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THEORETICAL STUDY OF RESONANT STIFFENING DEVICES

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THEORETICAL STUDY OF RESONANT STIFFENING DEVICES

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ABSTRACT

The vibration properties of a plate or a rod with attached mechanical structures, called resonators, are studied by transfer matrix and Green's function techniques. The plate may be in a vacuum, or acoustically coupled to water on one side. The effects of an added resonator at a given frequency depend only on its response strength in each of its modes of vibration. Expressions are derived for the strengths of several useful resonator types. Boundary conditions for the plate, such as clamping or simple support, are satisfied by the mathematical device of attaching resonators of infinite strength. A piston cut out of the plate material is handled similarly. Formulas for the interactions ("influence coefficients") between the modes of different resonators are given. Expansions are developed for the transcendental integrals that arise in the problems treated.

1. INTRODUCTION

Mechanical devices attached to a plate serving as a baffle were introduced as part of the CONTACT sonar system under development at TRG Incorporated, in order to make the flexible plate behave like a perfectly rigid wall for a selected frequency range. The resonators (see Fig. 1-1, at the end of this report) are attached to the back of the baffle, and at their resonant frequency have the same effect on the plate motion as though they were almost infinite masses. Earlier calculations of their effects were given in Appendix XIII of TRG-142-TR-2 (1963), based on the assumption that they were spaced so close together that an average density of resonators could be used. It has been determined more recently that the sea chests which house the transducer elements of the sonar system affect the plate motion in roughly the same way as the resonators attached to stiffen the plate, and may be even more important, since the total mass of the sea chests is comparable to that of the resonators. Thus it appears desirable to find exact and general methods for analyzing a whole class of similar problems, involving structures attached to and interacting with a plate.

The present report gives methods of calculating the effect of resonators of different types attached to a plate in various geometries. An associated report in this series, TRG-142-TN-64-11, by Yarmush and aronson, presents experimental results on the vibration properties of bars as modified by attached resonators.

A resonator attached to a plate at a point can vibrate in either the axial mode, in which compressional waves are set up in its material, or in a transverse mode, corresponding to a flexure or swaying of the resonator. (See Fig. 1-2.) Similarly, various modes exist for sea chests. The strength of a resonator is defined in Section 2, and then methods of calculating it are discussed. Section 3 gives the Green's functions for a few kinds of resonators, that is, the effect a resonator has on the plate when it is caused to vibrate with unit excitation.

A resonator attached at a point acts as a secondary source that produces an outgoing displacement wave in response to the displacement at its base (in the axial mode) or in response to the slope at its base (in the transverse mode). If there are several resonators, these secondary waves will produce additional responses at the attachment points and so on. Section 4 gives formulas for determining the net effect of all these waves, when a driving force is imposed at a single point.

In the earlier Sections (3 and 4) only two-way infinite rods, or plates extending indefinitely in all directions, are discussed. These are considered to be in free space. A plate carries either a finite number of resonators, or an infinite number of identical types in a regular array.

Treatment of a rod of finite length is reduced to treatment of a two-way infinite rod by an artifice, developed in Sections 5 and 6, in which non-physical resonators are attached to an infinite rod and have the effect of breaking it. These sections also contain a calculation of the strength of a transverse resonator of several segments and two different treatments for a rod having a thickened portion.

A traveling pressure wave on a rod is considered in Section 7. Infinitely many resonators are attached to the rod in a repeating pattern. There may be several different kinds of resonators in the repeating arrangement.

The effects of an infinite ocean of water on one side of a plate are discussed in Section 8. It turns out that if there are resonators attached to the plate, one can first find the properties of the system without resonators and then, so to speak, attach the resonators to the coupled plate-water system. In many cases, the formulas in the wave number domain that are valid in the presence of water differ from those without water in a simple way - the term that represents the inertia of an element of the plate vibrating at the given frequency must be increased by an "added mass" term due to the water that participates in the motion of the plate. However, at certain wave numbers the "added mass" is imaginary, which means that a disturbance is propagated through the water. This section also considers systems of parallel plates immersed in water, with upper and lower plates bounding the fluid.

Further kinds of resonators are discussed in Section 9, and also pistons, which are analyzed mathematically as the result of superimposing several kinds of fictitious or generalized resonators.

Finally, Section 10 contains evaluations for many of the integrals that arise in earlier sections.

In summary, Sections 2, 3 and 4 contain a more elementary treatment than the rest, and Section 10 is purely mathematical. The effects of water coupled to the plate are not discussed until Section 8. It has been convenient to introduce many kinds of mathematical resonators in order to satisfy various boundary conditions.

Much of the treatment speaks of a homogeneous rod, rather than a homogeneous plate, but every rod problem is analogous to a plate problem. Thus a rod stretched along the x-axis and vibrating in the xz-plane satisfies the same equation of motion as a plate whose median surface is in the xy-plane, if the plate executes vibrations such that the displacement is independent of y. The plate stiffness must of course be used instead of the rod stiffness coefficient. The reverse relation between plates and rods is of course not true - a plate problem with cylindrical symmetry is not the direct analog of any rod problem.

In a report of this size, it is impossible to avoid conflicts in notation and errors in proofreading. It can only be hoped that these have been kept to a minimum. Certain less pertinent material and involved algebraic calculations have been put in the Addenda.

2. TYPES OF RESONATORS; CALCULATION OF THEIR STRENGTHS

A single physical resonator can have several modes of vibration, but it is more convenient to speak of fictitious resonators, each oscillating in only one mode. Then a real one is the result of superposition of several such fictitious types.

2.1 Domain of Sensitivity and Response

A physical resonator is attached to a plate by a weld over some area. In the mathematical treatment, this can be idealized in several ways: A standard TRG resonator can be considered as exerting its force on one point. On the other hand, a seachest is attached over an annulus. When this is approximated as a circle, it can be called a rim resonator. It will also be useful to introduce a disc resonator, which is attached at all the points within a circle. Thus it is convenient to speak of the domain of sensitivity of a resonator - either a point, a curve, or an area. If the domain is not a point, then the resonator need not be sensitive to the plate displacement to the same extent at different points of the doman. Thus we introduce a weighting function $w^*(r)$, for sensitivity, and we normalize w^* by setting the integral of $(w^*(r))^2$ over all points r of the domain equal to unity. All the significant features of an axial or force resonator, insofar as it will affect the plate motion at a fixed frequency ω, are summed up in the following definition:

An axial resonator is a device which in response to a weighted average of the displacement over its domain, exerts a force on the plate at each point of the domain. In general, the force is not uniform and is given by $F(\omega)\ w(r)$, where $w(r)$ is a weighting function for response, (normalized so that the integral of $(w(r))^2$ is unity), and $F(\omega)$ is the strength of the resonator at frequency ω.

Similarly a transverse resonator responds to a weighted average of the slope by exerting a moment $G(\omega)w(r)$ at each point r of its domain. Here G is also called the strength. For a resonator whose domain is a point, it is necessary to specify the direction of the moment. Thus physically there will be a different mode for each direction. However, all such modes can be viewed as the result of superposition, with appropriate coefficients, of any arbitrarily chosen two of them.

The reciprocity relation between sources and responses, when applied to the resonator materials, leads to the conclusion that the weighting functions for sensitivity and response must be equal. Nevertheless a non-physical type of resonator will be introduced later to handle a special problem, and it will be necessary to use different functions in that case. Thus we have made the distinction from the beginning.

2.2 Strength of An Axial Resonator with a Uniform Weight Function.

The response in the simplest axial mode is due to compressional vibrations set up in the resonator material due to displacement of the base. The strength is the magnitude of the total force exerted in opposing unit displacement of the bar or plate. This will now be computed for a resonator made up of several segments, each segment being homogeneous and having a uniform cross-section. Note that a disturbance with planar wavefronts, such as we assume, does not satisfy the exact boundary conditions on the sides of segments, but the error introduced is negligible.

2.2.1 One-Segment Axial Resonator

We first consider a resonator that is a cylinder of cross-section area S attached to the plate along its base. The resonator material has Young's modulus E and density ρ. Hence the speed of sound in it is

$$c = \sqrt{E/\rho} \quad .$$

Let L be the length of the cylinder, and $\eta(y)$ the local displacement of the material at height y above the base.

The force exerted by the resonator back on the plate is

$$ES \left.\frac{\partial \eta}{\partial y}\right]_{y=0} .$$

But $\left.\frac{\partial \eta}{\partial y}\right]_{y=0}$ is proportional to $\eta(0)$. The strength of the resonator $F(\omega)$, is defined as

$$F(\omega) = - \frac{ES \frac{\partial \eta(0)}{\partial y}}{\eta(0)} .$$

We shall determine $F(\omega)$ as a special case of a more general result in Section 2.2.2. Here, we may notice that Final Report TRG-142-TR-2 vol. III, Appendix XII contains a calculation of $F(\omega)$ by Victor Mangulis. He finds

$$F(\omega) = - \frac{ES\ \omega}{c} \tan\ (\omega L/c) .$$

When $\omega L/c$ is small, the resonator behaves as a rigid rod, and F reduces to

$$F(\omega) \approx - \rho\ SL\ \omega^2 .$$

The factor ρ SL is of course the mass of the resonator.

When $\frac{\omega L}{c} = (n + \frac{1}{2})\pi$, there is a resonance, and the formula above predicts infinite strength, which means that an infinite force will be produced in response to any non-zero displacement of the plate at the attachment point. Actually, dissipation in the resonator material and non-linear effects will keep the strength finite, but there will be a phase change of π as ω moves across the resonant frequency.

. . .

2.2.2 Axial Resonator Made Up of Several Segments

The above-mentioned report also gives a derivation for the case of two segments; here a generalization is presented for a resonator made up of any number of segments. The segment nearest the plate will be numbered 1 (see Fig. 2-1). Consider a typical segment, say the i-th, (not the last) and let the zero of the y-coordinate correspond to its base (the end nearer the plate). From the equation for compressional vibrations of a cylindrical rod, it follows that the displacement η will be sinusoidal:

$$\eta(y) = A \cos ky + B \sin ky,$$

where

$$k = \omega/c \ .$$

Then

$$\eta' = \frac{d\eta}{dy} = - Ak \sin ky + Bk \cos ky \quad .$$

Let η_o and η'_o denote the values at the base of the i-th segment. Then

$$\eta_o = A$$

$$\eta'_o = kB$$

and so

$$\eta = \eta_o \cos ky + \frac{1}{k} \eta'_o \sin ky$$

$$\eta' = - \eta_o k \sin ky + \eta'_o \cos ky \quad .$$

In particular, this must be true at the outer end of the segment (of length L), where the displacement and derivative are η_e, η'_e:

$$\eta_e = \eta_o \cos L^*\omega + \frac{1}{k} \eta'_o \sin L^*\omega$$

$$\eta'_e = - \eta_o k \sin L^*\omega + \eta'_o \cos L^*\omega \quad .$$

Here we have introduced the abbreviation

$$L^* = \frac{L}{c} = [L\sqrt{\rho/E}\,]$$

Consider all the segments more remote than the i-th. These may be considered when taken together as forming a resonator of strength F_{i+1}. When the i-th segment is added, one has a new resonator of strength F_i. A formula expressing F_i in terms of F_{i+1} will now be derived. It will then be possible to compute the strength of a resonator of any number of segments, by adding one segment at a time, starting at the outer free end with a fictitious extra segment of zero length and zero strength.

By definition of the strength of a resonator, F_{i+1} is the force exerted across the junction between the i-th and (i+1)-th segments, divided by the displacement there. The force is given by the expressions

$$E_i S_i \frac{\partial \eta_i}{\partial y} = E_{i+1}\, S_{i+1} \frac{\partial \eta_{i+1}}{\partial y}$$

where the derivatives are evaluated on the appropriate sides of the junction. It should be noted that at a junction between segments of different cross-sections, a plane compressional wave in the segment with smaller cross-section will produce a complicated combination of compressional and shear waves in the other segment, since the condition of zero pressure along the portions of the base not in contact with the other segment must be satisfied. However, the shear waves and non-uniform compressional waves are attenuated extremely rapidly with depth into the segment. One can also say that the equation above comes from a macroscopic view, because it is not true that the pressure is uniform across the interface.

We can write

$$F_{i+1} = - \frac{E_i S_i \eta_e'}{\eta_e}$$

. . .

and also

$$F_i = \frac{E_i S_i \eta_o'}{\eta_o}$$

We now have four equations connecting η_c', η_e, η_e', F_i, and F_{i+1}, and can solve for F_i in terms of F_{i+1}. If we introduce the abbreviations:

$$S^* = ES/c = S\sqrt{E\rho} \quad ,$$

then we find

$$F_i = -\omega S^* \frac{F_{i+1} - \omega S^* \tan L^*\omega}{F_{i+1} \tan L^*\omega + \omega S^*} \quad .$$

Writing $K = \arctan (F_{i+1}/\omega S^*)$,

this becomes

$$F_i = -\omega S^* \tan (L^*\omega + K) \quad .$$

The strength of a resonator of N segments can thus be written in a convenient form

$$\begin{aligned} F = -\omega S_1^* \tan \Big(-L_1^*\omega + \arctan \Big(\frac{1}{S_1^*} \times \\ S_2^* \tan(-L_2^*\omega + \arctan \Big(\frac{1}{S_2^*} \times \\ \cdots \\ S_N^* \tan(-L_N^*\omega)) \ldots \Big). \end{aligned}$$

It is noteworthy that the four constants needed to give the properties of a segment appear only in the two combinations:

$$S_i^* = S_i \sqrt{E_i \rho_i}$$

$$L_i^* = L_i \sqrt{\rho_i / E_i}$$

2.2.3 Condition for Resonance in a Limiting Case

The resonator will have infinite strength at frequency ω if

$$\omega S_1^* + F_2 \tan L_1^*\omega = 0 \quad .$$

When there are only two segments, then

$$F_2 = -\,\omega S_2^* \tan L_2^*\omega$$

and so the condition becomes

$$\tan L_1^*\omega \tan L_2^*\omega = S_1^*/S_2^*$$

Now for a range of dimensions that we have been concerned with (S_1^*/S_2^* very small and L_1^* comparable to L_2^*), the lowest mode occurs when $L_1^*\omega$ and $L_2^*\omega$ are both small, and so the tangent can be replaced by a linear approximation, yielding

$$\omega^2_{\text{fund}} \cong \frac{S_1^*}{S_2^*}\,\frac{1}{L_1^*\,L_2^*}$$

$$\cong \frac{S_1}{S_2}\,\frac{E_1}{\rho_2}\,\frac{1}{L_1\,L_2}$$

Let D_1 and D_2 be the diameters of the segments, assumed to have circular cross-sections. Then

$$\omega_{\text{fund}} = \frac{D_1}{D_2}\sqrt{\frac{E_1}{\rho_2}}\,\frac{1}{\sqrt{L_1L_2}}$$

Note that E_2 and ρ_1 do not appear here. The outer segment behaves only as a mass and the inner one only as a spring.

2.3 Transverse Mode of a Point Resonator

A physical resonator can also undergo flexural vibrations, similar to those of a rod with one end clamped and the other free. On a plate, the vibration need not take place all on one plane, but mathematically we can superimpose two transverse point resonators, each of which is allowed to vibrate in one plane only (the planes of the resonators being perpendicular to each other). The strength of a transverse resonator is the magnitude of the moment it exerts in opposing a unit change of the slope at the resonator attachment point. Transverse oscillations of a many-segment resonator can be handled by introducing a matrix which gives the displacement and its first three derivatives at a point in terms of the displacement and first three derivatives at a neighboring point. This will be done in Section 5.2.

A simpler derivation for a single segment was given by V. Mangulis in the report already cited. We merely quote his results:

Let

ρ = density

S = cross-sectional area

E = Young's Modulus

I = moment of inertia of a cross-section of the cylinder

$$k^4 = \frac{\omega^2 \rho S}{EI} = \frac{\omega^2}{c^2} \frac{S}{I} .$$

Then the strength $G(\omega)$ is given by:

$$G(\omega) = EIk \frac{\cos kL \sinh kL - \sin kL \cosh kL}{1 + \cos kL \cosh kL}$$

2.4 More Complicated Types Attached at Points

At an earlier stage, resonators were considered having the cross-section shown in Fig. 2-2. This can be approximated by combining segments that separately oscillate in compressional or transverse modes.

Thus segment 1 is a cylinder that executes axial vibrations. Segment 2 is a circular disc vibrating transversely, interacting at its center and along its rim. Segment 3 is a cylinder that behaves as an axial rim resonator (see Section 2.5)

One can also conceive of resonators which are not circularly symmetrical, but have a helical ridge, as a screw with large threads. Then a displacement of the plate will excite complicated torsional vibrations in the resonator, which in turn induce torsional vibrations in the plate.

2.5 Axial or Force Resonators Attached Along a Rim

Resonators that are sensitive to the displacement along a circle of radius R can be thought of as idealizations of pipe sections welded to the plate. Because of the linear nature of the response, the total effect can be analyzed into the response for the individual Fourier components of the plate displacement along the circle. We consider that the resonator itself has circular symmetry. Let r and θ be polar coordinates using the center of the attachment circle as origin. Then in response to the displacement

$$\eta(R,\theta) = a_o + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin \theta) \quad ,$$

a physical resonator will exert the force per unit arc-length

$$\frac{1}{2\pi R}\left[F_o a_o + \sum_{n=1}^{\infty} F_n(a_n \cos n\theta + b_n \sin n\theta)\right].$$

Here F_n, for $n = 0,1,2,\ldots$, is the strength of the physical resonator in the n-th mode. We shall indicate how the F_n can be computed if the physical resonator is a circular pipe section of length L and inner, outer radii R_i, R_o. Then the attachment radius R is taken as an average between R_i and R_o.

2.5.1 The Azimuthally-Independent Mode n = 0

The term for n = 0 in $\eta(R,\theta)$ corresponds to a uniform displacement of the points of the attachment circle. In respo.ise to this, the physical resonator will behave as a rim axial or force resonator. That is, planar compressional waves will be set up, and the strength can be computed by the methods of Section 2.2, using the cross-section S:

$$S = \pi(R_o^{\,2} - R_i^{\,2}) \quad .$$

2.5.2 The Tilting or Swaying Mode

The value $n = 1$ corresponds to a displacement of the form

$$\eta = \cos \theta$$

or

$$\eta = \sin \theta$$

The response to such a displacement will be a rocking or tilting motion. Computation of the force exerted back on the plate is difficult without the following simplifications:

We assume that each cross-section of the pipe remains circular during the motion, and think of the pipe as made up of elements parallel to its axis. Then the stretching of the elements is a linear function of the distance from the neutral median surface, and so the arguments used in thin-rod theory can be applied, using the moment of inertia appropriate to the hollow cross-section. The pipe thus becomes a transverse resonator of one segment, which would respond to the slope of the plate at the center of the attachment circle by exerting a moment at the center. This moment must be replaced by an equivalent distributed force on the rim, varying as $\cos \theta$ or $\sin \theta$.

The approximations will be close when L/R_o is large and R_i/R_o is small. If the pipe is thin-walled one might expect distortion of the cross-section.

2.5.3 Shear Modes

For n greater than 1, the displacement $\eta = \cos n\theta$ or $\eta = \sin n\theta$ will set up shearing motions such that the elements of the cylindrical segment slide parallel to each other. Furthermore, it is probably inadequate to assume that the cross-sections remain circular.

In our application, the seachests are so massive that they effectively produce the boundary condition that the attachment rim must remain planar, even though the plane of the rim is displaced from equilibrium and tilts back and forth. As will be discussed later, this is equivalent to requiring a seachest to have infinite strength in each mode corresponding to a displacement $\eta = \cos n\theta$ or $\eta = \sin n\theta$, for $n = 2,3,\ldots$

2.6 Transverse or Moment Resonators Attached Along a Rim

The transverse modes of a rim resonator exert moments pointing radially on the plate at the points of the attachment circle. These arise in response to a weighted average of the slope along the circle, where the slope is always measured in a plane that contains the center of the attachment circle and is perpendicular to the plate.

2.6.1 Azimuthally-Independent Mode

The simplest mode, that for $n = 0$, results from a dishing or cupping of the plate that is circularly symmetrical. This tends to produce a "breathing" mode of the pipe (Fig. 2-3), that is, the oscillations of each particle of the pipe are radial. Thus they can only be realized physically by stretching or compressing the pipe material.

For our seachests, the strength in this mode is not sufficiently large to be considered infinite. The response back on the plate will also depend on the elastic properties of the solder of the weld, and the shape of the footing. We may assume that the plate and seachest meet at 90° at each point of the rim.

2.6.2 Modes with Azimuthal Dependence

The mode for $n = 1$ corresponds to a slope measured radially that varies as $\cos\theta$ or $\sin\theta$. This slope can be obtained by a tilt of the plate, regarded as planar, over the attachment circle and within it. Thus the mode is not distinct from the $n = 1$ axial mode (the swaying mode), if the plate is nearly planar over the attachment circle.

The modes with higher n are difficult to analyze but are expected to be strongly coupled to the corresponding axial modes.

2.7 Resonators Responding Over an Area

Several kinds of physical resonators that are sensitive over an area will be discussed in Section 9. One would like to decompose such systems into a complete set of ideal resonators that correspond to normal modes of vibration. However, modes that do not interact with each other for a resonator in isolation will interact through the plate after attachment. The plate interactions cannot be calculated until the results of Section 4 are available. Thus consideration of these resonators must be deferred.

3. GREEN'S FUNCTIONS FOR INFINITE RODS AND PLATES WITHOUT RESONATORS

We consider a homogeneous two-way infinite rod, or a homogeneous isotropic plate, extending to infinity in all directions.

3.1 Force Concentrated at a Point

A concentrated unit force, oscillating with angular frequency ω radians/second, is applied at point r^*. The resulting displacement of the plate or bar is observed at point r. This defines the Green's function $Q(r^*,r)$ of the system. Since we assume homogeneity and isotropy, Q depends only on the vector distance between r^* and r.

The letter Q will be reserved to denote Green's functions of systems without resonators, and the letter G for systems with resonators (to be studied beginning in Section 4).

3.1.1 Infinite Uniform Rod

The equilibrium position of the axis of the rod is taken as the x-axis. It executes vibrations in the xy-plane. The equation of motion of the rod is

$$EI \frac{\partial^4 \eta}{\partial x^4} - Z = 0$$

where

$\eta(x)$ = rod displacement at x

E = Young's modulus of the rod material

I = moment of inertia of a cross-section about an axis normal to the xy-plane (it is assumed that this is a principal axis).

$Z\,dx$ = non-elastic force on an element of the rod.

EI is called the flexural rigidity of the rod for vibrations in the xy-plane. The force on the rod is due to the inertia of the rod material, plus the impressed force. The inertial force is clearly

$$- \rho \, S \, \frac{\partial^2 \eta}{\partial t^2} \quad ,$$

where ρ is the rod density and S is the cross-section area.

The Green's function $Q(r^*,r)$ is defined as the displacement at r when a force varying as $e^{-i\omega t}$ is applied at r^*. Thus we obtain:

$$EI \frac{\partial^4 Q(x^*,x)}{\partial x^4} \; e^{-i\omega t} - \rho S \omega^2 Q(x^*,x) e^{-i\omega t} = \delta(x^*-x) e^{i\omega t}$$

or

$$\frac{\partial^4 Q}{\partial x^4} - \frac{\rho S \omega^2 Q}{EI} = \frac{\delta(x^*-x)}{EI}$$

The boundary conditions are that the displacement at infinity is finite, and there is no incoming wave from infinity.

Define the wave-number k by

$$k^4 = \rho S \omega^2 / EI \; .$$

Then it can easily be checked that

$$Q(x^*,x) = \frac{1}{4EIk^3} \left[-e^{-k|x^*-x|} + i e^{ik|x^*-x|} \right] \; .$$

A formal derivation is given in Addendum 1. The real part of the bracket is the negative of

$$B(ky) = e^{-k|y|} + \sin k|y|$$

where $y = x^*-x$. The power series expansion for B:

$$B(w) = i - 1 - (i+1)w^2/2 + |w|^3/3 + \ldots$$

shows that the first and second derivatives of B are continuous at $y = 0$, but the third is discontinuous there. The imaginary part of Q has continuous derivatives of all orders everywhere.

Another useful way of decomposing Q is into the real exponential term, which can be called the near field, and the complex exponential, or far field term.

3.1.2 Infinite Membrane Under Uniform Tension

The problem of a membrane is considered here as it provides the simplest illustration for some of the mathematical tools that will be used later.

The equation of motion under a sinusoidal applied force is

$$T\nabla^2\eta - \rho\frac{\partial^2\eta}{\partial t^2} = P(x,z)e^{-i\omega t}$$

Here the membrane is in the xz-plane, and

η = displacement out of equilibrium plane
T = tension on the membrane
ρ = mass of the membrane per unit area

The Green's function is defined as the displacement η that results when the pressure P is concentrated so as to be a delta-function $\delta(x^*-x)\delta(z^*-z)e^{-i\omega t}$.

We transform to polar coordinates r, θ, with the force point as origin, assume a response of the form

$$Q(r)e^{-i\omega t} \quad ,$$

and set

$$k = \omega/c = \omega/\sqrt{T/\rho} \quad .$$

Then the equation of motion becomes

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\ \frac{\partial}{\partial r} + k^2\right] Q = \frac{\delta(r)}{2\pi\ Tr}$$

after cancelling the time-dependent factor. Note that the product $\delta(x^*-x)\delta(y^*-y)$ is replaced by $\delta(r)/2\pi r$. This is dimensionally consistent, since a delta-function of a length has the dimensions of an inverse length.

The equation above will be solved by using the Hankel transform of order zero. As the precise form of this transform is not standardized in the literature we give the definition that is used throughout this report.

If $Q(r)$ is defined for $0 \leq r < \infty$, then the transform $\tilde{Q}(p)$ of Q is defined by

$$\tilde{Q}(p) = \int_0^\infty Q(r)r\ J_o(pr)\,dr \quad ,$$

where J_o is the Bessel function of the first kind of order zero. The transform is self-reciprocal, that is,

$$Q(r) = \int_0^\infty \tilde{Q}(p)p\ J_o(rp)\,dp$$

It can be easily seen that the Hankel transform of order zero, when applied to

$$\frac{d^2Q(r)}{dr^2} + \frac{1}{r}\ \frac{dQ(r)}{dr}$$

produces

$$-p^2\ \tilde{Q}(p) \quad .$$

Another result that will be useful is:

$$\int_o^\infty J_o(ax)\; J_o(bx)x\; dx = \delta(a-b)/a$$

This is a form of Hankel's theorem, which is usually stated in an integral form that avoids mentioning a delta-function. If $b = 0$, it becomes

$$\int_o^\infty J_o(ax)x\; dx = \delta(a)/a,$$

which shows that the transform of $\delta(r)/r$ is the constant 1. Now applying the Hankel transform to the equation of motion, we obtain

$$-\,p^2\; Q(p) + k^2\;\tilde{Q}(p) = 1/2\pi T$$

or

$$\tilde{Q}(p) = \frac{-1}{2\pi T(p^2 - k^2)}$$

and then inverting the transform,

$$Q(r) = \frac{-1}{2\pi T}\int_o^\infty \frac{p\; J_o(pr)\,dp}{p^2 - k^2} \quad .$$

The integral must be interpreted as a contour integral to be well-defined. The real part can, however, be considered as the Cauchy principal value (often denoted by putting a bar through the integral sign). But from standard tables, we find

$$\int_o^\infty \frac{J_o(ax)x\; dx}{x^2 - y^2} = -\,(\;\;/2)Y_o(ay)$$

and so

$$\mathrm{Re}\; Q(r) = (1/4T)\,Y_o(kr) \quad .$$

There will also be a pure imaginary contribution to $Q(r)$ from an indentation in the path to go around the pole at $p = k$. This term is $\pm(i/4T)J_0(kr)$, where the sign depends on whether the path goes below or above the pole. Because of our choice of time dependence $e^{-i\omega t}$, an outgoing wave must have the form of a constant multiple of $H_0(kr) = J_0(kr) + iY_0(kr)$. Thus we obtain finally

$$Q(r) = (-i/4T)H_0(kr) \quad ,$$

where H_0 is the Hankel function of the first kind of order zero. For typographical convenience, the supercript (1) usually affixed to H will be omitted in this report.

3.1.3 Infinite Isotropic Plate

The plate equation is

$$\frac{Eh^3}{12(1-\sigma^2)} \nabla^4\eta - \rho h\omega^2\eta = P(x,z)$$

where

E = Young's modulus
σ = Poisson's ratio
h = plate thickness
ρ = plate density

The following abbreviations will be introduced

$$D = \frac{Eh^3}{12(1-\sigma^2)} = \text{flexural rigidity}$$

$$k = \left(\frac{12(1-\sigma^2)\rho\omega^2}{Eh^2}\right)^{1/4} = \text{free wavenumber}$$

The Green's function for a concentrated force at (x^*,z^*) then satisfies:

$$(\nabla^4 - k^4)Q(x,z) = \delta(x^*-x)\delta(z^*-z)/D$$

After transforming to polar coordinates, and applying the Hankel transform, as for the membrane, we obtain

$$Q(r) = \frac{-1}{2\pi D}\int_0^\infty \frac{pJ_o(pr)\,dp}{p^4-k^4} \quad .$$

Then breaking up $1/(p^4-k^4)$ into partial fractions,

$$Q(r) = \frac{1}{4\pi k^2 D}\left[-\int_0^\infty \frac{pJ_o(pr)\,dp}{p^2-k^2} + \int_0^\infty \frac{pJ_o(pr)\,dp}{p^2+k^2}\right] .$$

The first integral has already appeared in the membrane problem; the second integral can be derived from it by replacing r by ir and then making ip the variable of integration. Unfortunately, however, the standard definition of the modified Bessel function K_o of the third kind inserts an extra factor $\pi/2$:

$$K_o(z) = (\pi i/2)H_o(iz) \quad ,$$

and thus Q(r) becomes:

$$Q(r) = \frac{-i}{4k^2 D} H_o(kr) + \frac{1}{2\pi k^2 D} K_o(kr) \quad .$$

The first term is, as mentioned earlier, an outgoing wave; the second represents a standing wave. The logarithmic infinities at $r = 0$ in K_o and in the Y_o part of H_o cancel each other, leaving a discontinuity in the third derivative of Q.

3.2 Force on a Rim

The displacement of a plate in response to a distributed force applied on a circle can be found by integrating the Green's function just given for a concentrated force. However, when the strength of the force is a sinusoidal function of the angle, the displacement can be found directly from the equation of motion.

3.2.1 Uniform Strength

The force integrated over the circumference of radius R will be normalized to unity. On transformation to polar coordinates, the equation for the Green's function $Q_R(r)$ becomes

$$(\nabla^4 - k^4)\, Q_R(r) = \frac{\delta(r-R)}{2\pi \; DR}$$

As earlier, we take the Hankel transform of both sides. For the right side, we use the relation already given in Section 3.1.2. Then we find

$$Q_R(r) = \frac{1}{2\pi} \int_0^\infty f^{-1}(p) J_o(pR) J_o(pr) p \; dp \quad ,$$

where we have introduced the abbreviation

$$f(p) = D(p^4 - k^4) \quad .$$

The integral for $Q_R(r)$ is a special case of a more general form for which the explicit evaluation is given in Section 4.4.3. See also 10.1.

From the form of the integrand, we note that

$$Q_R(r) = Q_r(R)$$

This is an example of a <u>reciprocity</u> relation.

3.2.2 Sinusoidal Variation of Force

The force on the circle is now assumed to vary as $\cos n\theta$, for $n = 1,2,3,\ldots$. The case $n = 1$ is of primary interest, since the reaction back on the plate of a seachest oscillating in the tilting mode has this form. To normalize the force, we require that the <u>square</u> of its magnitude, integrated over the circle, be unity. This will introduce an extra factor of 2, as compared to the case of a uniform distribution. The equation of motion is then

$$(\nabla^4 - k^4)Q(r,\theta) = \frac{(\cos n\theta)\,\delta(r-R)}{\pi DR}$$

where $Q(r,\theta)$ has the form:

$$Q(r,\theta) = Q_{R,n}(r)\cos n\theta$$

This time, Hankel transforms of order n are taken, and the right side is evaluated by using the relation:

$$\int_0^\infty J_n(ax)J_n(bx)\,x\,dx = \delta(a-b)/a \quad .$$

Then we obtain:

$$Q_{R,n}(r) = \frac{1}{\pi DR}\int_0^\infty f^{-1}(p)J_n(pR)J_n(pr)p\,dp \quad .$$

The integral is evaluated in section 10.1.

3.3 Concentrated Moment on a Bar or Plate

As a preliminary to the later investigation of the effect of a transverse point resonator, it will be useful to find the displacement produced on a bar or plate without resonators by a concentrated moment of unit strength.

In this simple situation the displacement at r due to a moment of unit strength at r* is found by differentiating the corresponding Green's function for a concentrated force. Thus for a plate, let $\theta = 0$ be the direction of the moment in the plane of the plate. Then differentiating the integral of section 3.1.3 under the sign of integration, we obtain the following formula for the Green's function Q^T:

$$Q^T(r,\theta) = \frac{\cos\theta}{2\pi D} \int_0^\infty f^{-1}(p) J_1(rp) p^2 dp \quad .$$

or differentiating the evaluation,

$$Q^T(r,\theta) = \frac{i \cos\theta}{4kD} H_1(kr) - \frac{\cos\theta}{2\pi kD} K_1(kr) \quad .$$

3.4 Moments Distributed on a Circle of Radius R and Pointing Radially.

3.4.1 Uniform Distribution of Moments

In this configuration each point of the plate on the circle experiences a moment point outward of strength $1/2\pi$ per unit arclength. This moment distribution can be regarded as the limit, as ϵ goes to zero, of two uniform force distributions with opposite senses on circles of radii $R-\epsilon$ and $R+\epsilon$. Since the Green's function of Section 3.2.1 has been normalized to unit total force, the expression there given for $Q_R(r)$ can be differentiated with respect to R to obtain the desired Green's function Q_R^T:

$$Q_R^{\,T} = \frac{-1}{2\pi} \int_0^\infty f^{-1}(p) J_1(pR) J_o(pr) p^2 dp \quad .$$

3.4.2 Moment Strength Proportional to cos θ

Inasmuch as the strengths are always measured with the radially outward direction as positive, the component parallel to $\theta = 0$ of the moment at each point is positive. This moment distribution can be synthesized just as in Section 3.4.1, from two force distributions on circles of radii $R-\epsilon$ and $R+\epsilon$, each varying as $\cos\theta$.

3.5 Reciprocity Relations

Consider an infinite plate with a point force applied at r*. We will later want to know the average displacement of the plate on the circumference of a circle of radius S, the center being at s. That is, using r as a vector variable, we will want

$$A = \frac{1}{2\pi} \int Q(|r^*-r|)\delta(|r-s|-S)\,dr$$

where Q is as in Section 3.1.3. It can be shown (see Addendum 3) by using the idea of a two-dimensional convolution, that

$$A = Q_S(|r^*-s|)$$

where Q_S is the Green's function for a rim force resonator.

More generally, consider a plate on which a distributed force of total magnitude unity is applied uniformly on a circumference of radius R and center r*. The average displacement on another circle of radius S and center s is

$$\int_0^\infty f^{-1}(p) J_o(pR) J_o(pS) J_o(p|r^*-s|) p\, dp \quad ,$$

and this expression is symmetrical between the circles. Thus it equals the average displacement on the circle of radius R due to a unit force evenly distributed on the circumference of radius S.

These results are obviously generalizations of the simple reciprocity relacion for the point-force, point-displacement Green's function of Section 3.1.3.

4. GREEN'S FUNCTION FOR A ROD OR PLATE WITH ATTACHED RESONATORS

4.1 Equation of Motion

In deriving the equation of motion for a plate bearing both axial and transverse point resonators, it is convenient to think of continuous distributions of each of these types.

Let $A(x,z)$ be the density of axial point resonators at the point (x,z). Each resonator is taken as of unit strength. This means that the force exerted on an element of area $dx\,dz$ in response to the displacement η is $\eta\, A(x,z)dx\,dz$. (It is understood that the time dependence has been factored out of η, and that A depends on the frequency.) Let a distributed pressure $P(x,z)$ be applied to the plate. Then the equation for η is

$$D(\nabla^4\eta - k^4\eta) = P(x,z) + A(x,z)\eta \quad .$$

where D and k were defined in section 3.1.3.

For transverse resonators, the motions in perpendicular planes will be handled independently. That is, each physical resonator, which can vibrate in many planes, is considered as resulting from the superposition of two kinds, say the xy and zy kinds. An xy resonator responds only to the slope of the cross-section of the plate that lies in the xy plane, and the moment it exerts on the plate lies in the same plane. Similarly for the zy resonators. These two types are assumed to have the same density $T(x,z)$ over the plate. (This corresponds to physical resonators which are made up of segments of circular cylinders. One can imagine resonators for which the cross-sections parallel to the plate are elliptical. Then the densities of the two component types of resonators would be unequal.)

We first study the effect of the xy resonators only. Then the dimensionality of the system is lowered, and thin-rod theory can be used.

Consider a beam with a continuous distribution of transverse resonators. The distribution is described by a density function

$T(x)$ whose units are moment/unit rotation/unit length. On an increment dx of the beam the increment of moment dm is therefore:

$$dm = T(x) \frac{d\eta}{dx} dx \quad .$$

But from simple beam theory,

$$EI \frac{d^2\eta}{dx^2} = m \quad ,$$

and therefore the contribution of the transverse resonators to the shear force per unit length is

$$EI \frac{d^4\eta}{dx^4} = \frac{d}{dx}\left(\frac{dm}{dx}\right) = \frac{d}{dx}\left(T(x) \frac{d\eta}{dx}\right) \quad .$$

For the zy resonators, the contribution is, by a similar argument,

$$\frac{d}{dz}\left(T(x,z) \frac{d\eta}{dz}\right) \quad .$$

The sum of the two terms can be written in the form

$$\text{div}\,(T(x,z)\ \text{grad}\ \eta(x,z)) \quad .$$

Therefore the equation of motion for a plate bearing axial point resonators with strength-density $A(x,z)$ and transverse point resonators with strength-density $T(x,z)$ is

$$D(\nabla^4 - k^4)\eta = P + A\eta + \text{div}\,(T\ \text{grad}\ \eta) \quad ,$$

where P is the driving pressure distribution. The same equation will hold for a uniform rod, provided that ∇^4 is interpreted as d^4/dx^4, and D is set equal to EI. Of course, the term involving T must be interpreted as the lower-dimensional analog:

$$\frac{d}{dx}\left(T\frac{d\eta}{dx}\right) \quad .$$

4.2 A Plate or Rod with I Axial Point Resonators

The i-th resonator, of strength $F_i(\omega)$, is attached to the plate or rod at the point specified by the vector s_i, $i = 1,\ldots, I$. Thus the density function A becomes

$$A(r) = \sum_{i=1}^{I} F_i(\omega)\delta(r-s_i) \quad .$$

A unit oscillating force is applied at r^*.

The Green's function $G(r^*,r)$ for the system is then by definition the displacement at r. The equation of motion for the plate becomes

$$D(\nabla^4-k^4)G(r^*,r) = \delta(r^*-r) + \sum_{i=1}^{I} F_i(\omega)\delta(r-s_i)G(r^*,r) \quad .$$

4.2.1 Assumed Form for G

It is clear that each resonator on a plate will act as the origin of a new circularly symmetric disturbance, and each resonator on a rod will similarly be the center of a disturbance with right-left symmetry. Thus one expects that G has the form

$$G(r^*,r) = Q(r^*-r) + \sum_i C_i\, Q(s_i-r) \quad ,$$

where Q is the Green's function in the absence of resonators that was discussed in Section 3.1.1 (for the case of a rod) or 3.1.3 (for a plate). On the other hand, a Green's function should be symmetric in its variables, and so we assume the following bilinear form:

$$G(r^*,r) = Q(r^*-r) - \sum_{i,j} Q(r^*-s_i)M_{ij}Q(r-s_j) \quad ,$$

where the M_{ij} are undetermined coefficients, and we substitute into the equation of motion. As this procedure will be repeated for many different kinds of resonators, the details have been relegated to Addendum 2.

4.2.2 Explicit Evaluation of the Green's Function

The results can be expressed as follows: Define the matrix N by

$$N_{ij} = Q(s_i-s_j) - (\delta_{ij}/F_i(\omega)) \quad .$$

Then

$$M = N^{-1} \quad .$$

Thus the Green's function for the plate with resonators depends on the inverse of the matrix N, which involves all the interactions $Q(s_i-s_j)$ between pairs of resonators.

Note that N and M are symmetrical in their indices. As their elements are in general complex, they are not Hermitian.

