SOME RESULTS AND PROBLEMS
IN STOCHASTIC LINEAR PROGRAMMING

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SUMMARY

Results and problems in the ordinary "here—and—now" and "wait—and—see" stochastic linear programming problems are described. A general formulation of the "here—and—now" problem is presented, and an approach for solving a special kind of "here—and—now" problem is suggested.

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1. INTRODUCTION

As evidenced by the number of papers presented recently
(e.g., [3], [9], [11], [13], [16], and [20]) whose subjects
fall under the general rubric of "stochastic linear programming,"
there seems to be a current flourishing of interest in this
area. It appears to me that work is being done on various sub-
problems in this area, but that the area in its entirety has
never been given suitable attention, or its problems even
formulated concisely. It is our hope to discuss the broad area
of stochastic linear programming, establish some consistent
terminology, point out where current work fits into this broad
picture, and present some basic problems which need solution
and suggested avenues of solution.

2. STOCHASTIC LINEAR PROGRAMMING SITUATIONS

The usual linear programming problem is to find the
activity vector x^* which minimizes $c^{*'}x^*$ subject to the
conditions $A^*x^* = b^*$ and $x^* \geq 0$. Let us rewrite the problem
as: find the decision vector x and the slack vector y which
minimizes $c'x + f'y$ subject to the conditions $Ax + By = b$,
$x \geq 0$, $y \geq 0$. In stochastic linear programming, some or all
of the matrices A and B and vectors b, c, and f are random.

Let us distinguish two stochastic linear programming situations, the "wait-and-see" and "here-and-now" situations. In the "wait-and-see" situation, one first observes the random elements and then solves the (nonstochastic) linear programming problem of finding $x \geq 0$ and $y \geq 0$ which minimizes $c'x + f'y$ subject to $Ax + By = b$.

In the "here-and-now" situation, one must (1) find the slack vector y as a function of x and of the random elements which optimizes some criterion and (2) determine a value of the decision vector x which is independent of y and the to-be-observed random elements and which optimizes some criterion which is independent of the observed value of the random elements, from among the vectors x and y such that the probability of feasibility is at least some specified P, $0 \leq P \leq 1$. After these vectors are determined, the random elements are observed and the value of y is determined. Following are some examples of "here-and-now" problems.

<u>Example I</u>:   Among all x and y whose probability of feasibility is at least P, find the y which minimizes $c'x + f'y$, and determine the value of x which minimizes $E \min_{y} (c'x + f'y)$.[1]

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[*]Unless otherwise specified, the expectation or probability is with respect to <u>all</u> the random elements.

Example II: Among all x and y whose probability of feasibility is at least P, find the y which minimizes $c'x + f'y$, and determine the value of x which maximizes $\Pr\left\{\min_{y} (c'x + f'y) \leq k\right\}$ for some fixed preassigned k.

Example III: Among all x and y whose probability of feasibility is at least P, find the y which minimizes $c'x + f'y$, and determine the value of x which maximizes $\Pr\left\{\min_{y} (c'x + f'y) \leq k \text{ and } x, y \text{ feasible}\right\}$ for some fixed preassigned k.

In the following we shall refer as a practical example to this simple case: A factory has 100 items on hand which may be shipped to an outlet at the cost of $1 apiece to meet an uncertain demand d. In the event that the demand should exceed the supply, it is necessary to meet the unsatisfied demand by purchases on the local market at $2 apiece. The equations that the system must satisfy are

$$100 = x_{11} + x_{12}$$
$$d = x_{11} + x_{21} - x_{22} \qquad (x_{1j} \geq 0)$$
$$C = x_{11} + 2x_{21}$$

where

x_{11} = number shipped from the factory,

x_{12} = number stored at factory;

x_{21} = number purchased on open market,

x_{22} = excess of supply over demand;

$$d = \text{unknown demand uniformly distributed between}$$
$$70 \text{ and } 120;$$
$$C = \text{total costs.}$$

It is clear that whatever be the amount shipped and whatever be the demand d, it is possible to choose x_{21} and x_{22} consistent with the second equation. The unused stocks $x_{12} + x_{22}$ are assumed to have no value or are written off at some reduced value (like last year's model automobiles when the new production comes in). In this example, $x' = (x_{11}, x_{12})$, $y' = (x_{21}, x_{22})$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$, $c' = (1, 0)$, $f' = (2, 0)$, and $b' = (100, d)$.

