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# Group Report

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## An Estimate of the Absorption Mean Free Path for a Photon in a Gas

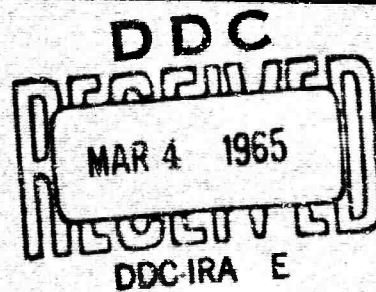
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AN ESTIMATE OF THE ABSORPTION MEAN FREE PATH  
FOR A PHOTON IN A GAS

*H. E. MOSES*

*Group 35*

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## ABSTRACT

Let us consider a photon which is emitted by an excited atom, the atom being surrounded by like atoms in the ground state. In the present paper we make an estimate of how far the photon can travel before it is absorbed by a ground state atom. In making this estimate we make use of the selection rules of the atom and thereby take into account the Doppler shift in energy and angular momentum of the photon.

A discussion is presented which indicates how the mean free path gives us an indication of whether or not the gas is optically thick. On the basis of the formula which is derived for the mean free path, it is shown that in general gases are more transparent to the shorter wavelengths than the longer ones.

Accepted for the Air Force  
Stanley J. Wisniewski  
Lt Colonel, USAF  
Chief, Lincoln Laboratory Office

## 1. INTRODUCTION

In the present report we wish to consider a gas which consists of atoms or molecules, some of which may be in an excited state. When these excited atoms drop to the ground state, they will emit photons whose frequency is determined by the usual Bohr rule in terms of the energy difference between the excited state and the ground state. Some of these photons may escape the gas without exciting any of the atoms to the ground state. But there is a probability that the photons will excite some of the atoms in the ground state, since the photons have the proper frequency to do so. The photons so absorbed are "trapped" until the atoms which are excited in this way drop down to the ground state. Such trapping will slow down the cooling rate of the gas due to the escape of the photons. If the trapping is considerable, the gas is said to be "optically thick" and in some circumstances the radiation in an optically thick gas approaches thermodynamic equilibrium. On the other hand, if there is no trapping, the radiation escapes freely from the gas and each ground state atom has no effect on any radiating atom other than possible resonance scattering. In such circumstances the radiation is not characterized in any way whatever by thermodynamic equilibrium.

The object of the present report is to estimate the mean distance  $L$ , through which a photon, which is emitted by an excited atom, can travel before it is absorbed by an atom in the ground state. If  $L$  is greater than or of the same order as the smallest linear dimension  $d$  of the mass of gas which we are considering, the gas is clearly optically thin. If on the other hand,  $L$  is many orders of magnitude less than  $d$ , some trapping is possible and more detailed calculations are called for.

Our considerations are valid for gases which consist of mixtures of two different gases. We could, for example consider

the radiative cooling of hydrogen atoms (using the Lyman- $\alpha$  line), where the hydrogen is only one of the constituents of the gas which might be largely nitrogen, and oxygen. It is clear that the nitrogen and the oxygen would play no role, provided the densities are low enough to preclude pressure broadening.

We shall be concerned essentially only with dipole radiation, though the considerations herein can be extended to forbidden lines also.

Our approach is based on the idea that both energy and angular momentum are conserved in the emission and absorption of a photon. Energy conservation is indeed usually taken into account in the sense that the frequency of the emitted and absorbed photon satisfies the Bohr condition. However, it does not appear that in most treatments that the selection rules are taken into account. These selection rules are the consequence of the conservation of angular momentum. When a photon is emitted by an atom, the photon carries away with it the angular momentum difference of the atom between the excited and ground states. Conversely, when the photon is absorbed by an atom, the photon wave function must have within it a finite probability of having enough angular momentum to make up the difference in angular momentum between the ground state and the excited state of the atom. We are thus led to the problem of finding the probability that a photon has a given angular momentum in a frame of reference (the absorbing atom) when in another frame of reference (the emitting atom) it is in an eigenstate of angular momentum.

This problem has been solved in reference 1. We shall use the results of this reference in our calculations. We shall find that the use of the concept of conservation of angular momentum or, equivalently, the use of selection rules takes into account the Doppler effect for both the frequency and angular momentum.

Finally, we want to emphasize the provisional character of the present treatment. To find a mean free path accurately would require knowledge of the emission and absorption coefficients of the atoms and the averaging over gas configurations in each of which the path of each photon would have to be traced by some random walk procedure. This procedure is extremely difficult in practice. However, despite the provisional character of the present calculations, we believe that we obtain at least some idea of the numbers involved.

## 2. CALCULATION OF THE MEAN FREE PATH FOR ABSORPTION

We shall denote the ground state of the atom by the subscript 1 and the excited state by the subscript 2. We denote by  $E_1$  and  $E_2$  the energy levels of the ground and excited states respectively.