It is helpful to write out the explicit form of the Green's function for one resonator of strength F:

$$G(r^*,r) = Q(r^*-r) - Q(r^*-s) \frac{1}{Q(0)+F^{-1}} Q(s-r) \quad .$$

4.2.3 Algebraic Transformation of the Expression for G

There is an interesting modification of the formula for G which may be useful for certain kinds of large-scale problems for electronic computers.

Suppose that one is interested in the Green's function when the source is at any of P possible points r^*_1, $r^*_2,\ldots,r^*_P$, and the observation point can be any of the same points . We combine these P points with the I resonator points, distinguishing between them by the index. Thus we write

$$s_{I+1} = r^*_1 ,\ldots, s_{I+P} = r^*_P$$

and we define an augmented matrix N*, having I+P rows and columns:

$$N^*_{ij} = Q(s_i - s_j) - (\delta_{ij}/F_i) \quad , \qquad i,j = 1,2,\ldots,I+P$$

where $F = \infty$ if $i = I+1,\ldots,I+P$. Thus the upper left block of N*, of N rows and N columns, is the same as the matrix N previously defined.

The formula given earler for the Green's function is equivalent to the following prescription:

Find the inverse M* of N*. Extract the lower right block of M* containing P rows and P columns. Invert this block. The (p,q)-th element of the inverse is $G(r^*_p, r^*_q)$.

This procedure may be symbolized as follows:

$$G^{-1} = (\underline{\underline{0}}\ \underline{\underline{I}}) \begin{pmatrix} Q_{ij} & Q_{i,I+q} \\ Q_{I+p,j} & Q_{I+p,I+q} \end{pmatrix}^{-1} \begin{pmatrix} \underline{\underline{0}} \\ \underline{\underline{I}} \end{pmatrix}$$

Here $\underline{\underline{I}}$ is the unit matrix of P rows and P columns. $\underline{\underline{0}}$ is the zero matrix,(with I rows and P columns at the appearance on the left, but P rows and I columns at the appearance on the right). G is the matrix with elements $G(r^*_p, r^*_q)$

4.3 A Resonator on a Rod as Vibration Isolator

It is now possible to discuss an isolation effect observed when a single axial resonator is attached between the source and the observation point on a rod. For this case, we recall that

$$Q(x) = \frac{1}{4Dk^3}\left[- e^{-k|x|} + ie^{ik|x|}\right] \quad .$$

Now assume that

$$k|r^*-s|, \quad k|s-r| \gg 1 \quad ,$$

that is, the distances from the resonator to the source and the observation point are large compared to the wavelength of free vibrations at the frequency of interest. Then the decreasing exponential term in Q can be discarded except in Q(0). That is, we retain only the far field term. Thus

$$G(r^*,r) \cong \frac{1}{4Dk^3}\, ie^{ik|r^*-r|}$$

$$- \frac{1}{4Dk^3}\, ie^{ik|r^*-s|} \frac{1}{\frac{-1+i}{4Dk^3} + \frac{1}{F(\omega)}} \frac{1}{4Dk^3}\, ie^{ik|s-r|}$$

Introduce the assumption that s is between r* and r:

$$e^{ik|r^*-s|} e^{ik|s-r|} = e^{ik(r^*-r)} \quad .$$

Then

$$G(r^*,r) = \frac{i}{4Dk^3} e^{ik(r^*-r)}\left[1 - \frac{1}{\frac{-1+i}{4Dk^3} + \frac{1}{F(\omega)}} \frac{i}{4Dk^3}\right] \quad .$$

The bracket can be rewritten as:

$$\frac{F(\omega) - 4Dk^3}{F(\omega) - 4Dk^3 - iF(\omega)} \quad .$$

Now there will be at least one value of ω for which

$$F(\omega) = 4Dk^3 \ ,$$

and then the value of the bracket is zero. Thus at this frequency the far field produced by the resonator cancels the far field produced directly by the source, and the resonator acts as a vibration isolator.

If one retains the near field produced by the source, then the frequency of isolation is shifted somewhat, and there is an out-of phase component that is not canceled.

4.4 Plate Bearing Azimuthally-Independent Rim Resonators

Rim force resonators of strength $F_1(\omega), \ldots, F_N(\omega)$ are attached to an infinite plate. The i-th resonator is sensitive to the average displacement on a circle of radius R_i with center s_i, and exerts its force uniformly on this circle.

Some of the resonators can actually be point resonators. Then the limit of zero radius must be taken in the formulas that follow.

Once more we assume that the applied force is concentrated at the point r*. The Green's function G(r*,r) is defined to be the displacement at r due to a point source at r*.

By analogy with the arguments of Section 4.1, the equation of motion of the plate can be written:

$$D(\nabla^4 - k^4) G(r^*, r) = \delta(r^* - r)$$
$$+ \sum_j F_j \left[\frac{1}{2\pi R_j} \int G(r^*, r') \delta(|r' - s_j| - R_j) \, dr' \right] \frac{1}{2\pi R_j} \delta(|r - s_j| - R_j) \ .$$

This is actually an integral equation.

4.4.1 Assumption of a Form for G

We now apply the results of Section 4.2 to guess the general form of the solution. There will be a term $Q(r^*-r)$ due to the direct effect of the source on r. Each rim resonator will act as the source of a circular wave which will be proportional to $Q_{R_i}(s_i-r)$, the Green's function in the absence of resonators for an applied force uniformly distributed on a circle (see Section 3.2.1). The proportionality constant will involve interaction constants between the resonators, and the average displacement on the i-th resonator circle due directly to the source. In view of the reciprocity relation discussed in Section 3.5, the average displacement is the same as $Q_{R_i}(r^*-s_i)$. Thus we assume the following form for the Green's function of the entire system:

$$G(r^*,r) = Q(r^*-r) - \sum_{i,j}^{N} Q_{R_i}(r^*-s_i)M_{ij}\, Q_{R_j}(s_j-r) \quad .$$

4.4.2 Results of Substitution

Once more the details appear in Addendum 2, for a very general case. One defines an interaction integral $Q(R_i,R_j,|s_i-s_j|)$ for the i-th and j-th resonators. It is the average displacement on a circle of radius R_i due to an oscillating force of unit strength distributed uniformly on a circle of radius R_j, when the distance between centers is $|s_i-s_j|$. By reciprocity, the indices i and j can be interchanged in this description. It is shown in Addendum 2 that

$$Q(R_i,R_j,|s_i-s_j|) = \int Q_{R_j}(s_j-r')\,\delta(|r-s_i|-R)\,dr'$$

$$= \int f^{-1}(p)\,J_o(pR_i)\,J_o(pR_j)\,J_o(p|s_i-s_j|)\,p\;dp$$

Now define a matrix N by

$$N_{ij} = Q(R_i,R_j,|s_i-s_j|) - (\delta_{ij}/F_i) \quad .$$

Then it is found thac

$$M = N^{-1} \ ,$$

in complete analogy with the earlier result for point axial resonators.

4.4.3 Va ıe of the Interaction Integral

The .ntegral is evaluated in Section 10.1 as

$$Q(R_i,R_j, s_i-s_j) = \frac{1}{2Dk^2}\left[\frac{\pi i}{2}J_o(kR_i)J_o(kR_j)H_o(k|s_i-s_j|) - I_o(kR_i)I_o(kR_j)K_o(k|s_i-s_j|)\right] \quad ,$$

provided that $i \neq j$ and

$$R_i + R_j < |s_i - s_j|$$

as will be true for two physical seachests.

If $i = j$, then

$$Q(R_i,R_i,0) = \frac{1}{2Dk^2}\left[\frac{\pi i}{2} J_o(kR_i)H_o(kR_i) - I_o(kR_i)K_o(kR_i)\right] \quad .$$

4.5 Rim Resonators with Azimuthal Dependence

We now consider a more general system in which there are N resonators whose weight functions vary with azimuth. The i-th resonator is sensitive to the weighted integral

$$\frac{1}{2\pi} \int \cos n_i\theta_i \; \eta(R_i,\theta_i)\, d\theta_i \quad ,$$

where θ_i is the polar angle of a coordinate system with the center s_i of the i-th attachment circle as origin, and $\eta(R_i,\theta_i)$ is the displacement at a point on this circle. The force exerted on the points of the circle is $F_i(\omega)\cos n_i\theta_i$. The equation of motion can be written down by analogy with earlier forms. Again we seek a solution of the form

$$G(r^*,r) = Q(r^*-r) - \sum_{i,j}^{N} Q_{R_i,n_i}(r^*,s_i)\cos n_i\theta_i^* \, M_{ij} Q_{R_j,n_j}(s_j,r)\cos n_j\theta_j$$

where Q_{R_i,n_i} is the Green's function of Section 3.2.2. It is found (see Addendum 2) that

$$M = N^{-1} \; ,$$

where

$$N_{ij} = V_{ij} - (\delta_{ij}/F_i) \quad .$$

V_{ij} is the interaction constant between the i-th and j-th resonators. It is the average of the displacement on the i-th circle, weighted according to the i-th weight-function, due to an applied unit force on the j-th circle varying as the j-th weight-function. Because of reciprocity, the roles of the two resonators can be interchanged in this description. An integral form for V_{ij} is given in Addendum 2.

In general, the polar coordinate systems set up at the different resonators will have different prime directions, but as discussed earlier, one need only consider modes which have axes either parallel or perpendicular to a fixed direction.

We specialize to the case of tilting rim types, i.e., $n_i = 1$ for all i, each having its axis of tilt perpendicular to a fixed direction $\theta = 0$. Let Ω_{ij} be the angle between the vector $s_i - s_j$ and $\theta = 0$. Then the interaction integral V_{ij} is given by:

$$V_{ij} = (-1/2)(\cos 2\Omega) \int_0^\infty f^{-1}(p) J_1(pR_i) J_1(pR_j) J_2(p|s_i - s_j|) p \, dp$$

$$+ (1/2) \int_0^\infty f^{-1}(p) J_1(pR_i) J_1(pR_j) J_o(p|s_i - s_j|) p \, dp \quad .$$

The integrals are evaluated in Section 10.1.

4.6 Transverse Point Resonators

A transverse resonator can be approximated as a very close pair of axial resonators that exert equal and opposite forces at the attachment points. The pair is sensitive to the difference in the displacements at the two points. The transverse resonator is obtained in the limit as the separation goes to zero while the strength of both members of the pair goes to infinity in such a way that the product remains finite.

The Green's function for a rod or plate bearing transverse resonators at $s_1, \ldots, s_I$ will be

$$G(r^*, r) = Q(r^* - r) + \sum_{ij} Q^T(r^* - s_i) M_{ij} Q^T(s_j - r) \quad ,$$

where $Q^T(x) = \frac{\partial}{\partial x} Q(x)$. Setting $M = N^{-1}$, we will have

$$N_{ij} = V_{ij} - (\delta_{ij}/G_j) \quad ,$$

where G_j is the transverse strength of the j-th resonator.

On a rod, the interaction constant V_{ij} between two resonators at s_i and s_j can be written very simply as

$$\frac{\partial}{\partial s_i \partial s_j} Q(s_i - s_j) = -\frac{\partial^2}{\partial s^2} Q(s_i - s_j)$$

On a plate, the derivatives must be taken along the directions of the moments, which will not in general coincide with the direction of $s_i - s_j$. It is more convenient in this case to consider the transverse point resonator as the limiting case of a rim force resonator with weight function cos $(\theta + a)$, for some angle a. As noted earlier, the form with arbitrary angle a results from superposition of two standard modes with angles zero and $\pi/2$.

Thus, taking the equation in Section 4.5 for V_{ij}, and letting R_i and R_j approach zero, we obtain the following limiting form, appropriate to two transverse point resonators, each with tilt axis perpendicular to $\theta = 0$:

$$V_{ij} R_i R_j \rightarrow (-1/8)(\cos\ 2\Omega) \int_0^\infty f^{-1}(p) J_2(p|s_i - s_j|) p^3 dp$$

$$+ \ (1/8) \int_0^\infty f^{-1}(p) J_o(p|s_i - s_j|) p^3 dp \quad .$$

After evaluation, this becomes a constant multiple of

$$-\ (\cos\ 2\Omega)(H_2 - i(2/\pi) K_2) + H_o - i(2/\pi) K_o \quad ,$$

where the functions are all evaluated for the argument $k|s_i - s_j|$.

4.7 Rim Moment Resonators with Constant Weight Functions

Physically, we are dealing with the "breathing" modes of a set of seachests. The interaction integral V_{ij} between two of these can be found from the interaction integral for two rim force resonators by differentiating with respect to the radii of both resonators:

$$V_{ij} = \frac{\partial^2}{\partial R_i \partial R_j} Q(R_i, R_j, |s_i - s_j|)$$

$$= \frac{1}{2D}\left[\frac{\pi i}{2} J_1(kR_i) J_1(kR_j) H_o(k|s_i - s_j|)\right.$$

$$\left. - I_1(kR_i) I_1(kR_j) K_o(k|s_i - s_j|)\right] \quad .$$

4.8 Mixed Types on a Plate

A plate bearing physical point resonators and seachests must be handled by considering all possible interactions between the various modes. A physical point resonator has one axial and two transverse modes (in perpendicular plates); a seachest has two infinite series of modes, one set responding to the displacement, and the other set to the radial slope.

The interaction integral for a rim moment resonator and any other resonator is obtained by differentiation, with respect to the radius, of the integgral for the corresponding rim force resonator. A point axial resonator is a limiting case of a rim force resonator with uniform weighting; a point transverse resonator is the limiting case for cos θ or sin θ weighting. Thus all integrals that arise can be ultimately thrown back onto the interaction integral for two rim force resonators with weight functions cos $n_1\theta_1$ and cos $n_2\theta_2$. The integtal is written down in Addendum 2 and is evaluated in Section 10.1.

5. MATRIX METHODS FOR RESONATORS ON A ROD

5.1 Transfer Matrices Based on Derivatives

5.1.1 A Homogeneous Rod

An infinite rod bearing resonators may execute free vibrations of arbitrary frequency in the absence of any impressed force. If there are no resonators, these are sinusoidal with wave-number $k(\omega)$. When resonators are attached, the motion can be studied conveniently using a matrix that relates the displacement and its first three derivatives at one point to the same four properties at another point.

As fundamental solution of the equation of motion of the homogeneous rod without impressed forces

$$\frac{d^4\eta}{dx^4} - k^4\eta = 0 \quad .$$

where $k^4 = \omega^2\rho h/D$ as usual, we take

$$H(x) = \frac{1}{2k^3}(\sinh kx - \sin kx)$$

$$= \frac{1}{k^3}\left[\frac{(kx)^3}{3!} + \frac{(kx)^7}{7!} + \frac{(kx)^{11}}{11!} + \ldots\right]$$

H has the convenient properties

$$H(0) = H'(0) = H''(0) = 0$$

$$H'''(0) = 1 \quad .$$

The first three derivatives of H can be taken as the remaining linearly independent solutions. Thus the first derivative I of H(x) has the properties

$$I(0) = I'(0) = 0$$

$$I''(0) = 1, \; I'''(0) = 0$$

Similarly for the second and third derivatives of H.

Let Y(x) be the column vector whose four components are the displacement η at x and its first three derivatives:

$$Y(x) = \begin{pmatrix} \eta(x) \\ \eta^{(1)}(x) \\ \eta^{(2)}(x) \\ \eta^{(3)}(x) \end{pmatrix}$$

$Y(x_o)$ at a point x_o determines Y(x) at an arbitrary point x through the matrix relation

$$Y(x) = M(x - x_o)Y(x_o)$$

where M is the 4 x 4 matrix whose elements are

$$M(x-x_o) = \begin{pmatrix} H^{(3)} & H^{(2)} & H^{(1)} & H \\ H^{(4)} & H^{(3)} & H^{(2)} & H^{(1)} \\ H^{(5)} & H^{(4)} & H^{(3)} & H^{(2)} \\ H^{(6)} & H^{(5)} & H^{(4)} & H^{(3)} \end{pmatrix},$$

in which the argument of all the functions is $x - x_o$. Of course, the fourth and higher derivatives can be simplified, since $H^{(4)} = k^4 H$. Clearly

$$M(a + b) = M(a)M(b) \quad .$$

The inverse of M(a) will be M(-a), which is obtained from M(a) by changing the signs in M in a checkerboard manner (that is, inserting minus signs in front of H and its even-order derivatives).

5.1.2 Transfer Matrices for Resonators

Now consider an axial resonator on a rod. One can define a transfer matrix that gives the displacement and its first three derivatives immediately to the right of the resonator in terms of these quantities immediately to the left.

The axial resonator produces a force proportional to the displacement. This force is proportional to the change in the third derivative across the resonator

$$D\left[\text{jump in } \frac{d^3\eta}{dx^3}\right] = F(\omega)\,\eta$$

Therefore the transfer matrix across the attachment point is

$$M_{ax} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{F(\omega)}{D} & 0 & 0 & 1 \end{pmatrix}$$

A transverse resonator produces a moment proportional to the local slope. This moment is proportional to the change in the second derivative.

$$D\left[\text{jump in } \frac{d^2\eta}{dx^2}\right] = G(\omega)\frac{d\eta}{dx}$$

As a physical resonator is the result of superimposing axial and transverse resonators, its transfer matrix is

$$M_{phys} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{G(\omega)}{D} & 1 & 0 \\ \frac{F(\omega)}{D} & 0 & 0 & 1 \end{pmatrix}$$

Now consider a segment of the rod of length L, with a physical resonator at its extreme right end. Because of the ordinary convention on matrix multiplication, the matrix for the combined rod plus segment is

$$M_{phys}\, M(L) \quad .$$

In this manner, one can write down the transfer matrix for a finite rod with any number of attached resonators.

5.1.3 Infinite Array of Resonators; Free Vibrations

One case of interest is an infinite rod bearing a regular array of identical resonators with spacing L between them. The matrix for N segments of length L, each with its resonator at the right hand end, is of course

$$(M_{phys}\ M(L))^N \quad .$$

If one considers N segments having resonators attached at their mid-points, then the matrix is

$$\left[M(L/2) M_{phys}\ M(L/2)\right]^N \quad .$$

A segment in this sense is symmetrical between right and left.

Another way of producing symmetry is to split each resonator into two halves, and attach the halves to the segments that meet at the attachment point. If we write M_o for the matrix formed from M_{phys} by replacing all the diagonal elements by zero, that is,

$$M_o = M_{phys} - I \quad ,$$

then the transfer matrix for a segment of length L bearing half of a physical resonator at each end is Z:

$$Z = \left[I + (M_o/2)\right] M(L) \left[I + (M_o/2)\right] \quad .$$

The four eigenvalues of Z must occur in two reciprocal pairs, say

$$\lambda, \frac{1}{\lambda}, \mu, \frac{1}{\mu} \quad .$$

This follows from the right-left symmetry of a segment. If a disturbance is multiplied by λ at one end of the segment, as compared to the other, then there is another disturbance, obtainable by applying the right-left symmetry, that is reduced by the factor $1/\lambda$. Similarly for the other pair. The eigenvalues will not be real, in general.

5.1.3.1 Free Vibrations

One can now study the existence of free or self-sustained vibrations of the infinite bar with resonators by considering the limit of Z^N as N goes to infinity. No energy is applied to the bar at any point, hence the transfer matrix is sufficient to describe the behavior.

Suppose one eigenvalue λ of Z has absolute magnitude greater than 1. Then Z^N has the eigenvalue λ^N, which goes to infinity as N increases indefinitely. This is not allowed by the boundary condition that the displacement of the bar must be finite even at infinite distances. If $1/\lambda$ has absolute value greater than 1, the same argument applies. The conclusion is: a free vibration of the infinite rod with resonators can exist only if Z has an eigenvalue with absolute value 1.

5.1.3.2 Procedure to Determine the Eigenvalues of Z

Let P be the diagonal matrix whose diagonal elements are, in order, -1, +1, -1, +1. Then

$$M^{-1}(L) = P\ M(L)\ P$$

This is the algebraic formulation of the verbal prescription for $M^{-1}(L)$ given at the end of Section 5.1.1. Furthermore

$$\begin{aligned} Z^{-1} &= (I + M_o/2)^{-1} M^{-1}(L)(I + M_o/2)^{-1} \\ &= (I - M_o/2) P\ M(L)\ P(I - M_o/2) \\ &= \left[P(I + M_o/2)P\right] PM(L)P\left[P(I + M_o/2)P\right] \\ &= PZP \end{aligned}$$

Let Y be an eigenvector of Z corresponding to the velue λ:

$$ZY = \lambda Y$$

Dividing by λZ, we have

$$PZPY = Z^{-1}Y = \lambda^{-1}Y$$

and then summing,

$$(PZP + Z)Y = (\lambda^{-1} + \lambda)Y \quad .$$

Half the elements of $PZP + Z$ will be zero, namely those for which the sum of the row and column indices is odd.

Let Q be the permutation matrix that applied on the left interchanges the second and third rows, and Q^T its transpose. Then $\lambda^{-1} + \lambda$ will be an eigenvalue of $Q(PZP + Z)Q^T$, which will have the following structure in terms of its 2 x 2 quarters:

$$\begin{pmatrix} C & O \\ O & D \end{pmatrix}$$

(O is the 2 x 2 zero matrix). Thus one obtains two lower-order eigenvalue conditions:

$$\det\,(C - (\lambda^{-1} + \lambda)I) = 0$$
$$\det\,(D - (\lambda^{-1} + \lambda)I) = 0 \quad ,$$

and as a matter of fact, these will turn out to be the same equation.

5.1.3.3. Calculations of Eigenvalues

For convenience, we write

$$f = F/2D$$

$$g = G/2D$$

and we break up $M(L)$ and M_{phys} into 2 x 2 blocks:

$$M(L) = \begin{pmatrix} U & V \\ k^4V & U \end{pmatrix} \qquad M_{phys} = \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$$

where

$$U = \begin{pmatrix} H^{(3)} & H^{(2)} \\ H^{(4)} & H^{(3)} \end{pmatrix} \qquad V = \begin{pmatrix} H^{(1)} & H \\ H^{(2)} & H^{(1)} \end{pmatrix}$$

(the argument of all the functions is L), and

$$S = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}$$

Then we find, from the definition of Z,

$$Z = \begin{pmatrix} U + VS & V \\ k^4V + SU + US + SVS & U + SV \end{pmatrix} .$$

The elements of PZP + Z are either twice the corresponding elements of Z, or else zero. After multiplying by Q and Q^T, we obtain the following determinantal equation from the upper left corner:

$$\det \begin{pmatrix} H^{(3)}+fH-(\lambda^{-1}+\lambda)/2 & H^{(1)} \\ k^4H^{(1)}+gH^{(4)}+fH^{(2)}+fgH^{(1)} & H^{(3)}+gH^{(2)}-(\lambda^{-1}+\lambda)/2 \end{pmatrix} = 0$$

The lower right quarter yields the same determinant with the diagonal elements interchanged. We then obtain

$$\lambda^{-1}+\lambda = 2H^{(3)} + fH + gH^{(2)}$$

$$\pm\sqrt{(fH-gH^{(2)})^2 - 4H^{(1)}((k^4+fg)H^{(1)} + gk^4H+fH^{(2)})} \quad .$$

5.2 The Strength of a Multi-Segment Transverse Resonator

The strength of a compound transverse resonator will now be computed by regarding it as made up of segments of thin rods and introducing a transfer matrix for each segment. For concreteness, we assume three segments (Fig. 5-1).

The x-axis is now taken along the resonator axis, and the displacement η is measured perpendicular to it. The transfer matrix for a segment of length L and wavenumber k will be written $M_k(L)$.

5.2.1 Junction Conditions

At the junction P_2 between segments 2 and 3, we apply the conditions of continuity of displacement and slope. We also equate the macroscopic moments and forces on opposite sides of P_2. Note that we cannot inquire into the microscopic shear-forces per unit cross-section area at J_2 without running into contradictions; thin-rod theory does not give an exact description very close to an abrupt change in cross-section. (See Section 5.5.2 for a more extensive discussion.) The transfer matrix J_2 that gives displacement and derivatives at the left side of P_2 in terms of those on the right is

$$J_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & r_2 & \\ & & & r_2 \end{pmatrix}$$

where

$$r_2 = \frac{E_2 I_2}{E_3 I_3} \quad .$$

The transfer matrix M from the right end of the outer segment to the left end of the inner one is then

$$M = M_{k_1}(L_1)\, J_1\, M_{k_2}(L_2)\, J_2\, M_{k_3}(L_3) \quad .$$

The boundary conditions at the outer (free) end are of course $\eta'' = \eta''' = 0$. At the inner (attached) end one can, for greatest simplicity, use the clamping conditions $\eta = \eta' = 0$. However, we shall discuss more general conditions.

5.2.2 Explicit Formulas for a Two-Segment Resonator

For each i, we break up $M_{k_i}(L_i)$ into quarters, with the notation

$$M_{k_i}(L_i) = \begin{pmatrix} U_i & V_i \\ W_i & X_i \end{pmatrix}$$

From the form given in Section 5.1.1, we have of course

$$X_i = U_i$$
$$W_i = k^4 V_i$$

The junction matrix J is similarly broken up into 2 x 2 blocks:

$$J = \begin{pmatrix} I & 0 \\ 0 & rI \end{pmatrix}$$

where I is the 2 x 2 unit matrix, and 0 the zero matrix. Then

$$M = M_{k_1}(L_1) J_1 \; M_{k_2}(L_2)$$

$$= \begin{pmatrix} U_1 \; U_2 + V_1 \; rW_2 & U_1 \; V_2 + V_1 \; rX_2 \\ W_1 \; U_2 + X_1 \; rW_2 & W_1 \; V_2 + X_1 \; rX_2 \end{pmatrix}$$

where now

$$r = \frac{E_2 I_2}{E_1 I_1} \quad .$$

Let Y_b be the vector whose components are the displacement and its derivatives at the base (the left end of segment 1) and Y_e the same for the outer free end. Then from the relation

$$Y_b = M \, Y_e$$

and the boundary conditions $\eta_e'' = \eta_e''' = 0$ for a free end, we obtain

$$\begin{pmatrix} \eta_b \\ \eta_b' \end{pmatrix} = (U_1U_2 + rV_1W_2) \begin{pmatrix} \eta_e \\ \eta_e' \end{pmatrix}$$

$$\begin{pmatrix} \eta_b'' \\ \eta_b''' \end{pmatrix} = (W_1U_2 + rX_1W_2) \begin{pmatrix} \eta_e \\ \eta_e' \end{pmatrix} ,$$

from which we have

$$\begin{pmatrix} \eta_b'' \\ \eta_b''' \end{pmatrix} = (W_1U_2 + rX_1W_2)(U_1U_2 + rV_1W_2)^{-1} \begin{pmatrix} \eta_b \\ \eta_b' \end{pmatrix} .$$

The strength of a transverse resonator was defined as the moment exerted in response to unit slope of the plate at the base point. If we assume that the resonator always meets the base at right angles, then η_b' equals this slope, and in any case, η_b'' is proportional to the moment exerted on the base. Thus to determine the strength, we must have an equation of the form

$$\eta_b'' = \text{const.}\ \eta_b' \quad .$$

Instead, we have two linear equations in four unknowns. In an elementary treatment, another equation is adjoined by assuming that the displacement η_b is very small, and so can be set equal to zero. Then η_b'' becomes proportional to η_b', and so the strength as a transverse resonator is well defined.

It may be noted that if one is concerned only with the condition for a resonance of the resonator, it is not necessary to introduce the assumption that η_b is negligible. The resonance condition is that a finite slope produces an infinite moment. This becomes

$$\det\ (U_1U_2 + rV_1W_2) = 0.$$

Note that η_b''' will also be infinite if this condition is satisfied. Thus at resonance there is infinite resistance to a non-zero slope at the base and also to displacement of the base point.

5.2.3 An Elastic Solid as Base

Bycroft (Proc. Roy Soc. 248A, p. 548,[1956]) has treated the problem of a rigid circular disc resting on a semi-infinte elastic solid or an elastic stratum, and subjected to oscillating forces of various kinds. In particular, he considers forces producing motion about a horizontal axis passing through the center of the disc, and traction forces parallel to the surface. From his analysis, one can determine two numbers e_1 and e_2 such that

$$\begin{pmatrix} \eta_b \\ \eta_b' \end{pmatrix} = \begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix} \begin{pmatrix} \eta_b'' \\ \eta_b''' \end{pmatrix} .$$

He remarks that the elements of the matrix that are here written as zero are actually non-zero, but are very small in comparison with e_1 and e_2. Using this new relation between the four derivatives, we can write

$$\begin{pmatrix} \eta_b'' \\ \eta_b''' \end{pmatrix} = (W_1U_2 + rW_1W_2)(U_1U_2 + rV_1W_2)^{-1} \begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix} \begin{pmatrix} \eta_b'' \\ \eta_b''' \end{pmatrix} ,$$

which is a homogeneous system of two linear equations in two unknowns. The condition for resonance is that there must be non-zero solutions, or:

$$\det \left\{ \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} (k_1^4V_1U_2 + rk_2^4U_1V_2) - (U_1U_2 + rk_2^4V_1V_2) \right\} = 0$$

Here the W_i and X_i have been replaced by the corresponding expressions in U_i and V_i, which are given explicitly at the beginning of 5.2.2.

Although the condition for resonance of a transverse resonator on an elastic layer is comparatively simple, the determination of the moment and force delivered to the interior of the layer is more complicated. This information is necessary if we wish to define the strength of the resonator on the elastic base. Another way of looking at the problem is this: The elastic material in the immediate neighborhood of the attachment area is in imagination separated from the rest of the base, and considered as some new kind of segment of the resonator, between the old segment 1 and what is now considered as a rigid base. In view of the complication, further results (which could be derived from Bycroft's work) have not been obtained.

5.2.4 Values for e_1 and e_2

Bycroft's formulas for e_1 and e_2 can be simplified to the following, for small radii a:

$$\eta' = \frac{0.187\ EI}{\mu a^3}\ \eta''$$

$$\eta = \frac{(3 + t^2)EI}{16\mu a}\ \eta'''$$

where E = elastic modulus of segment 1

I = moment of inertia of segment 1

a = radius of segment 1

$t^2 = \frac{\mu}{2\mu + \lambda}$

μ, λ = Lamé's constants for the elastic base

5.3 Transfer Matrix for a General Obstacle on a Rod

Transfer matrices can be defined for inhomogeneities in the structure of an infinite rod that are more general than resonators. One simple configuration is the following: An infinite bar of uniform thickness h has a step of height ah, where $0 < a < 1$, as shown in Figure 5-2. A similar configuration consists of a rod with two equal triangular notches on opposite sides, as in Fig. 5-3. This configuration will be called a <u>pinch</u>.

Both types of inhomogeneities have the common property that they can be considered as localized at a point. They are invariant under rotation of the figures through 180°, and they are both characterized by a single dimensionless number a which can range from zero to one. However, we shall not exclude other types of obstacles, in particular, a thickening of the rod over a finite length.

5.3.1 Existence of a Transfer Matrix for an Obstacle

Consider an obstacle applied at a point s, as for instance a step. Let r* be to the left of s, and r to the right. Let E(r*,r) be the transfer matrix from r* to r; that is:

$$\eta^{(i)}(r) = \sum_{j=0}^{3} E_{ij}(r^*,r)\,\eta^{(j)}(r^*) \qquad i = 0,1,2,3.$$

Let M(L) be the transfer matrix for a homogeneous rod segment of length L, and B(a) the transfer matrix that will represent the effect of the obstacle. Then

$$E(r^*,r) = M(s-r)B(a)M(r^*-s)$$

or

$$B(a) = M^{-1}(s-r)E(r^*,r)M^{-1}(r^*-s)$$

This can be regarded as the definition of B(a). E and M are physically well-defined; in order to be meaningful B(a) must not depend on the exact position of s between r* and r.

5.3.1.1 Need for Finite Separation Between Obstacles

The existence of a matrix B(a) for some obstacles depends on the fact that the problem has been linearized, and the thickness h is taken to be much smaller than any length of interest in the x-direction. Consider a rod with two steps of height ah and bh with $a + b < 1$, separated by a distance d (Fig. 5-4). Then the transfer matrix across the steps is B(b)M(d)B(a). Now suppose that one could allow d to go to zero. Then the two steps would merge into a single step of height a + b, and we would have

$$B(a + b) = B(b)B(a).$$

Any solution of this functional equation in matrices must have the form

$$B(a) = \exp aC \quad ,$$

where C is some constant matrix, and the exponential is defined by

$$\exp X = 1 + X + \frac{X^2}{2} + \frac{X^3}{3!} + \ldots$$

Now this solution will not behave in a physically acceptable manner as $a + b$ approaches 1, because in the limit the physical system consists of two detached semi-infinite rods. The root of the difficulty is that there are two limiting processes involved, namely $h \to 0$ and $d \to 0$, and it is not permissible to interchange the order; d must always be large compared to h.

5.3.1.2 Dimensionless Number a Close to Unity

If a is small for a step or pinch, one would equate displacements and slopes across the step: $B_{11}=1$, $B_{22}=1$. However if a is nearly equal to 1, it is not physically realistic to equate the slopes on opposite sides of a step, because there will be a hinge effect (see Figure 5-5). The displacements of the mid-lines can also not be equated, as there will be an oscillatory motion of the median line on one side relative to the other.

5.3.1.3 Distributed Obstacle

For an obstacle distributed between s_1 and s_2, such as a thickening (see Fig. 5-6), we define a transfer matrix B giving the derivative on the right of s_2 in terms of the derivatives on the left of s_1:

$$Y_R(s_2) = B_L Y\ (s_1)$$

Then we again have

$$E(r^*,r) = M(s_2-r)B\ M(r^*-s_1) \quad .$$

5.3.2 Consequences of Reciprocity

Certain properties of transfer matrices for concentrated or distributed obstacles can be deduced without any significant knowledge of the obstacle structure. That is, the special form of M(L) for a homogeneous rod and the reciprocity property of Green's functions lead to restrictions on the form of the transfer matrix.

5.3.2.1 Reflection in Minor Diagonal

The superscript S on a symbol for a matrix will be used to indicate reflection about its minor or non-principal diagonal. Then from the explicit form of $M(x-x_o)$ given in Section 5.1.1, we see that

$$M^S(L) = M(L)$$

A fundamental relation is

$$(C_1C_2)^S = C_2^S\, C_1^S$$

for two arbitrary 4 x 4 matrices C_1 and C_2. An easy way to verify this relation is to take the transpose (indicated by T) of both sides:

$$(C_1C_2)^{ST} = C_1^{ST}C_2^{ST} \quad .$$

Now the combined operation ST applied to a matrix is a reflection through the center. That is, the i-th row becomes the (5 - i)-th row, and similarly for the columns. Thus ST simply renumbers rows and columns in the same way, thus preserving the meaning of matrix multiplication. This establishes the last-written relation. On taking its transpose, we obtain the desired result.

5.3.2.2 Inverse of E

We now show that the transfer matrix E for a segment bearing inhomogeneities anywhere along its length satisfies the condition

$$E^{-1} = P\, E^{S}\, P$$

where P is the diagonal matrix with diagonal elements -1,1, -1,1. We first observe that

$$E^{-1}(r^*,r) = E(r,r^*) \quad .$$

Let D_r be the column vector whose elements are, in order, the differential operators 1, $\partial/\partial r$, $\partial^2/\partial r^2$, $\partial^3/\partial r^3$. Then there must be a row vector $\overline{e}$ of four functions:

$$\overline{e} = (e_3(r^*,r),\ e_2(r^*,r),\ e_1(r^*,r),\ e_o(r^*,r))$$

such that we can write, in operator form:

$$E(r^*,r) = D_r\, \overline{e} \quad .$$

Each $e_i(r^*,r)$, considered as a function of r, satisfies the rod equation of motion on both sides of the point $r = r^*$. At that point it has a discontinuity of magnitude + 1 in the derivative with respect to r of order 3 - i. Furthermore, e_i satisfies a boundary condition that there are no incoming waves at the ends of the segment. This condition makes each e_i unique, and then because of the relationship between the orders of the discontinuous derivatives we have:

$$e_i(r^*,r) = \frac{\partial^i}{\partial r^{*i}}\, e_o(r^*,r) \qquad i = 1,2,3.$$

Let Δ_{r^*} be the row vector of derivative operators with respect to r^*

$$(\partial^3/\partial r^{*3},\ \partial^2/\partial r^{*2},\ \partial/\partial r^*,\ 1) \quad .$$

Then

$$\overline{e} = \Delta_{r^*}\, e_o(r,r^*)$$

and so

$$E(r^*,r) = D_r \Delta_{r^*} e_o(r,r^*) \quad ,$$

where $D_r\Delta_{r*}$ is a 4 x 4 matrix of differential operators.

$$\frac{\partial^{i-j+3}}{\partial r^i \partial^{3-j} r*}$$.

(The indexing for the rows and columns starts from zero, rather than one.) In an analogous way, we find

$$E(r,r*) = D_{r*}\Delta_r \, e_o^*(r,r*) \quad ,$$

where e_o^* is the element in the upper right corner of $E(r,r*)$.

The equation

$$e_o^*(r,r*) = e_o(-r*, -r)$$

expresses the reciprocity property of the Green's function for a force applied at a point: the displacement at r due to a unit force applied at r* is equal to the displacement at r* due to a unit force applied at r(cf. Morse and Feshbach, Methods of Theoretical Physics, pp. 870-3). For obstacles such as pinches, steps or thickenings, which are exactly described by the full three-dimensional theory of elastic solids with self-adjoint boundary conditions, this reciprocity holds.

We observe that

$$(D_{r*}\Delta_r)^S = D_r\Delta_{r*} \quad .$$

Hence

$$E(r,r*) = (D_r\Delta_{r*})^S \, e_o(-r*, -r) \quad .$$

One can now verify, by writing out a typical differential operator explicitly, that the effect of the minus signs in the arguments of e_o will be to produce an alternating pattern of signs:

$$E(r,r*) = P[(D_r\Delta_{r*})^S \, e_o(r*,r)]P$$

and our desired result is proved.

5.3.2.3 Cross-Symmetry Property of an Obstacle Matrix B

It is clear from the form of M given in Section 5.1.1 that

$$M^S = M$$

and

$$M^{-1} = P\ M\ P \quad ,$$

so that

$$M = P(M^{-1})^S P$$

Then from the form for B(a) given at the beginning of 5.3.1, we find

$$\begin{aligned} B^{-1} &= M(r*-s)E^{-1}(r*,r)M(s-r) \\ &= P(M^{-1}(r*-s))^S P\ PE^S P\ P(M^{-1}(s-r))^S P \\ &= P(M^{-1}(s-r)E\ M^{-1}(r*-s))^S P \\ &= P\ B^S P \end{aligned}$$

5.3.2.4 Determinant of B

Taking determinants of both sides, and noting that det P = 1, we find det B = $(\det B)^{-1}$, that is, det B = +1 or -1. A continuity argument can be used to show that the plus sign must always be taken. This choice is certainly correct for a rod without any obstacle. Given a physical obstacle, such as a step of height $a_o h$, there is a family of obstacles (in this case, steps with values of a less than a_o) which form a continuous transition to the homogeneous rod. Clearly det B will not have a discontinuity for such a series. But it can only take on one of the values +1 or -1. Thus we have in general

$$\det B = 1.$$

5.3.3 Symmetric Obstacles

If an obstacle has right-left symmetry, there is an additional relation

$$B^{-1} = P\ B\ P \quad .$$

This can be seen by considering the effect of changing the sign of the length coordinate along the rod. This will not effect the displacement η and its second derivative, but will change the signs of first and third derivative.

Combining with the relation of 5.3.2.3, we obtain

$$B^S = B \quad .$$

Thus a symmetric obstacle has a matrix which is symmetric about its minor diagonal.

This shows that at most 10 of the elements of B are distinct. There is furthermore the relation det B = 1, so that at most 9 of the 16 (real) elements of B can be specified independently.

5.4 The Wave Basis for Transfer Matrices

We can also define a different type of transfer matrix that relates the magnitudes of the incoming and outgoing waves at one end of a rod to similar quantities at the other end. In mathematical terminology, the new matrix will be the result of applying a similarity transformation to the transfer matrix as originally defined in Section 5.1.

5.4.1 Waves on a Rod

We introduce the standard complex representation for waves on a rod. The four linearly independent solutions of the equation of motion of a homogeneous rod will be taken as (when the time dependence $e^{-i\omega t}$ is included):

$$\eta_1 = e^{i(kx-\omega t)}$$
$$\eta_2 = e^{-kx-i\omega t}$$
$$\eta_3 = e^{i(-kx-\omega t)}$$
$$\eta_4 = e^{kx-i\omega t} \quad .$$

The collection of functions (η_1, η_2, η_3, η_4) will be called the wave basis for describing the motion.

Suppose that in a neighborhood of the point x, the rod displacement η is given by

$$\eta = a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + a_4 \eta_4.$$

Then η is of course specified near x by the vector $A = (a_1,a_2,a_3,a_4)$. We have previously introduced the vector Y of derivatives of η at x:

$$Y(x) = (\eta(x), \eta^{(1)}(x), \eta^{(2)}(x), \eta^{(3)}(x)) \quad .$$

Clearly there will be a 4 x 4 matrix C that produces Y when applied to A:

$$Y(x) = C\,A(x) \quad ,$$

and C will be independent of x for a homogeneous rod, finite or infinite.