3. THE HERE-AND-NOW PROBLEM

3a. Example I

Work has been done on Example I by Beale [2], Dantzig [7], and Madansky [13], in the case where P = 1 and only b is random. The basic consideration of Beale was to consider how one might utilize the solution of the nonstochastic problem where b is replaced by Eb in order to iterate to a solution of the stochastic problem. He suggests that due to certain convexities in the problem, this solution be utilized as a first approximation in an application of the Newton–Raphson method to minimizing $E \min_y (c'x + f'y)$ with respect to x.

Dantzig considered a multistage problem where a decision x_1 has to be made at stage one, then the (random) right–hand side of the first stage of the problem, b_1, is observed, after

which a decision x_2 has to be made at stage two, etc., for m

stages. He found that $E_{b_1,\ldots,b_m} \min_y (c'x_1 + f'y)$ is convex

in x_1 [1] (where y includes $x_2, \ldots, x_m$, as well as slack), and

so suggests using appropriate techniques ([5], [6]) to

determine x_1, and hence x_2 after b_1 is observed, etc.[2]

Madansky, in connection with work on inequalities for

$\min_x E \min_y (c'x + f'y)$, the value of the objective function in

the "here-and-now" problem of Example I, and $E \min_x \min_y (c'x + f'y)$,

the expected value of the objective function of the "wait-and-see"

problem, found the following sufficient (though not necessary)

condition for the solution of the stochastic problem to be the

solution of the nonstochastic problem where b is replaced by Eb.

Theorem. If $\min_y (c'x + f'y)$ can be written as

$C_1(b, x) + C_2(b)$, where $C_1(b, x)$ is linear in b, then the

x which solves the nonstochastic problem where b is replaced

by Eb also solves the "here-and-now" problem of Example I

where P = 1 and only b is random.

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[1]Cf. also Theorem 2 of [2].

[2]Cf. also [4], where by imposing the condition that for
each i x_i be linear in the b_j, $j = 1, \ldots, i-1$, the quantity
$E_{b_1,\ldots,b_m} \min_y (c'x_1 + f'y)$, as interpreted in the context of
[7], is found to be convex in x_1.

This theorem generalizes work of Simon [13] and Theil [18] on what is known in the literature as the use of "certainty equivalents" in stochastic problems.[1]

The only other results on the version of Example I of the "here-and-now" problem are the inequalities of Madansky [13] on the value of the objective function when only b is random. Let $C(b, x) = \min_{y} (c'x + f'y)$. Then in general

$$E_b \min_{x} C(b, x) \leq \min_{x} E_b C(b, x) \leq E_b C(b, \bar{x}(Eb)),$$

where $\bar{x}(Eb)$ is the vector which solves the nonstochastic linear programming problem where $b = Eb$. Also, in the special case considered by Dantzig discussed above, there is the additional inequality

$$\min_{x} E_b C(b, x) \geq \alpha_0 + \sum_{j=1}^{r} \alpha_j \bar{x}_j = G(\bar{x})$$

where $\bar{x}$ is the r-dimensional vector which solves the "here-and-now" problem,

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[1]Reiter [14] studies the general problem of finding sufficient conditions for the solution of a stochastic problem to be that of a "surrogate" problem where the probability distribution is replaced by something simpler. His condition, in our case, is that $C(b, x) = \sum_{i=1}^{n} A_i(x)B_i(b)$, where $A_i(x) > 0$, $B_i(b) > 0$, in which case the x which minimizes $\sum_{i=1}^{n} A_i(x)EB_i(b)$ solves the stochastic problem. If we let $n = 3$, $B_1(b) = 1$, $B_2(b) = b$, $B_3(b) = C_2(b)$, and $A_3(x) = 1$, then $C(b, x) = C_1(b, x) + C_2(b) = A_1(x) + A_2(x)b + B_3(b)$, so that our theorem is a special case of Reiter's where the "surrogate" is Eb.