When a photon is emitted by an atom, in addition to having the angular momentum which is the difference between the angular momentum of the excited state of the atom and the ground state, it will have a natural line breadth. This breadth is usually expressed in probabilistic terms by saying that the probability that the emitted photon has a frequency outside the range  $\nu_0 - b < \nu < \nu_0 + b$  where  $\nu_0 = (E_2 - E_1)/h$  is essentially zero. (See reference 2. We are idealizing the situation somewhat.) The quantity  $b$  is very small compared to  $\nu_0$  and is called the half natural line breadth (in frequency terms).

The photon will not have a definite linear momentum, since the linear and angular momentum operators do not commute. Instead the components of the linear momentum will have a certain expectation value which can readily be calculated from the results of reference 1. In order to have a phenomenological treatment we shall consider an ensemble of photons which are continually being emitted by excited atoms located at the origin. On averaging over all orientations of

the photons, the expectation value of any component of linear momentum can be shown to be zero. Thus for the ensemble we may regard the problem as having spherical symmetry. We regard the photons as being emitted continuously in a radial direction from the origin, all of which have angular momentum one with respect to a frame of reference at the origin.

In the phenomenological model these photons stream through a gas of atoms in the ground state, this gas being uniformly distributed throughout space. The velocities of the atoms are distributed in accordance with the Maxwell-Boltzmann law.

Let us denote the density of ground state atoms by  $N$ , and the density of photons by  $n(r)$ . As indicated by the notation the density of photons will be a function of the distance  $r$  from the origin because absorptions take place as the photons move through the gas. We shall eliminate the diminution of  $n(r)$  due to spreading by using the flux.

We shall let  $S(r)$  be the flux of photons through a sphere of radius  $r$ . Clearly

$$S(r) = 4\pi r^2 cn(r), \quad (2.1)$$

where  $c$  is the velocity of light.

We shall require that  $S(r)$  decay exponentially,

$$S(r) = e^{-Wr} \quad (2.2)$$

where  $W$  is a constant which we wish to determine. Then the mean free path  $L$  is given by

$$L = \frac{1}{W} \quad (2.3)$$



As we shall see, the requirement (2.2) puts a severe assumption on the nature of the absorption of a photon of specific angular momentum. Still, since this absorption has not been calculated, equation (2.2) seems to be a reasonable requirement.

Let us now consider the change in flux as the photons pass through a shell of thickness  $\Delta r$ :

$$S(r+\Delta r)-S(r) \cong \frac{d}{dr} S(r) \Delta r. \quad (2.4)$$

The difference in the flux must equal the negative of the number of absorptions of photons per second in the shell. We must now make an assumption about the probability per unit time that an atom in the shell will absorb a photon. Since the atom will not be able to absorb the photon unless, as seen in a frame of reference whose origin coincides with the atom, the frequency lies in the range  $\nu_0 - b < \nu < \nu_0 + b$  and the angular momentum of the photon is  $l$ , we shall assume this probability is proportional to the number density of photons which satisfy the frequency and angular momentum requirements. This number density is  $n(r)P(r, \nu)$  where  $P(r, \nu)$  is the probability that the photons have the correct angular momentum and frequency as observed by an atom in the shell when the atom moves with the velocity  $v$  with respect to the origin.

Thus the probability per unit time that an atom in the shell will absorb a photon will be written

$$K(r)n(r)P(r, \nu)$$

where  $K(r)$  is a "constant" of proportionality. We make  $K(r)$  a function of  $r$  to take into account the possibility that  $K$  depends on the distance of the atom from the origin, which in a sense, is a point source of radiation. In principle,  $K(r)$  can be calculated

from quantum mechanics and, indeed, we intend to carry out this calculation ultimately. However, for the purposes of the provisional calculation of our paper we shall use phenomenological methods of obtaining it.

Then the number of absorptions of photons per unit time in the shell of thickness  $\Delta r$  is

$$4\pi r^2 N \Delta r K(r) n(r) P(r, \nu) = \frac{S(r)}{c} N K(r) P(r, \nu) \Delta r \quad (2.5)$$

On equating the change in flux (2.4) with the negative of the number of absorptions (2.5) we obtain a differential equation for  $S(r)$ :

$$\frac{d}{dr} S(r) = - \frac{1}{c} N K(r) P(r, \nu) S(r) \quad (2.6)$$

which can be integrated readily if  $K(r)$  and  $P(r, \nu)$  are known explicitly as a function of  $r$ . However, the assumption of exponential decay of  $S(r)$  given by (2.2) states that

$$\begin{aligned} W &= \frac{N}{c} K(r) P(r, \nu) \\ &= \frac{N}{c} K(0) P(0, \nu) \end{aligned} \quad (2.7)$$

Hence, as mentioned above, the absorption constant  $K(r)$  has its character determined by the exponential law (2.2).

Since  $P(0, \nu)$  is given in reference 1, we need only find  $K(0)$  to obtain  $W$  and hence the mean free path from (2.3).