From the definition of the η_i, the form of C can be written down immediately:

$$C = K\,S \quad ,$$

where

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k^2 & 0 \\ 0 & 0 & 0 & k^3 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & -i & -i & 1 \\ -1 & 1 & -1 & 1 \\ -i & -1 & i & 1 \end{pmatrix}$$

Clearly, this use of the letter S can always be distinguished from the earlier use as a superscript. It is easy to verify that

$$S^{-1} = \frac{1}{4} S^H$$

where S^H is the Hermitian transpose of S, i.e., the transpose after replacing each element by its complex conjugate. Thus

$$C^{-1} = \frac{1}{4} S^H K^{-1}$$

A transfer matrix E as previously defined for a rod with inhomogeneities will now be said to be appropriate to the <u>derivative basis</u> $(\eta, \eta^{(1)}, \eta^{(2)}, \eta^{(3)})$. To find the transfer matrix E^* appropriate to the new or wave basis, we write

$$Y_L = CA_L$$
$$Y_R = CA_R$$
$$Y_R = EY_L$$

from which follows

$$A_R = C^{-1}E\,CA_L$$

Therefore $E^* = C^{-1}EC$.

The special use of M(L) to denote the transfer matrix (in the derivative basis) for a rod of length L will be extended by writing $M^*(L)$ for the corresponding matrix in the wave basis. It is clear that M^* is diagonal:

$$M^*(L) = \begin{bmatrix} e^{ikL} & & & \\ & e^{-kL} & & \\ & & e^{-ikL} & \\ & & & e^{kL} \end{bmatrix}$$

which is considerably simpler than the form for M given in Section 5.1.1. On the other hand, the transfer matrix for a very short segment bearing an axial or a transverse resonator is simple in the derivative basis, but more complicated in the wave basis.

If one wishes to talk about the transfer properties of a rod, without specifying a particular basis, then properly one should adopt the terminology of other branches of physics, and speak of a transfer <u>tensor</u>.

5.4.2 Sources in Derivative and Wave Bases

It will be convenient to carry over some terminology from electric circuit theory, and say that an obstacle or inhomogeneity is a passive element. There are then two physical kinds of active elements - very short segments to which either a driving force is applied, or else a driving moment.

A source is represented in the derivative basis by a vector with four components which give the increases in the displacement and its first three derivatives across the point of application. For an oscillating force of strength F, this vector is (0, 0, 0, F/D), where D is the flexural rigidity. For an applied moment of strength G, the source vector is (0, 0, G/D, 0).

The vectors for these sources in the wave basis have nonzero elements in all four positions, and it may be more convenient to think of two kinds of ideal sources, which are linear combinations with complex coefficients of the two types of physical sources. Thus one can introduce a source that produces pure outgoing oscillatory motion on the left and standing waves decaying with distance on the right. This can be realized by superimposing physical force and moment sources with the appropriate phase relation. A second source type arises from the reversal of right and left in the above description.

It is clear why there are only two types of physical sources, and not four. The missing two would be sinks of incoming waves, and "sources" of standing waves that increase exponentially with the distance. Such behavior is excluded by the boundary conditions.

5.4.3 Transmission and Reflection Matrices

Introduction of the wave basis allows the definition of certain 2 x 2 matrices that characterize the transmission and reflection properties of finite segments of rods bearing resonators or other obstacles. As the exponentially decaying or growing waves η_2 and η_4 present a minor difficulty, we shall first consider a simpler problem, a uniform string under tension. Then only the oscillatory solutions, corresponding to η_1 and η_3, will exist. In this case, the 2 x 2 matrices reduce to numbers.

Consider a wave on an infinite string coming in from the left. We fix our attention on a very small portion of the string, on which a weight or another "obstacle" is attached. (These behave analogously to resonators on a rod.) The obstacle will act as a source of outgoing waves, but there is another way of describing the situation. Part of the incoming wave will be reflected at the obstacle, and part transmitted across the obstacle. There will also be phase changes. If the obstacle is symmetrical between right and left (as is true for a weight) then the transmission and reflection properties will not depend on whether the wave is incoming from the right or left. Thus the effect of the obstacle is completely specified by a transmission coefficient T and a reflection coefficient R. These are usually complex numbers (to take account of the phase shift). If the obstacle is unsymmetric, then there are different coefficients for waves coming in from the right and the left.

We now return to the problem of an infinite rod bearing an obstacle concentrated at a point. We must allow as "waves" incoming from the left not only η_1, a true oscillatory motion, but also η_2, which increases exponentially to the left. η_2 would be produced by a source to the left of the obstacle. On the other hand, η_4 is not allowed, since it could be produced only by a source to the right. Similarly, we allow η_3 and η_4 as "waves" incoming from the right. The obstacle or resonator will act as a source of outgoing waves only. To the left of the obstacle, these are given by η_3 and η_4. Thus transmission or reflection by the obstacle can be specified by a 2 x 2 matrix T or R with complex elements. T indicates how the two kinds of "incoming waves" on the left of the obstacle are transmuted into the two kinds of "outgoing waves" to the right of the obstacle. If the obstacle is not symmetric between right and left, then there will be a different transmission matrix T' for waves incoming from the right. Similarly for R and R'.

Suppose the rod can be described immediately to the right and to the left of the obstacle by the vectors A_R and A_L, where

$$A_R = (a_{R1}, a_{R2}, a_{R3}, a_{R4})$$

and A_L is analogous. Then the precise definitions of T and R are:

$$\begin{pmatrix} a_{R1} \\ a_{R2} \end{pmatrix} = T \begin{pmatrix} a_{L1} \\ a_{L2} \end{pmatrix} \qquad \begin{pmatrix} a_{L3} \\ a_{L4} \end{pmatrix} = R \begin{pmatrix} a_{L1} \\ a_{L2} \end{pmatrix}$$

Similarly for waves coming in from the right, we have:

$$\begin{pmatrix} a_{L3} \\ a_{L4} \end{pmatrix} = T' \begin{pmatrix} a_{R3} \\ a_{R4} \end{pmatrix} \qquad \begin{pmatrix} a_{R1} \\ a_{R2} \end{pmatrix} = R' \begin{pmatrix} a_{R3} \\ a_{R4} \end{pmatrix}$$

It can be shown (see Addendum 4) that the 4 x 4 transfer matrix B in the wave basis can be expressed in terms of T, T', R, and R' as a partitioned matrix:

$$B^* = \begin{pmatrix} T - R'T'^{-1}R & R'T'^{-1} \\ -T'^{-1}R & T'^{-1} \end{pmatrix}$$

The inverse of B* is then

$$B^{*-1} = \begin{pmatrix} T^{-1} & -T^{-1}R' \\ RT^{-1} & T'-RT^{-1}R' \end{pmatrix}$$

For a symmetric obstacle, there are many relations connecting the 8 complex elements of T and R, since it was shown that 9 real quantities are sufficient to specify B in this case.

5.4.4 Source and Obstacles on an Infinite Rod

Knowledge of the transfer matrix for a resonator and the jump-vector for a source are not sufficient in themselves to determine the displacement of a point on an infinite rod to which these are applied. The boundary conditions of finite displacement everywhere, and no incoming disturbance at either end, must be used to pick out the physically allowed solution of the equations of motion. These conditions are easy to express in the wave basis, but complicated in the derivative basis.

Let E_o^* be the transfer matrix for a segment that has only passive elements. If homogeneous segments of length L are attached both at the left and right, then the transfer matrix for the combination is of course $M^*(L)E_o^*\ M^*(L)$. Now imagine that L goes to infinity, and apply the boundary conditions. In the absence of a source anywhere along the infinite length, the displacement must be zero everywhere. (Free standing waves of an infinite homogeneous rod are excluded because in terms of the wave basis they must be considered as the sum of right- and left-going waves of equal amplitudes, and the traveling waves are incoming either on the right or the left.)

To obtain a non-zero displacement, an active element must be inserted somewhere along the rod. Consider that a source (say a force) is applied at the left end r^* of the original rod segment (see Fig. 5-7). Let $A_L(r^*)$, $A_R(r^*)$ be the vectors of wave-coefficients appropriate to the left and the right sides of the point r^*. Since there are no incoming waves immediately to the left of the source, $A_L(r^*)$ must have the form $(0,0,w,x)$ where w and x are not known. Immediately to the right of the segment's right end, at p, there are again no incoming waves. Hence the wave-coefficient vector $A(p)$ has the form $(s,t,0,0)$ where s and t are unknown. But by the definition of transfer matrix,

$$A(p) = E_o^*\ A_R(r^*) \quad .$$

Now the vector difference $A_R - A_L$ is the jump-vector for the source; it follows that

$$E_o^{*-1}\begin{pmatrix} s \\ t \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ w \\ x \end{pmatrix} = C^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \\ F/D \end{pmatrix}$$

For a moment source, the right-hand is changed suitably.

If we introduce the diagonal matrices I_o and I^o, with diagonal elements 1, 1, 0, 0 and 0, 0, 1, 1 respectively, then the last equation can be written

$$(E_o^{*-1} I_o - I^o) \begin{pmatrix} s \\ t \\ w \\ x \end{pmatrix} = (F/D)C^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and s, t, w, x can be found by solving a system of four linear equations. Once these are known, the displacement of the rod can be determined at each point outside the original segment. The displacement will of course be proportional to the applied force F.

5.4.5 Explicit Computation

If there is only one resonator on an infinite homogeneous rod, explicit formulas can easily be found using the transmission and reflection matrices defined in Section 5.4.3.

Let there be a source at r*, and a resonator at s, to the right of r*. We shall determine the displacement at a point r to the right of s. Consider a wave from r* incident on s from the left. The reflected wave produced at s will never undergo another reflection as it travels back to the left, because the source at r* does not reflect. Therefore, to determine what is transmitted to a point immediately to the right of s, we need only know the transmission matrix for the segment between r* and s, and the transmission matrix T_o across s. Thus the transmission matrix between r* and r is

$$T(r^*,r) = T(s-r)\,T_o\,T(s-r^*) \quad ,$$

where

$$T(L) = \begin{pmatrix} e^{ikL} & 0 \\ 0 & e^{-kL} \end{pmatrix}$$

T_o will now be computed by making use of the form given at the end of 5.4.3. This shows that the transmission matrix for a symmetric obstacle can be found from the 4 x 4 wave-transfer matrix B* by extracting the lower right 2 x 2 matrix and inverting. The transfer matrix for an axial resonator in the derivative basis was given explicitly as M_{ax} in Section 5.1.1. In the wave basis, it becomes (see Section 5.4.1)

$$M^*_{ax} = C^{-1} M_{ax} C = \frac{1}{4} S^H K^{-1} M_{ax} K S$$

If 0 and I represent the zero and unit matrics of order 2 x 2, then the extraction of the lower right quarter can be represented as

$$\frac{1}{4} (0 \quad I) S^H K^{-1} M_{ax} K S \begin{pmatrix} 0 \\ I \end{pmatrix}$$

We find that this is a 2 x 2 matrix of the form

$$I - F X ,$$

where X is a 2 x 2 matrix not depending on F. T_o is then found by taking the inverse:

$$T_o = (I - FX)^{-1} = I - X(IF^{-1} + X)^{-1}$$

and then we find

$$T(r^*,r) = \begin{pmatrix} e^{ik(r-r^*)} & 0 \\ 0 & e^{-k(r-r^*)} \end{pmatrix}$$

$$- \begin{pmatrix} e^{ik(r-s)} & 0 \\ 0 & e^{-k(r-s)} \end{pmatrix} X(F^{-1}I+X)^{-1} \begin{pmatrix} e^{ik(s-r^*)} & 0 \\ 0 & e^{-k(s-r^*)} \end{pmatrix}$$

In Sections 3 and 4, we were concerned with a Green's function, that is, the displacement at r due to a force applied at r^*, in the presence of a resonator at s. The matrix $T(r^*,r)$ gives more information, inasmuch as the coefficient at r of the propagating wave appears separately from the coefficient of the decaying or evanescent "wave," and these are given for independent excitation of the two types at r^*. (The resonator at s responds to both kinds, and produces new propagating and evanescent waves, so that the total displacement at r can be visualized as a sum of four terms.) Thus the Green's function originally found in Section 3 can be recovered by adding together the elements of $T(r^*,r)$ using appropriate coefficients.

It is possible to extend the computation just given to a rod with several resonators between r^* and r, but the algebra becomes very tedious.

5.5 Thickenings On a Rod

5.5.1 Methods for Treating Thickened Portions

Finite-length thickenings can be handled by at least three different methods. In Section 5.5.4, transfer matrices are used. In Section 6.2.1, a new kind of non-physical resonator will be introduced, which does not satisfy the reciprocity relation. By also introducing a change of length-scale, comparatively simple formulas for thickenings are obtained in 6.2.2. A third method considers the thickening as the result of superimposing infinitely many resonators with orthogonal weight-functions. It can be used for thickenings on a plate bounding a semi-infinite ocean of water, for which problem the first two methods fail. It is unneccessarily complicated in the absence of water, and so the presentation will be delayed until Section 9.

5.5.2 Junction Between Dissimilar Segments

Consider two segments of different D and k joined together as in Fig. 5-8. A precise treatment of conditions near the junction would involve the exact theory of elasticity. However, we shall only need the accuracy that is consistent with the use of thin-rod theory away away from the junction.

The conditions that displacement and slope are unchanged across the junction can be written

$$\eta_L = \eta_R$$

$$\left.\frac{d\eta}{dx}\right]_L = \left.\frac{d\eta}{dx}\right]_R$$

From elementary physical arguments, the total transverse shear force and the total moment must match across s, whether or not the simplifications of thin-rod theory can be made. An argument from equality of the total moments to a relation between the second derivatives on opposite sides of s must be given, since thin-rod theory does not hold near a change in cross-section. Thus thin-rod theory could not predict any difference between a symmetrical junction (Fig. 5-8), and an unsymmetrical junction (Fig. 5-9), for equal thicknesses in the two figures.

The general nature of such a justification can be understood by going over to the higher dimensional analog. A semi-infinite rod then becomes an elastic layer that is semi-infinite in its own plane. Now consider two of these having different thicknesses, attached to each other along their edges. When the junction is examined microscopically, the forces on opposite sides must be equal at each point, although the magnitude varies with distance from the median plane. Lamb waves of all modes can be excited in each semi-infinite layer by shear force and moment transmitted across the junction, but all except two modes in each half-layer decay with distance very rapidly - these two being the "outgoing waves" discussed in Section 5.4. Now we return to the lower dimensional analog. There will be a point p to the left of the junction s such that the distance $|p - s|$ is very small compared to a free wavelength, and $D \frac{\partial^3 \eta}{\partial x^3}$ evaluated at p is nearly equal to the shear force across the junction. There is a similar point p' on the other side of s; and the same properties hold for the moments at p and p'.

5.5.3 Transfer Matrix Across a Junction

If two points r_1 and r_2 are on the same side of the junction point s, then transfer matrices can be defined as earlier in either derivative or wave basis (but now the wave basis on the left involves the appropriate wavenumber k_L, and on the right k_R is involved). If r_1 and r_2 are on opposite sides of the junction, it will clearly be simpler to use the derivative basis, in view of the change in wavenumber. Thus we return to the formulation of Section 5.2.1. If the ratio

$$\gamma = D_L/D_R$$

is introduced, and the transfer matrix Γ for the junction is defined by

$$Y_R = \Gamma \, Y_L \quad ,$$

then on equating force and moment on opposite sides, we have

$$\Gamma = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \gamma & \\ & & & \gamma \end{bmatrix} .$$

5.5.4 Source and Finite Thickening Handled by Transfer Matrices

Consider a source at r^* on a bar thickened from s_1 to s_2 (Fig. 5-10). The finite section can actually be thinner than the rest of the rod. Then one speaks of a neck in the rod. The subscript b will be used for quantities pertaining to the finite section and a for the rest of the rod. Let r be an observation point on the other side of the thickening from r^*. The transfer matrix $E(r^*,r)$ between r^* and r can be written (in the derivative basis)

$$E(r^*,r) = M_a(s_2-r)\Gamma^{-1} M_b(L)\Gamma\, M_a(r^*-s_1)$$

where $L = |s_2-s_1|$. We now set

$$B_o = \Gamma^{-1} M_b\Gamma \quad .$$

B_o behaves as the transfer matrix for an obstacle on an infinite homogeneous rod. Therefore it must satisfy all the conditions derived for such an obstacle. In particular, $B_o^S = B_o$, since the "obstacle" is symmetric. B_o must also be related to 2 x 2 transmission and reflection matrices.

Invoking the same argument as was used in Section 5.4.5, the transmission matrix from r^* to r is the product of the transmission matrices for the two segments and the obstacle.

$$T(r^*,r) = T(r-s)T_o\, T(s-r^*) \quad .$$

The outer factors were evaluated earlier; they are given by the formula

$$T(x) = T_a(x) = \begin{pmatrix} \exp\, ik_a x & 0 \\ 0 & \exp\, -k_a x \end{pmatrix}$$

The transmission matrix T_o for the obstacle is found by converting B_o to the wave basis (wavenumber k_a) extracting the lower right 2 x 2 matrix, and inverting. Using * to indicate the wave basis, one obtains

$$B_o^* = C^{-1} B_o C$$

$$= C^{-1} \Gamma^{-1} C\, C^{-1} M_b(L) C\, C^{-1} \Gamma C$$

$$= S^{-1} K_a^{-1} \Gamma^{-1} K_a S\, M_b^*(L) S^{-1} K_a^{-1} \Gamma K_a S$$

$M^*(L)$ was given Section 5.4.1.

K_a and Γ are both diagonal, hence they commute with each other. Therefore

$$B_o^* = G^{-1} M_b^*(L)\; G$$

where

$$G(\gamma) = S^{-1} \Gamma(\gamma)\; S = \frac{1}{4} S^H \Gamma(\gamma)\; S \quad .$$

Note that G is Hermitian, and that

$$G^{-1}(\gamma) = S^{-1} \Gamma^{-1}(\gamma)\; S = G(\gamma^{-1}) \quad .$$

Adding and subtracting 1 from the third and fourth elements on the diagonal of G, we find, in terms of 2 x 2 blocks,

$$G = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \frac{\gamma-1}{4} S^H \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} S$$

Extraction of the lower right quarter of B_o^* corresponds to forming

$$T_o^{-1} = (0 \quad I) B_o^* \begin{pmatrix} 0 \\ I \end{pmatrix} = \left[(0 \quad I)\; G^{-1}\right] M_b^* \left[G\begin{pmatrix} 0 \\ I \end{pmatrix}\right] .$$

Setting for the moment

$$S = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \qquad S^H = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

we find that

$$G \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} + \frac{\gamma-1}{4} \begin{pmatrix} qz \\ sz \end{pmatrix} = \frac{1+\gamma}{2} \begin{pmatrix} 0 \\ I \end{pmatrix} + \frac{\gamma-1}{4} \begin{pmatrix} U \\ U^T \end{pmatrix}$$

where

$$U = \begin{pmatrix} 0 & -1 + i \\ -1 - i & 0 \end{pmatrix}$$

and U^T is the transpose of U. Furthermore

$$(0 \quad I)\, G^{-1} = \frac{1+\gamma}{2\gamma} (0 \quad I) + \frac{1-\gamma}{2} (U \quad U^T) \quad .$$

M_b^* can be written in terms of 2 x 2 blocks as

$$M_b^* = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix}$$

where

$$M_1 = \begin{pmatrix} \exp\ ik_b L & 0 \\ 0 & \exp\ -k_b L \end{pmatrix}$$

$$M_4 = \begin{pmatrix} \exp\ -ik_b L & 0 \\ 0 & \exp\ k_b L \end{pmatrix} \quad .$$

Then

$$T_o^{-1} = \frac{(1-\gamma)^2}{16\gamma}\, U\, M_1\, U$$

$$+ \frac{1}{16\gamma} \left(2(1+\gamma) I + (1-\gamma) U^T\right) M_4 \left(2(1+\gamma) I - (1-\gamma) U^T\right)$$

Then we find the following explicit forms for the elements of $8\gamma T_o^{-1}$, after dropping the subscript b on k:

(1,1) element:

$$-(1-\gamma)^2 e^{-kL} + 2(1+\gamma)^2 e^{-ikL} - (1-\gamma)^2 e^{kL}$$

(1,2) element:

$$(-1-i)(1-\gamma^2)(e^{kL} - e^{-ikL})$$

(2,1) element:

$$(+1-i)(1-\gamma^2)(e^{kL} - e^{-ikL})$$

(2,2) element:

$$-(1-\gamma)^2 e^{ikL} + 2(1+\gamma)^2 e^{kL} - (1-\gamma)^2 e^{-ikL} .$$

6. BOUNDARY CONDITIONS ON A ROD OR PLATE HANDLED BY MEANS OF RESONATORS

6.1 Standard Boundary Conditions for a Rod

There are four standard boundary-value conditions that can be imposed at a point of a rod:

(1) Simple support at an internal point.
(2) Simple support at an end.
(3) Clamping at an end.
(4) A free end.

We shall show that each of these conditions can be produced by attaching one or two resonators - not necessarily of the types that have been already discussed - to a two-way infinite rod, and then letting the strength of the resonators go to infinity. Thus a problem involving boundary conditions at several points of a rod, with any number of axial or transverse resonators between them, can be handled by a uniform technique which treats the boundary conditions as simply due to the effects of additional resonators.

Problem (2), end support, has a well-known elementary solution, obtained by introducing a negative mirror-image source for each physical source on the rod. It is important to observe that (2) is not the same problem as (1), internal support. This arises because the slope must be continuous at a support at an internal point of a rod. Then two points on the same side of the support point can not only influence each other directly, but also by a route that, so to speak, crosses to the other side and returns.

In the problems with clamped and free ends we artificially introduce another half-rod on the other side of the end. Clearly two half-infinite bars with abutting free ends will not influence each other. A source applied to one will produce zero displacement for the other rod. Similarly, a two-way infinite bar clamped at a point behaves effectively as two independent clamped semi-infinite bars since a force applied on one half has no effect on the other.

6.1.1 Internal Simple Support

It is clear that if one attaches an axial resonator of infinite strength at the point s on an infinite rod, then the displacement is constrained to be zero. The Green's function (displacement at r due to a source at r*) is obtained by letting 1/F go to zero in our earlier formulas:

$$G(r^*,r) = Q(r^* - r) - \frac{Q(r^*-s)Q(s-r)}{Q(0)}$$

Clearly $G(r^*,s) = 0$.

6.1.2 Infinite Rod Clamped at a Point

Consider an infinite clamped at the point s. Then we must have

$$\eta(s) = 0 \quad .$$

$$\frac{d\eta(s)}{dx} = 0 \quad .$$

The condition $\eta = 0$ can be realized by attaching an axial resonator of infinite strength at s. Similarly, the condition $d\eta/dx = 0$ is realized by attaching an infinitely strong transverse resonator. The interaction matrix V_{pq} is then

$$\lim_{\substack{F \to \infty \\ G \to \infty}} \begin{bmatrix} Q(0) + \frac{1}{F} & 0 \\ 0 & -Q^{(2)}(0) + \frac{1}{G} \end{bmatrix}$$

The off-diagonal terms are zero because the two types do not interact if they are attached at the same point; the axial resonator has a Green's function that is an even function of the distance, and the other, an odd function. Therefore, the Green's function for the rod clamped at s is:

$$G(r^*,r) = Q(r^*-r) - \frac{Q(r^*-s)Q(s-r)}{Q(0)} - \frac{Q^{(1)}(r^*-s)Q^{(1)}(s-r)}{Q^{(2)}(0)}$$

One can easily verify that $\eta = 0$ and $d\eta/dx = 0$ as x approaches s on either side, and that G = 0 if s lies between r* and r.

If an infinite rod is clamped at s, then the two halves behave independently of each other; a source on one side produces no displacement on the other side. Therefore the treatment can be considered as applying to a semi-infinite rod clamped at its end.

6.1.3 <u>A Rod Clamped at Two Points; Free Vibrations</u>

The treatment for two clamped points is obvious. the interaction matrix V is now 4 x 4 and not diagonal.

$$V = \begin{pmatrix} Q(0) & 0 & & \Delta^* \\ 0 & -Q^{(2)}(0) & & \\ & & Q(0) & 0 \\ & \Delta & 0 & -Q^{(2)}(0) \end{pmatrix},$$

where

$$\Delta = \begin{pmatrix} Q(s_2-s_1) & Q^{(1)}(s_2-s_1) \\ -Q^{(1)}(s_2-s_1) & -Q^{(2)}(s_2-s_1) \end{pmatrix},$$

and Δ^* differs from Δ by interchange of the signs of the off-diagonal elements.

Then

$$G(r^*,r) = Q(r^*,r) - RV^{-1}C,$$

where R is the row vector

$$R = \left| Q(r^* - s_1),\ -Q^{(1)}(r^* - s_1),\ Q(r^* - s_2),\ -Q^{(1)}(r^* - s_2) \right|$$

and C is the column vector with elements $Q(s_1 - r)$, $Q^{(1)}(s_1 - r)$, $Q(s_2 - r)$, $Q^{(1)}(s_2 - r)$. Note that two elements of R have negative signs, because all quantities for a moment resonator are derived from those for a force resonator by applying the operator d/ds_i. It is now convenient to reintroduce the diagonal matrix P with elements -1, 1, -1, 1. Then

$$RV^{-1}C = -RP(-VP)^{-1}C$$

and -VP has the following form, when partitioned into quarters:

$$-VP = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$$

with

$$X = \begin{bmatrix} Q(0) & 0 \\ 0 & Q^{(2)}(0) \end{bmatrix},\quad Y = \begin{bmatrix} Q(s_2 - s_1) & Q^{(1)}(s_2 - s_1) \\ Q^{(1)}(s_2 - s_1) & Q^{(2)}(s_2 - s_1) \end{bmatrix}$$

Then $(VP)^{-1}$ can be conveniently determined from the relation, valid for any square matrices A and B, with A non-singular:

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}^{-1} = \left(I - (A^{-1}B)^2\right)^{-1} \begin{bmatrix} A^{-1} & -A^{-1}BA^{-1} \\ -A^{-1}BA^{-1} & A^{-1} \end{bmatrix}$$

where I is the unit matrix of the same size as A or B. This relation can be verified by direct multiplication.

These formulas can be used to determine the conditions for free vibrations of the rod. It will be recalled that a mechanical structure may have frequencies such that an arbitrarily small applied force will produce finite displacements. Of course, the energy for these displacements is the result of the summation of the very small increments put into the system during each of the infinitely many earlier cycles of the force.

If the structure is idealized by assuming that the displacement is always exactly proportional to the applied force, then the condition for the existence of free vibrations is that there is an infinite displacement in response to a finite force. This linearization assumption has been made for our thin rod. Then from the form of $G(r^*,r)$, it is clear that a free vibration can take place only if V is a singular matrix, that is, if det V = 0, or equivalently det(VP) = 0.

In view of the last equation, this implies

$$\det\left|I - (X^{-1}Y)^2\right| = \det(I - X^{-1}Y)\det(I + X^{-1}Y) = 0$$

Thus one of the factors on the right must be zero. Then multiplying by $\det X^{-1}$, it is clear that either

$$\det(Y - X) = 0$$

or

$$\det(Y + X) = 0$$

On substituting the values of X and Y, and setting $d = s_s - s_1$, we obtain

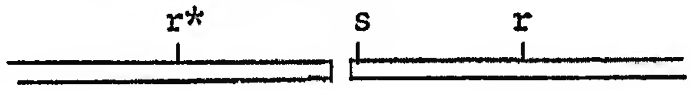
$$\det\begin{pmatrix} Q(d) \pm Q(0) & Q^{(1)}(d) \\ Q^{(1)}(d) & Q^{(2)}(d) \pm Q(0) \end{pmatrix} = 0$$

or

$$Q(d)\,Q^{(2)}(d) - \left|Q^{(1)}(d)\right|^2$$
$$\pm Q(0)\left[Q(d) + Q^{(2)}(d)\right] + \left|Q(0)\right|^2 = 0$$

The equations for the two choices of signs correspond to modes of vibration that are symmetrical and antisymmetrical about the midpoint of the segment.

6.1.4 <u>A Break in an Infinite Rod; Non-Physical Resonators</u>



The conditions for a break at s are:

$$\frac{d^2\eta}{dx^2} = 0$$

$$\frac{d^3\eta}{dx^3} = 0 \quad ,$$

on both sides of s. Of course, η and $d\eta/dx$ need not be continuous across s.

6.1.4.1 <u>Transfer Matrix Across s</u>

By analogy with the results for clamping at s, it would be expected that the transfer matrix across the break is realizable only as a limit after certain elements are allowed to become infinite. Now the transfer matrix for clamping at s could be visualized as

$$\lim_{\substack{F \to \infty \\ G \to \infty}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & G & 1 & 0 \\ F & 0 & 0 & 1 \end{pmatrix}$$

This suggests that the transfer matrix for a break should be taken as:

$$\lim_{\substack{H \to \infty \\ J \to \infty}} T_B$$

where

$$T_B = \begin{pmatrix} 1 & 0 & 0 & J \\ 0 & 1 & H & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It will now be verified that the conditions at s will be satisfied using T_B. We recall that the vector of derivatives on the right Y_R is related to that for the left Y_L by

$$Y_R = T_B \, Y_L$$

The upper row of this relation yields

$$\eta_R = \eta_L + J\eta_L^{(3)}$$

Now all this solutions of the wave equation have finite displacement at finite points. Hence if $J \to \infty$, it must be that $\eta_L^{(3)} \to 0$. Similarly, from

$$\eta_R^{(1)} = \eta_L^{(1)} + H\eta_L^{(2)}$$

we deduce that $\eta_L^{(2)} \to 0$ when $H \to \infty$.

Further, Y_L can be expressed in terms of Y_R:

$$Y_L = T_B^{-1} Y_R \quad ,$$

where

$$T_B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -J \\ 0 & 1 & -H & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

It follows that

$$\eta_R^{(3)} \rightarrow 0 \qquad \text{as } J \rightarrow \infty$$

$$\eta_R^{(2)} \rightarrow 0 \qquad \text{as } H \rightarrow \infty$$

6.1.4.2 New Kinds of Resonators

One can interprete T_B as the transfer matrix that results from superimposing two unusual kinds of resonators, one of which contributes the element H and the other, element J. Neither corresponds to a physical object. H would be the strength of a resonator which responds to a non-zero second derivative of η by producing a finite discontinuity in the slope. It can be visualized as the result of allowing two transverse resonators of the same strength but opposite orientations to approach each other, at the same time as the common strength goes to infinity in such a way that the produce of distance and strength remains constant. If one thinks of a moment applied to the rod at a point as a dipole of force, then one can speak of a transverse resonator as a dipole resonator. The new kind of object that produces the H term is then a quadrupole resonator. In the same way, the entity that produces the J element responds to a non-zero third derivative by producing a finite discontinuity in the displacement, and can be called an octupole resonator.

Attaching a quadrupole resonator of infinite strength at s corresponds to breaking the bar at s and reconnecting the ends with a hinge. There is no simple interpretation for an infinite octupole resonator.

6.1.4.3 Green's Functions and Interaction Constants

Once one accepts these ideal resonators, it becomes possible to define Green's functions for each of them. It is clear that for a quadrupole resonator, this must be $Q^{(2)}(x)$, where x is the distance between source and object, and for an octupole resonator, it is $Q^{(3)}(x)$. The interaction integrals can be evaluated in the general form:

$$V_{pq} = \int Q^{(p)}(x-s_1)\delta^{(q)}(s_2-x)\,dx$$
$$= Q^{(p+q)}(s_2-s_1) \quad .$$

There is an ambiguity of sign in this formula if the resonators are attached at the same point ($s_1 = s_2$) and p+q is odd. But if p+q = 1 or 5, then $V_{pq} = 0$. The case p+q = 3 presents real difficulties of physical interpretation (see the discussion in 6.1.7).

The Green's function for a break at s is then

$$G(r^*,r) = Q(r^*-r) - \frac{Q^{(2)}(r^*-s)Q^{(2)}(s-r)}{Q^{(4)}(0)}$$
$$- \frac{Q^{(3)}(r^*-s)Q^{(3)}(s-r)}{Q^{(6)}(0)}$$

Of course, we have

$$Q^{(4)}(0) = k^4Q(0)$$

$$Q^{(6)}(0) = k^4Q^{(2)}(0) \quad .$$

6.1.5 A Rod of Finite Length

We conceive of a finite rod as arising from an infinite rod by breaks at the points s_1 and s_2, where $|s_1-s_2|$ equals the rod length L. Let a force be applied at r*, which as a matter of fact need not lie between s_1 and s_2. Then the Green's function is

$$G(r^*,r) = Q(r^*-r)$$

$$- \left(Q^{(2)}(r^*-s_1) - Q^{(3)}(r^*-s_1) \quad Q^{(2)}(r^*-s_2) - Q^{(3)}(r^*-s_2)\right)$$

$$\times V^{-1} \times \begin{pmatrix} Q^{(2)}(s_1-r) \\ Q^{(3)}(s_1-r) \\ Q^{(2)}(s_2-r) \\ Q^{(3)}(s_2-r) \end{pmatrix}$$

where

$$V = \begin{bmatrix} Q^{(4)}(0) & 0 & Q^{(4)}(L) & -Q^{(5)}(L) \\ 0 & -Q^{(6)}(0) & Q^{(5)}(L) & -Q^{(6)}(L) \\ Q^{(4)}(L) & Q^{(5)}(L) & Q^{(4)}(0) & 0 \\ -Q^{(5)}(L) & -Q^{(6)}(L) & 0 & -Q^{(6)}(0) \end{bmatrix}$$

It is clear that $V = k^4$ times the interaction matrix for a rod clamped at s_1 and s_2. Therefore the condition det $V = 0$ produces the same equation as the similar condition for the clamped finite rod, which indeed is obvious from first principles. A free vibration for the finite rod satisfies the conditions

$$\nabla^4\eta - k^4\eta = 0$$
$$d^2\eta/dx^2 = 0$$
$$d^3\eta/dx^3 = 0$$

therefore the second derivative of η will satisfy the conditions for a clamped rod.

6.1.6 Simple Support at the End of a Semi-Infinite Rod

An axial resonator of infinite strength attached at s leads to the conditions

$$\eta = 0$$
$$\frac{d^2\eta}{dx^2} = 0$$

We now attach a quadrupole resonator of finite strength H at the same point. Assume a source at r* to the left of s. Then as earlier we have:

$$\eta_R^{(1)} = \eta_L^{(1)} + H\eta_L^{(2)}$$

Now as H goes to infinity, we may have $H\eta_L^{(2)}$ going to a finite limit; thus there can be a finite change in the slope $\eta^{(1)}$ at s. That is, a transfer matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & H & 0 \\ 0 & 0 & 1 & 0 \\ F & 0 & 0 & 1 \end{pmatrix}$$

where F and H are indefinitely large, produces discontinuities in $\eta^{(1)}$ and $\eta^{(3)}$ at s, as well as constraining η and $\eta^{(2)}$ to be zero.

Thus if we think of an infinite rod with axial and quadrupole resonators at s, then as their strengths go to infinity, the two halves become decoupled from each other, while the conditions $\eta = 0$ and $\eta^{(2)} = 0$ are realized more and more exactly. Thus either half can be viewed as a semi-infinite rod with a simply supported end.

Of course, a finite rod simple supported at both ends can be handled by applying the above treatment at each end.

6.1.7 Summary of Boundary Conditions

We summarize the physical effects of attaching one or two resonators of infinite strength at a point.

1) A single resonator attached at s.
   Two kinds lead to physically interesting situations:
   a) Axial-------simple support at an internal point
   b) Quadrupole----An internal hinge in the rod.

2) Two resonators attached at s.
   Of the six combinations of two types, only three are of physical interest:
   a) Axial and Transverse-----a clamped end
   b) Axial and Quadrupole-----a supported end
   c) Quadrupole and Octupole-----a free end

For a symmetrical resonator or obstacle attached to an homogeneous rod, it is known that the determinant of the transfer matrix must equal unity:

$$\det M_{res} = 1 \quad .$$

The three combinations above satisfy this condition, at least before one goes to the limit of infinite strength for the resonators. Of the three remaining pairs, two do not satisfy the condition above, namely a transverse combined with a quadrupole resonator, and an axial combined with an octupole resonator. The corresponding transfer matrices are:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & H & 0 \\ 0 & G & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & J \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ F & 0 & 0 & 1 \end{pmatrix}$$

The determinant of the first has a term GH, and the second a term FJ. The remaining combination has the matrix

$$\begin{pmatrix} 1 & 0 & 0 & J \\ 0 & 1 & 0 & 0 \\ 0 & G & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is not excluded by the argument above, but it is difficult to see a physical realization for it.

It is noteworthy that one cannot assign a definite meaning to attaching a transverse and quadrupole resonator at exactly the same point - one must be to the right of the other. In other words, if there is an internal hinge at s, the condition of zero derivative at s is ambiguous; one must specify either the derivative on the right or the left.

6.2 Thickenings on a Rod

A junction between two rod segments having different rigidities $D = EI$ and free wavenumbers k can be regarded as producing a special form of boundary value problem. We shall show that this can be handled by introducing new kinds of resonators, which do not satisfy the reciprocity relation -- that is, the weight function for sensitivity is not the same as that for response.

6.2.1 Non-Reciprocal Resonators

We will use a resonator considered as centered at s which has the sensitivity function $w^*(r-s)$, but responds by exerting a force distributed as $w(s-r)$. That is, the equation of motion of a rod bearing such a resonator of strength F, is:

$$D(\nabla^4 - k^4)\,\eta(r) = P(r) + F\left[\int dx\; w^*(x-s)\,\eta(x)\right] w(s-r)\,,$$

where $P(r)$ is the applied force per unit length. If the rod is divided up into segments with different values of D, this equation does not apply at the junction points.

When there are many resonators on a rod, we assume that the attachment domain of the i-th resonator lies entirely within a segment for which the rigidity has the constant value D_i. We further restrict the applied force to a single segment of rigidity D_o. Then the equation of motion becomes, after dividing by D:

$$(\nabla^4 - k^4)\,\eta(r) = P(r)/D_o$$

$$+ \sum_i S_i\left[\int dx\; w_i^*(x-s)\,\eta(x)\right]\; w_i(s-r)$$

where

$$S_i = F_i/D_i \quad .$$

Now we make the assumption that w_i^* is related to w_i:

$$w_i^* = C_i(w_i)$$

where C_i is a linear differential or integral operator. Let Q_i be the Green's response-function for w_i, defined by

$$0(Q_i(x)) = w_i(x) \quad ,$$

where 0 is the operator $\nabla^4 - k^4$. Then there is an associated Green's sensitivity function $(C_i(Q_i(x))$, the result of applying C_i to Q_i.

Because O is a differential operator, it commutes with C_i, and so we have

$$O(C_i(Q_i(x))) = C_i(w(x)) .$$

Note that Q_i differs by the factor $1/D_i$ from the form that would be expected if the procedure of section 3 had been followed exactly. This of course arises because the equation of motion was divided by D.

For the moment, we specialize to the case of a junction at s between two segments having equal wave-numbers, but unequal rigidities. We shall define pseudoresonators of two new types R_3 and R_4, such that the effects of the junction are equivalent to the attachment at s of an R_3 and an R_4-type resonator. An R_3 type of strength S has the transfer matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1+S & \\ & & & 1 \end{bmatrix}$$

and an R_4 type of strength S has the matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1+S \end{bmatrix} .$$

To produce agreement with the Γ matrix defined in Section 5.5.3, we must set

$$1 + F = \gamma$$

or

$$S = D_R/(D_R-D_L) .$$

where D_R and D_L are the rigidities on the right and left of s.

An R_4 resonator responds to the third derivative of the displacement by exerting a force, while an ordinary axial resonator responds with a force to the displacement itself. Thus the C-operator for an R_4 resonator consists of three differentiations:

$$C_4 \equiv \frac{d^3}{dx^3}$$

By a similar argument involving a transverse resonator, the C-operator for an R_3 resonator corresponds to a single differentiation:

$$C_3 \equiv \frac{d}{dx}$$

The restriction that the wavenumbers are equal on opposite sides of the junction will now be relaxed. In order to do this, we agree to measure all lengths along the rod in terms of the local wavenumber. The measurement of lengths transverse to the rod is not affected. Thus the distance between two arbitrary points will be expressed as so many wavelengths (better, as so many radians).