$$\alpha_j = \frac{\partial E_b\, C(b,\, x)}{\partial x_j}\bigg|_{x_j = \overline{x}_j(Eb)}, \qquad j = 1, \ldots, n,$$

$$\alpha_0 = E_b\, C(b,\, \overline{x}(Eb)) - \sum_{j=1}^{r} \alpha_j \overline{x}_j(Eb),$$

and the derivative of $E_b\, C(b,\, x)$ exists in the neighborhood of $\overline{x}(Eb)$. Hence, one can determine the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_r$ without solving Dantzig's "here—and—now" problem and, if one has bounds on $\overline{x}$, can utilize this information to get a lower bound on $G(\overline{x})$. This then yields a lower bound on the value of the objective function of Dantzig's "here—and—now" problem which is independent of the solution of the problem.

As shown in [13], a sufficient condition for $\min_x E_b\, C(b,\, x) = E_b \min_x C(b,\, x)$ is that $C(b,\, x)$ be a linear function of b. A sufficient condition for $\min_x E_b\, C(b,\, x) = E_b\, C(b,\, \overline{x}(Eb))$ is that $E_b\, C(b,\, x)$ be of the form $x_j^2 - 2\overline{x}_j(Eb)x_j + k$ in each x_j, for then $\alpha_j = 0$ for $j = 1, \ldots, r$, so that $G(\overline{x}) = E_b\, C(b,\, \overline{x}(Eb))$ and $E_b\, C(b,\, \overline{x}(Eb)) \leq \min_x E_b\, C(b,\, x) \leq E_b\, C(b,\, \overline{x}(Eb))$. Examples can be constructed such that $G(\overline{x}) > E_b \min_x C(b,\, x)$ and $G(\overline{x}) < E_b \min_x C(b,\, x)$, so that one cannot compare these bounds in general.

One can generalize the inequalities further, to the case where all the coefficients in the problem (i.e., the elements

of A, B, b, c, and f) are random, for even then

$$E \min_{x} \min_{y} (c'x+f'y) \leq \min_{x} E \min_{y} (c'x+f'y) \leq E\left[\min_{y} (c'x+f'y) \Big| x = \bar{x}(\mathcal{E})\right],$$

where $\mathcal{E}$ denotes (EA, EB, Eb, Ec, Ef). However, the conditions
for equality do not generalize.

The major problem area in Example I is that of finding a
technique which solves the general problem. The only techniques
known are those of Dantzig for the special problem he considered[1]
and the use of "certainty equivalents" in the case covered by
the above Theorem. It would be of interest to know for what
real situations C(b, x) is of the form required by the above
Theorem. Also, a possible area of research is on iterative
techniques such as that suggested by Beale. The inequalities
of Madansky may be used to provide benchmarks on the number of
iterations needed to arrive at a solution.

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[1]Charnes and Cooper [3] suggest that one break the problem
into two parts, that of determining the distribution of the
optimizing x_i's (since they are functions of the b_j's,
$j = 1, \ldots, i-1$) and that of approximating the x_i's by linear
functions, say, of the b_j's, the coefficients being those for
which the distribution of this linear function best approximates
the distribution of the optimizing x_i's. It is unclear from
[3], though, how the distribution of the optimizing x_i's can
be determined without actually solving the "here-and-now"
problem. Elmaghraby [9] studies the use of the gradient method
to solve the problem of [8].

It would be of interest to know for what situations other than that studied by Dantzig is E C(b, x) a convex function of x, so that the techniques of [5] and [6] may be used. Also, what is the probability that the "certainty equivalent" solution is feasible in various situations? The answer to this question may be useful in helping someone decide whether he will be willing to suboptimize and solve a solvable problem.