To obtain  $K(0)$  we shall consider the special situation in which the temperature of the gas is very low so that we may set  $\nu = 0$ . Let us consider an atom which is in the ground state and

which is very near the origin. An observer attached to this atom will observe that all the photons which are being emitted from the origin have their frequencies in the range  $\nu_0 - b < \nu < \nu_0 + b$  and will have angular momentum equal to 1. Upon the original radiation we could superpose radiation of any other frequency or angular momentum and not affect the probability that the atom absorbs one of the original photons. We see that by superposing the additional radiation in a sufficiently clever manner we could simulate, as far as the absorbing atom is concerned, a black body environment. Then the probability per unit time that the atom will absorb the radiation in the range  $\nu_0 - b < \nu < \nu_0 + b$  is

$$B_{12} \rho_\nu$$

where  $\rho_\nu$  is the energy density per unit frequency of the radiation and  $B_{12}$  is the Einstein coefficient for absorption.

Thus in a shell of the low temperature gas which is near the origin the number of absorptions per unit time is

$$4\pi r^2 B_{12} \rho_\nu N \Delta r = 4\pi r^2 K(0) N n(r) \Delta r P(0,0). \quad (2.8)$$

In (2.8) we are really considering the limit of very small  $r$ . Hence we have used  $r = 0$  in  $K(r)$  and  $P(r,0)$  since these functions exist in the limit.

From reference 1,  $P(0,0) = 1$ . Also  $\rho_\nu$  which is a function of  $r$  can be expressed in terms of  $n(r)$  by

$$\rho_\nu = \frac{n(r) h \nu_0}{2b}. \quad (2.9)$$

Equations (2.8) then yields the result:

$$K(0) = B_{12} \frac{h \nu_0}{2b}. \quad (2.10)$$

In (2.10) we have dropped the subscript zero on  $\nu_0$  because we shall no longer have to distinguish the center of the spectrum line from its extremities.

Then

$$W = NB_{12} \frac{h\nu}{2bc} P(0,\nu). \quad (2.11)$$

Further simplifications of (2.11) are possible. Let  $A_{21}$  be the Einstein coefficient of spontaneous emission. Then from the usual Einstein relations

$$B_{12} = \frac{c^3 g_2}{8\pi h\nu^3 g_1} A_{21}, \quad (2.12)$$

where  $g_i$  is the statistical weight of the  $i$ 'th energy level of the atom. Also we have from the Heisenberg uncertainty principle

$$2b \approx A_{21}. \quad (2.13)$$

From reference 1

$$P(0,\nu) = \frac{b^2}{\nu^2} \lambda^2, \quad (2.14)$$

where  $\lambda$  is the wave-length of the photon. For our purposes we shall replace  $\nu$  by the average velocity of an atom in a Maxwellian gas of temperature  $T$ . Thus

$$\nu^2 = \frac{2RT}{\mu}, \quad (2.15)$$

where  $\mu$  is the atomic or molecular weight.

Then on substituting (2.12)-(2.15) into (2.11) we obtain

$$W = \frac{N \lambda^4 A_{21}^2 \mu g_2}{64 \pi R T g_1} \quad (2.16)$$

Thus on using numerical values for the universal constants we obtain our final expression for the mean free path of a photon in a gas:

$$L = 1.67 \times 10^{10} \frac{g_1 g_2 T}{\lambda^4 \mu N (g_2 A_{21})^2} \quad (2.17)$$

In reference 3 experimental values are given for  $g_2 A_{21}$ . Thus for a given line  $L$  can be computed from (2.17) explicitly. As an example, for the sodium line 5889.95A we have

$$L = .56 \times 10^{10} \frac{T}{N} \quad (2.18)$$

### 3. A SUGGESTED TECHNIQUE FOR VIEWING GLOWING GASES

When a glowing gas is examined spectroscopically, one sees either a continuous spectrum, single lines, or sometimes lines superposed upon the continuous spectrum. Usually a spectrum has a different one of these characteristics in different frequency ranges.

The continuous spectrum arises in those frequency ranges for which the gas is optically thick while the single spectrum lines, which can be used to identify the components of the gas and whose doppler breadth can be used to obtain temperature estimates, occur at frequencies for which the gas is optically thin.

An examination of the expression (2.17) for the mean free path of a photon shows that the mean free path is inversely

proportional to the fourth power of the wavelength of the photon. It would therefore seem, on the whole, that to see single lines one should go toward the shorter wavelengths in spectroscopic measurements. For given values of  $A_{21}$  the gas is much more transparent to short wavelengths than long ones.

If one knows the dimensions of the glowing gas it is possible to use (2.17) to make estimates of  $T/N$  in the following way. One looks for a wavelength for which the gas is optically thick, i.e. where single spectrum lines can be seen. One knows for this wavelength  $L$  is much less than the dimension of the gas. One then looks for a wavelength for which the gas is optically thin. For this wavelength  $L$  is greater than the dimension of the gas. From (2.17) one can then put upper and lower bounds on  $T/N$ . If one has an independent way of estimating  $T$ , such as through the Doppler shift, one can then put bounds on  $N$ .

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