The transfer matrix across a junction will now have the form

$$\Gamma = \begin{bmatrix} 1 & & & \\ & \rho & & \\ & & \rho^2\gamma & \\ & & & \rho^3\gamma \end{bmatrix}$$

where ρ is the ratio of the wavenumbers on two sides. This form makes clear the need to introduce still another kind, the R_2 resonator. The transfer matrix for an R_2 matrix of strength S will be

$$\begin{bmatrix} 1 & & & \\ & 1+S & & \\ & & 1 & \\ & & & 1 \end{bmatrix} .$$

To determine the corresponding C-operator, we observe that a quadrupole resonator responds to the second derivative of displacement by exerting a force quadrupole. An R_2 resonator responds similarly to the <u>first</u> derivative of displacement. Thus C_2 consists of a single integration:

$$C_2 = \int dx \ldots$$

6.2.2 Interaction Between R_1-resonators

Addendum 2 shows that the displacement of a rod bearing several not-necessarily-reciprocal resonators is given by a formula analogous to that for the simpler case of reciprocity, but the distinction between a Green's response-function Q_p and sensitivity-function $C_p(Q_p)$ must be observed:

$$G(r^*,r) = Q(r^*-r)$$

$$- \sum_{p,q} C_p(Q_p(r^*-s_p))M_{pq}\, Q_q(s_q-r)$$

where $Q(x)$ is the Green's function for an axial resonator, that is,

$$O(Q(x)) = \delta(x)$$

and

$$M = \left[V_{pq} - (\delta_{pq}/S_p) \right]^{-1}$$

$$V_{mn} = \int dx\ C_n(w_n(x-s_n))Q_m(s_m-x)$$

We now indicate how the interaction integrals V_{mn} can be evaluated in some typical cases. The factor 1/D will be omitted.

Consider an R_3 resonator at s and an axial resonator at t. Then the interaction integral (<u>from</u> axial <u>to</u> R_3 resonator) is

$$\int dx \left(\frac{\partial}{\partial x}\,\delta(x-s)\right)\;Q(t-x)$$

$$= -\int dx\,\frac{\partial}{\partial s}\,\delta(x-s)\;Q(t-x)$$

$$= -\frac{\partial}{\partial s}\,Q(t-s)$$

$$= \left.\frac{\partial Q(y)}{\partial y}\right]_{y=t-s}$$

Next, consider two R_3 resonators at s and t. Using primes to indicate differentiation we find the interaction integral is

$$-\frac{\partial}{\partial s}\int dx\;\delta'(x-s)\;Q'(t-x) = Q'''(t-s) \quad .$$

If s approaches t, there is an ambiguity of sign, which is apparently real, and results from the need to specify which resonator is regarded as affecting the other.

More generally, the interaction between an R_i-type at s and an R_j-type at t is found to be the (j-i+3)-th derivative of Q(x) evaluated at x = t-s, for i,j = 2,3,4. Note again that the order is significant.

6.2.3 Computational Usefulness

Although the introduction of R_i-resonators is not very useful for hand calculations, it has considerable advantages in programming for computing machines, espcially when clamped or free ends, or supports are also handled by the method of section 6.1. (The main problem is often not to reduce the number of arithmetic operations to a minimum, but rather to arrange the calculation as systematically as possible.) All the interactions between the inhomogeneities are handled in the same way, and the only functions of distance L that appear are Q(L) and its first three derivatives.

The higher derivatives can be replaced by using the rod equation.

There is a saving, as compared to the use of transfer matrices, or transmission matrices, in that it is not necessary to keep separate track of true waves and evanescent waves at each step.

As an example of how these non-physical resonators might be used, consider the situation of Fig. 6-1.

This can be handled by 10 resonators and pseudoresonators as in Fig. 6-2. The resulting 10 x 10 matrix is easily inverted on a present-day computing machine. Furthermore, many of its elements are equal to each other. The only quantities that must be computed are γ and ρ at each junction, and Q and its first three derivatives for six combinations of non-dimensionalized distances; namely

$$k_iL_i, \quad i = 1,2,3,$$

$$k_1L_1 + k_2L_2$$

$$k_2L_2 + k_3L_3$$

$$k_1L_1 + k_2L_2 + k_3L_3 \quad .$$

Of course, ordinary axial or transverse resonators can be attached to any of the three segments.

6.3 Infinite Plate with Boundary Conditions Along a Line

The four standard boundary conditions of Section 6.1 can also be imposed at the points of a plate that lie on a given straight line or curve. If the curve is a circle, then the techniques used for the case of a rod can be modified very easily. Thus to produce internal support, one attaches an infinite series of rim force resonators, with every possible azimuthal dependence, i.e., weight functions $\cos n\theta$ for $n = 0,1,2,\ldots$, and $\sin n\theta$ for $n = 1,2,3,\ldots$.

Each resonator has infinite strength. To produce the clamped condition, it is also necessary to attach rim moment resonators of infinite strength on the same circle. Resonators having different azimuthal dependence will not interact with each other (that is, the corresponding element of the interaction matrix V will be zero). However, a rim force type and a rim moment type with the same angular dependence do have a non-zero interaction. For external support or a free edge, there is a minor modification of the procedures of Sections 6.1.6 and 6.1.4, because of the effect of Poisson's ratio. The explicit formulas for a free edge are given in Section 9, in connection with the study of circular pistons.

An infinite plate can also be supported or clamped at a point. Conditions at a point can be easily handled by axial and transverse point resonators of infinite strength.

The only other problem which is at all amenable to treatment arises when conditions are imposed along a straight line. Even for this apparently simple situation, it seems to be impossible to evaluate all the pertinent integrals that arise. In the present Section 6.3, integral forms for various Green's functions are derived. These integrals are then discussed in 6.4.

6.3.1 Simple Support Along an Infinite Straight Line

6.3.1.1 Point Resonators Along The Line

We first conceive of a regularly-spaced infinite array of axial resonators along the line $z = 0$. They have strength F and are separated by the distance s. Simple support along the line results when F goes to infinity and s goes to zero.

The interaction matrix V becomes in the limit the kernel of an integral operator, and the summations on i and j become integrations. Thus the displacement $G(r^*,r)$ at $r = (x,z)$ due to a unit point force at $r^* = (x^*,z^*)$ is

$$G(r^*,r) = Q(|r^*-r|)$$

$$- \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; Q\left(\sqrt{(x^*-u)^2 + z^{*2}}\right) K(u-v) Q\left(\sqrt{(x-v)^2 + z^2}\right)$$

where K(u-v) is the kernel that corresponds to the inverse of the limiting form of V. The double integral obviously has the form of a convolution; thus it is appropriate to use the Fourier cosine transform.

We introduce a cap ^ over a function of x to indicate taking the Fourier cosine transform with respect to x. Thus $\hat{Q}(p,z)$ is the one-dimensional cosine transform of Q with respect to x:

$$\hat{Q}(p,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(\sqrt{x^2+z^2}) \cos px \, dx \quad .$$

$\hat{Q}$ is evaluated in Section 6.4.1.

The cosine transform of K will be the reciprocal of the transform of Q for the special value z = 0:

$$\hat{K}(p) = 1/\hat{Q}(p,0) \; .$$

We now use the following standard result for convolutions:

$$\int_{-\infty}^{\infty} F(x-y)G(y)\,dy = \int_{-\infty}^{\infty} \hat{F}(p)\hat{G}(p) \cos px \, dp \quad .$$

Then we can write down an algebraic equation for $\hat{G}(p,z^*,z)$ by applying the transform to the equation for G:

$$\hat{G}(p,z^*,z) = \hat{Q}(p,z^*-z) - \hat{Q}(p,z^*)\hat{Q}^{-1}(p,0)\hat{Q}(p,z) \quad .$$

The resemblance to the Green's function for a rod internally supported at z = 0 is apparent. The reason for this will be clear: Since the boundary condition along the line z = 0 is independent of x, the various Fourier components, which have the form cos px multiplying a function of z* and z, do not interact and so are decoupled from each other. Thus the original point excitation of z* is analyzed into Fourier components, and each is affected in a different way by the support at z = 0. Then the displacement at z is found by an integration.

6.3.1.2 Resonators with Sinusoidal Weight-Function

The mathematical process of taking the Fourier cosine transform can be replaced by an argument that has more physical content.

Instead of using point resonators, we work with a new class of resonators that exert their forces along the whole line $z = 0$. For every positive wavenumber p, we introduce an axial resonator that exerts force proportional to $\cos px$ at each point $(x,0)$ of the edge. We would like to say that this resonator responds to the integral

$$\int \eta(x,0) \cos px \, dx$$

taken over the edge. However, this integral may not converge. Therefore we refine our definition as follows. An axial resonator of wavenumber p and strength $F(p)$ responds to the average weighted displacement $\bar{\eta}_\epsilon(p)$,

$$\bar{\eta}_\epsilon(p) = \left(1/N(\epsilon)\right) \int_{-\infty}^{\infty} e^{-\epsilon |x|} \eta(x,0) \cos px \, dx,$$

where

$$N(\epsilon) = \int_{-\infty}^{\infty} e^{-\epsilon |x|} \cos px \, dx$$

by exerting the force $F(p)\bar{\eta}_\epsilon(p) \cos px$ at each point $(x,0)$. Now we let ϵ go to zero, and allow the strength of each resonator to go to infinity, without specifying precisely how rapidly each strength increases.

The great advantage in introducing these resonators is that they do not interact with each other. It will be realized that this is just a physical way of expressing the earlier mathematical derivation. To avoid having to use resonators responding as $\sin px$, the source must be placed on the line $x = 0$.

6.3.1.3 Effect of Sinusoidal Resonators

A force varying as $\cos px$ applied along $x = 0$ can be expected to produce a displacement at (x,z) that is proportional to the sum of terms of the form

$$\eta = e^{-uz} \cos px$$

where u is to be determined. Substituting this form into the plate equation (valid away from $z = 0$):

$$D(\nabla^4 - k^4)\eta = 0$$

we find that

$$u = \sqrt{p^2 - k^2}\,, \qquad u = \sqrt{p^2 + k^2}$$

Thus if $p > k$, two exponentially decaying motions are allowed; for $0 \leq p < k$, decaying and propagating modes appear. When p goes to zero, these have the form expected for a problem with η independent of x. The relative amplitudes with which the two modes are excited can be determined by an analysis similar to Addendum 1. However, another evaluation is given in 6.4.1.

One can think of the problem as being decomposed into an infinite number of one-dimensional problems. For each fixed number p, one has the analog of a rod, if the dependence on x is neglected. The Green's function depends on p. A unit force at a point on a rod is now replaced by a force applied along a line $z = c$.

6.3.1.4 Rod Welded to Plate

It is clear that the condition of simple support can be generalized by considering an infinite elastic rod welded to the plate along the line $z = 0$. The critical idea is that the different Fourier components will remain decoupled. The net effect on the formula for $\hat{G}(p,z^*,z)$ in 6.3.1.1 will be to add a term to $\hat{Q}(p,0)$, before the inverse is taken.

6.3.2 Zero Normal Derivative

The analogy with point resonators on a rod enables us to write down the Green's function for other types of resonators immediately.

Thus, suppose that on the line z = 0 the plate satisfies the condition of zero normal derivative:

$$\frac{\partial \eta(x,z)}{\partial z} = 0 \quad , \qquad\qquad z = 0 \quad .$$

Then the cosine transform of the Green's function is

$$\hat{G}(p,z^*,z) = \hat{Q}(p,z^*-z)$$
$$+ \frac{\partial \hat{Q}(p,z^*)}{\partial z^*} \left[\frac{\partial^2 \hat{Q}(p,0)}{\partial z^2} \right]^{-1} \frac{\partial \hat{Q}(p,z)}{\partial z}$$

Of course we can think of the boundary condition as being achieved by the attachment of very many transverse resonators along the line, and letting the total strength per unit length go to infinity.

Similarly to the discussion for axial resonators, we can introduce a transverse resonator of wavenumber p and strength G(p) that responds to the "average normal derivative" $\frac{\partial \bar{\eta}(p)}{\partial z}$,

$$\frac{\partial \bar{\eta}(p)}{\partial z} = \lim_{\epsilon \to 0} \left(1/N(\epsilon)\right) \int_{-\infty}^{\infty} e^{-\epsilon |x|} \eta(x,0) \cos px \, dx$$

by exerting the moment $G(p)\frac{\partial \bar{\eta}(p)}{\partial z}\cos px$ at each point (x,0).

It is clear that if all the G(p) are allowed to go to infinity (the rate possible depending on p) then the normal derivative of the plate will be constrained to be zero at each point (x,0).

6.3.3 Clamped Edge

The condition of a clamped straight edge can be achieved by superimposing axial and transverse resonators. Since these do not interact, the two resonator terms simply add together.

6.3.4 Free Edge

The boundary conditions for a free edge are

$$\frac{\partial^2 \eta}{\partial z^2} + \sigma \frac{\partial^2 \eta}{\partial x^2} = 0$$

$$\frac{\partial^3 \eta}{\partial z^3} + (2-\sigma)\frac{\partial^3 \eta}{\partial z \partial x^2} = 0$$

where σ is Poisson's ratio.

We first consider a plate satisfying the first condition only. Let $G(x^*-x, z^*, z)$ be the Green's function. Then from the representation as a Fourier integral

$$G(x^*-x, z^*, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{G}(p, z^*, z) \cos p(x^*-x)\, dp$$

We see that the operator $\frac{\partial^2}{\partial z^2} + \sigma \frac{\partial^2}{\partial x^2}$ corresponds to

$$D_2 = \frac{\partial^2}{\partial z^2} - \sigma p^2$$

in the transform domain. We can then write down the cosine transform $\hat{G}$ for a plate satisfying the first condition only;

$$\hat{G}(p, z^*, z) = \hat{Q}(p, z^*-z) - \frac{\left[\frac{\partial^2}{\partial z^{*2}} - \sigma p^2\right]\hat{Q}(p, z^*)\left[\frac{\partial}{\partial z^2} - \sigma p^2\right]\hat{Q}(p, z)}{\left[\frac{\partial^2}{\partial z^2} - \sigma p^2\right]^2 \hat{Q}(p, 0)}$$

This may be readily verified by applying the operator $\frac{\partial^2}{\partial z^2} - \sigma p^2$ to $\hat{G}$, and then letting z go to zero.

The second boundary condition is handled in an entirely analogous way. The corresponding differential operator is

$$D_3 = \frac{\partial^3}{\partial z^3} - (2-\sigma)p^2 \frac{\partial}{\partial z}$$

Because D_2 is an even operator in z, and D_3 is odd, they do not interact, and the Green's function for a free edge is found by simply summing the separate terms for D_2 and D_3.

In the analogous problem for a rod, we considered a free edge as resulting from the superposition of "quadrupole" and "octupole" resonators, both of infinite strength. We can also adopt this manner of speaking for a plate, but now these non-physical objects have more complicated properties, since they depend on Poisson's ratio. That is, a quadrupole resonator is sensitive to the value of

$$\left(\frac{\partial^2}{\partial z^2} - \sigma \frac{\partial^2}{\partial x^2}\right)\eta = \left(\nabla^2\eta + (1-\sigma)\frac{\partial^2}{\partial x^2}\right)\eta$$

at a point on the edge, and an octupole resonator to

$$\left(\frac{\partial}{\partial z}\nabla^2 - (1-\sigma)\frac{\partial^3}{\partial z \partial x^2}\right)\eta \qquad .$$

Now if we were to think of attaching one of these at an internal point on the plate, the terms in $(1-\sigma)$ would have to be dropped. This paradox however does not interfere with any physical application of the non-physical resonators, and we can say that a free edge results from the superposition of a set of quadrupole resonators and a set of octupole resonators, each member of both sets having infinite strength.

6.3.5 Conditions Along Several Parallel Straight Lines

6.3.5.1 Simple Support Along Any Finite Number of Parallel Straight Lines

Suppose there are N lines of support, with coordinates $z_1, z_2, \ldots, z_N$. Then decomposing into the non-interacting Fourier components, we can use familiar arguments to write down the cosine transform $\hat{G}$ of the Green's function in terms of the interaction matrix M for the supports. The (i,j)-th element of M is actually an integral operator on an x-coordinate difference, and involves z_i and z_j as parameters. After taking the cosine transform $\hat{M}$ of M, each element of $\hat{M}$ is a function of the transform variable p, and not an operator.

$$\hat{G}(p,z^*,z) = \hat{Q}(p,z^*-z)$$

$$- \sum_{ij} \hat{Q}(p,z^*-z_i)\hat{M}_{ij}\,\hat{Q}(p,z_j-z)$$

where

$$\hat{N}_{ij} = (\hat{M}^{-1})_{ij} = \hat{Q}(p,z_i-z_j) \quad .$$

6.3.5.2 Two Parallel Free Edges

An infinitely long bar of finite width 2w, with both edges free, yields a problem of sufficient interest to justify the explicit presentation of the formulas. Then N is the same as the interaction matrix V, and is now 4 x 4:

$$\hat{N} = \begin{bmatrix} D_2^2\hat{Q}(p,0) & 0 & D_2^2\hat{Q}(p,2w) & -D_2D_3\hat{Q}(p,2w) \\ 0 & -D_3^2\hat{Q}(p,0) & D_2D_3\hat{Q}(p,2w) & -D_3^2\hat{Q}(p,2w) \\ D_2^2\hat{Q}(p,2w) & D_2D_3\hat{Q}(p,2w) & D_2^2\hat{Q}(p,0) & 0 \\ -D_2D_3\hat{Q}(p,2w) & -D_3^2\hat{Q}(p,2w) & 0 & -D_3^2\hat{Q}(p,0) \end{bmatrix}$$

M, the inverse of N, is multiplied by the following column vector on the right:

$$\begin{bmatrix} D_2\hat{Q}(p,w-z) \\ D_3\hat{Q}(p,w-z) \\ D_2\hat{Q}(p,-w-z) \\ D_3\hat{Q}(p,-w-z) \end{bmatrix}$$

and on the left by the row vector:

$$\begin{bmatrix} D_2^*\hat{Q}(p,z^*+w) & D_3^*\hat{Q}(p,z^*+w) & D_2^*\hat{Q}(p,z^*-w) & D_3^*\hat{Q}(p,z^*-w) \end{bmatrix}$$

where D_2^* and D_3^* indicate the differential operators in which z* replaces z.

If $z^* \neq z$, then $\hat{G}(p,z^*,z)$ can also be written in terms of the inverse of a 6 x 6 matrix N** that results from bordering N with two rows and two columns (see Section 4.2.3).

6.4 Evaluation of Green's Functions

6.4.1 Evaluation of $\hat{Q}(p,z)$.

The fundamental Green's function for a plate depends of course on whether the plate is in a vacuum or bounds a fluid medium on one side, but it is given by the formula

$$Q(r) = \frac{-1}{2\pi} \int_0^\infty f^{-1}(t) J_o(tr) t \, dt$$

where $f(t) = D(t^4-k^4)$ in the absence of water. The general form of f(t) will be discussed in Section 8. $\hat{Q}(p,z)$, the cosine transform of Q(x,z) is given by:

$$\hat{Q}(p,z) = \frac{-1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} dx \int_0^\infty dt \; f^{-1}(t) J_o(t\sqrt{x^2+z^2}) t \cos xp$$

The integration over x can be performed (see Erdelyi et al., Tables of Integral Transforms, Vol. 1, p. 55):

$$\hat{Q}(p,z) = \frac{-2}{(2\pi)^{3/2}} \int_p^\infty dt\ f^{-1}(t)\,t\ \frac{\cos z\sqrt{t^2-p^2}}{\sqrt{t^2-p^2}}$$

Let $\quad q = \sqrt{t^2-p^2}$. Then

$$\hat{Q}(p,z) = \frac{2}{(2\pi)^{3/2}} \int_0^\infty f^{-1}\left(\sqrt{q^2+p^2}\right)\cos zq\ dq$$

If we now specialize to $f(x) = D(x^4-k^4)$, this becomes

$$\hat{Q}(p,z) = \frac{1}{(2\pi)^{3/2}Dk} \int_0^\infty \left[\frac{1}{q^2+p^2-k^2} - \frac{1}{q^2+p^2+k^2}\right] \cos zq\ dq$$

$$= \frac{1}{2^{5/2}\pi^{1/2}Dk^2} \left[\frac{e^{-|z|v}}{v} - \frac{e^{-|z|u}}{u}\right]$$

where the abbreviations

$$u = \sqrt{p^2+k^2}$$

$$v = \sqrt{p^2-k^2}$$

have been introduced. The absolute value signs on z will often be omitted. We observe that

$$1/\hat{Q}(p,0) = 2^{5/2}\pi^{1/2}Dk^2 \left[\frac{1}{v} - \frac{1}{u}\right]^{-1}$$

$$= 2^{3/2}\pi^{1/2}D(p^4-k^4)\left[\frac{1}{u} + \frac{1}{v}\right]$$

6.4.2 G for a Supported Plate

We can now write out explicitly the integral for $G(x^*,z^*,x,z)$ for an infinite plate in a vacuum, supported along the line $z = 0$.

$$G(x^*,z^*;x,z) = Q\left(\sqrt{(x^*-x)^2 + (z^*-z)^2}\right)$$

$$- \frac{1}{2^5\pi Dk^4} \int_{-\infty}^{\infty} dp\,(\cos(x^*-x)p)\left[\frac{e^{-z^*v}}{v} - \frac{e^{-z^*u}}{u}\right]$$

$$\times \quad (p^4-k^4)\left[\frac{1}{u} + \frac{1}{v}\right]\left[\frac{e^{-zv}}{v} - \frac{e^{-zu}}{u}\right]$$

The three brackets can be multiplied out to produce eight terms, of which two lead to forms that can be integrated explicitly, namely the two terms

$$u^{-3}e^{-(z^*+z)u} \quad \text{and} \quad v^{-3}e^{-(z^*+z)v}.$$

Thus

$$\int_{-\infty}^{\infty} dp\,(\cos(x^*-x)p)\,(p^4-k^4)\,(p^2+k^2)^{-3/2}\,e^{-(z^*+z)\sqrt{p^2+k^2}}$$

$$= \int dp\,\cos(x^*-x)p\left[p^2+k^2-2k^2\right]\frac{e^{-(z^*+z)\sqrt{p^2+k^2}}}{\sqrt{p^2+k^2}}$$

$$= \left[\frac{\partial^2}{\partial z^2} - 2k^2\right]\int_{-\infty}^{\infty} dp\,\cos(x^*-x)p\,\frac{e^{-(z^*+z)\sqrt{p^2+k^2}}}{\sqrt{p^2+k^2}}$$

The integral turns out to be a multiple of $K_0\left(k\sqrt{(x^*-x)^2+(|z^*|+|z|)^2}\right)$. Since z and z^* appear here with absolute value signs around them, this term is related to an image source on the other side of the support line. The term $v^{-3}e^{-(z^*+z)v}$ in the analogous way produces the contribution

$$\text{const.} \left[\frac{\partial^2}{\partial z^2} - 2k^2\right] H_o(k\sqrt{(x^*-x)^2+(|z^*|+|z|)^2})$$

However, the constants in front of K_o and H_o do not have the ratio appropriate to an ordinary source, and the second derivative terms are not expected in any simple picture in terms of mirror images.

The other six terms that arise from expanding the brackets cannot be evaluated simply. In a numerical evaluation of integral it would apparently be better not to separate out the two terms mentioned above.

It may be noted that the integral for G will converge if either z* or z is different from zero. The integral is not well defined until branch cuts for the radicals u and v have been specified.

6.4.3 Green's Functions for Other Types of Boundary Conditions

For the boundary condition of zero normal derivative, we need the functions:

$$\frac{\partial\hat{Q}(p,z)}{\partial z} = \frac{1}{2^{5/2}\pi^{1/2}Dk^2}\left[e^{-zu} - e^{-zv}\right]$$

$$\frac{\partial^2\hat{Q}(p,0)}{\partial z^2} = \frac{1}{2^{5/2}\pi^{1/2}Dk^2}\left[v-u\right]$$

Therefore

$$1/\frac{\partial^2\hat{Q}(p,0)}{\partial z^2} = -\,2^{3/2}\pi^{1/2}\,D\left[v+u\right]$$

The Green's function for zero normal derivative along the line z = 0 is then

$$G(x^*,z^*,x,z) = Q(\sqrt{(x^*-x)^2+(z^*-z)^2})$$

$$-\frac{1}{2^5\pi Dk^4}\int_{-\infty}^{\infty}dp\ \cos(x^*-x)p\left[e^{-z^*u}-e^{-z^*v}\right]\left[v+u\right]\left[e^{-zu}-e^{-zv}\right]$$

For a free edge, we need the expressions

$$D_2\hat{Q}(p,z) = \left[\frac{\partial^2}{\partial z^2} - \sigma p^2\right]\hat{Q}(p,z)$$

$$= \frac{1}{2^{5/2}\pi^{1/2}Dk^2}\left\{(v - \frac{\sigma p^2}{v})e^{-zv} - (u - \frac{\sigma p^2}{u})e^{-zu}\right\}$$

and

$$D_3\hat{Q}(p,z) = \left[\frac{\partial^3}{\partial z^3} - (2-\sigma)p^2\frac{\partial}{\partial z}\right]\hat{Q}(p,z)$$

$$= \frac{1}{2^{5/2}\pi^{1/2}Dk^2}\left[(v^2-(2-\sigma)p^2)e^{-zv} - (u^2-(2-\sigma)p^2)e^{-zu}\right].$$

The more complicated cases involving several supports, or two free edges, etc. are handled in the same way.

6.5 Free Edge Waves; Bar of Finite Width

An infinitely long free edge of a plate can sustain vibrations that are analogous to Rayleigh waves propagated along the surface of an elastic medium. In addition, there are motions that decay exponentially with distance along the edge from a source. We shall show that the wavenumber for the propagating edge wave is not the free wavenumber of the plate, because of Poisson's ratio effects.

6.5.1 Semi-Infinite Plate with a Free Edge

In view of the discussion of 6.3.4, the wavenumbers for free edge vibrations are the values of p for which $D_2^2\hat{Q}(p,0)$ or $D_3^2\hat{Q}(p,0)$ are equal to zero, since these expressions appear in the denominators of the terms of $\hat{G}(p,z*,z)$. They can now be calculated:

$$D_2^2 = \frac{(v^2-\sigma p^2)^2}{v} - \frac{(u^2-\sigma p^2)^2}{u}$$

$$D_3^2 = v(v^2-(2-\sigma)p^2)^2 - u(u^2-(2-\sigma)p^2)^2$$

We readily see that

$$(u^2 - (2-\sigma)p^2)^2 = (v^2 - \sigma p^2)^2$$

so that

$$D_3^2Q(p,0) = - uv\ D_2^2Q(p,0)$$

and the zeros of D_3^2 are the same as those of D_2^2, plus possibly the points $p = \pm k$ and $p = \pm ik$.

We now write out the condition $D_3^2 = 0$ explicitly:

$$\sqrt{p^2-k^2}\ (p^2(1-\sigma) + k^2)^2 = \sqrt{p^2+k^2}\ (p^2(1-\sigma) - k^2)^2$$

Squaring each side, we obtain an equation for p which is apparently of fifth degree in p^2, but the terms in p^{10} will cancel after expansion. Further, only terms of order odd in k^2 will survive cancellation. Thus the equation becomes, after division by k^{10},

$$(p/k)^8\left[-(1-\sigma)^4 + 4(1-\sigma)^3\right] + (p/k)^4\left[4(1-\sigma) - 6(1-\sigma)^2\right] - 1 = 0$$

This is a quadratic equation in $(p/k)^4$, with roots

$$p = \frac{k}{\sqrt{1-\sigma}}\sqrt[4]{\frac{1-3\sigma \pm 2\sqrt{1-2\sigma+2\sigma^2}}{3+\sigma}}$$

All four possibilities for the fourth root are allowable, as well as the arbitrary choice of sign in front of the inner radical. The ambiguity in $\sqrt{1-\sigma}$ is absorbed into that for the fourth root. Thus there are eight possible values that satisfy the algebraic equation for p.

It is easy to verify that:

$$\frac{1}{2} < 1-2\sigma + 2\sigma^2 < 1 \qquad \text{if} \qquad 0 < \sigma < 1.$$

Therefore the inner radical is always real. Furthermore, $1 - 3\sigma \pm 2\sqrt{1-2\sigma + 2\sigma^2}$ is negative for $0 < \sigma < 1$ if the minus sign is taken, but is positive if the plus sign is taken. Therefore four of the values of p have equal moduli and angles $0, \pi/2, \pi, 3\pi/2$. The other four have equal moduli (different from the first set) and have angles $(2n + 1)\pi/4$, for $n = 0,1,2,3$.

Thus when $\sigma = 0$, the possible values of p are

$$p = \pm k, \pm ik, \pm(1+i)k/\sqrt[4]{12}, \pm(1-i)k/\sqrt[4]{12}$$

It is not entirely clear that all the eight values of p given explicitly for general σ are roots of the equation $D_3^2 = 0$, because some roots may have been introduced by the squaring step. However, the value of p close to k must certainly correspond to a physical wave. For this particular p, an expansion of p/k in powers of σ shows that the coefficients of the first three powers of σ are zero:

$$p = k(1+(\sigma^4/16) + \ldots) .$$

6.5.2 Bar of Finite Width with Free Edges

The waves propagated along the two edges of an infinite bar of finite width will interact with each other if the width is less than several wavelengths. This is analogous to the interaction between Rayleigh waves on the two faces of an elastic layer. In both problems, one obtains modes of vibration which are either symmetrical or anti-symmetrical about the median line (or plane). One symmetric mode becomes the ordinary flexural vibration of a rod, in the limit as the width w goes to zero.

For an infinite bar of width 2w, the free vibrations will occur for values of the wavenumber p such that

$$\det N(p) = 0 \quad ,$$

where N(p) is the 4 x 4 matrix of Section 6.3.5.2. A similar algebraic problem was discussed in Section 6.1.3.

If P is the diagonal matrix with elements -1, 1, -1, 1, then $-NP$ can be written as a partitioned matrix with the form

$$-NP = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where now

$$A = \begin{bmatrix} D_2^2\,\hat{Q}(p,o) & 0 \\ 0 & D_3^2\,\hat{Q}(p,o) \end{bmatrix} = D_2^2\,\hat{Q}(p,o) \begin{bmatrix} 1 & 0 \\ 0 & -uv \end{bmatrix}$$

$$B = \begin{bmatrix} D_2^2 & D_2D_3 \\ D_2D_3 & D_3^2 \end{bmatrix} \hat{Q}(p,2w)$$

B is found from the expressions of 6.4.3. The equation $\det NP = 0$ can be factored into two equations, corresponding to symmetric and antisymmetric modes:

$$\det\,(A - B) = 0$$

or

$$\det\,(A + B) = 0$$

If we introduce the abbreviations

$$s = p^2(1 - \sigma) + k^2$$

$$d = p^2(1 - \sigma) - k^2,$$

then after changing signs in the lower rows, $\det\,(\pm A + B) = 0$ can be written as

$$\det \left\{ \pm\left(\frac{d^2}{v} - \frac{s^2}{u}\right)\begin{bmatrix} 1 & 0 \\ 0 & uv \end{bmatrix} + e^{-2wv}\begin{bmatrix} \frac{d^2}{v} & -sd \\ sd & -vs^2 \end{bmatrix} - e^{-2wu}\begin{bmatrix} \frac{s^2}{u} & -sd \\ sd & -ud^2 \end{bmatrix} \right\} = 0$$

or explicitly:

$$(ud^2-vs^2)^2 \pm (u^2d^4-v^2s^2)(e^{-2wv} - e^{-2wu})$$

$$+ (ud^2-vs^2)^2 e^{-2w(u+v)} + 2d^2s^2uv(e^{-4wv} + e^{-4wu}) = 0 \quad .$$

Even in the special case $\sigma = 0$, this equation is very difficult to work with, comparable to the difficulty in determining the wavenumbers of Lamb waves in an elastic layer of finite thickness.

6.6 Boundary Conditions Along a Semi-Infinite Line

6.6.1 Infinite Plate Internally Supported Along the Half-Line $z = 0$, $x \leq 0$.

We shall investigate the Green's function $G(r^*,r)$ only for points r^*,r which are both on the other half of the line $z = 0$. G cannot be obtained explicitly, but a one-dimensional integral equation for it will be derived. The corresponding integral equation for the case of an infinite membrane held down along a half-line is a standard problem for the Wiener-Hopf technique.

We make use of the transformation of Section 4.2.3 generalized to apply to infinitely many observation points on the half-line $z = 0$, $x > 0$. $G(r^*,r)$ can be computed, for r^* and r restricted to the half-line, from a knowledge of $Q(|r^*-r|)$, where r^* and r are here allowed to range over the whole line $z = 0$. The standard Green's function Q is of course known from 3.1.3. The prescription given in Section 4.2.3 requires us to first determine the inverse of Q, considered as a linear integral operator on functions of x^*-x. In order to do this, we take the cosine transform of Q, and obtain what has been called $\hat{Q}(p,0)$ in Section 6.4. The transform of the inverse of Q will be the reciprocal of $\hat{Q}(p,0)$, provided of course that no divergences arise. Thus we have (see the end of 6.4.1):

$$\hat{Q}^{-1}(p,0) = C_1(p^4-k^4)\left[(p^2-k^2)^{-1/2} + (p^2+k^2)^{-1/2}\right] \quad ,$$

where C_1 is a constant. The Fourier inversion integral to obtain $Q^{-1}(x^*,x)$, the inverse of Q, from this expression does not converge, but proceeding formally, we see that the factor p^4 under the integral sign is equivalent to a differentiation outside. Thus we can write

$$Q^{-1}(x^*,x) = C_2\left(\frac{d^4}{dx^4} - k^4\right)\int_{-\infty}^{\infty} \cos p(x^*-x) \left[(p^2-k^2)^{-1/2}+(p^2+k^2)^{-1/2}\right] dp$$

$$= C_3\left(\frac{d^4}{dx^4} - k^4\right)\left[iH_o(k|x^*-x|) + (2/\pi)K_o(k|x^*-x|)\right] .$$

Note that $Q^{-1}(x^*,x)$ is the inverse of Q with respect to the line $z = 0$; if we were considering a plate simply supported along a finite or infinite portion of an arbitrary curve $z = f(x)$, then we would have to consider a different inverse of Q, depending on the function f.

The second part of the prescription is now invoked: The Green's function $G(x^*,x)$ results from restricting the variables of $Q^{-1}(x^*,x)$ to correspond to points of the unsupported portion of the line $z = 0$, and then inverting the resulting linear integral operator. Thus we obtain the following integral equation, which holds for $x,x^* \geq 0$:

$$\left(\frac{\partial^4}{\partial x^4} - k^4\right) \int_0^{\infty} dy \left[iH_o(k|x-y|) + (2/\pi)K_o(k|x-y|)\right] G(x,x^*) = C_4 .$$

If the plate is supported at each point along the line except for a number of segments from a_i to b_i, $i = 1,2,\ldots I$, then the integral from 0 to ∞ is replaced by a sum of integrals over the segments.

6.6.2 Internal Support Along Two Parallel Half-Lines

In this case, we must consider two source points, r_1^* on the line $z = -w$, and r_2^* on the line $z = w$. Similarly for observation points r_1 and r_2. Now a 2 x 2 matrix of four functions is involved:

$$\begin{bmatrix} Q(|x_1^*-x_1|) & Q\sqrt{(x_2-x_1^*)^2 + (2w)^2} \\ Q\sqrt{(x_2^*-x_1)^2 + (2w)^2} & Q(|x_2^*-x_2|) \end{bmatrix}$$

After taking the cosine transform of each element, the resulting matrix must be inverted. It does not seem possible to obtain a tractable system of integral equations.

7. TRAVELING PRESSURE WAVES ON INFINITE RODS AND PLATES

In the problems discussed so far, the applied force was concentrated at a point. We will now consider traveling pressure waves on a rod or plate bearing resonators. The rod problem is not of direct physical application, but it is mathematically equivalent to the higher dimensional analog, namely, a plate with attached beams or stiffeners on one side, all infinitely long and parallel to each other. The traveling pressure wave can be explained by assuming that the side without resonators is in contact with an ocean in which acoustic waves are propagating. However, the treatment in this section will not take account of the reaction of the plate back on the water. Inclusion of this effect is postponed until Section 8.

7.1 An Infinite Regular Array of Axial Resonators on a Rod

Consider a rod with axial resonators of strength F attached at uniform intervals along its length. A sinusoidal traveling pressure wave is applied along the rod. This will be written in complex form as

$$P_o e^{i(vx-\omega t)}$$

where v = wavenumber of the incident pressure wave
ω = frequency

The resonator spacing will be denoted by s, and we also write

$$u = 2\pi/s \quad .$$

This makes u analogous to a wavenumber, if one thinks of the resonator spacing as analogous to a wavelength. The equation for vibration of the rod is, after taking out the time-dependence $e^{-i\omega t}$,

$$\frac{\partial^4 \eta}{\partial x^4} - k^4 \eta = (P_o/EI)e^{ivx} + (F/EI)\eta \, \Sigma \, \delta(x-ns)$$

where

$$k^4 = \omega^2 \rho S/EI$$

7.1.1 Displacement at an Arbitrary Point

We now give a prescription for writing down solutions of equations of this general form; of course such prescriptions are not obvious until after solutions to typical problems have been obtained by some more laborious process.

7.1.1.1 No Resonators

One first looks for a solution that holds in the absence of any resonators. The corresponding mode of vibration of the bar may be called sympathetic. It is given by:

$$\eta_{sym} = (P_o/EI)\,e^{ivx}\,(v^4-k^4)^{-1} \quad .$$

Of course, the general solution of

$$\frac{\partial^4\eta}{\partial x^4} - k^4\eta = (P_o/EI)\,e^{ivx}$$

has four arbitrary constants; it is

$$\eta = ae^{-kx} + be^{kx} + ce^{ikx} + de^{-ikx} + \eta_{sym} \quad .$$

The first and second terms produce infinite displacement at infinity, and so are excluded by the boundary conditions. The third and fourth terms correspond to permitted free modes of vibration of the bar, which would not be excited by the applied pressure wave, unless it happens that $k = v$.

7.1.1.2 Function With Uniformly-Space Discontinuities

The effects of the regular array of resonators can be handled by looking for a function $M(x)$, which also depends on u and v as parameters, such that

$$EI\,\left(\frac{\partial^4}{\partial x^4} - k^4\right)M(x)\,e^{ivx} = \Sigma\,\delta(x-ns)\,e^{ivx} \quad .$$

That is, when $M(x)e^{ivx}$ is acted upon by the rod differential operator, one obtains a δ-function singularity at each attachment point. It is easily verified that M is given by:

$$M(x) = \frac{u}{2\pi} \sum_{p=-\infty}^{\infty} \frac{e^{ipux}}{EI\left((pu+v)^4-k^4\right)} \quad ,$$

since

$$EI\left(\frac{\partial^4}{\partial x^4} - k^4\right)M\, e^{ivx} = \frac{u}{2\pi} \sum_{p=-\infty}^{\infty} e^{ipux}e^{ivx} = \sum_{p=-\infty}^{\infty} \delta(x-ns)\,e^{ivx}$$

Note that M has the period s as a function of x, since replacing x by x+s merely changes a term in the sum for M into the next term.

7.1.1.3 Explicit Form of Displacement

The solution to the full equation for the plate displacement is now taken as the sum of the sympathetic vibration and the discontinuity-producing term with an unknown coefficient:

$$\eta = P_o\left[\frac{e^{ivx}}{EI(v^4-k^4)} - AM(x)\; e^{ivx}\right] \quad .$$

On substitution in the equation of motion, we find

$$e^{ivx} + A\,\Sigma\,\delta(x-ns)\; e^{ivx}$$

$$= e^{ivx} + F\left[\frac{1}{EI(v^4-k^4)} + AM(x)\right] e^{ivx}\;\Sigma\,\delta(x-ns)$$

As this must hold for every x, the coefficients of the δ-functions must be equal. Since $M(ns) = M(0)$ for integral n, we obtain

$$A = \frac{1}{EI(v^4-k^4)}\;\frac{1}{M(0)-(1/F)}$$

and so

$$\eta = \frac{P_o\; e^{ivx}}{EI(v^4-k^4)}\left[1 - \frac{M(x)}{M(0) - 1/F}\right]$$

$$= \frac{P_o\; e^{ivx}}{EI(v^4-k^4)}\left[1 - \frac{1}{T_o-(1/F_o)} \sum_{p=-\infty}^{\infty} \frac{e^{ipux}}{EI\left([pu+v]^4-k^4\right)}\right]$$

where

$$T_o = \sum_{p=-\infty}^{\infty} \frac{1}{EI((pu+v)^4-k^4)}$$

and

$$F_o = uF/2\pi = F/s \quad .$$

Note that F_o is the resonator strength per unit length of rod.

The infinite sum that defines M(x) can be expressed in finite terms, but there is a different explicit form for each segment between resonators. For each segment considered by itself, the equation of motion is just that given in 7.1.1.1 for the sympathetic vibration. The general solution was written down there, in terms of four arbitrary coefficients a,b,c,d. These will differ for each rod segment, but the values are such that the displacement and its first two derivatives are continuous across the attachment points.