To illustrate the use of the known results for Example I, let us study the simple case given in Section 2. (This case is a single-stage problem of the multistage type which Dantzig studied.) It is clear that if supply exceeds demand $(x_{11} > d)$, that $x_{21} = 0$ gives minimum costs and, if $x_{11} \leq d$, that $x_{21} = d - x_{11}$ gives minimum costs. Hence

$$C(b, x) = \begin{cases} x_{11} & \text{if } x_{11} > d \\ x_{11} + 2(d - x_{11}) & \text{if } x_{11} \leq d \end{cases} .$$

Since d is assumed to be uniformly distributed between 70 and 120,

$$E_b \, C(b, x) = \begin{cases} -x_{11} + 190 & \text{if } x_{11} \leq 70 \\ 107.5 + (95 - x_{11})^2/50 & \text{if } 70 < x_{11} \leq 120 \\ x_{11} & \text{if } 120 < x_{11} \end{cases} .$$

Note that $E_b \, C(b, x)$ is convex in x. This function attains its minimum, 107.5, when $x_{11} = 95$. Hence, $\min_x E_b \, C(b, x) = 107.5$.

One also sees that min $C(b, x) = d$, so that $E_b \min_x C(b, x) =$
$\quad\quad\quad\quad\quad\quad\quad\quad x$
$Ed = 95$. Also, since $\bar{x}(Eb) = 95$,

$$C(b, \bar{x}(Eb)) = \begin{cases} 95 & \text{if } 95 > d \\ -95 + 2d & \text{if } 95 \leq d \end{cases} ,$$

and $E\, C(b, \bar{x}(Eb)) = 107.5$. Our inequalities yield

$$95 \leq \min_x E_b\, C(b, x) \leq 107.5,$$

and we note that the upper bound is sharp.

Since this example is a degenerate multistage problem of
the type studied by Dantzig, let us determine $G(\bar{x})$. Here

$$\frac{\partial E_b\, C(b, x)}{\partial x_{11}} = \begin{cases} -1 & \text{if } x_{11} \leq 70 \\ (x_{11} - 95)/25 & \text{if } 70 < x_{11} \leq 120 \\ 1 & \text{if } 120 < x_{11} \end{cases}$$

and $\bar{x}_{11}(Eb) = 95$, so that $\alpha_1 = 0$, $\alpha_0 = E_b\, C(b, \bar{x}(Eb)) = 107.5$
and $G(\bar{x}) = 107.5$. Hence this lower bound is sharp, as well,
in this case.

Note that the sufficient condition for $\min_x E_b\, C(b, x) =$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x$
$E_b\, C(b, \bar{x}(Eb))$ given above is satisfied in this case, so that
we expect both $G(\bar{x})$ and the upper bound to be sharp. Note
also that both the optimum x_{11} and $\bar{x}(Eb)$ are 95. This is to
be expected, because $C(b, x)$ is linear in b for each x, and
so satisfies the requirement of the Theorem stated above.
Finally, note that $C(b, x)$ is not linear in b for all x, so
the sufficient condition for $\min_x E_b\, C(b, x) = E_b \min_x C(b, x)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x\quad\quad\quad\quad\quad\quad\quad\quad x$
is not satisfied, and in fact in this example equality of these
values does not hold.

3b. Other Examples

I have seen no work on either of the other examples given in Section 2. A natural approach which might be taken in solving Example II where only b is random is to replace b by a vector b_γ where $\Pr\left\{b \leq b_\gamma\right\} = \gamma$, and solve the nonstochastic linear programming problem for x_γ, say. One might then search for the largest γ and concomitant x_γ such that x_γ and $y(x_\gamma)$ are feasible with probability P or more and such that $C(b_\gamma, x_\gamma) = k$. This decision solves the problem of Example II.

The problems involved in implementing this approach, aside from the "search" problem, are the following. In multidimensions, b_γ is not unique, and so one must first study each b_γ, for a given γ, such that $\Pr\left\{b \leq b_\gamma\right\} = \gamma$. Also, as γ increases, at least one component of b_γ increases. However, though x_γ is a continuous function of b_γ, it may not be the case that by increasing γ, $\Pr\left\{C(b_\gamma, x_\gamma) \leq k\right\}$ will increase.

Let us illustrate how, in a one-dimensional uncertainty problem, where these difficulties do not exist, use of the γ-th quantile of the distribution of b will yield the solution of the problem of Example II where only one component of b is random and where P = 1. We consider the illustration studied above.

