the higher-gain sidelobes of the Arecibo antenna.

On the basis of these analyses, we can feel reasonably confident that the time-delay experiments discussed here will provide a meaningful test of Einstein's theory of general relativity. Because the magnitude of the effect decreases slowly with the distance of closest approach of the radar wave to the sun (logarithmic dependence on d), we have the possibility of testing not only a single numerical prediction, but the functional form of $\Delta\tau_r$ as well. We require, of course, the ability to measure time delays at superior conjunction with an error of no more than about 10 μsec. The upgraded Haystack facility is expected to provide such a capability.[14] Repeated measurements extended over a period of several years should then enable the maximum effect to be determined to about 1 percent and the effect at greater distances of closest approach to correspondingly lower accuracies (see Fig. 2). We need not worry about constant biases in the radar system limiting our use of statistics to improve accuracy since, as shown above, constant biases serve only to introduce an error in the estimate of the planet radius.

In principle, this relativistic effect on time delay could also be observed either by using space probes in orbit about the sun, or by placing transponders on a planet. At present, each of these approaches would be more difficult and more costly to implement than the one discussed here.

*Since for $r > 20R_s$, N(r) decreases more rapidly with increasing r than is indicated in Eq. (4. 2), we might find $\Delta\tau_m$ for $d = 20R_s$ to be somewhat less than estimated.

†It may be possible to estimate the plasma effect on delay adequately by simultaneously observing the deflection of low-frequency (< 100 Mcps) signals from suitable radio stars near the line-of-sight to the target planet. However, a variety of plasma conditions having different effects on delay can lead to the same over-all deflection of the radio star emanations.

V. ADDITIONAL SCIENTIFIC "FALLOUT"

The time-delay measurements proposed to test the predictions of general relativity concerning the effect of gravitational fields on light rays also have other scientific implications. The orbital accuracies achievable will represent not only a very significant improvement over present knowledge, but will enable a somewhat independent and more accurate determination to be made of the precession of Mercury's perihelion position. With the measurements distributed fairly uniformly about the orbits of both earth and Mercury, the standard deviation $\sigma(\dot{\omega})$ of the error in perihelion precession is given by[15]

$$\sigma(\dot{\omega}) \approx \frac{2 \times 10^7}{(Nt^3)^{1/2}} \frac{1}{ae} \frac{\sigma(\tau)}{\tau_{min}} \text{ sec of arc/100 yr} \quad , \tag{5.1}$$

where N is the number of measurements made per year, t is the total time span of the observations in years, $\sigma(\tau)$ is the standard deviation of the single-measurement error, τ_{min} is the minimum range to Mercury, and a (in astronomical units) and e are the semimajor axis and eccentricity, respectively, of Mercury's orbit. (The numerical coefficient merely represents the conversion from radians per revolution to seconds of arc per century.) With N = 50, t = 3, $\sigma(\tau) = 10\,\mu$sec, $\tau_{min} = 350$ sec, a = 0.38, and e = 0.2, we find

$$\sigma(\dot{\omega}) \approx 0.2 \text{ sec of arc/100 yr} \quad , \tag{5.2}$$

which is considerably smaller than the error currently attributed to the conventional determination of the centennial value of the precession.

In order to determine to this accuracy the anomalous (non-Newtonian) contribution to the precession, we must of course know the Newtonian contribution to a comparable degree of accuracy. Two major impediments to this determination are the uncertainty in our knowledge of the sun's quadrupole moment and of Venus's mass.* The former problem has been discussed elsewhere.[16] The latter can probably be solved by using earth-Venus time-delay observations from which the perturbations of Venus on the orbit of earth should enable Venus's mass to be determined accurately enough. Knowledge of the mass of Mercury may also be improved significantly in virtue of its effect on the orbits of earth and Venus.

The determination of the radii of both Venus and Mercury will also be refined substantially, in fact by about two orders of magnitude. If the optical diameter of Venus were determinable with an accuracy of about ±5 km, then the height above the surface of the Venusian cloud layer could be deduced with reasonable accuracy; such a result would be of great importance for the study of planetary atmospheres.† The combination of accurate radii and mass determinations will, of course, yield correspondingly accurate densities for Mercury and Venus. These values are of interest to studies of planetary formation.

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* Although it is claimed by Jet Propulsion Laboratory analysts that their tracking data on Mariner II yielded five-place accuracy in the determination of the mass of Venus, there is the possibility that systematic errors may result in this accuracy being more apparent than real.

† There is also the possibility that the X-band radar waves are reflected from a cloud layer and the S-band and lower frequency waves from the surface. In such a case, simultaneous delay measurements at X-band and at a lower frequency might enable the height of the reflecting layer to be determined.

Highly accurate determinations of the mean anomalies of the inner planets will allow a reasonably strict limit to be placed on the possible time dependence of the gravitational constant G. [The theoretical conjecture (first made by Dirac[17]) that G may decrease with time is based partly on the apparent expansion of the universe.] If the mean anomalies could be determined each year with an error no greater than 0.02 seconds of heliocentric arc, which should be achievable if individual time-delay measurements are made with errors of 10 μsec or less, then a variation in G of about five parts in 10^{10} per year would be discernible after several years of observations.* Of course, one must also have clocks with the requisite long- and short-term stabilities. Since the theoretical estimates of the rate of change of G vary down to a part in 10^{11} per year (see, e.g., Dicke[18]), this test will probably not be crucial unless the measurements are either improved in accuracy or continued over many years.

As stated in Sec. IV, the solar corona affects time-delay measurements by an amount inversely proportional to the square of the radar frequency. In particular, accurate delay measurements made at about 400 Mcps, for example at Arecibo, would afford an excellent opportunity to study the solar corona. The integrated electron densities could be determined as a function of d and from these results the average radial electron density could be deduced. Short-term fluctuations could be studied by making frequent measurements and long-term trends observed by extending the study over the length of the solar cycle. The solar-corona plasma will also influence the Doppler shift of the radar waves via an effect analogous to the one discussed at the end of Sec. III. As in that discussion, we conclude that the time-delay differential holds more promise at present as a tool for investigating the corona. However, Doppler observations may enable the detection of dense plasma "wedges" moving perpendicular to the path of the radar wave. Bending caused by the plasma would be significant only close to the solar limb where measurements cannot be made. Hence this effect too seems to be of less interest than the time-delay measurements.

Faraday rotation effects, caused by the different propagation speeds of the ordinary and extraordinary modes, would also enable certain characteristics of the solar magnetic field to be investigated. The detailed deductions are somewhat complicated, however, because sign reversals in the magnetic field tend to cancel in their effect on Faraday rotation. In summary, such radar experiments could probably provide all the solar corona information currently expected from the forthcoming Sun-Blazer probes.

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* Note that the effect of a change in G on the mean anomaly is proportional to the square of the elapsed time, whereas the corresponding effect on the Astronomical Unit is merely proportional to the time. (In this discussion, we are, of course, referring to nongravitational (e.g., atomic) measures of time.)

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13. ABSTRACT

The predictions of general relativity regarding interplanetary time-delay measurements are explored in detail. We conclude that a fourth test of the theory is now feasible since the modifications of general relativity introduce an extra delay of about 200 μsec when radar waves are reflected from either Mercury or Venus near superior conjunction. The uses of such measurements to investigate the solar corona, a possible time dependence of the gravitational constant, and the precession of Mercury's perihelion are also discussed.

14. KEY WORDS

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