7.1.1.4 Plate Bounding an Ocean on One Side

The displacement can be written in the general form

$$\eta = \frac{P_o e^{ivx}}{f(v)} \left\{ 1 - \frac{1}{T_o - (1/F_o)} \sum \frac{e^{ipux}}{f(v+pu)} \right\}$$

where $f(v) = EIv^4 - \omega^2 \rho S$

$$T_o = \sum_{n=-\infty}^{\infty} \frac{1}{f(v+nu)} \quad .$$

It will be seen later than the same general formula holds for a plate with attached stiffeners when the reaction of the plate back on the water is taken into account, provided that f(v) is replaced by the form:

$$f(v) = \frac{Eh^3 v^4}{12(1-\sigma^2)} - \omega^2 h\rho_{plate} + \frac{i\omega^2 \rho_{water}}{\sqrt{k*^2 - v^2}}$$

where k* is the wavenumber for sound waves in the water.

7.1.2 Displacement of the Rod at the Midpoints Between Resonators

We have seen that the displacement for a rod covered with a regular array of axial resonators has the form

$$\eta = \frac{e^{ivx}}{EI(v^4-k^4)} X \quad ,$$

where X has period s in x. X was expressed above as a Fourier series, which did not exhibit the discontinuities in $X^{(3)}$ directly. It will be of interest to have explicit forms for X evaluated at special points, for instance at the resonator attachment points and half-way between them. One can then fit the first few terms of a trigonometric series through these points, but of course these terms are not the first terms of the Fourier expansion of X, which give a fit to X that is moderately good everywhere, but is not exact anywhere, except accidentally.

We first evaluate T_o. It is convenient to introduce

$$v' = v/u$$

$$k' = k/u$$

and to write

$$T(v',k') = u^4 \, EI \, T_o(v,k,u) = \sum_{p=-\infty}^{\infty} \frac{1}{[p+v']^4 - k'^4}$$

We now observe that $T(v',k')$ is periodic in v' with period 1, and has no essential singularities in the v'-plane. It has poles at

$$v = n \pm k' \text{ or } v = n \pm ik'$$

for $n = 0, \pm 1, \pm 2, \ldots$ It is therefore clear that $T(v',k')$ is the sum of terms of the form

$$\frac{a}{b - \cos 2\pi v'} ,$$

where a and b are independent of v'. Define c by $b = \cos 2\pi c$. Then $T(v',k')$ will have a pole when $v = c$ or $v = -c$. Thus all the poles are included in the sum S of two terms:

$$S = \frac{a_1}{\cos 2\pi k' - \cos 2\pi v'} + \frac{a_2}{\cos 2\pi i k' - \cos 2\pi v'}$$

To determine a_1, the residue of S at $v' = k'$ is evaluated and compared to that of T. The residue of S is

$$\frac{a_1}{2\pi \sin 2\pi k'}$$

The residue of

$$T(v',k') \text{ at } v' = k' \text{ is } \frac{1}{4k'^3} \quad .$$

Therefore

$$a_1 = \frac{2\pi \sin 2\pi k'}{4k'^3} \quad .$$

Similarly for a_2 . Therefore

$$T(v',k') = \frac{\pi \sin 2\pi k'}{2k'^3(\cos 2\pi k' - \cos 2\pi v')}$$

$$- \frac{\pi \sinh 2\pi k'}{2k'^3(\cosh 2\pi k' - \cos 2\pi v')}$$

and so

$$T_o(v,k,u) = \frac{s}{4k^3 EI}\left[\frac{\sin ks}{\cos ks - \cos vs} - \frac{\sinh ks}{\cosh ks - \cos vs}\right]$$

In order to compute the displacement at the midpoints between the resonators, we must evaluate:

$$T_{mid} = \sum_{p=-\infty}^{\infty} \frac{\exp i\, p\, u(s/2)}{EI((pu+v)^4 - k^4)}$$

$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^p}{EI((pu+v)^4 - k^4)}$$

We separate the terms with odd and even p:

$$T_{mid} = \sum_{q=-\infty}^{\infty} \frac{1}{EI((q(2u) + v)^4 - k^4)}$$

$$- \sum_{q=-\infty}^{\infty} \frac{1}{EI((q(2u) + v + u)^4 - k^4)}$$

$$= A_o - B_o$$

where

$$A_o = T_o(v,k,2u)$$

$$B_o = T_o(v+u,k,2u)$$

Note that $T_o(v,k,u) = A_o + B_o$.

Thus an approximate expression for X that is exact at the resonator attachment points and also half-way between them is

$$X = 1 - \frac{1}{(A_o+B_o) - (1/F_o)} (A_o + B_o \cos u\, x)$$

where

$$A_o = \frac{s}{8k^3 EI} \left[\frac{\sin (ks/2)}{\cos(ks/2) - \cos(vs/2)} \right.$$

$$\left. - \frac{\sinh (ks/2)}{\cosh(ks/2) - \cos(vs/2)} \right]$$

and B_o differs from A_o only by having plus signs in both denominators.

7.2 A Rod with Combined Axial-Transverse Resonators

As in Section 7.1, identical resonators are attached at points with spacing s. Their axial strength at frequency ω will be $F(\omega)$, and their transverse strength $G(\omega)$. The equation of motion, in response to a traveling pressure wave of unit peak magnitude is:

$$EI \frac{\partial^4 \eta}{\partial x^4} - \rho S \eta = e^{ivx} + F\eta \, \Sigma \, \delta(x-ns)$$

$$+ G \, \Sigma \frac{\partial}{\partial x} \delta(x-ns) \frac{\partial \eta}{\partial x}$$

If one attempts to carry out the general method of Section 7.1 by finding a function with suitable discontinuities at the attachment points, there is a difficulty, as terms such as $\frac{d\delta(x)}{dx} T(x)$ appear, where T is not sufficiently continuous at x = 0. It thus becomes impossible to interpret such terms. There is reason to believe that the trouble arises because of the idealizations used in Section 4.1 to derive the equation of motion. The original arguments assumed a continuous distribution of transverse resonators, but now a δ-function distribution is taken.

In order to avoid the problem of discontinuities, it is necessary to introduce a Fourier series expansion for the δ-function density, and also for the displacement η.

The algebra will be found in Addendum 5 . The final result is:

$$\eta = e^{ivx} f^{-1}(v) \left\{ 1 + \Delta^{-1}(F_o - v^2 G_o + u^2 F_o G_o T_2) \sum_p f^{-1}(pu+v) e^{ipux} \right.$$

$$\left. - \Delta^{-1}(uvG_o + u^2 F_o G_o T_1) \sum_p f^{-1}(pu+v) p e^{ipux} \right\}$$

where

$$f(w) = EI(w^4 - k^4)$$

$$F_o = F/s$$

$$G_o = G/s$$

$$T_i = \sum_p f^{-1}(pu+v) p^i \qquad i = 0,1,2$$

$$\Delta = 1 - F_o T_o + G_o(u^2 T_2 + 2uvT_1 + v^2 T_o)$$

$$+ F_o G_o (T_1^{\,2} - T_o T_2) u^2$$

and all sums run from p = -∞ to p = +∞.

If the time dependence is included, then η has the form

$$\eta = f^{-1}(v)e^{i(vx-\omega t)}Z(x),$$

where $Z(x)$ is independent of the time, and is periodic with period s. Thus the displacement of the rod with axial and transverse resonators is the product of a traveling wave factor with wavelength the same as that of the applied pressure, multiplying a function of space alone that has the same periodicity as the resonators.

7.3 Several Sets of Axial Resonators on a Rod

It is clear that the inclusion of the transverse mode results in a very significant increase in the complexity of the derivations and ultimate formulas. As the frequencies at which transverse vibrations are significant are apparently not important for our purposes, further calculations will be limited to the simpler axial mode.

Consider an infinite rod bearing J different kinds of resonators. The resonators of the j-th kind, for $j = 1,2,..,J$ have strength F_j and are spaced uniformly at $x_j \pm ns$, where $n = 0, \pm 1, \pm 2, \ldots$. Thus the resonator spacing s is the same for each kind of resonator (see Fig. 7-1). If there is an incident pressure wave $P_o e^{i(vx-\omega t)}$, then the equation of motion is

$$EI(\frac{\partial^4 \eta}{\partial x^4} - k^4\eta) = P_o e^{ivx} + \eta \sum_{j=1}^{J} F_j \sum_{n=-\infty}^{\infty} \delta(x-x_j - ns)$$

In analogy with the procedure of Section 7.1, we assume the form

$$\eta = P_o f^{-1}(v)e^{ivx} - \sum_{j=1}^{J} A_j M(x-x_j)e^{ivx} \quad ,$$

where $M(x-x_j)$ is defined as in Section 7.1.1.2 and the A_j are to be determined.

After substitution, the terms not containing a δ-function cancel. Then we can equate the corresponding coefficients of the δ-functions for the different attachment points. We thus obtain

$$-A_j = P_o^{-1}(v)F_j - \sum_{j'} A_{j'} M(x_{j'} - x_j)F_j$$

or

$$\sum_{j'} (M(x_j - x_{j'}) - (\delta_{jj'}/F_j))A_{j'} = P_o f^{-1}(v)$$

$$j = 1,2,\ldots,J$$

Thus the coefficient $A_{j'}$ is found by inverting the matrix N, where

$$N_{ij} = M(x_i - x_j) - (\delta_{ij}/F_i) \ ,$$

then finding the sum of the elements in the j'-th row of the inverse, and finally multiplying by $P_o f^{-1}(v)$. If one defines a matrix M by

$$M = N^{-1} \ ,$$

then

$$\eta = \frac{P_o e^{ivx}}{EI(v^4 - k^4)} \left\{ 1 - \sum_{p,q} M_{pq} \ M(x - x_p) \right\} .$$

This result is reminiscent of the form found in Section 4.2 for a finite number of axial resonators on a rod. However, the N and M matrices are now Hermitian and not complex symmetric. A justification for the resemblance is provided by the following argument:

Suppose that v is an integral multiple of u, that is, an exact number of pressure waves can fit between the attachment points of two consecutive members of one set of resonators. Then one can conceive of the rod with its several kinds of resonators as bent into a helix, and then all the turns of the helix can be thought of as identical, so that the rod becomes a ring, whose circumference equals the spacing s. It is meaningful to speak of the impressed pressure at a point on the ring, because of the assumption that v/u is an integer. (See Fig. 7-2.)

It thus becomes plausible that the interactions between the different infinite classes of resonators are given by formulas that are similar to those for a finite number of resonators on an infinite rod. The function $M(x_i - x_j)$ plays the same role as did the Green's function $Q(s_i - s_j)$ in Section 4.2. However, $M(-x)$ is the complex conjugate of $M(x)$, whereas $Q(x)$ is an even function of x.

We give the explicit form of η for the case of two sets of resonators with attachment points $x_1 + ns$ and $x_2 + ns$.

$$\eta = \frac{P_o e^{ivx}}{EI(v^4 - k^4)} \left\{ 1 - C^{-1} M(x - x_1)\left[M(0) - \overline{M}(d) - \frac{1}{F_2}\right] \right. \left. - C^{-1} M(x - x_2)\left[M(0) - M(d) - \frac{1}{F_1}\right] \right\}$$

where

$\overline{M}$ = complex conjugate of M

$d = x_2 - x_1$

$$C = (M[0])^2 - M(d)\overline{M}(d) - \frac{M(0)(F_1 + F_2)}{F_1 F_2} + \frac{1}{F_1 F_2}$$

7.4 Resonators Between Equally-Spaced Supports

A configuration having some practical interest is the following: There are M infinite sets of axial resonators, the sets being uniformly interspersed among each other. Thus the spacing between a resonator of the m-th set and the nearest one of the (m + 1)-th set is s/M. All the resonators, except for one set, have the same strength E. Those of the remaining set have the strength E + F. In the limit as F becomes infinite, one obtains an infinite rod simply supported at points spaced s units apart, which also bears M - 1 resonators of strength E between any two neighboring supports. In the higher dimensional analog, there is a plate to which are attached a series of parallel beams that act as supports. Smaller beams are attached parallel to the supports and act as resonators. (See Fig. 7-3.)

The derivation of the displacement η is given in Addendum 6. It is found that

$$\eta(x) = f^{-1}(v)\,e^{ivx}$$

$$+\ f^{-1}(v)E_o(1-E_oT(0))^{-1}\sum_r f^{-1}(Mru+v)\,e^{(Mru-v)ix}$$

$$+\ \frac{f^{-1}(v)F_o\sum(1-E_oT(i))^{-1}}{1 - F_o\sum T(i)(1-E_oT(i))^{-1}}\ \sum_j\left\{1+E_oT(j)(1-E_oT(j))^{-1}\right\}\ X$$

$$\sum_r f^{-1}((j+Mr)u+v)\,e^{((j+Mr)u+v)ix}$$

where

$$E_o = ME/s$$

$$F_o = F/s$$

$$u = 2\pi/s$$

$$T(j) = \sum_{n=-\infty}^{\infty} f^{-1}((j+Mn)u+v) \qquad j = 0,1,\ldots,M-1$$

The summation on j goes from 0 to M-1; those on r, from $-\infty$ to ∞. The terms in the expression for η can be interpreted as follows: The first term $f^{-1}(v)e^{ivx}$ is the response of the rod without any resonators. The second term gives the response of a regular array of resonators of strength E and spacing s/M. The third term shows the effect of the extra strength of the widely-spaced resonators. Note that E_o is the strength per unit length of the closely-spaced resonators, and F_o gives the extra strength per unit length of the widely spaced ones.

7.5 A Plate With a Finite Number of Resonators

Consider a plate to which N axial resonators, with strengths $F_1,\ldots,F_N$, are attached at $s_1,\ldots,s_N$. There is a traveling pressure wave of unit strength and wavenumber v moving parallel to the x-axis.

The equation of motion is then, after setting $s_i=(x_i,z_i)$,

$$D(\nabla^4-k^4)\,\eta(x,z) = e^{ivx} + \sum_{i=1}^{N} F_i\,\delta(x-x_i)\,\delta(z-z_i)\,\eta(x,z)$$

where D is the plate rigidity. It can be verified by the same techniques as used earlier that the displacement η is given by:

$$\eta(x,z) = \frac{e^{ivx}}{D(v^4-k^4)} - \sum_{ij} \frac{e^{ivx_i}}{D(v^4-k^4)}\,M_{ij}\,Q(|s_j-(x,z)|) \quad .$$

Here Q(r) is the displacement of a plate without resonators due to a unit force at distance r from the observation point (see Section 3.1.3), and M is the matrix of interaction terms which has been encountered earlier:

$$(M^{-1})_{mn} = Q(|s_m-s_n|) - (\delta_{mn}/F_m) \quad .$$

The formula for η also holds for problems with lower-dimensional geometry (i.e., each resonator exerts its force along a line parallel to the wave front) by using the appropriate Green's function Q.

7.6 A Lattice of Axial Point Resonators

A plate covered with a rectangular gridwork of identical axial resonators presents no significant new problem of treatment, but several additional parameters are necessary to specify a configuration.

7.6.1 Description of the Problem

The resonators, all of strength $F(\omega)$, are attached at the points

$$x = ms \quad , \qquad z = nt \quad , \qquad m,n = 0, \pm1, \pm2\ldots$$

A small modification of the treatment that follows would allow for a grid with axes oblique to each other, but this will not be discussed explicitly. The impressed pressure wave travels at angle θ with respect to the x axis. Therefore the equation of motion is

$$D(\nabla^4-k^4)\eta = P_o e^{iv(z \sin\theta+x \cos\theta)}$$

$$+ \eta F \sum_m \sum_n \delta(x-ms)\delta(z-nt)$$

where P_o is the peak pressure.

7.6.2 Generalization of the Solution for a Rod

As in Section 7.1.1 one can define a sympathetic mode of the plate

$$\eta_{sym} = P_o f^{-1}(v) e^{iv(z \sin\theta+x \cos\theta)}$$

where

$$f(v) = D(v^4-k^4) \quad .$$

The discontinuity function M is now required to satisfy:

$$D(\nabla^4-k^4)M(x,z)e^{iv(z \sin\theta+x \cos\theta)}$$

$$= \sum_{m,n} \delta(x-ms)\delta(z-nt)e^{iv(z\sin\theta+x \cos\theta)}$$

Therefore

$$M(x,z) = \frac{uw}{4\pi^2 D} \sum_{p,q} \frac{e^{ipux + iqwz}}{\left[(v \cos\theta+pu)^2 + (v \sin\theta+qw)^2\right]^2 -k^4}$$

where

$$u = 2\pi/s, \qquad w = 2\pi/t \quad .$$

As earlier, we assume

$$\eta = \eta_{sym} + A\, M(x,z)\, e^{iv(z \sin\theta + x \cos\theta)}$$

and substitute to determine A. We obtain in the end:

$$\eta = \frac{P_o\, e^{iv(z \sin\theta + x \cos\theta)}}{D(v^4 - k^4)} \left[1 - \frac{M(x,z)}{M(0,0) - (1/F)}\right]$$

where $$M(0,0) = \frac{uw}{4\pi^2 D} \sum_{p,q} \frac{1}{\left[(v \cos\theta + pu)^2 + (v \sin\theta + qw)^2\right]^2 - k^4}$$

7.6.3 A Crystallographic Analogy

Consider that only half of the plate ($z > 0$) is covered with a gridwork of axial resonators. Instead of a traveling pressure wave, we assume an oscillating point-source of pressure indefinitely far to the left. As noted earlier, every resonator acts as a new center for a flexural wave in the plate. Each resonator may be considered to be surrounded by other resonators in a way that is the same for all of them (except for a few layers near the boundary $z = 0$ of the lattice). Thus all the induced waves starting at the resonator attachment points have the same strength. The situation can be viewed as analogous to diffraction of X-rays by crystals, and one can apply the construction which is ordinarily used to find the diffracted X-rays in order to determine diffracted flexural waves that propagate into the resonator-free region $z > 0$ at an angle to the x-axis (see Fig. 7-4).

There is one feature which is unlike the crystallographic analog. In the X-ray treatment, the scattering strength is considered sufficiently small that multiple interactions can be ignored. For the plate with resonators, however, the displacement involves M, which has the form

$$M = (Q + F^{-1}\iota)^{-1}$$

where Q is understood as the matrix whose (i,j)-th element is $Q(|s_i - s_j|)$. Now if the strength F of each resonator is small, M can be expanded as a power series in F:

$$M = -F(1 - FQ)^{-1} = -F(1 + FQ + F^2Q^2 + \ldots) \quad .$$

Here each term represents an order of scattering. Thus the strength of the wave that originates at a resonator attachment point depends not only on the intrinsic strength F of an individual resonator, but also on the dimensions of the gridwork. If only one order of scattering were considered, this would not be so.

7.6.4 Several Interspersed Lattices of Resonators

If there are several superimposed grids, each consisting of one kind of resonator, and if the grids all have the same wave numbers u and w, then the technique of Section 7.3 yields the displacement.

8. PLATES ACOUSTICALLY COUPLED TO AN INFINITE OCEAN

A plate with attached resonators, bounding and acoustically coupled to an ocean that fills a half-space, can be handled by first studying the plate without resonators but coupled to the ocean. Then the effects of attaching resonators to the coupled plate-water system can be determined by the techniques already given.

As earlier $\eta(x,z)$ will represent the displacement of the plate, and $D = Eh^3/12(1-\sigma^2)$ its flexural rigidity. We also define

$u(x,y,z)$ = y - component of displacement of fluid from equilibrium

$p(x,y,z)$ = pressure in the fluid

$\phi(x,y,z)$ = velocity potential in the fluid

ρ^* = density of the fluid

ρ = density of the plate

k^* = wavenumber for sound waves in the water ($k^* = \omega/c$).

k = wavenumber for free flexural waves of the plate

The time dependence $e^{-i\omega t}$ has been suppressed in u, p and ϕ. These are related by the equations

$$p(x,y,z) = i\omega\rho^* \phi(x,y,z)$$

$$u(x,y,z) = \frac{-1}{i\omega} \frac{\partial\phi(x,y,z)}{\partial y}$$

8.1 Plane Acoustic Wave in an Ocean Bounded by a Plate

8.1.1 Expressions for Pressure and Displacement

The force driving the plate is taken to be a plane acoustic wave coming up obliquely from below. This causes a deformation of the plate which then acts as a source of secondary sound waves.

The planar wave fronts will be assumed parallel to the z-axis. Φ will be the angle between the plate and the normal to the wave fronts, and P_o the peak pressure that would be exerted by the acoustic wave in the absence of the plate.

The velocity potential for the incoming wave can be written as

$$\phi_{in} = \frac{-iP_o}{\omega\rho^*} \exp i(x\, k^* \cos \Phi + y\, k^* \sin \Phi - \omega t)$$

The wave number for the resulting pressure wave on the plate will be denoted by

$$v = k^* \cos \Phi$$

The reflected wave will be influenced by the plate motion. Therefore its velocity potential will be represented in the form

$$\phi_{re} = \frac{-iP_o B}{\omega\rho^*} \exp i\ (x\, k^* \cos \Phi - y\, k^* \sin \Phi - \omega t)$$

where B is a complex number to be determined. Thus the total velocity potential (without time dependence) is

$$\phi = \frac{-iP_o \exp ixv}{\omega\rho^*} \left[e^{iyk^* \sin \Phi} + B\, e^{-iyk^* \sin \Phi} \right]$$

and therefore the net pressure on the plate is

$$P = P_o\, e^{ixv} \left[1 + B \right] \quad .$$

The displacement at an arbitrary point is

$$u = \frac{P_o}{\omega^2 \rho^*} \quad (\exp ixv) \quad ik^* \sin \Phi \left[\exp (iyk^* \sin \Phi) \right.$$

$$\left. - B \exp (-iyk^* \sin \Phi) \right]$$

and since $\eta = u$ at the interface, we have

$$\eta = \frac{P_o}{\omega^2 \rho^*} (\exp ixv)\ ik^* \ (1-B)\ \sin \Phi \ .$$

Now applying the equation of motion of the plate, we find that

$$B = \frac{Dv^4 - \omega^2 \rho\ h + i\omega^2 \rho^*/k^* \sin \Phi}{Dv^4 - \omega^2 \rho\ h - i\omega^2 \rho^*/k^* \sin \Phi}$$

After replacing $k^* \sin \Phi$ by its equivalent in terms of k^* and v, the displacement of the plate can be written as

$$\eta = 2P_o\ e^{ixv} / \ f(v)$$

where

$$f(v) = Dv^4 - \omega^2 \left[\rho\ h + \rho^* (v^2 - k^{*2})^{-1/2} \right] ,$$

$$= D \left[v^4 - k^4 - e(v^2 - k^{*2})^{-1/2} \right] .$$

Here $\qquad e = \omega^2 \rho^* / D.$

The net pressure at a point of the interface is

$$P = 2\ P_o\ e^{ixv}\ D(v^4 - k^4)\ / f(v).$$

8.1.2 <u>Interpretation of the Denominator f(v)</u>

The expression for f(v) is of great interest in that each term has an immediate interpretation as a force (or rather, a pressure) on a small strip of the plate. Thus Dv^4 gives the net strength of the shear forces transmitted across the boundaries of the small strip by its neighboring strips. This can be called the elastic term. Furthermore, $-\omega^2 \rho\ h$ or $-Dk^4$ is the force on the strip due to its own mass. This will be called the <u>inertial</u> term. Finally, $-De(v^2 - k^{*2})^{-1/2}$ is the pressure of the water on the strip.

Many of the approximations that are often introduced in vibration problems can be understood as simplifications of the form of $f(v)$. Thus, if the plate is considered as locally reacting, the term Dv^4 is dropped. That is, the elastic forces are disregarded in comparison with the inertial forces. Again, retardation is sometimes neglected. This corresponds to making the speed of sound infinite. Then the pressure term simplifies to $-De/v$. A rigid plate is obtained by going to the limit $\rho \to \infty$, while the other quantities are allowed to remain finite. Finally, by letting ρ^* go to zero while P_o remains finite (which means that ϕ and u must become infinite) one obtains the system studied in Section 7, that is, a rod in a vacuum subject to impressed traveling waves.

8.2 Oscillating Line Source on Underside of Plate

One can think of the preceding problem as involving a driving pressure in the form of a traveling wave of strength $2P_o e^{i(xv-\omega t)}$ that arises in some unknown manner on the underside of the plate.

In a problem where the driving pressure at the interface is not a simple harmonic function of distance, a Fourier analysis can be performed. Thus, suppose the pressure can be represented as:

$$P(x,t) = \int_{-\infty}^{\infty} dv\ \hat{P}(v)\ e^{i(xv-\omega t)}$$

and assume the resulting displacement can be represented similarly:

$$\eta(x,t) = \int_{-\infty}^{\infty} dv\ \hat{\eta}(v)\ e^{i(xv-\omega t)}$$

Then we will have:

$$\hat{\eta}(v) = \hat{P}(v)\ /f(v).$$

Although it is not necessary to specify how the driving force on the plate is produced, it is certainly necessary to distinguish carefully between the driving pressure and the net pressure, to which the induced plate motion contributes.

Now consider an oscillating line source of pressure on the underside. Then

$$P(x,t) = \delta(x)\, e^{-i\omega t}$$

and hence $\hat{P}(v) = \frac{1}{2\pi}$, independent of v.
Therefore the displacement is

$$\eta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{-1}(v)\, e^{i(xv - \omega t)} dt.$$

It appears that this integral cannot be evaluated in finite form, because of the radical. If retardation is neglected, however, the path of integration can be deformed to infinity, and one is left with the residues at the zeros of f(v). Computation of the zeros is discussed in Section 10.2.

8.3 Oscillating Ring Source on Underside of Plate

Problems with cylindrical symmetry can be handled by using the Hankel transform of order zero, which will be denoted by a tilde. The precise forms that we use were indicated in Section 3.1.2. In analogy with the result for two-dimensional symmetry, we have

$$\tilde{\eta}(v) = \tilde{P}(v) \,/ f(v)$$

Consider an oscillating source exerting force uniformly over a ring of radius S. Let the total force be unity. We use the Hankel transform of $\delta(r-R)/R$ (see Section 3.1.2), in order to write the pressure as:

$$P(r) = (1/2\pi) \int_{0}^{\infty} dv\, v\, J_0(Sv)\, J_0(rv)$$

Therefore

$$\eta_{ring}(r) = (1/2\pi) \int_0^\infty dv\, f^{-1}(v)\, J_o(Sv) v\, J_o(v,r)$$

For the special case of an oscillating bubble at the interface. S = 0, and so

$$\eta_{bubble}(r) = (1/2\pi) \int_0^\infty dv\, f^{-1}(v)\, v\, J_o(vr)$$

8.4 <u>Traveling Pressure Wave on Upper Face of Plate</u>

It will be essential to keep in mind the distinction between a pressure or force on the upper side of the plate and one on the lower, and between a driving pressure and a secondary pressure. As an example, we consider a problem similar to that of Section 8.1: An infinite flexible plate floats on an ocean of water. But now, a traveling pressure wave of the form $T = P_o e^{i(vx-\omega t)}$ is applied to the <u>upper</u> side. We shall compute the pressure on the lower side of the plate, that is, at the interface.

The equation of motion of the plate is now

$$D(\nabla^4 - k^4)\, \eta = P(x,z) + T,$$

where P(x,z), the pressure at the interface, is unknown. As earlier, one introduces the velocity potential $\phi e^{-i\omega t}$. ϕ must satisfy the Helmholtz equation $\nabla^2\phi + k^{*2}\phi = 0$, and so ϕ will have the form

$$\phi(x,y.z) = A \exp\left(i(vx-\omega t) + y\,(k^{*2} - v^2)^{-1/2}\right)$$

Then the displacement u at the interface is given by

$$u = \frac{Ai}{\omega}\,(k^{*2} - v^2)^{-1/2}\, e^{i(vx-\omega t)}$$

and the pressure at the interface is

$$P = i\omega\rho^* A e^{i(vx-\omega t)}$$

Substitution in the equation of motion determines A. We then find the pressure on the underside is

$$P = \frac{i\omega^2\rho^* P_o}{-\sqrt{k^{*2} - v^2}} \frac{e^{ixv}}{f(v)}$$

The displacement η is given by the same formula as in Section 8.1.1 except for the absence of the factor 2.

In the Section 8.1 the condition

$$k^* > v$$

was always satisfied, since v was there defined by $v = k^* \cos \Phi$. In the present problem, the magnitudes of k^* and v are independent, and so the radical can be imaginary. When $v > k^*$ an acoustic wave will be propagated into the fluid. If one now compares the pressure term in f(v), namely, $-\omega^2\rho^* (v^2 - k^2)^{-1/2}$, with the inertial term $-\omega^2\rho h$, one sees that the effect of the water can be thought of as adding the mass $\rho^*(v^2 - k^{*2})^{-1/2}$ per unit area to the plate. However, this mass depends on the wavelength, and for $v < k^*$ becomes pure imaginary, so that the water term becomes dissipative.

8.5 Attachment of Resonators to the Water-Coupled Plate

8.5.1 Oscillating Line and Ring Sources on the Upper Side of the Plate

One can easily derive the effects of stationary sources on the upper side in complete analogy with the treatment of Sections 8.2 (for line sources at the interface) and 8.3 (for ring sources). That is,one decomposes a stationary line source into traveling waves by means of a Fourier transform, and the Hankel transform is used for the ring source. The displacements will be given by the formulas of Sections 8.2 and 8.3, but the pressures are determined from integrals which differ from those given earlier, since they contain the additional factor $i\omega^2\rho^*(k^{*2}-v^2)^{-1/2}$. Thus:

$$P_{line}(x) = i\omega^2\rho^* \int_{-\infty}^{\infty} \frac{e^{ixv}\, dv}{(k^{*2} - v^2)^{1/2}\, f(v)}$$

$$P_{ring}\ (r) = i\omega^2\rho^* \int_{0}^{\infty} \frac{J_o(Rv)\ J_o(rv)\ v\ dv}{(k^{*2} - v^2)^{1/2}\, f(v)}$$

8.5.2 A Resonator as a Secondary Source

The displacement in a problem involving a traveling pressure wave, or a stationary oscillating force, on the upper side of a plate coupled to water will differ from the displacement for a plate in free space only in that $D(v^2 - k^4)$ will be replaced by $f(v)$, after the appropriate transform has been taken. In other words, it is sufficient to add the pressure term $-De\ (v^2 - k^{*2})^{-1/2}$ to each denominator.

The force produced by a resonator is simply a special type of impressed force. Thus all the earlier results on Green's functions for a resonator on a plate can be carried over, for any type of resonator attached at a point, on a circle, or on a line.

8.5.3 Interaction of Several Resonators

Consider M axial point resonators, with strengths $F_1, \ldots, F_M$, attached at $s_1, \ldots, s_M$ to a water-coupled plate. The Green's function $G(r^*, r)$ for a source at r^* is defined by the equation of motion

$$D(\nabla^4-k^4)\ G = \delta(r^* - r) + \sum_{m=1}^{M} \eta\ F_m\ \delta(r - s_m) + P^*(G)$$

where $P^*(G)$ is the pressure due to the flexure of the plate. P^* is a functional of G, that is, its value depends on the values of $G(r^*, r)$ at all points r. This term is taken to the left side of the equation, and we then define a linear operator O, which when applied to a function η of position on the plate produces a new function of position:

$$O(\eta) = D(\nabla^4 - k^4)\eta - P^*(\eta).$$

O is linear, and is homogeneous and isotropic in the space variable. In other words, in order to compute $O(\eta)$ when η is known it is not necessary to specify a particular point or a particular direction. Addendum 2 can now be applied once more, and indicates that the displacement in the presence of the resonators is:

$$G(r^*,r) = Q(r^* - r) - \sum_{i,j=1}^{M} Q(r^*-s_i)\ M_{ij}\ Q(s_j-r),$$

where $Q(r^* - r)$ is the displacement at r, in the absence of resonators, due to a source at r^*, and is the same as $\eta_{bubble}(r^*-r)$ given at the end of Section 8.3. The matrix M is the inverse of a matrix N:

$$N_{pq} = Q(s_p-s_q) - (\delta_{pq}/F_p).$$

In the absence of the water, $f(v)$ reduces to $D(v^4 - k^4)$, and the above formulas reduce to those of Section 4.2.

For an axial rim resonator of radius R, the Q function is (compare the result without water in Section 3.2.1),

$$Q_R\,(x) = Q_R\,(|x|) = \int_0^\infty f^{-1}(v)\,J_o(vR)\,J_o\,(v|x|)\,v\,dv$$

The elements of the N matrix involve interaction integrals, which were denoted by $Q(R_i, R_j, |s_i - s_j|)$ in Section 4.4:

$$N_{ij} = \int_0^\infty f^{-1}(v)\,J_o(vR_i)J_o(vR_j)J_o(v|s_i-s_j|)v\,dv - (\delta_{ij}/F_i).$$

It is thus reasonably clear that all the results obtained earlier in the absence of water are valid when the plate is acoustically coupled to an ocean on one side, if the denominator is modified in each case by adding the pressure term $-De(k^{*2} - v)^{-1/2}$.

8.6 Pressure on a Plate Bearing Resonators

The pressure at a point on the underside of a plate can be found from the formula for the displacement by determining which terms are due to forces on the upper side of the plate, and which to driving pressures in the water. The need for this distinction was explained in Section 8.5.

8.6.1 Point Force, and one Axial Point Resonator

We assume that the force is impressed on the upper side of the plate. A displacement expressed in the wave-number domain is converted to a pressure by multiplying by $ie(k^{*2}-v^2)^{-1/2}$.

Thus we define

$$P_u(r^*-r) = \frac{1}{2\pi}\int_0^\infty \frac{ie}{\sqrt{k^{*2}-v^2}}\;\frac{vJ_o(v|r^*-r|)\,dv}{f(v)}$$

P_u will be the pressure at r on the plate without resonators due to a point source at r* on the upper side of the plate.

Then the total pressure due directly to a source at r^* and to a resonator at s is

$$P_u(r^*-r) - Q(r^*-s) \frac{1}{(Q(0) - (1/F))} P_u(s-r)$$

In some applications one wants the integrated force on a small area of the plate, thought of a a piston. Thus the total force P_{int} on a circle of radius R with center at t due to a source at the origin is

$$P_{int} = \int P(r)\, W_R\, (t-r)\, dr$$

where

$$W_R\,(x) \begin{cases} = 1 & \text{if } |\, x \,| < R \\ = 0 & \text{if } |\, x \,| > R \end{cases}$$

By using the result of Addendum 3 on two-dimensional convolutions, this becomes

$$P_{int} = 2\pi \int \widetilde{P}\,(v)\ \widetilde{W}_R\,(v)\ v\ J_o\,(v|\,t\,|)\ dv$$

But $\widetilde{W}_R\,(v) = R\ J_1(vR)/v,$

and so

$$P_{int}(t) = i\ eR \int_0^\infty \frac{J_1(vR) J_o\ (v|\,t\,|)\ dv}{(k^{*2} - v^2)^{1/2}\ f(v)}$$

P_{int} is the total force on the underside of the plate; the driving force at the origin is not included.

The integrated pressure on a circle centered at t when a unit force is applied at r^* and a resonator is attached at s is then

$$P_{int}(r^*-t) = Q(r^*-s)\ \frac{1}{Q(0) - (1/F)}\ \ P_{int}\ (s-t)$$

Generalizations to many point resonators, or to combinations of rim and point resonators, are obvious.

8.6.2 Acoustic Wave, and an Infinite Array of Axial Resonators

There is a acoustic wave in the water, striking the plate obliquely at such an angle that the driving pressure on the plate is $2P_o e^{ivx}$. The resonators are infinitely long stiffeners, all parallel to the wave fronts. The same geometry was considered in Section 6.1, but the reaction of the plate back on the water was neglected. In view of the preceding discussion, the displacement taking account of the reaction can be written down immediately:

$$\frac{\eta}{2P_o} = \frac{e^{ivx}}{f(v)} + \frac{e^{ivx}}{f(v)}\ \frac{1}{(s/F)-T_o}\ \sum_p \frac{e^{ipux}}{f(v+pu)}$$

where s is the spacing, and $u=2\pi/s$. The first term is the displacement in the absence of resonators, and the second gives their effect. The pressure resulting from the first term can be written as (see Section 8.1)

$$P_1 = 2P_o\ e^{ixv}\ D(v^4-k^4)/f(v) \quad .$$

The second term in $\eta/2P_o$ analyzes the effect of the resonators into an infinite series of traveling waves $e^{i(v+pu)x}$. The displacement due to such a term must be multiplied by the factor $i\omega^2\rho^*(k^{*2} - (v+pu)^2)^{-1/2}$ to obtain the pressure. Thus the total pressure due to all such waves is

$$\frac{2\,P_o e^{ivx}}{f(v)} \quad \frac{i\omega^2\rho^*}{(s/F)-T_o} \sum_p \frac{e^{ipux}}{(k^{*2} - (v+pu)^2)^{1/2} f(v+pu)}$$

The expression for P_1 can be transformed by writing

$$D(v^4-k^4) = f(v) - \frac{i\omega^2\rho^*}{\sqrt{k^{*2}-v^2}}$$

Then the total pressure becomes

$$2\,P_o e^{ivx}\left[1 - \frac{i\omega^2\rho^*}{(k^{*2}-v^2)^{-1/2} f(v)} + \frac{i\omega^2\rho^*}{f(v)} \quad \frac{1}{(s/F)-T_o} \quad X \right.$$

$$\left. \sum \frac{e^{ipux}}{(k^{*2}-(v+pu)^2)^{1/2} f(v+pu)}\right]$$

This formula has been programmed for computation on the IBM 7090. In the intended application, the ratio u/v is large, and so only a few terms of the infinite sum on p need be taken.

8.7 Dynamical Analogy for a Plate Immersed in Water

To prepare for the work of Section 8.8 on Green's functions for submerged acoustic sources, we will construct a mathematical analogy for a plate in an infinite ocean filling all space. As usual, $\eta(x,z)$ will be the departure of the median surface of the plate from the equilibrium plane, which is taken as $y = c$. There are sources in the water at finite distances, but no source applying force directly to the plate.

8.7.1 Conditions for $\emptyset$.

The equation of motion of the plate is

$$D(\nabla^4 - k^4)\ \eta(x,z) = P(x,z)$$

where P is the difference between the pressures on the upper and lower sides of the plate. These are due to the sources and the resultant flexure of the plate. The Green's function for the same plate in free space satisfies the equation

$$D(\nabla^4 - k^4)\ Q(r^* - r) = \delta(r^* - r)$$

together with boundary conditions at infinity (no incoming waves, and $Q(r^*-r) \to 0$ as $r \to \infty$). The displacement $\eta(r) = \eta(x,z)$ can be written in terms of Q as

$$\eta(r) = \int dr'\ Q(r'-r)\ P(r')$$

where $\int dr'$ indicates a two-dimensional integral extending over the plane $y = c$. In much of what follows, no properties of Q are used except the above relation, so that the plate could be replaced by an infinite membrane under uniform tension, or any other two-dimensional system.

We introduce the notation Δ to indicate the difference between the values of a function of x,y,and z at the upper and lower faces of the plate. Thus, for instance

$$\Delta \phi(x,z) = \lim_{\epsilon \to 0}\left[\phi(x,y+\epsilon,z) - \phi(x,y-\epsilon,z)\right]$$

We have apparently assumed that the thickness of the plate is infinitesimal, but as a matter of fact this assumption is not essential, as one can consider that the y-coordinate does not measure a true distance perpendicular to the plate, but only the portions of the perpendicular that are in the water.

In view of our sign conventions, we have

$$P = -\Delta p$$

On introducing the connections between p and ϕ, we obtain

$$\eta(r) = -i\,(De/\omega) \int dr'\; Q(r'-r)\,\Delta\phi(r')$$

where

$$e = \omega^2 \rho^*/D$$

Note that we write $\Delta p(r')$ for $\Delta p(x',z')$, where $r' = (x',y')$. Since the plate is assumed to maintain a constant thickness while it executes flexural oscillations, the vertical displacement of the fluid at both upper and lower interfaces must equal η. But

$$u = \frac{-1}{i\omega}\frac{\partial\phi}{\partial y},$$

and so $\partial\phi/\partial y$ must have the same value on both sides of the plate:

$$\left.\frac{\partial\phi(x',y',z')}{\partial y'}\right]_{y'=c} = -De \int d\,r''\; Q(r''-r')\;\Delta\phi(r'')$$

Thus $\phi(x,y,z)$ is determined by the following conditions:

1) ϕ is defined, is continuous and has a continuous derivative at each point in the water.

2) $(\nabla^2 + k^{*2})\, \phi(x,y,z) = 0$ (Helmholtz equation for sound waves) at each point in the water not occupied by a source.

3) ϕ satisfies boundary conditions at infinity ($\phi \to 0$ if x,y, or z goes to plus or minus infinity; no incoming waves).