Let us write $d_\gamma = 70 + 50\gamma$, $0 \leq \gamma \leq 1$. We note that, as a function of d,

$$\min_{x} C(b, x) = \begin{cases} 0 & d \leq 0 \\ d & 0 < d \leq 100 \\ 2d - 100 & 100 < d \end{cases},$$

or rewriting in terms of γ,

$$\min_{x} C(b, x) = \begin{cases} 0 & \gamma \leq -1.4 \\ 70 + 50\gamma & -1.4 < \gamma \leq .6 \\ 40 + 100\gamma & .6 < \gamma \end{cases}.$$

Note also that $\min_{x} C(b, x) = 70 + 50\gamma$ when only x_{11} and x_{12} are the only (possible) non-zero components of the solution vector, and $\min_{x} C(b, x) = 40 + 100\gamma$ when only x_{11} and x_{21} are the only (possible) non-zero components of the solution vector.

Let us say we wish $C(b, x) \leq k$. Hence

$$x = \begin{cases} x_{11}, x_{12} \text{ if } \gamma \leq (k-70)/50 \quad \text{and} \quad 0 \leq \gamma \leq .6 \\ x_{11}, x_{21} \text{ if } \gamma \leq (k-40)/100 \quad \text{and} \quad .6 < \gamma \leq 1 \end{cases},$$

and finding x which maximizes $\Pr\left\{C(b, x) \leq k \mid P = 1\right\}$ is the same as finding x which maximizes γ subject to the above restraints on γ. For example, if $k = 110$,

$$x = \begin{cases} x_{11}, x_{12} \text{ if } \gamma \leq .8 \text{ and } 0 \leq \gamma \leq .6 \\ x_{11}, x_{21} \text{ if } \gamma \leq .7 \text{ and } .6 \leq \gamma \leq 1 \end{cases},$$

i.e.,

$$x = \begin{cases} x_{11}, x_{12} \text{ if } \gamma \leq .6 \\ x_{11}, x_{21} \text{ if } \gamma \leq .7 \end{cases}.$$

Hence γ is maximized if $x = (x_{11}, x_{21})$, in which case $\gamma = .7$, $d_\gamma = 105$, $x_{11} = 100$, and $x_{21} = 105 - 100 = 5$. For the decision $x_{11} = 100$, $x_{21} = 5$, the probability of feasibility is 1 and the probability of the objective function being less than or equal to 110 is maximized.

4. THE WAIT—AND—SEE PROBLEM

Tintner [19] has considered the problem where one wishes to optimize, over a set of possible distributions of the random elements of the problem, a "preference functional" of the (random) value of the objective function of the "wait—and—see" problem. Examples of such a preference functional are $E(\min_x \min_y (c'x + f'y))$ and $\Pr\left\{\min_x \min_y (c'x + f'y) \leq k\right\}$. He suggests an approximate procedure for obtaining the distribution of the value of the objective function in case the random elements are normally distributed.

The only other work on the "wait—and—see" linear programming problem are the inequalities of Madansky [13] when only b is random, and unpublished work of Talacko and Rockafellar (cf. [17]).[1] It has been shown in [13] that

$$\min_x C(Eb, x) \leq E_b \min_x C(b, x) \leq \min_x E_b C(b, x),$$

and also, that if b is an n—dimensional vector with finite

[1]See [10] for a special type of "wait—and—see" problem which reduces to a "wait—and—see" quadratic programming problem.

range, $[\beta_1, \beta_2]$, then

$$E_b \min_x C(b, x) \leq \sum_\phi \prod_{j=1}^{n} \frac{(-1)^{\phi_j} (\beta_{\phi_j j} - Eb_j)}{(\beta_{2j} - \beta_{1j})} \min_x C(\beta_{\bar\phi_1 1}, \ldots, \beta_{\bar\phi_n n})$$

$$= H^*(Eb)$$

where ϕ_j $(j = 1, \ldots, n)$ takes on the values 1 and 2, $\bar\phi_j = 3 - \phi_j$, ϕ is the set of 2^n n-vectors of 1's and 2's, and $\beta_{\phi_j k}$ is the k-th element of the vector β_{ϕ_j}. The inequality $\min_x C(Eb, x) \leq E_b \min_x C(b, x)$ was first derived, in a different manner from that of [13], by Vajda [20].

A necessary and sufficient condition is given in [13] for $H^*(Eb) = E_b \min_x C(b, x) = \min_x C(Eb, x)$, namely that $\min_x C(b, x)$ be linear in b. Also, it is shown that a sufficient condition for $\min_x E_b C(b, x) = E_b \min_x C(b, x) = \min_x C(Eb, x)$ is that $C(b, x)$ be linear in b. Examples have also been constructed where $H^*(Eb)$ is greater than $\min_x E_b C(b, x)$ and where it is less than $\min_x E_b C(b, x)$.

Talacko and Rockafellar [17] have studied the problem of obtaining confidence limits on $\min_x C(b, x)$ by using quantiles of the distribution of b. They also have generalized these confidence limits to the case where A, B, c, and f are also random. As yet, this work is unpublished.

A variant of the use of "certainty equivalents" was studied in Babbar in [1]. He considered the modified "wait-and-see" problem (where the criteria are those of Example I)

where the activity vector x but not its value must be
determined before the random elements are observed, and after
the random elements are observed the value of x must be
determined so as to "best" satisfy the resulting restraints
of the problem. More precisely, he studied the approximate
distribution of the value of x, where the vector x is that
vector which solves the nonstochastic problem where the
random elements are replaced by their expected values.
Unfortunately, as Wagner [21] noted, this vector may not be
feasible after the random elements are observed, and so the
distribution studied is not necessarily the distribution of
a vector which even satisfies the problem.

Also, we note from [13] that the expected value of the
objective function of the modified "wait—and—see" problem is
not only at least as great as the expected value of the
objective function of the "wait—and—see" problem, which is to
be expected, but also is at least as great as the value of
the objective function of the "here—and—now" problem. Hence,
one is better off in the modified "wait—and—see" situation of
Babbar to recast the problem as a "here—and—now" problem
(if, of course, the "here—and—now" problem can be solved).
Other comments on Babbar's procedure are made in [21].

The outstanding problems in the "wait—and—see" area are
those of determining the distribution, or at least the mean
and variance, of the objective function and of the value of
the optimizing vector under various distributional assumptions

about the random variables. It may also be fruitful to look
further at Tintner's approximations in the case he considered
and derive approximate means and variances of these quantities.

One should also note from Theorem 3 of [2] that if both
A and b are random, then $\min_y (c'x + f'y)$ is a convex function
of both A and b. Let $C(A, b, x) = \min_y (c'x + f'y)$, in this
case. Then, by the same argument as in [12], one can show
that $\min_x C(EA, Eb, x) \leq E_{A,b} \min_x C(A, b, x) \leq \min_x E_{A,b} C(A, b, x)$
and, if the elements of both A and b are distributed over a
finite range, then one can determine an upper bound of the
form of $H^*(Eb)$, say $H^*(EA, Eb)$, by the method of [12].

To return once again to the simple case studied earlier,
we record once again that $E_b \min_x C(b, x) = 95$, which was less
than $\min_x E_b C(b, x) = 107.5$. When d = Ed = 95,

$$C(Eb, x) = \begin{cases} x_{11} & \text{if } x_{11} > 95 \\ -x_{11} + 190 & \text{if } x_{11} \leq 95 \end{cases}$$

and $\min_x C(Eb, x) = 95$. Finally, in this case

$$H^*(Eb) = \frac{(-1)(70 - Ed)}{(120 - 70)} \min_x C(120, x) + \frac{(-1)^2(120 - Ed)}{(120 - 70)} \min_x C(70, x)$$

$$= Ed = 95.$$

Hence $95 \leq E_b \min_x C(b, x) \leq 95$. But this was to be expected,
since $\min_x C(b, x) = d$ and hence satisfies the necessary and
sufficient condition for equality of these three quantities.

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