4) $\partial\phi/\partial y$ is continuous at $y = c$ across the plate, but ϕ itself has a discontinuity, and the two are related by the equation above.

8.7.2 Application of Hankel Transform

It will now be assumed that all sources are on the line $x = z = 0$, so that there is cylindrical symmetry. Then the results of Addendum 3 on two-dimensional convolutions can be applied to the last equation above. As earlier, a tilde over a function symbol will indicate the Hankel transform of order zero with respect to the radial variable. The corresponding transform variable will be v. The wavenumbers are decoupled in the transformed equation:

$$\left.\frac{\partial\tilde{\phi}\ (v,y')}{\partial y'}\right]_{y' = c} = -\ 2\pi\ De\ \tilde{Q}\ (v)\ \Delta\ \tilde{\phi}\ (v)$$

or

$$\frac{\Delta\ \tilde{\phi}\ (v)}{\partial\tilde{\phi}/\partial y'} = -\ 2\pi\ De\ \tilde{Q}\ (v)$$

This suggests, by comparison with much of our earlier work, that the plate behaves for each v as some kind of fictitious resonator in a process such that $\tilde{\phi}(v,y)$ is the analog of a Green's function and 2π De $\tilde{Q}(v)$ is the analog of a resonator strength.

8.7.3 Change to $\tilde{u}$ as Analog Quantity

It turns out that a more transparent analogy is produced if we use $\tilde{u}$ rather than $\tilde{\phi}$. Clearly

$$\tilde{u}\ (v,y) = \frac{-1}{i\omega}\ \frac{\partial\tilde{\phi}(v,y)}{\partial y}\ .$$

The conditions (1) - (4) of Section 8.7.1 are carried over to $\tilde{u}$ in the following way: Clearly (1), the condition of continuity within the water, is not modified. But (2) becomes, for each y not corresponding to a point of the plate,

$$\left(\frac{\partial^2}{\partial y^2} + k^{*2} - v^2\right) \quad \tilde{u}\ (v,y) = 0.$$

It is thus clear that the analog problem that generates $\tilde{u}(v,y)$, for $k^* > v$, is just that of waves on a uniform string along the y-axis. The free wavenumber is $(k^{*2} - v^2)^{1/2}$. For $k^* < v$, a pure imaginary wavenumber must be allowed. Condition (3) becomes: $\tilde{u}\ (v,y) \to 0$ if $y \to \pm \infty$, together with a condition in the y-variable of no incoming waves. The discontinuity conditions for u are, however, not the same as those of $\emptyset$, which were given by (4). Thus $\tilde{u}(v,y)$ is continuous at $y = c$. However, its derivative is discontinuous there. To see this we observe that

$$\frac{\partial \tilde{u}}{\partial y} = \frac{\partial}{\partial y}\left[\frac{-1}{i\omega} \quad \frac{\partial \tilde{\emptyset}}{\partial y}\right] = \frac{-1}{i\omega} \ \frac{\partial^2 \tilde{\emptyset}}{\partial y^2} = \frac{(k^{*2} - v^2)\ \tilde{\emptyset}(y)}{i\omega}$$

Thus

$$\begin{aligned}
\Delta(\partial\tilde{u}/\partial y) &= \frac{(k^{*2} - v^2)\ \Delta\ \tilde{\emptyset}(y)}{i\omega} \\
&= \frac{(k^{*2} - v^2)}{i\omega} \ \frac{\partial \tilde{\emptyset}}{\partial y} \quad \frac{-1}{2\pi \mathrm{De}\ \tilde{Q}(v)} \\
&= \frac{(k^{*2} - v^2)\ \tilde{u}}{2\pi\ \mathrm{De}\ \tilde{Q}(v)}
\end{aligned}$$

or

$$\frac{\Delta(\partial\ \tilde{u}/\partial y)}{\tilde{u}} = \frac{(k^{*2} - v^2)}{2\pi\ \mathrm{De}\ \tilde{Q}(v)}$$

8.7.4 Analog for the Plate

We now determine the nature of the structure on the string that will correspond to the plate.

If a concentrated force of magnitude M is applied to a string at s, there is a discontinuity at s in the derivative of the displacement, of magnitude - M/T, where T is the tension of the string. A point axial resonator, of the type first introduced in Section 2, can be attached to a string just as to a rod. An ordinary mass is the simplest example. Consider now that the resonator has strength F. This means, by definition, that in response to the displacement $\tilde{u}$, it exerts the concentrated force $F\tilde{u}$. Thus

$$\frac{\Delta\tilde{u}}{\tilde{u}} = \frac{-F}{T}$$

Thus for each v the analog of the plate is a resonator on the string of strength F(v), where $F(v)/T(v) = -(k^{*2} - v^2)/2\pi\ \omega^2\rho^*\tilde{Q}(v)$.

If the plate is rigid, then $D = \infty$, and hence $F/T = \infty$. Thus an infinite mass is attached to the string, corresponding to a boundary condition of zero displacement.

It is to some extent arbitrary how to break up the expression for F/T into a strength factor and a tension factor. However, the strength should depend only on properties of the plate, and the tension only on the water. Thus we set

$$F = -1/2\pi\ \tilde{Q}(v)$$

$$T = \omega^2\rho^*/(k^{*2} - v^2)$$

8.8 Green's Function for a Submerged Source in the Presence of Several Plates

Consider now a single acoustic source of unit strength at $R^* = (0,y^*,0)$, and J parallel plates at the heights $y_1,\ldots,y_J$.

The j-th plate has rigidity D_j, free wavenumber k_j, and the Green's function Q_j.

8.8.1 Ocean Filling All Space

By taking the Hankel transform, the following analog problem is obtained for each v:

There is an infinite string stretched along the y-direction, with tension T(v) and free wavenumber $(k^{*2} - v^2)^{1/2}$. A resonator of strength $F_j(v)$ is attached at y_j, for j = 1,...J.

Let $S_v(y^*-y)$ be the Green's function for a homogenous infinite string, that is, the displacement at y due to a unit force at y*. Then

$$T \left(\frac{\partial^2}{\partial y^2} + k^{*2} - v^2\right) S_v(y^*-y) = \delta(y^*-y).$$

Let W(R*,R) be the Green's function for an acoustic source in an infinite three-dimensional medium — that is, the pressure at R = (x,y,z) due to a source of unit strength at R* = (x*,y*,z*). We will also use the notation

$$W(R^*,R) = W(r^*-r \; , \; y^*-y),$$

where r* and r are two dimensional vectors. $\widetilde{W}(v,y^*-y)$ will be the Hankel transform of W. A three-dimensional source at R* produces, on application of the Hankel transform, a one-dimensional source at y* for each value of the transform variable v. The strength in the analog problem is $\widetilde{W}(v,0)$ multiplied by a factor c independent of v that arises from the possibly different normalizations for water, plate and string.

In analogy with the treatment of Section 4, the Hankel transform $\tilde{u}(v,y)$ of the displacement at the height y can be written as:

$$\tilde{u}(v,y) = c\,\tilde{W}(v,0)\left[S_v(y^*-y) - \sum_i \sum_j S_v(y^*-y_i)M_{ij}(v)\;S_v(y_j-y)\right]$$

where M(v) is the inverse of a matrix N(v):

$$N_{ij}(v) = S_v(y_i-y_j) + \delta_{ij}\,2\pi\;\tilde{Q}_i(v)$$

8.8.2 Ocean Bounded by a Plate

If the ocean fills only the lower half of space, the string analog must be modified correspondingly.

Consider first that there is a layer of water on top of the uppermost plate, and vacuum above the layer. Then the pressure at the surface is zero, which implies that $\partial u/\partial y = 0$ there. If this layer is made arbitrarily thin, there will be a negligible difference on the motion of this system, as compared to a system with no water at all above the plate. The condition of zero normal derivative must still be satisfied.

To realize the corresponding condition $\partial\tilde{u}/\partial y = 0$ for the analog problem, we employ a standard technique and introduce mirror image sources and mirror image resonators (all with positive sign) on an infinite homogeneous string. Note that the reflection is with respect to the water surface, even if the layer is infinitesimally thin. Thus the uppermost resonator in this case must be doubled. This does not correspond to a plate of twice the thickness; slippage is permitted in the infinitesimal water layer.

8.8.3 Layer of Water Between Two Plates

If the fluid medium is bounded from above and below by plates, it is necessary to introduce mirror images with respect to both plates, and then there will be secondary mirror images of these, and so on. In order to avoid the infinite regression, we proceed in analogy with Section 6.1, and define a dipole resonator for a string.

Since a homogeneous string is governed by a second-order differential equation, it is sufficient to specify the displacement and its first derivative at a point in order to know the same quantities at any other point. Thus a transfer matrix will be 2 x 2. The matrix across a concentrated inhomogeneity such as a weight will be

$$\begin{pmatrix} 1 & 0 \\ F/T & 1 \end{pmatrix}$$

where F is the strength of the inhomogeneity as an axial resonator.

A dipole resonator on a string can be defined in terms of a pair of axial resonators of equal and opposite strengths that approach each other while the strengths go to infinity. The transfer matrix across this system becomes, in the limit, of the form

$$\begin{pmatrix} 1 & G/T \\ 0 & 1 \end{pmatrix}$$

A similar limiting process for a rod produces a transverse resonator, which is an allowable mode of a physical resonator, once the idealization of a point attachment is accepted. However, a dipole resonator on a string is not physical, any more than an octupole resonator on a rod. Attachment of a dipole resonator of infinite strength has the effect of breaking the string, and imposing the condition of zero normal derivative on each side of the break.

A plate which serves as an upper bound for a layer of water can be represented in the string analog by a dipole resonator of infinite strength immediately <u>above</u> the ordinary or monopole resonator which is the analog to a plate in an infinite medium. Similarly, a plate bounding the fluid from below is analogous to a dipole resonator just <u>underneath</u> a monopole resonator.

Setting equal to zero the interaction matrix between four resonators in these positions, one obtains the condition for free waves in a water layer bounded by two flexible plates. The explicit algebraic form is exhibited in Section 8.9.2.

In the prescription above, we could not allow a monopole and dipole resonator to be attached at the same point; the interaction constant V for such a pair is not zero, as would be expected from symmetry considerations, but nonzero and ambiguous in sign. It will be recalled (see Section 6.1.7) that an axial and an octupole resonator could not be attached at the same point of a rod, since this would violate the condition that a transfer matrix has unit determinant. Similarly, a monopole and a dipole resonator attached at the same point on a string would produce the transfer matrix

$$\begin{pmatrix} 1 & G/T \\ F/T & 1 \end{pmatrix}$$

whose determinant is not unity.

The physical interpretation of this difficulty is perfectly clear. If the water extends almost up to, but does not quite reach a plate above it, then the plate has no effect on the water. The plate is coupled to the water as soon as the water height is infinitesimally greater than the plate height.

8.8.4 Plate Separating Different Fluid Media

Different fluid media will correspond to strings with different tensions and free wavenumbers. Then the pseudoresonators used in Section 5.5 to handle thickenings on rods can be introduced in the string analog problem. Since displacement is always continuous across a plate, there will be only one type, with a transfer matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$$

8.9 Evaluation of Relevant Transforms

8.9.1 Green's Function for the String

S_v is ready found to be

$$S_v(y^* - y) = -i(K/2\omega^2\rho^*)\ \exp iK\ |y^* - y|$$

where

$$K = \begin{cases} (k^{*2} - v^2)^{1/2} & \text{if } k^* > v \\ i(v^2 - k^*)^{1/2} & \text{if } k^* < v \end{cases}$$

The positive sign for $k^* < v$ satisfies the finiteness condition as y goes to infinity.

From this result, and the form for G in 8.8.1, one can already draw a consequence that is not entirely obvious. If there is a single plate in an ocean filling all space, and the observation point (x,y,z) is on the opposite side of the plate from the source at (x^*,y^*,z^*), then u(x,y,z) depends only on $|y^* - y|$ and not on y and y^* separately. To see this, it is sufficient to write out the term in $\tilde{u}(v,y)$ that is due to the plate. This is proportional to

$$\frac{S_v(y^*-c)\ S_v(c-y)}{S_v(0)\ -\ (1/F)}$$

where c is the plate height. But clearly $S_v(y^*-c)S_v(c-y)$ is independent of c, if y^* and y are on opposite sides of c.

8.9.2 Green's Function for a Plate

It is known from Section 3.1.3 that

$$\tilde{Q}(v) = \frac{-1}{2\pi\ D(v^4 - k^4)}$$

A typical diagonal term of N can now be computed as

$$S_v(0) - 2\pi Q(v) = \frac{-iK}{2\omega^* \rho(v^4 - k^4)} \left[v^4 - k^4 + 2ie/K \right],$$

Note that the quantity in brackets is not $f(v)/D$, where $f(v)$ was introduced in Section 8.1, because of the presence of the extra factor 2 in front of term involving $e = \omega^2 \rho^*/D$. But it is clear why this arises. There is water on both sides of the plate, and so the "added mass" due to the water will be twice what it is for a plate with water on only one side.

In the image methods used in Section 8.8.2 for a semi-infinite ocean, the need to insert a duplicate of the bounding plate will have the effect of reducing the factor 2 to 1. In other words, the plate and its image become, in the dynamical analogy, two resonators very close together. These are equivalent to a single resonator of twice the strength. The strength F appears in the diagonal element in the form $S_v(0) - (1/F)$. Thus the term involving e will be halved, as compared to a plate with water on both sides.

We return to the problem considered in 8. 8. 3 of a layer of water between two plates separated by a distance L between their inner faces. Primes will be used for quantities pertaining to the lower plate. The attachment points of the resonators will be indicated as follows:

b - upper dipole resonator
a - upper monopole
a' - lower monopole
b' - lower dipole resonator

To fulfill the analogy, we must have b>a and a'>b. In the limit, these will become equalities. The four rows and columns of N will be arranged according to the order of the resonators given above.

Then N has the following form (where the subscript v on S has been dropped):

$$\begin{bmatrix} -S^{(2)}(0) & \partial S(b-a)/\partial b & \partial S(b-a')/\partial b & \partial^2 S(b-b')/\partial b\partial b' \\ \partial S(a-b)/\partial b & S(0) - (1/F) & S(a-a') & \partial S(a-b')/\partial b' \\ \partial S(a'-b)/\partial b & S(a'-a) & S(0) - (1/F') & S(a'-b')/\partial b' \\ \partial^2 S(b'-b)/\partial b\partial b' & \partial S(b'-a)/\partial b' & \partial S(b'-a')/\partial b' & -S^{(2)}(0) \end{bmatrix}$$

We allow a and b to coalesce, as well as a' and b', and introduce the abbreviations

$$E = \exp iKL$$

$$H = ie/K(v^4 - k^4)$$

Let Q be the permutation matrix that interchanges columns 3 and 4; then Q^T interchanges rows 3 and 4. Furthermore, let X be the diagonal matrix with elements 1, -1, 1, 1. Then

$$XQ^TNQX = \frac{-iK}{2T\omega^2\rho^*} \begin{bmatrix} K^2 & -iK & K^2E & iKE \\ -iK & 1+2H & iKE & -E \\ K^2E & iKE & K^2 & -iK \\ iKE & -E & -iK & 1+2H' \end{bmatrix}$$

which can be written in partitioned form

$$\begin{bmatrix} A & B \\ B & A' \end{bmatrix}$$

where

$$A = \begin{bmatrix} K^2 & -iK \\ -iK & 1+2H \end{bmatrix} \qquad B = \begin{bmatrix} K^2E & iKE \\ iKE & iE \end{bmatrix}$$

and A' differs from A in that H' replaces H. Assume now that $H' = H$, which implies that the two plates have the same values of D and k - that is, they have the same elastic constants. Then, the condition det $N = 0$ implies by the arguments of Section 6.1.3 that either $\det(A - B) = 0$ or $\det(A + B) = 0$. These become after taking out K^2 factors:

$$\det\left\{\begin{bmatrix} 1 & -1 \\ -i & 1+2H \end{bmatrix} \pm \begin{bmatrix} E & iE \\ iE & -E \end{bmatrix}\right\} = 0,$$

and after simplification,

$$1 + H = \pm E\,(1 - H)$$

Then using the definitions of E and H,

$$(v^4 - k^4 + ie/K) = \pm(\exp i\,KL)\;(v^4 - k^4 - ie/K).$$

If k^* has a small positive imaginary part so that there is dissipation in the water, and L goes to infinity, then exp i KL goes to zero and this condition becomes $f(v)/D = 0$. Thus we recover our earlier result for free waves of a plate floating on an infinite ocean.

In the general case in which the two plates are not similar, the condition for free waves must be derived from the more general relation that

$$\det\left[I - A^{-1}BA'^{-1}B\right] = 0$$

In this case, no simple factorization is possible.

8.9.3 Green's Function for Water

For our purposes, a source within the fluid must be normalized so that the total normal force exerted on a plane passing very close to the source is unity.

It is well known that the Green's function W for a three-dimensional acoustic source is, except for dimensional factors A,

$$W(R^*,R) = W(r^*-r, y^*-y) = \frac{A \exp ik^* |R^*-R|}{|R^* - R|}$$

Then from Erdelyi et al, Table of Integral Transforms, vol.II, p.9, no. 26, we have

$$\tilde{W}(v, y^*-y) = \frac{Ai \exp iK |y^*-y|}{K}$$

where K is as in 8.9.1.

We can now exhibit explicitly the results of the image technique of 8.8.2; we will give the displacement at $R^* = (x^*,y^*,z)$ due to a source at $R = (x,y,z)$, in the presence of a flexible plate at $y = 0$ bounding the semi-infinite ocean. $(y^*, y < 0)$. R' will be the position $(x^*, -y^*, z^*)$ of the image source. We observe that

u = Direct term from R^* to R
+ Direct term from R' to R
+ Term from R^* mediated through the plate
+ Term from R' mediated through the plate

The last two terms are equal, because the source and its image affect the plate in the same way. In the analog, this simply means that $S_v(y^*) = S_v(-y^*)$. Thus we have

$$u(R) = \frac{1}{\omega^2 \rho^*} \left[\frac{\partial W(R^*-R)}{\partial y} + \frac{\partial W(R'-R)}{\partial y} \right]$$

$$- \frac{2cA}{\omega^2 \rho^*} \int_0^\infty \frac{W(v,0)\ S_v(y^*)\ S_v(y)\ v\ J_o(v\ r)\ dv}{S_v(0) - (1/2\ F(v))}$$

The reason for taking the strength as 2F(v) was explained in 8.9.2. On substituting all the relevant expressions, the last term becomes

$$\frac{cA}{\omega^2 \rho^*} \int_0^\infty \frac{(v^4 - k^4)\ (\exp i\ K\ (|y^*| + |y|))\ v\ J_o(vr)dv}{(v^4 - k^4\ \ + ie/K)}$$

9. Resonators Attached Over an Area; Pistons

The concept of an axial resonator sensitive to the displacement over an area was first introduced in Section 2. Several special cases of this type of resonator will be studied in detail in the present section.

Let $\eta(r,\theta)$ be the displacement of the plate, where r and θ are polar coordinates. Then a resonator of strength F is defined to respond to the weighted integral of the displacement over the attachment domain:

$$\int \int w(r,\theta) \ \eta \ (r,\theta) r \ dr \ d\theta$$

by exerting the force $Fw(r,\theta)$at each point (r,θ). Note that w applies for both sensitivity and for response.

The weight function w need not be a real quantity. Then the force exerted by the resonator will vary in phase over the domain. This general case will not be considered here. In any event, the normalization condition for w can be taken as

$$\int \int |w(r,\theta)|^2 \ r \ dr \ d\theta$$

where the integral is taken over the domain.

In Addendum 2, it is shown that the interaction constant for two resonators with weight functions $w_1(r,\theta)$ and $w_2(r,\theta)$ when their origins are at the points given by the vectors s_1 and s_2, is

$$V_{12} = \int dx \ \ w_1(x-s_1) \ Q_2 \ (s_2-x)$$

$$= \int dx \ Q_1(x-s_1) \ w_2 \ (s_2-x)$$

where x represents a vector variable, and the integration is taken over the domain. Q_i, for $i = 1$ or 2, is the unique solution of

$$O(Q_i(x)) = w_i(x)$$

that satisfies the usual conditions of finiteness and no incoming waves at infinity. (Here O is the plate operator, which for a plate in a vacuum is $D(\nabla^4 - k^4)$).

9.1 Physical Resonators Analyzed Into Many Ideal Resonators

Many physical systems attached to plates over an area, for instance, solid cylinders of plastic-like material, or thickening discs welded to a plate, can be considered as resulting from the superposition of the ideal resonators just mentioned. All of the component resonators will have the attachment area as their domain of sensitivity and response.

In the special case of a physical resonator having circular symmetry, it is immediately clear that there will be a two-fold infinity of ideal resonators, since an arbitrary displacement of the circular base can be represented in the form

$$\eta(r,\theta) = \sum_i \sum_j b_i(r) \cos j\theta,$$

where the $b_i(r)$, for $i = 0, 1, 2, \ldots$ are a set of functions suitable for expanding an arbitrary function of one variable. We thus expect that in the absence of symmetry it will be necessary to use a two-fold inf'nity of component resonators.

All of these will be considered as having a common "origin" at the point s. The Green's function giving the displacement at a typical point caused by a concentrated force will involve the inverse of a matrix N, whose elements are the interaction constants between all possible pairs of component ideal resonators (minus diagonal terms containing the strengths of the resonators). It would obviously be convenient if N were a diagonal matrix, that is, if two distinct component resonators has zero interaction. If this is the case, we say that the component resonators are orthogonal to each other.

It is clear from symmetry considerations that two resonators with common origin will not effect each other if their weight-functions have the factors $\cos n\theta$ and $\cos m\theta$, with $m \neq n$. In the discussion of rim resonators in Sections 3 and 4, this condition was automatically fulfilled. For physical resonators on an area, the orthogonality problem must be studied explicitly. Thus even for the case of circular symmetry, there are an infinite number of modes with different radial dependence, for each fixed azimuthal dependence $\cos n\theta$ or $\sin n\theta$.

The orthogonality condition $\int dx\, w_i Q_j = 0$ makes reference to the properties of the plate alone. Suppose a set of w_i satisfying this relation has been found. If the base of a physical resonator is given the displacement $\eta(r,\theta) = w_i(r,\theta)$, there is no guarantee that the physical resonator will respond with a force varying proportionally to w_i. That is, the decomposition of the vibrations of a resonator into normal modes is in general not related to the orthogonality condition for weight functions on the plate.

We shall see, however, that in the absence of the water there are fortunate cases where the two methods of decomposition agree, namely when the physical resonator is a solid cylinder of plastic material, or a column of water in a tube with rigid side walls, or a thickening of the plate.

9.2 Cylinders of Plastic and Columns of Water

9.2.1 Plastic-Cylinder Resonator

The resonator is a solid cylinder of plastic, of length L and radius R. The material is assumed to be compressible, but unable to support shear waves. The boundary condition of zero pressure on the curved surface is taken. Introduce a cylindrical coordinate system based on the axis of the resonator. (See Fig. 9-1). Then the pressure at the base of the cylinder can be expanded in Bessel functions:

$$p(r,\theta) = \sum_n \sum_i c_{ni} J_i (k_{ni} r) e^{in\theta} \qquad ,$$

where each k_{ni} satisfies:

$$J_i(k_{ni}R) = 0$$

This condition is of course motivated by the boundary condition on the curved surface above the plate, but it has the effect that at the intersection of the curved surface and the plate, $p(R,\theta) = 0$ for each θ. This is exactly so for the pressure inside the cylinder very close to the plate, but not the force that the plate would exert on a non-yielding material. Thus there will be a singularity in the displacement of the plastic at the lower rim of the cylinder. However, this is a common feature of similar problems, and means simply that many terms must be taken to obtain accurate values of pressure near the circular edge of the cylinder.

Each term of the expansion above generates a traveling pressure wave that moves up the cylinder. The complete expression for a pressure wave in the plastic is

$$p_{ni}(r,\theta,z) = \text{const. } J_i(k_{ni}r)e^{in\theta}\ e^{-\sqrt{k_{ni} - k'^2}\, z} ,$$

where k' is the wavenumber for a plastic-filled infinite space. This pressure wave is reflected from the upper face of the cylinder. The boundary condition of zero pressure will produce a free upper face. The alternative condition of zero normal derivative of pressure can also be used.

On the passage outward, the wave is attenuated by the factor $e^{-\sqrt{k_{ni}^2 - k'^2}L}$, and by the same factor on the passage back. Thus the physical cylinder has been represented as a sum of a doubly infinite set of Bessel function resonators, of which the (n,i)-th has the weight function

$$w_{ni}(r) = \text{const. } x\ J_n(k_{ni}r),\ r < R$$

and the strength

$$F_{ni} = \text{const. } x\ e^{-2\sqrt{k_{ni} - k'^2}\, L} \qquad \begin{matrix} i = 1,2,3,\ldots \\ n = 0, \pm 1, \pm 2,\ldots \end{matrix}$$

In some cases of interest R is considerably smaller than the free wave length in the plastic, and L is large. Then all the waves will be strongly attenuated by the passage out and back, but the one with the smallest value of k_{ni} will be least attenuated. The least value occurs for n=0 and i=1. If only F_{01} is significantly different from zero, we see that the plastic cylinder can be approximated by a single ideal resonator whose weight-function is a truncated Bessel function.

9.2.2 Column of Water

A very similar problem arises when the solid cylinder is encased in a rigid cylindrical wall that produces the condition of zero normal displacement. Slippage is allowed at the inner surface of the wall. Thus the material is effectively a liquid. The wall is assumed weightless, and does not affect the plate directly. In an idealization closer to a physical system, the wall would be replaced by a rim axial resonator, such as has been studied in Sections 3 and 4.

The pressure at the base can once more be expanded in Bessel functions, but this time the boundary condition is

$$d\, J_i(k_{ni}R)/dR = 0.$$

The other details of the treatment are the same.

9.2.3 Verification of the Plate Orthogonality Condition in Absence of Water

In each of the problems above, the displacement of the attachment area was expanded in the set of orthogonal solutions of the membrane equation

$$(\nabla^2 + k'^2)\, p = 0$$

for the appropriate boundary conditions. We generalize now to consider cylinders standing on an arbitrary area A, and introduce expansions in terms of the solutions U_i of the membrane equation that satisfy the general linear homogeneous condition

$$aU_i + b\,(\partial U_i/\partial n) = 0,$$

where a and b are quantities that may depend on the boundary point, and $\partial/\partial n$ indicates differentiation with respect to the outward normal. Let k_i be the eigenvalue corresponding to U_i. We observe that

$$O(U_i) = D(\nabla^4 - k^4)\, U_i = D(k_i^4 - k^4)\, U_i ,$$

and therefore

$$\int_A dx\, U_j O(U_i) = D(k_i^4 - k^4) \int_A dx\, U_j\, U_i \quad .$$

But as is well known, any two distinct solutions of the membrane equation that satisfy the boundary conditions above are orthogonal. That is, the integral on the right is zero. Thus we see: if w_i is taken as a normalization factor N_i times U_i, then Q_i will equal $N_i U_i / D(k_i^4 - k^4)$, and the plate orthogonality condition will be satisfied.

9.3 Uniform Thickening Handled by Analysis into Resonators

Consider an infinite plate that has thickness h except over a finite area A, where the thickness is h'. The corresponding flexural rigidity will be denoted by D', and the free wavenumber by k'. Suppose further that there is a unit source at (r^*, θ^*), which is not in A. Then the displacement is of course given by:

$$D(\nabla^4 - k^4)\eta(x,z) = \delta(x^* - x)\delta(z^* - x), \quad (x,z) \text{ not in A}$$
$$D'(\nabla^4 - k'^4)\eta(x,z) = 0, \quad (x,z) \text{ in A}$$

together with junction conditions on the boundary of A. Let $X_A(x,z)$ be the function that is 1 if (x,z) is in A, and zero otherwise. Then we can combine the equations above as

$$D(\nabla^4 - k^4)\eta(x,z) = \delta(x^* - x)\delta(z^* - z) + $$
$$+ X_A(x,z)\, [(D - D')\nabla^4 - (Dk^4 - D'k'^4)]\eta(x,z)$$

for (x,z) not on the boundary of A. The junction conditions are that η and $\partial\eta/\partial n$ are continuous across the boundary, but $\nabla^2\eta$ and $(\partial/\partial n)\nabla^2\eta$ are multiplied by a common factor γ (see section 5.5.1).

If A is circular, then the discontinuities on the boundary can be produced by attaching devices which are generalizations of the pseudoresonators of types R_3 and R_4 introduced in section 6.2. These new forms will exert their effects on <u>rims</u> rather than <u>points</u>. The displacement due to such resonators need not be discussed in detail, since the Green's functions can be obtained by differentiations from $Q_R(r)$ for a rim axial resonator; similarly for the interaction integrals (including those with types whose domain is an area).

We shall therefore simplify our problem by assuming that η satisfies the above equation at <u>all</u> points, <u>including</u> the boundary of A.

The second term on the right of the last equation will now be transformed so that it has the form appropriate to an infinite collection of axial resonators, each of which has the domain A.

Consider the plate equation for an arbitrary wavenumber λ, in the absence of forces:

$$(\nabla^4 - \lambda^4)\, U\,(x,z) = 0,$$

and impose any of the standard conditions (free edge, simple support, clamping) on the boundary of A. Let the U_i be the set of eigenfunctions, that is, normal modes, and the λ_i the set of corresponding eigenvalues. (Properly speaking, there should be two subscripts on U and λ, but this is inconsequential.) As is well known, the U_i are orthogonal to each other, and form a complete set for the expansion of any function over A. They may be normalized to unity:

$$\iint_A U_i^2\, dx\, dz = 1$$

Now expand $\eta(x,z)$ in the $U_i(x,z)$, for (x,z) in A:

$$\eta(x,z) = \sum_i c_i\, U(x,z),$$

where the expansion coefficients are given by

$$c_i = \iint_A U_i(x,z)\; \eta\,(x,z)\; dx\, dz\; .$$

It follows that

$$[(D - D')\nabla^4 - (Dk^4 - D'k'^4)]\eta$$

$$= \sum_i [(D-D')\lambda_i^4-(Dk^4-D'k'^4)][\iint_A U_i(x',z')\eta(x',z')dx'dz']U_i(x,z)$$

Then the equation of motion has the form appropriate to an infinite plate bearing an infinite collection of resonators with weight-functions $U_i(x,y)$ and strengths

$$F_i = (D-D')\lambda_i^4 - (Dk^4-D'k'^4).$$

The choice of the boundary conditions remains open. Clearly they should be chosen so that the F_i form a rapidly convergent series. Another criterion is that the expansion $\eta = \Sigma\, c_i\, U_i$ must converge quickly. Thus the U_i should approximate as well as possible the motion of the thickened portion when it is part of the plate. If h'/h is very large, then the use of free edge conditions will ensure this. If h'/h is very small, then clamping conditions are appropriate. More general boundary conditions would be desirable for less extreme situations.

9.4 Green's Functions for Special Types of Ideal Area Resonators

9.4.1 Disc Resonators

A disc resonator is defined as having a weight-function that is constant within a circle of radius R, and zero outside the circle. Thus it responds to the average displacement on its attachment area by exerting a uniformly-distributed pressure. Since the total pressure is normalized to unity,

$$w(r) = \begin{cases} 1/\pi R^2 & \text{if } r < R \\ 0 & \text{if } r > R \; . \end{cases}$$

The equation $O(Q_R^D) = w$ for the corresponding Green's function can be solved by the familiar technique of taking the Hankel transforms of both sides. Then we have

$$\tilde{Q}_R^D(v) = f^{-1}(v)\, \tilde{w}(v),$$

where the tilde indicates the transform as usual. The function $f(v)$ was defined in section 8.1, for the case of water on one side of the plate, but we do not exclude the case of a plate in vacuum. A standard formula for Bessel functions shows that

$$\tilde{w}(v) = \int_0^R r\, J_0(vr)\, dr = R\, J_1(vR)/v.$$

Then the Green's function might be obtained by inverting the $\tilde{Q}_R^D$, but it can also be obtained from the Green's function Q_R for a force uniformly distributed over a rim (see section 3.2.1) by integrating $R'Q_{R'}$ from $R' = 0$ to $R' = R$. Note that by reciprocity, $Q_R^D(r)$ equals the displacement of the plate, averaged over a disc of radius R, when a point force is applied at distance r from the center.

The interaction integral between two disc resonators can be evaluated by a double integration of the corresponding result for a pair of rim axial resonators.

9.4.2 Truncated Bessel Function Resonators

The problem of the plastic cylinder leads naturally to the truncated Bessel resonators, defined by

$$w_n(r) = \begin{cases} (1/N_n) \ J_n \ (\lambda_n r) \cos n\theta, & r < R \\ 0 & r > R \end{cases}$$

where n is zero or a positive integer, $1/N_n$ is a normalization factor, and λ_n, is for the moment, an unspecified constant. There will of course be another series with sines instead of cosines (except that no sine resonator exists for n = 0). The Green's function Q for the n-th cosine resonator can be expected to have the form

$$Q_n(r) \cos n\theta \quad .$$

Applying the Hankel transform of order n to the defining equation

$$O(Q) = w_n \ ,$$

we see once more that

$$\tilde{Q}_n(v) = f^{-1}(v) \ \tilde{w}_n(v) \quad ,$$

where the tilde now indicates the transform of order n

A standard argument based on the differential equation for Bessel's functions now shows that

$$N_n\tilde{w}_n(v) = \int_o^R J_n(\lambda R) \ J_n(vr) r \ dr$$

$$= \frac{1}{\lambda^2 - v^2} \left(- \frac{d \ J_n(\lambda R)}{dR} \ J_n(vR) + \frac{d \ J_n(vR)}{dR} \ J_n(\lambda R) \right)$$

In the special case where λ is determined by the condition $J_n(\lambda R) = 0$ (see Section 9.3.1), one term disappears, and then $Q_n(v)$ has a comparatively simple form:

$$N_n Q_n(r) = -(dJ_n(\lambda R)/dR)\int \frac{J_n(vR)J_n(vr)v\,dv}{f(v)(\lambda^2-v^2)}$$

When there is no water, that is to say $f(v) = D(v^4-k^4)$, then $1/D(v^4-k^4)(\lambda^2-v^2)$ can be expanded in partial fractions, and then the resulting integrals are forms that have already been encountered. (They are special cases of the integral of 10.1, for $i = n$, $j = -n$, $k = 0$, $T = 0$).

The integral that appears when λ satisfies the condition $dJ_n(\lambda R)/dR = 0$ is found from the form for $J_n(\lambda R) = 0$ by differentiating with respect to R.

The interaction integral V between two truncated Bessel resonators can be written down easily by reference to Addendum 3 and the form for $\tilde{w}_n(r)$ given above. In the absence of water a partial fraction expansion is again possible. The algebra becomes lengthy however. Thus for two resonators with circular symmetry (n_1, $n_2 = 0$) having radii R_1, R_2, centers s_1, s_2, constants λ_1, λ_2, and satisfying

$$J_o(\lambda_i R_i) = 0, \qquad i = 1,2,$$

we have

$$N_1 N_2 V_{12} = \lambda_1 J_1(\lambda_1 R_1)\lambda_2 J_1(\lambda_2 R_2)\int \frac{J_o(vR_1)J_o(vR_2)J_o(v|s_1-s_2|)\,v\,dv}{f(v)(\lambda_1^2 - v^2)(\lambda_2^2 - v^2)}$$

But in view of Section 9.2.3, V_{12} will be zero if $s_1 = s_2$, $R_1 = R_2$, and $\lambda_1 \neq \lambda_2$.

9.4.3 Resonators Corresponding to Normal Modes of a Circular Plate

The decomposition of a circular thickening according to the procedure of Section 9.3 produces ideal resonators whose weight-functions have the form

$$w(r) = \begin{cases} a\,J_o(\lambda r) + b\,I_o(\lambda r) & r < R \\ 0 & r > R \end{cases},$$

as well as types with an azimuthal factor cos nθ or sinθ. These can be treated in the same way as the truncated Bessel resonators, but of course there are twice as many terms in the Green's functions and four times as many in the interaction integrals.

9.5 Piston Cut Out of a Flexible Baffle

We consider an infinite flexible plate bounding on ocean on one side. In the two-dimensional configuration to be discussed in Section 9.5.1, an infinite strip is severed from the rest of the plate by two parallel, infinitely long, cuts and an oscillating force is applied to the strip. In the three-dimensional problem of Section 9.5.2 a circular disc is severed from the rest of the plate, and a circularly symmetric force distribution is applied on the disc. Then more general force distributions are considered.

9.5.1 An Infinite-Strip Piston

A line source applies an oscillating force along the midline of an infinite strip cut out of a plate. Let s_1 and s_2 be the ends of a typical cross-section of the piston (see Fig. 9-2). Consider first that there are no breaks at s_1 and s_2. By attaching quadrupole and octupole resonators of infinite strength at those points, the portion between them can then be detached from the rest of the cross-section. Thus the formula for the Green's function $G(r^*,r)$ in terms of the point-to-point function $Q(x)$ will be exactly the same as for a rod of length $|s_1-s_2|$ in air (Section 6.1.5), except of course that the meaning of Q is now different. That is, $Q(x)$ has the form

$$\int_0^\infty \frac{\cos xp\, dp}{f(p)}$$

which cannot be evaluated in finite terms. The interaction matrix will have exactly the form given in 6.1.5.

It must be verified that the boundary conditions for a break in the plate are not modified by the presence of the water. To see this, we imagine that the water extends just up to the upper surface of the plate. Now $\partial^3\eta/\partial x^3$ is proportional to the shear force delivered across the boundary of a strip of the plate. However, the water cannot support a shear force, since we assume it to be perfectly non-viscous. Therefore $\partial^3\eta/\partial x^3 = 0$ at a break. The same reasoning applied to the moment shows that $\partial^2\eta/\partial x^2 = 0$.

The crack between strip and baffle can be considered as the limit of a gap of finite width. An arbitrary boundary condition might be imposed on the water surface in such a gap, in particular, either zero pressure or zero displacement. At the edge of the plate, $\partial^2\eta/\partial x^2 = 0$ and $\partial^3\eta/\partial x^3 = 0$ would still hold. A piston is obtained in the limit only if the boundary condition is taken as zero displacement, that is, only if there is no pressure release at the crack. The horizontal component of the water velocity should become infinite there, but not the vertical component. Now the construction of quadrupole and octupole resonators from a rim axial resonator requires only differentiations normal to the crack. It is thus reasonably clear that the vertical component of velocity will be finite in the presence of these types. However, the repeated differentiations will cause singularities in the horizontal component of velocity, since the Green's function for a rim axial resonator has a discontinuous third derivative.

9.5.2 Problems with Circular Symmetry

We first treat a simpler problem which does not involve a piston.

9.5.2.1 Flexible Plate Clamped Along a Circumference $|x - s| = S$

We consider a force applied along a circle of radius R^*. The displacement can be constrained to be zero along $|x-s| = S$ by attaching a rim axial resonator of infinite strength. Let $Q_o(R_1, R_2)$ be the displacement at radius R_2 due to a unit force applied on a concentric circle of radius R_1. Then the Green's function $G(R^*, R)$ when the resonator is attached is

$$G(R^*,R) = Q_o(R^*,R) - \frac{Q_o(R^*,S)Q_o(S,R)}{Q_o(S,S)}$$

To constrain the slope to be zero on $|x-s| = S$, we must attach a rim moment resonator of infinite strength. The Green's function in the presence of this alone is

$$G(R^*,R) = Q_o(R^*,R) - \frac{\dfrac{\partial Q_o(R^*,S)}{\partial S} \; \dfrac{\partial Q_o(S,R)}{\partial S}}{\lim\limits_{S^* \to S} \dfrac{\partial^2 Q_o(S^*,S)}{\partial S^* \partial S}}$$

This may be verified by differentiating with respect to R^*, and then setting R^* equal to R.

When both force and moment resonators are attached at the same radius, there is a non-zero interaction between them, and the Green's function involves the inverse of a 2 x 2 interaction matrix:

$$G(R^*,R) = Q_o(R^*,R) - \begin{bmatrix} Q_o(R^*,S) & \dfrac{\partial Q_o(R^*,S)}{\partial S} \end{bmatrix} \begin{bmatrix} Q_o(S,S) & \lim\limits_{S^* \to S} \dfrac{\partial Q_o(S^*,S)}{\partial S} \\ \lim\limits_{S^* \to S} \dfrac{\partial Q_o(S^*,S)}{\partial S} & \lim\limits_{S^* \to S} \dfrac{\partial^2 Q_o}{\partial S^* \partial S} \end{bmatrix}^{-1} \begin{bmatrix} Q_o(S,R) \\ \dfrac{\partial Q_o(S,R)}{\partial S} \end{bmatrix}$$

The results hold of course whether or not there is water on one side of the plate. However, an explicit expression for Q_o in terms of Bessel functions is available only in the absence of water.

9.5.2.2 Piston of Radius S with Force Applied Uniformly Over a Circumference of Radius R.

The Poisson's ratio effects cause the treatment to more complicated than for a clamped edge.

The force at the edge, in the case of circular symmetry, is proportional to $\partial^3\eta/\partial S^3$. Thus the force can be made zero by attaching a resonator which behaves as an octupole resonator at each point along the rim. The Green's function for displacement is then

$$G(R^*,R) = Q_o(R^*,R) - \frac{\dfrac{\partial^3 Q_o(R^*,S)}{\partial S^3} \quad \dfrac{\partial^3 Q_o(S,R)}{\partial S^3}}{\lim\limits_{S^*\to S} \dfrac{\partial^6 Q_o(S^*,S)}{\partial^3 S^* \partial^3 S}}$$

The moment at a free edge is proportional to

$$\nabla^2 \eta + \frac{\sigma - 1}{r} \frac{d\eta}{dr} = \frac{d^2\eta}{dr^2} + \frac{\sigma}{r} \frac{d\eta}{dr} \quad .$$

If we use $D_r^{(2)}(\eta)$ to indicate this expression, in the form of an operator $D_r^{(2)}$ applied to a function η of r, then the condition of zero force and zero moment on the edge are satisfied by:

$$G(R^*,R) = Q_o(R^*,R)$$
$$- \left[D_S^{(2)} Q_o(R^*,S) \quad \frac{\partial^3 Q_o(R^*,S)}{\partial S^3} \right] M \begin{bmatrix} D_S^2 Q_o(S,R) \\ \dfrac{\partial^3 Q_o(S,R)}{\partial S^3} \end{bmatrix}$$

where

$$M^{-1} = \lim_{S^*\to S} \begin{bmatrix} D_S^{(2)} \; D_S^{(2)} & D_{S^*}^{(2)} \dfrac{\partial^3}{\partial S^3} \\ \dfrac{\partial^3}{\partial S^{*3}} D_S^{(2)} & \dfrac{\partial^6}{\partial^3 S^* \partial^3 S} \end{bmatrix} \quad Q_o(S^*,S)$$

In the absence of water this gives the Green's function for a circular disc with a free edge of radius S, when a force is applied uniformly over a circle of radius R^*.

9.5.3 Circular Piston Cut Out of Flexible Baffle; Azimuthally-Varying Force

Problems with azimuthal variation can be handled by the standard procedure of composition into Fourier series. Thus consider a circular piston of radius S. An oscillating force is applied at each point at distance R* from the center, the magnitude varying as cos nθ.

If there were actually no circular cut, so that we had instead an infinite homogeneous plate, then the displacement at the point with polar coordinates (R,θ), due to the azimuthally-varying force on the circle of radius R* would be

$$Q_n(R^*,R)\ \cos n\theta \quad ,$$

where

$$Q_n(R^*,R) = \int_0^\infty f^{-1}(p)\,J_n(R^*p)\,J_n(Rp)\,p\ dp \quad .$$

This differs from the quantity $Q_{R^*,n}(R)$ of Section 3.2.2 only in a normalization factor. To simulate the effect of the cut, we add two fictitious rim resonators that vary azimuthally; neither is a pure quadrupole or pure octupole resonator because of the Poisson's ratio effect, but one results in zero moment at each point of the rim, and the other produces zero shear force.

To do this systematically, we introduce a differential operator $M^{(n)}$, the moment operator of index n:

$$M_r^{(n)}\ (Z(r)) = \left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{\sigma-1}{r}\left[\frac{d}{dr} - \frac{n^2}{r}\right]\right\} Z(r)$$

and $F_r^{(n)}$, the force operator of index n:

$$F_r^{(n)}(Z(r)) = \left\{\frac{d}{dr}\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right] + \frac{n^2}{r^2}\left[(2-\sigma)\frac{d}{dr} - (3-\sigma)\frac{1}{r}\right]\right\} Z(r)$$

If the displacement of the piston is of the form

$$\eta = Z(r)\ \cos n\theta$$

Then the conditions of zero moment and force can be written as

$$M_r^{(n)}\ Z(r) = 0$$

$$F_r^{(n)}\ Z(r) = 0, \qquad\qquad \text{for } r = S \quad .$$

Then by generalizing the argument of 9.4.2,the displacement at a typical point (R,θ), either on the piston or the surrounding baffle is given by:

$$G(R,\theta,n) = G_n(R^*,R)\ \cos n\theta \quad .$$

Here

$$G_n(R^*,R) = Q_n(R^*,R)$$

$$- \begin{bmatrix} M_S^{(n)} Q_n(R^*,S) & F_S^{(n)} Q_n(R^*,S) \end{bmatrix} M \begin{bmatrix} M_S^{(n)} Q_n(S,R) \\ F_S^{(n)} Q_n(S,R) \end{bmatrix}$$

where

$$M^{-1} = \lim_{S^* \to S} \begin{bmatrix} M_{S^*}^{(n)}\ M_S^{(n)} & M_{S^*}^{(n)}\ F_S^{(n)} \\ F_{S^*}^{(n)}\ M_S^{(n)} & F_{S^*}^{(n)}\ F_S^{(n)} \end{bmatrix} Q_n(S^*,S)$$

If we write

$$W = \begin{bmatrix} 1 & \dfrac{\partial}{\partial S} \\ \dfrac{\partial}{\partial S^*} & \dfrac{\partial^2}{\partial S^* \partial S} \end{bmatrix} Q_n(S^*,S)$$

and

$$C = \begin{bmatrix} \frac{n^2}{r} & -1 \\ -\frac{n^2}{r^2} & \frac{n^2}{r} \end{bmatrix} \left(\frac{1-\sigma}{r}\right), \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then we have also

$$M^{-1} = \lim_{\substack{S^* \to S \\ r \to S}} (C\ I) \begin{bmatrix} W & \nabla_S^2 W \\ \nabla_{S^*}^2 W & \nabla_{S^*}^2 \nabla_S^2 W \end{bmatrix} \begin{pmatrix} C \\ I \end{pmatrix}$$

$$= \lim \left[CWC + C\ \nabla_S^2\ W + \nabla_{S^*}^2 WC + \nabla_S^2 \nabla_{S^*}^2\ W \right]$$

9.6 <u>Piston Not of The Same Material as the Surrounding Plate</u>.

The method of 9.5 can be generalized to handle a flexible circular disc acting as a piston in a flexible baffle of another material. As before, $M^{(n)}$ and $F^{(n)}$ resonators will represent the break. Area resonators of the types introduced in Section 9.3 are needed to give the effect of the change in D and k. The pseudo-resonators of types R_3 and R_4, which were mentioned in that section, are now unnecessary.

The weight functions for the area resonators should clearly be taken as proportional to the normal modes of the circular disc in a vacuum, with free edge conditions. These component resonators will now interact with each other <u>through the water</u>. The integrals that give the interactions can be written down from the discussions of 9.4.2 and 9.4.3. One can expect that the water effects will be small, and thus the interaction matrix is nearly diagonal.

10. EVALUATIONS OF INTEGRALS ENCOUNTERED EARLIER

Most of the integrals that have appeared in previous sections can be subsumed under two general forms: The integrals for displacement are special cases of

$$I = \int_0^\infty f^{-1}(p) J_i(Rp) J_j(Sp) J_k(Tp) p \, dp$$

where i, j and k are non-zero integers, and R.S.T. are ≥ 0. By taking i = 0 and R = 0 one of the Bessel functions disappears. On setting i = 1 and going to the limit $R \to 0$, one obtains a form involving p^2 rather than p. On the other hand, the integrals for pressure are similar except for an extra factor $(p^2 - k*^2)^{-1/2}$.

10.1 Absence of Water

We recall that

$$f(p) = D(p^4 - k^4 - e(p^2 - k*^2)^{-1/2}) \ .$$

Now if e = 0 (that is, the effect of the water can be neglected) then the I integral can be evaluated for the values of i,j,k,R,S, and T that have appeared in our work. In this case

$$\frac{1}{f(p)} = \frac{1}{2k^2 D} \left[\frac{1}{p^2 - k^2} - \frac{1}{p^2 + k^2} \right] .$$

We define

$$I* = \int_0^\infty (p^2 + k^2)^{-1} J_i(Rp) J_j(Sp) J_k(Tp) p \, dp$$

and I** as the analog with $p^2 - k^2$ in place of $p^2 + k^2$. Now we apply formula 9 on p. 430 of Watson, Bessel Functions, in a special case:

$$\int_0^\infty dx \frac{x}{x^2+k^2} J_i(Rx) \ J_i(Sx) \left[\cos \tfrac{1}{2} (2+i+j-k)\pi \ J_k(Tx) \right.$$
$$\left. + \sin \tfrac{1}{2} (2+i+j-k)\pi \ Y_k(Tx) \right]$$
$$= - I_i(Rk) \ I_j(Sk) \ K_k(Tk) \quad .$$

This is valid for non-integral i,j,k provided $T > R + S$ and $2 + i + j > k$. The following special case will be sufficient for us:

If $i + j - k$

is even

and $i + j - k > -2$

and

$T > R + S$

then

$$I^* = (-1)^{(i+j-k)/2} I_i(Rk) I_j(Sk) K_k(Tk) \quad .$$

I^* is an analytic function of the wavenumber k, except at $k = 0$, and so the form for I^* can be deduced from that for I^{**} by replacing k by -ik and using the formulas

$$I_n(z) = \exp(n\pi i/2) J_n(iz)$$

$$K_n(z) = (\pi i/2) \exp(n\pi i/2) H_n^{(1)}(iz)$$

Therefore

$$I^{**} = (-1)^{(i+j-k)/2} (\pi i/2) (\exp(i+j+k)\pi/2) J_i(Rk) J_j(Sk) H_k^{(1)}(Tk)$$

The choice of the first, rather than the second, Hankel function is of course due to our convention that the time dependence is $\exp(-i\omega t)$. Since

$$\exp(i+j+k)\pi/2 = (-1)^{(i+j+k)/2}$$

we have if $e = 0$,

$$I = \frac{1}{2k^2 D} \left[-I_i(Rk) I_j(Sk) K_k(Tk) \right.$$

$$\left. + (\pi i/2)(-1)^{(i+j+k)/2} J_i(Rk) J_j(Sk) H_k^{(1)}(Tk) \right]$$

10.2 Rational Contribution in the Presence of Water

It appears that the integral I cannot be evaluated exactly in the presence of the water, and this is true of the pressure integral even without water. However, they can be broken up into terms that can be interpreted physically in the various special cases of interest. Certain of these terms can be evaluated. We shall derive approximations for the others.

10.2.1 Rationalization of Denominators

10.2.1.1 Displacement Integral

We write:

$$\frac{D}{f(p)} = \frac{p^4 - k^4 + e(p^2-k^{*2})^{-1/2}}{(p^4-k^4)^2 + e^2(k^{*2}-p^2)^{-1/2}}$$

$$= \frac{(p^4-k^4)(k^{*2}-p^2) - e(p^2-k^{*2})^{1/2}}{P(p^2)}$$

$$= \frac{(p^4-k^4)(k^{*2}-p^2)}{P(p^2)} - \frac{e(p^2-k^{*2})}{P(p^2)(p^2-k^{*2})^{1/2}} \ .$$

where

$$P(p^2) = (p^4-k^4)^2(k^{*2}-p^2) + e^2$$

$P(p^2)$ is a fifth-order polynomial in p^2. Let the roots be $p_1^2, \ldots, p_5^2$.

Then we can write

$$\frac{D}{f(p)} = \sum_{i=1}^{5}\left[\frac{A_i}{p^2-p_i^2} - \frac{e\,B_i}{(p^2-p_i^2)(p^2-k^{*2})^{1/2}}\right]$$

where A_i and B_i are evaluated by comparing residues at the poles:

$$A_i = -(p_i^4 - k^4)B_i$$

$$B_i = \frac{(p_i^2 - k*^2)}{\prod_{j\neq i} (p_j^2 - p_i^2)}$$

10.2.1.2 <u>Pressure Integrals</u>

Computation of pressure involves integrals of the form

$$P = \int_0^\infty \frac{J_i(pR)\, J_j(pS)\, J_k(pT)\, p\; dp}{g(p)}$$

where $g(p) = \sqrt{p^2-k*^2}\; f(p)$. The denominator is again rationalized:

$$\frac{D}{g(p)} = \frac{1}{\sqrt{p^2-k*^2}\,(p^4-k^4) - e}$$

$$= \frac{\sqrt{p^2-k*^2}\,(p^4-k^4) + e}{P(p^2)}$$

$$= \frac{e}{P(p^2)} + \frac{(p^4-k^4)\,(p^2-k*^2)}{P(p^2)\sqrt{p^2-k*^2}}$$

Now expanding in partial fractions, we have:

$$\frac{D}{g(p)} = \sum_{i=1}^{5} \left[\frac{eC_i}{p^2-p_i^2} + \frac{D_i}{(p^2-p_i^2)\sqrt{p^2-k*^2}} \right]$$

where

$$C_i = \frac{1}{\prod_{j\neq i} (p_j^2 - p_i^2)}$$

$$D_i = (p_i^4-k^4)\,(p_i^2-k*^2)\,C_i = -A_i$$

10.2.2 Recursive Computation of The Roots of $P(p^2)$

If $e = 0$, then the roots of $P(p^2)$ are $k*^2$, k^2, k^2, $-k^2$, $-k^2$. When e^2 increases from zero, each pair of roots will separate. Writing the equation $P(p^2) = 0$ in the form

$$p^2 - k*^2 = \frac{e^2}{(p^4-k^4)^2}$$

suggests a recursive procedure for p^2:

$$p^2_{(i+1)} = k*^2 + \frac{e^2}{(p^4_{(i)} - k^4)^2}$$

This will converge if e^2 is small enough and one starts with $p^2_{(0)} = k*^2$. We find

$$p^2_{(1)} = k*^2 + \frac{e^2}{(k*^4-k^4)^2} = k*^2 \left[1 + \frac{e^2}{k*^2(k*^2-k^2)^2} \right]$$

On the other hand, the form

$$p^4 - k^4 = \frac{\pm e}{\sqrt{p^2-k*^2}} \quad ,$$

yields the recursion relation

$$p^2_{(j+1)} = \pm\sqrt{k^4 \pm e(p^2_{(j)} - k*^2)^{-1/2}} \quad ,$$

where the $\pm$ signs are independent of each other, but the choices of signs must not be changed from one step of the recursion to the next. Thus four different series $p^2_{(j)}$ are possible.

If one starts with $p^2_{(0)} = k^2$ then the two procedures obtained by the positive choice of the left $\pm$ sign will yield real values at all stages, provided that e is sufficiently small and $k > k*$. In particular

$$p^2_{(1)} = k^2\sqrt{1 \pm \frac{e}{k^4\sqrt{k^2-k*^2}}} .$$

In a typical situation of practical interest, k/k* is about 4.

If the left ± sign is taken as negative, and one starts with $p^2_{(0)} = -k^2$, then one obtains

$$p^2_{(1)} = - k^2\sqrt{1 \pm \frac{e}{k^4\sqrt{k^2+k*^2}}}$$

and the $p^2_{(j)}$ are complex.

10.2.3 "Rational" and "Irrational" Parts of the Green's Funtion

It is convenient to refer the sum of the terms having the factor $1/\sqrt{p^2-k*^2}$ in the expression given in 10.2.1 for D/f as the "irrational" part of D/f, and the remaining terms as the "rational" part. Similarly for D/g. This terminology can also be extended to the integral I for the displacement and the corresponding integral for the pressure. When e is small, the rational part of the displacement will be larger then the irrational part. The situation is reversed for the pressure.

The common approximation of a "locally reactive" plate leads to a Green's function that is the sum of "rational" and "irrational" terms having the same forms as given above, but with a different root p_i and different coefficients. We recall that the assumption of "local reactivity" means that the elastic part of the plate response can be neglected, but the inertial term in f(p) is changed to allow a more general dependence on ω. Then we have

$$\frac{D}{f(p)} = \frac{1}{c^4 - (p^2-k*^2)^{-1/2}}$$

where c is a function of ω.

Now rationalizing, we have:

$$\frac{D}{f(p)} = \frac{c^4(p^2-k*^2)}{P_1(p^2)} - \frac{e(p^2-k*^2)}{P_1(p^2)\sqrt{p^2-k*^2}}$$

where

$$P_1(p^2) = c^8(p^2 0 k*^2) + e^2$$

Thus D/f is the sum of a rational and an irrational term of the kind encountered in 10.2.1. The root of P_1 is obviously

$$p^2 = k*^2 - e^2\ c^{-8}\ .$$

10.2.4 <u>Interpretation of the Roots of $P(p^2) = 0$</u>.

A root p_i^2 contributes the following term to the rational part of integral I:

$$I_i = \frac{A_i}{D} \int_0^\infty \frac{J_i(Rp)J_j(Sp)J_k(Tp)p\ dp}{p^2 - p_i^2}$$

If p_i^2 is real and positive, this has the same form as I**, discussed in Section 10.1.

It is interesting to examine the special case j = k = 0, S = T = 0, in which case the integral I is proportional the Green's function Q(R) for a plate without resonators (see Section 3.1.3). The term I_i then corresponds to

$$\frac{A_i}{2\pi D} \int_0^\infty \frac{J_o(rp)p\ dp}{p^2 - p_i^2} = \frac{A_i}{2\pi D} H_o(p_i r)$$

Such a term corresponds to an outgoing wave. When e is small, there will be three real roots p_i^2 of $P(p^2)$, of which one is extremely close to $k*^2$; the corresponding term must represent an acoustic wave in the water, as modified slightly by the presence of the plate. The other two real roots are close to k^2. Thus the water coupling apparently produces two distinct but close wavenumbers in the plate.

The remaining roots p_i^2 are complex conjugates and have small imaginary parts and real parts near $-k^2$. If the imaginary parts were zero, these would correspond to standing waves on the plate. The small imaginary parts, however, indicate that there is also a small propagation effect associated with each root, one having incoming characteristics, the other outgoing. However, these effects of the two conjugate roots largely cancel, and produce a standing disturbance. An approximation for the resultant behavior is given in Addendum 7.

10.2.5 Comparison with Treatment of Morse and Ingard

Morse and Ingard (in Encyclopedia of Physics, vol. XI/1, pages 108-116) consider the integral I for the special case S = T = 0, as well as the corresponding integral for the pressure. They do not rationalize the denominator of the integrand, as was done in 10.2.1. Then a zero of $P(p^2)$ will contribute a residue term only if it lies on the sheet of the Riemann surface containing the integration path. Or in simpler terms: Certain of the zeroes of $P(p^2)$ correspond to the wrong choice of sign in front of the square root that appears in f(p). Physically, there will be no outgoing wave for such a wavenumber. Morse and Ingard speak of three contributing values of p_i^2, which are near k^{*2}, k^2, and $-k^2$. This does not seem to be correct, as it is easy to show that the value near k^{*2} is non-contributing. We first observe that $D(p^4-k^4)$ is an increasing function of p, if $p > 0$, while $De(p^2-k^{*2})^{-1/2}$ is a decreasing function of p for $p > k^*$. The equation $f(p) = 0$ is the same as

$$D(p^4-k^4) = De(p^2-k^{*2})^{-1/2},$$

and so there is only one positive real root of $f(p) = 0$. It is always greater than k, and for very small ω, it is given by $p = e^{1/5}$.

There is a second root of $f(p)$ (i.e., a zero of $P(p^2)$ on the physical sheet) near $p = ik^*$. It has a small positive real part. There is also a non-physical root nearby, with negative real part.

In the range of frequencies and material parameters of interest to us, for which $k^* < k$, there are two further non-physical roots of $P(p^2)$ between $p^2 = k^{*2}$ and $p^2 = k^2$. They are roots of the equation

$$D(p^4-k^4) = - De(p^2-k^{*2})^{-1/2}$$

As the frequency increases, they approach each other, coalesce, and then become complex.

There seems to be another, unrelated, difficulty in Morse and Ingard's discussion of the related membrane problem, because they speak on p. 110 of the point near k^* as being a branchpoint, whereas it is a pole, and the branch point is exactly at k^*. One must expect that a large error will be introduced by such an approximation.

Nevertheless, one must reconcile the two roots that actually contribute with the five roots of $P(p^2)$ that appear in the formulas of 10.2.1. It is clear that after rationalization, the fraction $D/f(p)$ has the form $D\overline{f}(p)/f(p)\overline{f}(p)$, where the bar on f indicates taking the opposite sign of the square root. At a non-contributing root, $f(p) = 0$ but $\overline{f}(p) \neq 0$. Thus our form of the integrand is 0/0 at such a root. But we then break up $\overline{f}(p)$ into "rational" and "irrational" parts, which are each non-zero at the root. The value of the rational part is entirely due to the pole and thus is an outgoing wave, as indicated in 10.2.4. Furthermore, each of the five irrational integrals behaves asymptotically like an outgoing wave as T goes to infinity. For a root that actually is non-contributing, this asymptotic wave cancels with the true wave produced by the rational part. This is shown explicitly in section 10.3.9. All the irrational integrals remain finite as T goes to zero, and the displacement is also finite at $T = 0$. Since each rational integral produces a logarithmically infinite term proportional to $Y_o(pr)$, it follows that these must cancel, and therefore,

$$\sum_1^5 A_i = 0.$$

The same cancellation already occurs in the simple problem of the flexible plate in free space (Section 3.1.3).

The value of our treatment is that the original integral is rewritten as a sum of five integrals of relatively simple form, each a function of only one root p_i^2. The same form also appears in the problems of a locally-reactive boundary, or a membrane bounding a semi-infinite ocean. In our formulation, three extraneous waves appear in the intermediate steps of the calculation but these cancel in the final result.

10.3 "Irrational Part" of the Point-to-Point Green's Function

We use λ^2 instead of p_i^2 to represent a root of $P = 0$, and introduce the notation

$$h(p) = \sqrt{p^2 - k^{*2}}\,(p^2 - \lambda^2)$$

$$G_{ijk} = \int_o^\infty h^{-1}(p)\; J_i(pR)\; J_j(pS)\; J_k(pT)\; p\; dp$$

$$G_o(T) = \int_o^\infty h^{-1}(p)\; J_o(pT)\; p\; dp$$

10.3.1 Differential Equation for G

One can easily show that G_{ijk}, considered as a function of T, satisfies what may be called an inhomogeneous Bessel equation, that is Bessel's equation with a non-zero right-hand side. We first define ∇_T^2 by

$$\nabla_T^2 = \frac{\partial^2}{\partial T^2} + \frac{1}{T}\frac{\partial}{\partial T}$$

Then on applying ∇_T^2 to G_{ijk}, taking the operator under the integral sign, and using Bessel's equation for $J_k(pT)$, we have

$$(\nabla_T{}^2 - \frac{k^2}{T^2})\ G_{ijk} = - \int_0^\infty \frac{J_i(pR)\ J_j(pS)\ J_k(pT)\ p\ dp}{\sqrt{p^2 - k^{*2}}} - \lambda^2 G_{ijk}$$

G_{ijk} reduces to $G_o(T)$ in the special case $i = j = k = 0$, $R = S = 0$, and the integral on the right can then be evaluated. Thus $G_o(T)$ satisfies the equation

$$(\nabla_T{}^2 + \lambda^2)\ G_o(T) = \frac{\exp\ ik^* T}{iT}\ .$$

This equation can be further transformed into a homogeneous equation of fourth order by applying the operator (related to the 3-dimensional Laplacian) that annihilates the right-hand side. Thus

$$(\frac{d^2}{dT^2} + \frac{2}{T}\ \frac{d}{dT} + k^{*2})\ (\frac{d^2}{dT^2} + \frac{1}{T}\ \frac{d}{dT} + \lambda^2)\ G_o(T) = 0.$$

However, we shall not make any use of this form.

It can be seen from Addendum 3 that G_o is a multiple of the two-dimensional convolution of the functions $(\exp\ ik^* z)/z$ and $H_o^{(1)}(\lambda z)$. Thus G_o is directly related to the Green's function for a process in which there is <u>exactly</u> <u>one</u> interchange of energy between a membrane and water. However, the wavenumber for such a membrane is not k or ik, but is λ. The wavenumber of the water is not modified.

10.3.2 Solution by Formal Expansion

10.3.2.1 A Bessel Function Identity

The following formula is valid for any real ν:

$$\left(\frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} + 1 \right) y^{\nu} C_{\nu}(y) = 2\nu\, y^{\nu-1} C_{\nu-1}(y),$$

where C indicates a Bessel Function of the first or second kind. To show this, we write the parentheses on the left as:

$$\frac{1}{y} \frac{d}{dy} y \frac{d}{dy} + 1$$

On applying this to $y^{\nu} C_{\nu}(y)$, we obtain

$$y^{\nu} \left[C''_{\nu} + \frac{2\nu+1}{y} C'_{\nu} + C_{\nu} + \frac{\nu^2}{y^2} C_{\nu} \right]$$

Now subtract and add

$$\frac{2\nu}{y} C'_{\nu} + \frac{2\nu^2}{y^2} C_{\nu} ,$$

within the bracket, and combine terms to obtain

$$y^{\nu} \left[C''_{\nu} + \frac{1}{y} C'_{\nu} + \left(1 - \frac{\nu^2}{y^2} \right) C_{\nu} + \frac{2\nu}{y} \left(C'_{\nu} + \frac{\nu}{y} C_{\nu} \right) \right]$$

But the sum of the first three terms of the bracket is zero, since C_{ν} is a Bessel function. The desired formula then follows from the recurrence relation

$$C_{\nu-1}(y) = C'_{\nu}(y) + \frac{\nu}{y} C_{\nu}(y)$$

10.3.2.2 Formal Inversion of Differential Operator

We now observe that

$$\frac{\exp\, ik^*T}{iT} = \sqrt{\pi/2}\, k^* \, (k^*T)^{-1/2} \times$$

$$\left[J_{-1/2}\,(k^*T) + i\, Y_{-1/2}\,(k^*T)\right]$$

and further that

$$\nabla_T^2 + k^{*2} + \lambda^2 - k^{*2} = k^{*2}\,(\nabla_y^2 + 1 + t^2)$$

where $y = k^*T$, and we have introduced the important abbreviation

$$t = \sqrt{\lambda^2 - k^{*2}}\,/k^*.$$

Thus the equation for G_o in Section 10.3.1 can be written

$$(\nabla_y^2 + 1 + t^2)\; G_o = i\,\sqrt{\pi/2}\; k^{*-1}\, y^{-1/2}\left[J_{-1/2}(y) + i\, Y_{-1/2}(y)\right]$$

We can formally invert the operator in a power series, as follows:

$$(\nabla_y^2 + 1 + t^2)^{-1} = \frac{1}{(\nabla_y^2 + 1)\left[1 + t^2/\left(\nabla_y^2 + 1\right)\right]}$$

$$= \sum_{n=0}^{\infty} \frac{(-t^2)^n}{(\nabla_y^2 + 1)^{n+1}}$$

Thus

$$G_{series}(T) = i\sqrt{\pi/2}\; k^{*-1} \sum_{n=0} \frac{(-t^2)^n\; (k^*T)^{n+(1/2)}}{(2n+1)\;!!}\Big(J_{n+(1/2)}\,(k^*T)$$

$$+ i\, Y_{n+(1/2)}\,(k^*T)\Big).$$

where (2n+1) !! means the product of the odd integers from 2n+1 down to 1.

The J terms of the series converge for all t and y. The Y terms converge if $|t| < 1$, but diverge if $|t| > 1$ (see Section 10.3.2.5).

Note that we have obtained a particular solution of the differential equation of 10.3.1, and this apparently may differ from $G_o(T)$, which was defined as an integral, by $aJ_o(\lambda T) + bY_o(\lambda T)$. But the terms of the series above are outgoing waves, and on physical grounds $G_o(T)$ can have no incoming component. Thus b must equal ia. We shall see in Section 10.3.2.5 that a is actually zero.

10.3.2.3 Another Identity

One can also show that

$$(\nabla_y^2 + 1)\ y^{-\nu}\ C_\nu(y) = 2\nu\ y^{-\nu-1}\ C_{\nu+1}\ .$$

Because of the relations

$$Y_{-(n+(1/2))} = (-1)^n\ J_{n+(1/2)}$$

$$J_{-(n+(1/2))} = -\ (-1)^n\ Y_{n+(1/2)}\ ,$$

this is equivalent to the result of Section 10.3.2.1, when ν is a half-integer.

10.3.2.4 A Second Expansion

If we make the formal expansion in another way:

$$(\nabla_y^2 + 1 + t^2)^{-1} = \frac{1}{t^2(1 + (\nabla_y^2 + 1)/t^2)}$$

$$= \sum_{n=0}^{\infty} \frac{(\nabla_y^2 + 1)^n}{(-t^2)^{n+1}}$$

then in view of the result

$$\frac{\exp iz}{z} = i\sqrt{2/\pi}\quad z^{-1/2}\,(J_{1/2} + i\,Y_{1/2}),$$

we obtain

$$G_{series} = \sqrt{\pi/2}\;k^{*-1}\sum_{n=0}^{\infty}\frac{(-1)^n\,(2n-1)!!\left[J_{n+(1/2)}(k^*T)+iY_{n+(1/2)}(k^*T)\right]}{t^{2n+2}\,(k^*T)^{n+(1/2)}}$$

after using the relations between positive and negative half-integer orders.

10.3.2.5 Convergence of the Expansions

We shall use the following asymptotic evaluations, which are valid for fixed argument z and order p increasing indefinitely:

$$J_p(z) \longrightarrow z^p/2^p\,\Gamma(p+1)$$

$$Y_p(z) \longrightarrow -2^p\,\Gamma(p)/\pi\,z^p$$

The form given for J_p is, as a matter of fact, the leading term of the expansion in powers of z. If p is non-integral, then the expression for $Y_p(z)$ results similarly from the first term of the expansion for $J_{-p}(z)$ in the formula

$$Y_p = (J_p\cos p\,\pi - J_{-p})/\sin p\,\pi$$

after using the identity

$$\Gamma(p)\,\Gamma(1-p)\,\sin p\,\pi = \pi\,.$$

When these evaluations substituted in the expansion of 10.3.2.2, it is clear that the J terms in G_{series} converge for all t, but the Y terms diverge if $|t| > 1$.

To determine the coefficient a in

$$G_o(T) = G_{series} + a(J_o(\lambda T) + i\, Y_o(\lambda T)),$$

we let T go to zero. Then it is easy to see that the integral defining G_o goes to a finite limit. But G_{series} also goes to a finite limit. This can be seen by substituting the leading terms (given above) of the expansions of $J_{n+(1/2)}$ and $Y_{n+(1/2)}$. In particular, each J term goes to zero. Each term involving Y goes to a finite limit as T goes to zero, and the sum of the limits, namely

$$\frac{i}{\pi} \frac{2^{n+(1/2)} \; \Gamma(n+(1/2))}{(2n+1)!!} (-t^2)^n,$$

converges if $|t| < 1$. But $Y_o(\lambda T)$ becomes infinite as T goes to zero. It follows that $a = 0$ and so $G_o(T) = G_{series}$, if $|t| < 1$.

In regard to the expansion of 10.3.2.4, we see that the terms involving $J_{n+(1/2)}$ will converge if $|t| > 1$, but diverge otherwise. The terms involving $Y_{n+(1/2)}$ never converge.

10.3.3 Another Integral Satisfying the Differential Equation

Now consider the function

$$W(T) = \frac{-1}{\sqrt{\lambda^2 - k^{*2}}} \int_{k^*}^{\lambda} \frac{e^{iTu}\, du}{\sqrt{\lambda^2 - u^2}}$$

Then

$$(\nabla_T^2 + \lambda^2)W = \frac{-1}{\sqrt{\lambda^2 - k^{*2}}} \left[\frac{-i}{T} \sqrt{\lambda^2 - u^2} \; e^{iTu} \right]_{k^*}^{\lambda}$$

$$= \frac{\exp ik^*T}{iT}$$

Thus $W(T)$ is a particular solution of the second order equation for $G_o(T)$. The difference $W(T) - G_o(T)$ must be a solution of the homogeneous equation

$$(\nabla_T^2 + \lambda^2)\ y(T) = 0,$$

and so is of the form $a\ J_o(\lambda T) + b\ Y_o(\lambda T)$. Now $W(T)$ is defined as an integral over outgoing waves e^{iTu}. Thus by the same argument as earlier,

$$W(T) - G_o(T) = a(J_o(\lambda T) + i\ Y_o(\lambda T)).$$

We again determine the coefficient a by letting T go to zero in the integral defining W. Then

$$W(T) \to \frac{-\arccos\ (k^*/\lambda)}{\sqrt{\lambda^2 - k^{*2}}} \quad \text{as } T \to 0.$$

Since $G_o(T)$ is finite as $T \to 0$, the coefficient of $Y_o(\lambda T)$ must be zero, and so $W = G_o$.

When k^* goes to zero, $W(T)$ goes to

$$\frac{-\pi}{2\lambda} \left[J_o\ (\lambda T) + i\ \mathbf{H}_o\ (\lambda T) \right] \quad ,$$

where $\mathbf{H}_o(\lambda T)$ is the Struve function of order zero. (See Section 10.3.4)

Assuming that λ is real and $\lambda > k^*$, we also have

$$W(T) = \frac{-1}{\sqrt{\lambda^2 - k^{*2}}} \int_0^{\arccos(k^*/\lambda)} \exp\ (iT\lambda \cos \theta) d\theta$$

where $\theta = \arccos u/\lambda$. Therefore

$$\sqrt{\lambda^2 - k^{*2}}\ W(T) \to \frac{-\pi}{2\sqrt{2\lambda}} \quad (J_o(\lambda T) + iY_o(\lambda T))$$

as $k^* \to \lambda$.

10.3.4 Elementary Properties of Struve Functions

The Struve function of order p, can be defined for $p > -1/2$ by the integral representation

$$\underset{\sim}{H}_p(z) = \frac{2}{\sqrt{\pi}} \frac{(z/2)^p}{\Gamma(p+(1/2))} \int_0^{\pi/2} \sin(z \cos x) \sin^{2p} x \, dx$$

The asymptotic relation

$$\underset{\sim}{H}_p(z) = Y_p(z) + \frac{1}{\pi} \sum_{r=0} \frac{\Gamma(r+(1/2))}{\Gamma(p+(1/2)-r)} \left(\frac{2}{z}\right)^{1+2r-p}$$

valid for $|\arg z| < \pi$, shows that $\underset{\sim}{H}_p(z)$ differs from $Y_p(z)$ for large z by terms that increase as z^{-1+p}, if $p > 1$. If $p = -1/2, -3/2, \cdots$, then $\underset{\sim}{H}_p(z) = Y_p(z)$. The Struve function for positive half-integral order can also be expressed in terms of sines and cosines, e.g.,

$$\underset{\sim}{H}_{1/2}(z) = \sqrt{2/\pi z} \; (1-\cos z).$$

It will be useful to introduce a notation for the complex combination of J and $\underset{\sim}{H}$ that is analogous to a Hankel function of the first kind. No notation for this has been standardized in the literature. For arbitrary real p, we set

$$X_p(z) = J_p(z) + i \, H_p(z)$$

A well-known integral representation for J_p can be combined with the representation given above for $\underset{\sim}{H}_p$:

$$X_p(z) = \frac{2}{\sqrt{\pi}} \frac{(z/2)^p}{\Gamma(p+(1/2))} \int_0^{\pi/2} e^{iz \cos x} \sin^{2p} x \, dx.$$

We recall that $J_p(z) + i \, Y_p(z)$ goes to zero exponentially as z moves to infinity along the positive imaginary axis. Thus $X_o(z)$ goes to zero along this axis as $1/z$.

We have seen in Section 10.3.3 that in the limiting case of an incompressible fluid (that is, $k^* = 0$), $G_o(T)$ is a multiple of $X_o(\lambda T)$. The asymptotic form of $X_o(\lambda T)$ shows that it can be written as the sum of an outgoing wave part $J_o(\lambda T) + iY_o(\lambda T)$ and a pure imaginary part which can be thought of as a standing wave, or else as a disturbance propagated with infinite velocity. X_o can be thought of as a pseudowave, which behaves nearly like a true wave $J_o + iY_o$ for large T, since $J_o + iY_o$ decreases as $T^{-1/2}$ while the remainder (the standing wave) decreases as T^{-1}.

10.3.5 Expansion for $G_o(T)$ in Struve Functions

We rewrite the function W(T) introduced in Section 10.3.3 as

$$W(T) = \frac{-1}{\sqrt{\lambda^2 - k^{*2}}} \left[\int_o^{\lambda} - \int_o^{k^*} \right] \frac{e^{iTw}\, dw}{\sqrt{\lambda^2 - w^2}}$$

$$= \frac{-1}{\sqrt{\lambda^2 - k^*}} \left((\pi/2)\, X_o\, (\lambda T) - D \right),$$

where:

$$D = \int_o^{k^*} \frac{e^{iTw}\, dw}{\sqrt{\lambda^2 - w^2}}$$

10.3.5.1 Expansion of D

By setting $w = k^* x$, we can transform D into a power series in inverse powers of $t = \sqrt{\lambda^2 - k^{*2}}/k^*$:

$$D = (1/t) \int_o^1 \frac{e^{ik^*Tx}\, dx}{\sqrt{1 + (1-x^2)/t^2}}$$

$$= (1/t) \sum_{j=o} \frac{(-1)^j (2j-1)!!}{j!\, 2^j t^{2j}} \int_o^1 (1-x^2)^j e^{ik^*Tx}\, dx,$$

where

$$(-1)!! = 0.$$

The expansion of the radical will converge absolutely if $\left|(1-x^2)/t^2\right| < 1$ for all values of x in the range of integration. Then the integration can be performed term by term. Thus the series above converges absolutely if $|t| > 1$.

By making u = cos x the integration variable in the integral representation given in 10.3.4, we find that

$$z^{-j-(1/2)}\, X_{j+(1/2)}\,(z) = \sqrt{\frac{2}{\pi}}\; \frac{1}{2^j\; j!} \int_o^1 e^{izu}(1-u^2)^j\; du\ .$$

We then obtain directly

$$G_o(T) = W(T) = -\frac{\pi\, X_o(\lambda T)}{2\sqrt{\lambda^2-k^{*2}}}$$

$$+\sqrt{\frac{\pi}{2k^*T}}\;\frac{1}{k^*t^2}\;\sum_{j=o}\;\frac{(-1)^j(2j-1)!!\; X_{j+(1/2)}(k^*T)}{(t^2k^*T)^j}$$

10.3.5.2 Convergence of the Expansion

To investigate the convergence of the $\underset{\sim}{H}$-terms, we need the following asymptotic evaluation:

$$H_p(z) \longrightarrow z^{p+1}/\sqrt{\pi}\;\; 2^p\; \Gamma(p+(3/2)).$$

This is the first term in the series expansion for $\underset{\sim}{H}_p(z)$. After substitution of this form into the infinite summation, the ratio test shows that both the J and $\underset{\sim}{H}$ terms converge as fast as a geometric series with ratio t^2 if $|t| > 1$, but diverge if $|t| < 1$.

The asymptotic evaluation for $J_n(z)$ is a gross over-estimate if z is large (say $z > 20$), and $n < 3z/4$. Thus in many cases fewer J terms are significant than in the corresponding geometric series. On the other hand, the evaluation for $\underset{\sim}{H}_n(z)$ is reasonably close, for the same range of z and n. Thus for $\underset{\sim}{H}_{25}(50)$, it is in excess by less than a factor of 2. Therefore, the expansion will not be useful if $|t|$ is very close to 1.

10.3.6 Comparison and Physical Interpretation of the Power Series in t^2 and in t^{-2}

We have now obtained two different convergent expansions for $G_o(T)$, one in ascending powers of t^2, (Section 10.3.3.2), the other in descending powers (10.3.5), as well as one for which the Y terms never converge (Section 10.3.2.4).

10.3.6.1 The Convergent Series

The two convergent expansions for $G_o(T)$ both contain an infinite summation over half-order terms and are otherwise very similar.

The $X_o(\lambda T)$ term of Section 10.3.5 decreases as $T^{-1/2}$. This indicates that it represents a disturbance that is not propagated into the fluid, but spreads along the interface between water and plate. Thus it is the analog of a surface wave. The discontinuity between plate and water acts in effect as a waveguide or channel for the propagation of Rayleigh waves.

Each term in the infinite series of 10.3.5.1 decreases with increasing T as fast as T^{-1}. The series represents volume waves propagated into the water. Presumably the expansion in ascending powers of t^2 also decreases as T^{-1}.

The existence of separate expansions for different ranges of t is in hindsight not surprising.

One can think of λ/k^* as being in a rough sense the ratio of the "hardness" of the water to that of the plate vibrating with wavenumber λ. We have seen, however, that the quantity t is actually a better measure of the relation between water and plate. The "hardness" of the water arises from its near-incompressibility; the "softness" of the plate arises from the fact that it can sustain flexural waves. The plate is relatively soft for a wavenumber λ near its free wavenumber k, while the water is relatively soft for λ close to its free wavenumber k^*. (In our application, k is considerably greater than k^* for all frequencies of interest.)

If $|t| > 1$, then the water is sufficiently harder than the plate in its λ mode so that an approximation method for the irrational integral can start from incompressibility. This yields the surface disturbance term $X_o(\lambda T)$. There is also a series of correction terms proceeding in descending powers of t^2, namely, the volume disturbances with wavenumber k^*. The rational integral represents a true outgoing Rayleigh wave with wavenumber λ.

When $|t|$ becomes greater than one, the expansion based on an incompressible medium no longer converges. It is not possible to give a simple physical interpretation for the terms of the convergent expansion that proceeds in ascending powers of t^2, because they are unbounded as T goes to infinity. Nevertheless, there is convergence for every fixed T. The rational integral still gives a surface wave with wavenumber λ.

10.3.6.2 Relation Between Neumann and Struve Function Terms

The real part of the summation in Section 10.3.5 is the same as the real part (that is the terms involving J) of the sum in 10.3.2.4. The imaginary part of the latter, involving Y, diverges as has been noticed. At first sight, it is remarkable that the convergent series derived in 10.3.5 differs from the series that was derived in 10.3.2.4 by formal expansion, only in that the

Neumann function Y is replaced by the Struve function $\mathbf{H}$. We now make this plausible by using the concept of Hilbert transform, which will be denoted by $\mathcal{H}$. As is necessary in using this transform, the transformed variable k^* will range from minus to plus infinity.

We first observe from Erdelyi et al, Tables of Integral Transforms, vol. 2, page 252:

$$\mathcal{H}(\sin k^*T/k^*T) = \frac{\cos y T}{y T} - \frac{1}{y T}$$

and on page 255, valid for $p > -3/2$:

$$\mathcal{H}\left[(|k^*T|)^{-p} J_p(|k^*T|)\right] = - \operatorname{sgn} y \left(|y|T\right)^{-p} \mathbf{H}_p(|y|T)$$

On the assumption that k^* and λ are real, we write

$$G_R(T) = \operatorname{Re} G_o(T)/k^*$$

Then from 10.3.1,

$$(\nabla_T^2 + \lambda^2)\ G_R(T) = \sin k^*T/k^*T,$$

and taking the Hilbert transform

$$(\nabla_T^2 + \lambda^2)\ \mathcal{H}(G_R) = \frac{\cos y T}{y T} - \frac{1}{y T}\ .$$

Therefore, by renaming the transform variable,

$$(\nabla_T^2 + \lambda^2)\ (G_R - i\mathcal{H}(G_R)) = \frac{e^{ik^*T}}{ik^*T} + \frac{i}{k^*T} = k^{*-1}\ (\nabla^2 + \lambda^2)\ G_o(T) + \frac{i}{k^*T}\ .$$

It follows that

$$G_o(T) = k^* \left[G_R - i\ \mathcal{H}(G_R)\right] - g(T)$$

where $g(T)$ is a solution of

$$(\nabla_T^2 + \lambda^2)\ g(T) = iT^{-1}$$

This equation has the general solution

$$g(T) = aJ_o(\lambda T) + bY_o(\lambda T) - (i\pi/2\lambda)\ \mathbf{H}_o(\lambda T)\ .$$

It follows from the usual argument involving small T that b must be zero. But then the J_o and $\mathcal{H}_o$ terms must combine to behave asymptotically like an outgoing wave for large T.

Therefore,

$$g(T) = - (\pi/2\lambda)\ X_o(\lambda T) \ .$$

In the differential equation for $G_R(T)$ given above, the inverse of the operator on the left can be expanded by the method of 10.3.2.4. One then obtains the $J_{n+(1/2)}$ terms of the expansion given in that section. This series converges for $|t| > 1$, even though the $Y_{n+(1/2)}$ terms diverge. Then an expansion for $\mathcal{H}(G_R)$ is obtained by applying the transform term-by-term to the series. This produces terms involving $\mathcal{H}$. Then $G_R - i\mathcal{H}(G_R)$, and therefore $G_o(T)$, will be expressed as a sum of terms of the form

$$(k^*T)^{-(n+(1/2))}\ X_{n+(1/2)}(k^*T) .$$

10.3.7 <u>The Convergence Criterion $|t| = 1$</u>

In general λ will be complex, and so the condition $|t| = 1$ that separates the regions of convergence for the two expansions must be studied in the complex λ plane. It can be written as

$$\frac{|\lambda - k^*|\ |\lambda + k^*|}{k^{*2}} = 1$$

From this form, we see that the equation $|t| = 1$ defines a lemniscate (see Fig. 10-2), which has the property that the product of the distances from a point on the curve to the two points $-k^*$, k^* is the constant k^{*2}.

Of the five roots λ^2 of the equation $P(\lambda^2) = 0$ of section 10.2, the four near k^2 or $-k^2$ satisfy the condition $|t| > 1$, at least for the physical dimensions and range of frequencies of interest to us. The remaining root, which is very close to k^{*2}, does not satisfy the condition.

We recall that the free plate wavenumber k is a function of frequency of the form

$$k = A\sqrt{\omega}$$

and the water wavenumber is

$$k^* = B\omega$$

where A depends on the material and geometrical properties of the plate, and B is the reciprocal of the speed of sound in water.

Suppose that λ^2 is one of the two real roots near k^2, and further that $\lambda^2 - k^2$ is small enough to be ignored. Then the condition $|t| > 1$ becomes

$$\omega < A^2/2B^2$$

Thus, for any given plate the expansion in powers of t^{-2} will be divergent for sufficiently high frequencies.

For the two roots λ^2 near $-k^2$, the condition $|t| > 1$ is always satisfied, provided that the difference between λ^2 and $-k^2$ can be ignored.

For a more exact investigation of these questions, we return to the polynomial $P(p^2)$ of 10.2.1, and express p^2 in terms of t^2:

$$p^2 = k^{*2}(t^2 + 1)$$

Then $P(p^2) = 0$ becomes

$$k^{*2}t^2((t^2 + 1)^2\ k^{*4} - k^4)^2 = e^2$$

For a real root, the condition $|t| = 1$ separating the two regions of convergence is, of course, $t = 1$. Then if we set

$$e = C\omega^2\ ,$$

$P(p^2) = 0$ becomes

$$B\omega(4B^4\omega^2 - A^4) = C$$

A recursion relation of the form

$$\omega_{(i+1)} = \frac{1}{2B^2}\sqrt{A^4 + (C/B\omega_{(i)})}$$

is thus suggested. On substituting $\omega_{(o)} = A^2/2B^2$, we obtain, to first order in C:

$$\omega_{(1)} = \frac{A^2}{2B^2} + \frac{C}{2A^4B}$$

This is a more accurate evaluation of the critical frequency for the real root that is less than k^2, but not very close to k^{*2}.

10.3.8 Derivatives of G_o

It will be useful for several purposes to have explicit expansions for

$$\left(\frac{d}{y\ dy}\right)^m G_o(y).$$

10.3.8.1 Case 1. $|t| < 1.$

The series of 10.3.2.2 can be differentiated term-by-term, making use of the identity

$$\frac{d}{TdT}(k^*T)^p\ C_p(k^*T) = k^{*2}(k^*T)^{p-1}\ C_{p-1}(k^*T),$$

where C is J or Y. Thus

$$\left(\frac{d}{TdT}\right)^m G_o(T) = i\sqrt{\pi/2}\ k^{*2m-1}\ X$$

$$\sum_{n=0} \frac{(-t^2)^n (k^*T)^{n-m+(1/2)}}{(2n+1)!!}\left[J_{n-m+(1/2)}(k^*T) + iY_{n-m+(1/2)}(k^*T)\right]$$

It is easily checked that this expansion converges if $|t| < 1$.

10.3.8.2 Case 2. $|t| > 1$.

If $|t| > 1$, then the corresponding series, given in Section 10.3.5.1, involves the forms $(k^*T)^{-p}\ X_p(k^*T)$. Now for the Struve function, there is an algebraic term in the differential formula:

$$\frac{d}{zdz}\ z^{-p}\ \mathbf{H}_p(z) = -\ z^{-p-1}\ \mathbf{H}_{p+1}(z) - \frac{1}{2^p\sqrt{\pi}\ \Gamma(p+3/2)z}$$

Thus, when $\frac{d}{TdT}$ is applied to the series for $G_o(T)$, each term will produce a Struve function term and also an algebraic term proportional to $1/T$. Applying the operator a second time will result in terms in $1/T^3$ and $1/T$. Let S_n be the sum of the algebraic contributions of all the terms.

S_n can be found directly by the following argument: $\mathbf{H}_p(z)$ is nearly proportional to z^{p+1} when p is small. Therefore, $z^{-p}\ \mathbf{H}_p(z)$ approaches zero as z goes to zero, and further,

$$\frac{d}{zdz}\ z^{-p}\ \mathbf{H}_p(z)$$

behaves as const./z for small z. Thus the Struve function terms that arise after differentiation do not affect the behavior as $z \to 0$, which is determined entirely by the algebraic terms.

The part G_H of $G_o(T)$ corresponding to all the Struve function terms (both the zero-order and half-order terms) is given by the following modification of the W(T) integral of 10.3.5:

$$G_H = \frac{-i}{\sqrt{\lambda^2 - k^{*2}}} \int_{k^*}^{\lambda} \frac{\sin Tu\ du}{\sqrt{\lambda^2 - u^2}}$$

Then

$$\frac{d}{TdT} G_H(T) = \frac{-i}{T} \frac{1}{\sqrt{\lambda^2 - k^{*2}}} \int_{k^*}^{\lambda} \frac{u \cos Tu \, du}{\sqrt{\lambda^2 - u^2}}$$

$$= -iT^{-1} + R_1,$$

where R_1 is a remainder term that goes to zero as T goes to zero. Thus $S_1 = -iT^{-1}$. If the limit notation is used in an extended sense, we can write for the higher S_n:

$$S_2 = \lim \left(\frac{d}{TdT}\right)^2 G_H(T)$$

$$= \frac{i}{\sqrt{\lambda^2 - k^{*2}}} \lim \int_{k^*}^{\lambda} [T^{-3} u \cos Tu + T^{-2} u^2 \sin Tu] \frac{du}{\sqrt{\lambda^2 - u^2}}$$

$$= iT^{-3}$$

and

$$S_3 = -3iT^{-5} + iT^{-3} (2\lambda^2 + k^{*2})/3.$$

In general

$$S_n = (-1)^n (2n-3)!! \; T^{-2n+1}$$

$$+ \sum_{r=1} C_{nr} T^{-2n+2r+1}$$

where the summation includes all integers r for which $r < n/2$.

10.3.8 3 Differential Equation for Derivatives

It can be shown by mathematical induction that the derivatives of $G_o(T)$ satisfy the following equation:

$$\left(\nabla_T^2 + 2n \frac{d}{TdT} + \lambda^2\right) \left(\frac{d}{TdT}\right)^n G_o(T) = \left(\frac{d}{TdT}\right)^n \frac{\exp ik^*T}{iT}.$$

In analogy with the treatment of 10.3.2, one can now expand the differential operator on the left. This leads to the same series as was obtained in 10.3.8.1, for $|t| < 1$.

Using the second mode of expansion (Section 10.3.2.4), which is appropriate for $|t| > 1$, one obtains a series having $J_{n+(1/2)}$ terms that coincide with those of 10.3.8.2. However, it also has $Y_{n+(1/2)}$ terms that diverge, as might be expected.

10.3.9 Asymptotic Evaluation of $G_o(T)$

When T is very large, the only significant contributions to $G_o(T$ will come from the regions near the singularities at $p = k^*$ and $p = \lambda$ in the integral that defines $G_o(T)$. By taking slowing varying factors out of the integrand, $G_o(T)$ can be approximated as a sum:

$$G_o(T) \rightarrow \frac{1}{k^{*2} - \lambda^2} \int_o^\infty \frac{J_o(pT)\ p\ dp}{\sqrt{p^2 - k^{*2}}}$$

$$+ \frac{1}{\sqrt{\lambda^2 - k^{*2}}} \int_o^\infty \frac{J_o(pT)\ p\ dp}{p^2 - \lambda^2}$$

The first term equals $-e^{ik^*T}/t^2 k^{*2} T$, and thus represents a volume wave with wavenumber exactly k^*, that is propagated into the water. This has not appeared explicitly in our earlier expansions.

The second term can be evaluated (see Section 3.1.2) as

$$\frac{i\pi}{2\sqrt{\lambda^2 - k^{*2}}} \left| J_o(\lambda T) + i\ Y_o(\lambda T) \right| .$$

We now recall that λ is just another name for p_m. Then from Section 10.2.1.1 we see that the asymptotic evaluation of the sum of the rational and irrational integrals corresponding to p_m is

$$\left[A_m - \frac{e\ B_m}{\sqrt{p_m^2 - k^{*2}}} \right] \frac{i\ \pi (J_o + iY_o)}{2} .$$

On substitution of the explicit expressions for A_m and B_m given there, the bracket above becomes proportional to

$$-\ (p^4-k^4)\ -\ \frac{e}{\sqrt{p_m^2-k^{*2}}}$$

If the sign in front of the radical were changed, this expression would be $-f(p_m)$. That is: the expression above is zero when p_m^2 is a zero of $P(p^2)$ but p_m is <u>not</u> a root of $f(p)$ (see Section 10.2.5). This means physically that the true wave due to the rational integral cancels with the asymptotic wave due to the irrational integral, unless p_m is a contributing root.

10.4 <u>Expansion of the "Irrational Part" of the General Interaction Integral</u>

We set

$$G_{ij} = \int_0^\infty h^{-1}(p)\ J_i(pR)\ J_j(pS)\ J_{i+j}(pT)\ pdp$$

We shall assume that i and j are integers, and $i+j \geq 0$, although i or j can be negative or zero.

10.4.1 <u>Reduction of the Integral to a Form Involving G_o.</u>

The product of the three Bessel functions will be rewritten as

$$\frac{T^{i+j}\ L(p)\ p^{i+j}\ J_{i+j}(pT)}{(k^*T)^{i+j}}$$

in which

$$L(p) = k^{*(i+j)}\ \frac{J_i(pR)}{p^i}\ \frac{J_j(pS)}{p^j}$$

Since

$$\left(\frac{d}{TdT}\right)^m J_o(pT) = (-1)^m p^m T^{-m} J_m(pT)$$

for positive integral m, we can write

$$G_{ij} = k^{*(-i-j)} T^{i+j} \left(\frac{-d}{TdT}\right)^{i+j} G^*$$

where

$$G^* = \int_o^\infty h^{-1}(p) L(p) J_o(pT) pdp$$

We now expand L(p) in a power series in $(p^2-k^{*2})/k^{*2}$:

$$L(p) = \sum_n b_n \left((p^2-k^{*2})/k^{*2}\right)^n$$

The discussion that follows will apply to more general forms of L(p) than the product of Bessel functions. All that is needed is the possibility of the above expansion, together with appropriate convergence conditions.

When the series for L(p) is substituted in the integrand of G^*, the individual terms result in divergent integrals, if $n \geq 1$. To avoid this, a convergence factor exp (-cp) will be inserted in the integrand of G^*. Thus we define

$$G^*(c) = \int_o^\infty \exp(-cp)\, h^{-1}(p) L(p) J_o(pT) pdp$$

we now recall that

$$\nabla_T^2 J_o(pT) = -p^2 J_o(pT)$$

Therefore

$$G^*(c) = \sum_{n=o}^\infty (-1)^n b_n \int_o^\infty \exp(-cp)\, h^{-1}(p) X$$

$$\left((\nabla_T^2 + k^{*2})/k^{*2}\right)^n J_o(pT) pdp .$$

We have assumed that the infinite summation converges. This is not assured by the convergence of the original expansion for $L(p)$.

The differentiation operator can be taken out from under the integration sign. After setting

$$T^* = k^* T,$$

we observe that

$$(\nabla_T^2 + k^{*2})/k^{*2} = \nabla_{T^*}^2 + 1$$

and obtain

$$G^*(c) = \sum_{n=o}^{\infty} (-1)^n \, b_n (\nabla_{T^*}^2 + 1)^n \, G_o(c)$$

where

$$G_o(c) = \int_o^{\infty} \exp(-cp) h^{-1}(p) \, J_o(pT) \, pdp$$

Now let c go to zero. Then $G^*(c)$ becomes G^* in the limit. We now use the differential equation of 10.3.1 in the form

$$(\nabla_{T^*}^2 + 1) \, G_o(T) = -t^2 \, G_o(T) + \frac{1}{ik^*} \, \frac{\exp \, i \, T^*}{T^*}$$

$$= -t^2 \, G_o(T) + \frac{1}{k^*} \sqrt{\frac{2}{\pi}} \, T^{*-1/2} \, H_{1/2}(T^*)$$

and the identity of 10.3.2.3, to obtain

$$(\nabla_{T^*}^2 + 1)^n \, G_o(T) = (-1)^n \, t^{2n} \, G_o(T)$$

$$+ \frac{1}{k^*} \sqrt{\frac{2}{\pi}} \sum_{q=0}^{n-1} (-1)^q \, t^{2(n-q-1)} \, (2q-1)!! \, T^{*-(q+(1/2))} \, H_{q+(1/2)}(T^*) \; .$$

Here H indicates the Hankel function. Then

$$G^* = (\sum_n b_n t^{2n}) G_o(T)$$

$$+ \frac{1}{k^*} \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} (-1)^n b_n \sum_{q=0}^{n-1} (-1)^q t^{2(n-q-1)} (2q-1)!! \times$$

$$T^{*-(q+(1/2))} H_{q+(1/2)}(T^*)$$

The summation in the first term is equal to $L(\lambda)$. The order of integration of the double sum is now interchanged to produce

$$G^* = L(\lambda) G_o(T)$$

$$+ \frac{1}{k^* t^2} \sqrt{\frac{2}{\pi}} \sum_{q=0}^{\infty} (-1)^q t^{-2q} (2q-1)!! a_q T^{*-(q+(1/2))} H_{q+(1/2)}(T^*) ,$$

where

$$a_q = \sum_{n=q+1}^{\infty} (-1)^n b_n t^{2n} .$$

10.4.2 Evaluation of the a_q

10.4.2.1 Recursive Relation

The a_q can be computed conveniently from the b_q by descending on the index q:

$$a_{q-1} = a_q + (-1)^q b_q t^{2q}.$$

One must start with a q so large that a_q can be taken as zero without introducing any significant error.

10.4.2.2 Evaluation of the b_n

The expansion coefficients b_n, defined at the beginning of 10.4.1, can be evaluated by using the Lommel series (see Watson, Bessel Functions, p. 140) for the two factors in

$$L(p) = k^{*(i+j)} \frac{J_i(pR)}{p^i} \quad \frac{J_j(pS)}{p^j} \quad .$$

Thus from the Lommel expansion

$$J_i(pR)/p^i = k^{*-i} \sum_{m=\infty}^{\infty} \frac{(-1)^m (k^*R)^m J_{i+m}(k^*R)}{2^m m!} \left(\frac{p^2 - k^{*2}}{k^{*2}}\right)^m$$

and the similar one for $J_j(pS)/p^j$, it follows that

$$b_s = \frac{(-1)^s k^{*s}}{2^s} \sum_{m=0}^{s} \frac{R^m S^{s-m} J_{i+m}(k^*R) J_{j+s-m}(k^*S)}{m! (s-m)!}$$

10.4.3 Convergence of the Series for G^*

The infinite sum for G^* at the end of 10.4.1 is closely related to the sum in Section 10.3.2.4. It was noted in Section 10.3.2.5 that the $Y_{n+(1/2)}$ terms did not converge for any t. The form for G^* derived above contains an extra factor a_q in each term. For many forms of L(p), this decreases very rapidly with increasing n, and thus produces convergence.

10.4.3.1 A Special Case

In the special case i = j = 0 and S = 0, we have

$$b_s = \frac{(-1)^s}{2^s} \frac{(k^*R)^s J_s(k^*R)}{s!}$$

The $J_{n+(1/2)}$ terms in the series for G^* will clearly converge if the $Y_{n+(1/2)}$ terms converge. Thus it is sufficient to look at the Y terms. Using the asymptotic evaluations given in Section 10.3.2.5, and taking absolute values, we find that the Y part of the sum over q is less than

$$\sum_{q=0}^{\infty}\left(\sum_{n=q+1}^{\infty}\frac{(tk^{*}R)^{2n}}{(2n)!!^{2}}\right)t^{-2q}\,(2q-1)!!\;T^{*(-1-2q)}\;C_q$$

where

$$C_q = 2^{q+(1/2)}\,\Gamma(q+1)/\pi\ .$$

C_q can be approximated very closely as $c(2q-1)!!$, where c is a constant. On making $u = n - q$ the inner summation variable, the double sum becomes

$$\frac{c}{T^{*}}\sum_{q=0}^{\infty}\sum_{u=1}^{\infty}\frac{(2q-1)!!^{2}}{(2q+2u)!!^{2}}\;\frac{R^{2q}}{T^{2q}}\;(tk^{*}R)^{2u}$$

From the obvious relation

$$\frac{1}{(A+B)!!^{2}} < \frac{1}{A!!^{2}}\;\frac{1}{B!!}\ ,$$

the double sum is seen to be less than the product of two single summations

$$\frac{c}{T^{*}}\left[\sum_{q=0}^{\infty}\frac{(2q-1)!!^{2}}{(2q+2)!!^{2}}\;\frac{R^{2q}}{T^{2q}}\right]\left[\sum_{u=1}^{\infty}\frac{(tk^{*}R)^{2u}}{(2u-2)!!}\right]$$

The second series is the convergent expansion of $(tk^{*}R)^{2}\times \exp(tk^{*}R/2)$. The first series converges if $R < T$, faster than a geometric series with ratio $(R/T)^{2}$. It also converges, but slowly, if $R = T$.

10.4.3.2 S Different From Zero

To handle the more general case $i = j = 0$, but $S \neq 0$, we will use the following inequality for positive integral n:

$$J_n(x) < x^{n}/2^{n}\,n!\ ,\qquad x > 0.$$

That is, the first term of the power series expansion of $J_n(x)$ is always greater than $J_n(x)$, for real positive x. This can be proved from the standard representation

$$J_n(x) = \frac{x^n}{2^{n-1}\sqrt{\pi}\ \Gamma(n+1/2)} \int_0^{\pi/2} \cos(x \cos\theta)\, \sin^{2n}\theta\, d\theta$$

by replacing the integral by the following approximation in excess:

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2}\ \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n\ n!}\ .$$

When the above inequality is used in the expression for b_s given in 10.4.2.2, we obtain, assuming as usual that k^* is real,

$$|b_s| \leq \frac{k^{*2s}}{2^{2s} s!^2} \sum_{m=0}^{s} \frac{s!^2 R^{2m}\, S^{2(s-m)}}{m!^2\ (s-m)!^2}$$

$$\leq \frac{k^{*2s}}{2^{2s} s!^2} \left[\sum_{m=0}^{s} \frac{s!\, R^m\, S^{s-m}}{m!\ (s-m)!}\right]^2$$

$$\leq \frac{k^{*2s}\ (R+S)^{2s}}{2^{2s}\ s!^2}.$$

Then the discussion can be thrown back onto the special case treated in 10.4.3.1, if the former R is now replaced by R + S.

In summary: if i = j = 0, and $T \geq R + S$, the series for G^* converges. Presumably this result can be extended to non-zero i and j.

10.4.3.3 Speed of Convergence ; Special Summation Methods When T = R and S = 0.

Since the convergence proof involved a rearrangement of the terms of a double summation, we cannot draw any conclusions about the speed of convergence for the original form of G^*. Clearly the summation (Section 10.4.2) that gives the a_q from the b_q converges very rapidly. It appears that the G^* series converges faster than a geometric series with ratio $(R + S)^2/T^2$.

The special case R = T, S = 0 is very important, and it is desirable to have some method of accelerating convergence. Assume that q is large enough so that the leading term in the power series expansion of the first term of a_q is a good approximation to a_q. This can be true only if $q > k^*R$. Then the double sum reduces to a single sum

$$\frac{c}{T^*} tk^*R \sum_{q=0}^{\infty} \frac{(2q - 1)!!^2}{(2q + 1)!!^2}$$

The terms of the summation decreases approximately as $1/q^3$. This indicates that after a certain q_o is reached, the sum of the remaining terms can be estimated by finding constants A and B such that the last few computed terms can be fitted closely in the form

$$\text{q-th term} = A(q + B)^{-3}$$

Then the remainder can be evaluated as

$$\sum_{q_o+1}^{\infty} A(q + B)^{-3}.$$

10.4.4 Relation to the Work of Pritchard

Pritchard (J. Acoustical Society America, vol. 23, p. 591 (1951)) has derived expansions for integrals not containing the factor $(p^2 - \lambda^2)^{-1}$:

$$P = \int_{k^*}^{\infty} \frac{J_u(xp)\ J_v(yp)\ pdp}{p^{u+v}\ (p^2-k^{*2})^{1/2}} =$$

$$\sqrt{\frac{\pi k^*}{2y}}\ \frac{1}{k^{*(u+v)}}\ \sum_{r=0}^{\infty} \frac{(-1)^r\ \Gamma(r + (1/2))}{r!\ \Gamma\ (1/2)}\ \left|\frac{x}{y}\right|^r\ J_{u+r}(k^*y)\ J_{v-r-(1/2)}(k^*y)$$

valid for Re $(u+v) > 0$, and $x > y$. The formula is also valid in the limiting case $u = v = 0$.

This special case $u = v = 0$ was also derived by Pachner, p. 187-8 of the same volume. Pritchard's derivation can be simplified when $u = -v = i$, where i is an integer, by an argument very similar to the treatment of Section 10.4.1.

As earlier, we introduce a convergence factor $\exp(-cp)$ in the integrand, with the intention of ultimately letting c go to zero. Now write

$$P(c) = \int_{k^*}^{\infty} \exp\ (-cp)\ \frac{J_i(xp)}{p^i}\ \frac{p^i J_i(yp)\ pdp}{\sqrt{p^2 - k^{*2}}}$$

$J_i(xp)/p^i$ is now expanded in a Lommel series in $p^2 - k^{*2}$, and the summation is interchanged with the integration. The integrals have the general form

$$\int_{k^*}^{\infty} \frac{\exp\ (-cp)\ (p^2-k^{*2})^h p^i J_i(yp)\ pdp}{\sqrt{p^2 - k^{*2}}}$$

$$= y^i \left(\frac{-d}{ydy}\right)^i \int_{k^*}^{\infty} \frac{\exp(-cp)\ (p^2-k^{*2})^h J_o(yp)\ pdp}{\sqrt{p^2 - k^{*2}}}$$

$$= y^i \left(\frac{-d}{ydy}\right)^i (-1)^h\ (\nabla_y^2 + k^{*2})^h \int_{k^*}^{\infty} \frac{\exp(-cp)\ J_o(yp)\ pdp}{\sqrt{p^2 - k^{*2}}}$$

Now c can be set equal to zero, and the last integral becomes equal to $(\cos k^{*}y)/y$, which equals

$$\sqrt{\pi k^{*}/2}\; y^{-1/2}\; J_{-1/2}(k^{*}y),$$

and the differentiations can be expressed explicitly using the identity of 10.3.2.1.

10.5 Transformations of the Irrational Part of the Green's Function

Relations will now be derived between integrals over the two ranges 0 to k^{*} and k^{*} to ∞. It will be assumed that λ^2 has a positive imaginary part, possibly infinitesimal, in order to avoid any discussion of the path to be taken around the pole at $p=\lambda$.

10.5.1 Restriction to the Range 0 to k^{*}.

10.5.1.1 Introduction of the U Integral

We define

$$U = \int_{-\infty}^{\infty} h^{-1}(p)\; J_i(pR)\; J_j\;(pS)\; H_k(pT)\, p\; dp$$

where H_k indicates the Hankel function of the first kind.

To specify the path C of integration in the complex plane, we first introduce a branch cut from $-k^{*}$ to k^{*}, and distort it slightly upward as shown in Figure 10-1, so that it does not pass through the point $p = 0$. We choose the branch of $\sqrt{p^2 - k^{*2}}$ which is positive for $p > k^{*}$. Then $\sqrt{p^2 - k^{*2}}$ is positive imaginary for $-k^{*} < p < k^{*}$, and negative real for $p < -k^{*}$. The point $p = 0$ is a logarithmic branch point of the Hankel function. To make the integrand single valued, we introduce another cut extending from $p = 0$ downward along the negative imaginary axis, as shown. Then the path C of integration is taken as going below $p=-k^*$ and $p=k^*$, but above p=0. Our intention is to distort the path of integration upward to infinity. This will leave the value unchanged only for a path C' that goes above the branch cut. Let U' be the integral for such a path. Then we have

$$U = U' + 2 \int_{-k^*}^{k^*} h^{-1} J_i J_j H_k p \, dp$$

where the integral is taken along the lower side of the cut.

When p has a large imaginary part ip_o and zero real part, $J_i(pR)$ equals $I_i(p_oR)$, which behaves asymptotically like e^{p_oR} (multiplied by an algebraic factor, which will be neglected). $J_j(pS)$ behaves asymptotically like e^{p_oS}, while $H_k(pT)$ becomes $K_k(p_oT)$, behaving asymptotically like e^{-p_oT}. It follows that a countour that goes above the cut can be distorted upward to infinity, if $T > R + S$. Therefore, U' is given by the residue at the pole $p = \lambda$:

$$U' = 2\pi i \frac{J_i(\lambda R) \, J_j(\lambda S) \, H_k(\lambda T)}{2\sqrt{\lambda^2 - k^{*2}}}$$

10.5.1.2 Folding of Integration Path

We now rewrite U as an integral from 0 to ∞.

The branch cut from the origin downward has unambiguously fixed the phase angle of z. Then one can apply the general formula in Watson, Bessel Functions, p. 75:

$$H_k(ze^{m\pi i}) = e^{-mk\pi i} H_k(z) - 2e^{-k\pi i} \frac{\sin mk\pi}{\sin k\pi} J_k(z)$$

where m is an integer.

Case 1: $i + j$ even and k even.

Then from the above equation

$$H_k(z) = \begin{cases} J_k(|z|) + i\, Y_k(|z|) & \text{if } z > 0 \\ J_k(|z|) + i\, Y_k(|x|) & \text{if } z < 0 \end{cases}$$

Now observing that U has the form

$$U = \int_{-\infty}^{\infty} F(p)\, dp = \int_{-\infty}^{\infty} [F(p) + F(-p)]\, dp,$$

we find, since $i + j$ is assumed even,

$$U = 2 \int_{0}^{k*} h^{-1}(p)\, J_i(pR)\, J_j(pS)\, i\, Y_k\, (pT)\, p\, dp$$

$$+ 2 \int_{k*}^{\infty} h^{-1}(p)\, J_i(pR)\, J_j(pS)\quad J_k\, (pT) p\, dp$$

The integral from 0 to k^* is along the lower side of the cut.

Case 2: $i + j$ odd and k odd

Then

$$H_k(z) = \begin{cases} J_k(|z|) + i\, Y_k(|z|) & \text{if } z > 0 \\ J_k(|z|) - i\, Y_k(|z|) & \text{if } z < 0 \end{cases}$$

Using the hypothesis that $i + j$ is odd also leads to the expression given above for U.

Thus the formula for U holds if $i + j + k$ is even, which is satisfied in the cases of interest to us ($k=i \pm j$).

10.5.1.3 New Integral for G_{ijk}

Combining the expressions for U and U' we find in abbreviated form

$$2\pi i \frac{J_i(\lambda R)\ J_j(\lambda S)\ H_k(\lambda T)}{2\sqrt{\lambda^2 - k*^2}} + \int_{-k*}^{k*} JJH$$

$$= 2i \int_o^{k*} JJY + 2 \int_{k*}^{\infty} JJJ$$

where the integrals are taken along the lower side of the cut. The path for the integral on the left-hand side can be folded and then the real parts will cancel:

$$2 \int_{-k*}^{k*} JJH = 4i \int_o^{k*} JJY.$$

Now add and subtract $\int_{k*}^{\infty}$ JJJ from the equation above, and rearrange, to obtain

$$G_{ijk} = - \frac{\pi i\ J_i(\lambda R)\ J_j\ (\lambda S) H_k(\lambda T)}{2\sqrt{\lambda^2 - k*^2}}$$

$$+ \int_o^{k*} h^{-1}(p)\ J_i(pR)\ J_j(pS)\ H_k(pT)\ p\ dp$$

where the integration path is under the cut.

If the imaginary part of λ goes to zero, then the first term on the right is an outgoing wave in the distance T. The second term is an integral over outgoing waves $H_k(pT)$ with wavenumber equal to k^* or less. If we reintroduce the time dependence $e^{-i\omega t}$, we see that these all travel faster than ω/k^*, the speed of sound in water. There is no upper limit to the speed. Thus one cannot say that the waves are propagated primarily through the plate.

This apparent paradox can presumably be resolved by considering the cancellations between the waves, or else by distinguishing between phase velocity and group velocity.

10.5.2 Restriction to the Range k^* to Infinity

The integral defining G_o at the beginning of 10.3 can also be modified so that the integration runs from k^* to ∞. The new integrand involves the Struve function of order zero, in the combination $X_o(z) = J_o(z) + i\underset{\sim}{H}_o(z)$.

Consider the integral U^*

$$U^* = \int_{-\infty}^{\infty} h^{-1}(p) X_o(pT) p \, dp$$

where the path of integration is again C (see Fig. 10-1). (Since $\underset{\sim}{H}_o$ has no singularity, the vertical cut is not needed.) There is no change in the sign of Re $X_o(z)$ as z changes from positive to negative, but Im $X_o(z)$ changes sign. Then in analogy with the arguments of 10.5.1, we obtain

$$2\pi i \frac{X_o(\lambda T)}{2\sqrt{\lambda^2 - k^{*2}}} + 4 \int_0^{k^*} h^{-1}(p) \; J_o(pT) p \, dp$$

$$= 2 \int_0^{k^*} h^{-1}(p) \; J_o(pT) p \, dp + 2 \int_{k^*}^{\infty} h^{-1}(p) \, i\underset{\sim}{H}_o(pT) p \, dp$$

Now adding and subtracting $2 \int_{k^*}^{\infty} J_o$, we obtain after rearrangement:

$$G_o(T) = - \frac{\pi i \; X_o(\lambda T)}{2\sqrt{\lambda^2 - k^{*2}}} + \int_{k^*}^{\infty} h^{-1}(p) \; X_o(pT) p \; dp.$$

If we choose to interpret the factor p in the argument of $X_o(pT)$ as related to the "speed" of the pseudowave by the usual relation, then we can say that G_o has been analyzed into a residue term proportional to $X_o(\lambda T)$ and a sum over pseudowaves all traveling slower than $c = \omega/k^*$.

10.6 Related Integrals

10.6.1 Another Treatment of the Differential Equation for $G_o(T)$

The second-order equation for G_o in Section 10.3.1 can also be solved by introducing a Green's function $F(T,T^*)$ for it. F is defined by:

$$F(T^*,T) = \begin{cases} J_o(\lambda T^*) \; Y_o(\lambda T), & \text{if } T^* > T \\ J_o(\lambda T) \; Y_o(\lambda T^*), & \text{if } T > T^* \end{cases}$$

Then F is continuous, but has a discontinuous derivative at $T^*=T$. G_o is expressible as

$$G_o(T) = -i \int_{-\infty}^{\infty} F(T^*,T) \; T^{*-1} \exp i \; k \; T^* \; dT^*$$

The literature on related integrals is summarized by Luke, Y. L., in Integrals of Bessel Functions (McGraw-Hill, 1962). There are several tabulations (for references, see Luke, p. 251) of the Schwarz functions J_e, Y_e:

$$J_e(\lambda z) = \int_o^z e^{it} J_o(\lambda t)dt$$

$$Y_e(\lambda z) = \int_o^z e^{it} Y_o(\lambda t)dt$$

J_e and Y_e were first encountered in panel flutter problems.

10.6.2 Integrals Arising in Two-Dimensional Problems

The integrals that arise in considering water-coupled plates with a stiffening beam attached along an infinite line, or else with boundary conditions applied along a straight line, are all related to the general form (see Section 6.4 and 9.5.1).

$$\int \frac{\cos zp\, dp}{f(\sqrt{p^2+q^2})} = \frac{1}{D} \int_o^\infty \frac{\cos zp\, dp}{(p^2+q^2)^2 - k^4 + ie(p^2+q^2-k*^2)^{-1/2}}$$

Following the techniques of Section 10.2, this can be reduced to the sum of five integrals of the form

$$R(z) = \int_o^\infty \frac{\cos zp\, dp}{(p^2-\lambda^2)\sqrt{p^2-k*^2+q^2}}$$

$R(z)$ differs from $G_o(z)$ in having a cosine factor, instead of $pJ_o(zp)$, and in that the k^{*2} of the radical in G_o is replaced by $k^{*2} - q^2$. If $q^2 > k^{*2}$, there will be a qualitative difference in the nature of the solutions.

By applying the operator $\frac{\partial^2}{\partial z^2} + \lambda^2$, one obtains a differential equation for $R(z)$:

$$\left(\frac{\partial^2}{\partial z^2} + \lambda^2\right) R(z) = - \int_0^\infty \frac{\cos zp \, dp}{\sqrt{p^2 - k*^2 + q^2}}$$

$$= \pi H_o \left(z \sqrt{k*^2 - q^2}\right)$$

and so R(z) satisfies a fourth-order homogeneous equation:

$$(\nabla_z^2 + k*^2 - q^2) \left(\frac{\partial^2}{\partial z^2} + \lambda^2\right) R(z) = 0.$$

To solve by the Green's function technique, we define the function

$$S(z*, z) = \begin{cases} \cos \lambda z* \quad \sin \lambda z & \text{if } z* > z \\ \sin \lambda z* \quad \cos \lambda z & \text{if } z > z* \end{cases}$$

Then

$$R(z) = \text{const} \int S(z*, z) \; H_o(z* \sqrt{k*^2 - q^2}) dz*$$

and thus R(z) can be expressed in terms of four integrals which are the real and imaginary parts of J_e and Y_e.

By applying $\nabla_3^2 + \lambda^2$ to the fourth order equation (∇_3^2 is the 3-dimensional Laplacian), we see that

$$\left(\frac{\partial^2}{\partial z^2} + \lambda^2\right) R(z)$$

is a solution of the equation for G_o.

10.6.3 Integrals Involving a Depth Factor

A treatment was given in Sections 8.7 - 8.9 for a point source immersed in an infinite ocean covered by a flexible plate. This led to an integral of the following general form:

$$G(T,y) = \int_0^\infty e^{-\sqrt{p^2-k^{*2}}\,y}\, h^{-1}(p)\, L(p)\, J_0(Tp)\, p\, dp$$

In the problem actually considered, $L(p) = 1$. It is plausible that if a rim resonator were attached to the plate, with the center of its attachment circle directly over the source, the analysis would lead to integrals in which $L(p) = J_0(pR)$. More complicated forms would result from submerged ring sources, and off-center configurations.

The above integral cannot be evaluated as a special case of the form treated in Section 10.4, by combining $e^{-\sqrt{p^2-k^{*2}}\,y}\, L(p)$ into a new L, because the square root does not allow an expansion in powers of $(p^2-k^{*2})/k^*$.

10.6.3.1 L(p) Equal to Unity; $G = G_0(T,y)$

The approach of Section 10.3.1 can be easily modified to take advantage of the special form of the exponential factor. We first set

$$V^* = k^* \sqrt{T^2+y^2},$$

and observe that

$$\nabla^2_{T^*} = \frac{1}{T^*} \frac{d}{dT^*} T^{*2} \frac{1}{T^*} \frac{d}{dT^*}$$

$$= \frac{1}{V^*} \frac{d}{dV^*} (V^{*2}-k^{*2}y^2) \frac{1}{V^*} \frac{d}{dV^*}$$

$$= \nabla^2_{V^*} - (k^*y)^2 \left(\frac{d}{V^* dV^*}\right)^2 .$$

The differential equation derived for G_0 by the method of Section 10.3.1, namely

$$(\nabla^2_T + \lambda^2)\, G_0(T,y) = \frac{\exp i\, k^* \sqrt{T^2+y^2}}{i \sqrt{T^2+y^2}}$$

then becomes

$$\left[\nabla^2_{V^*} + 1 + t^2 - \left(\frac{k^* y d}{V^* dV^*}\right)^2\right] G_o(t,y) = \frac{1}{k^*} \frac{\exp i V^*}{i V^*}$$

The inverse of the operator on the left can now be expanded just as in 10.3.2.2, but with the term t^2 now replaced by the operator $t^2 - \left(\frac{k^* y d}{V^* dV^*}\right)^2$.

$$G_{series}(T,y) = i\sqrt{\pi/2}\; k^{*-1} X$$

$$\sum_{n=0} \frac{(-1)^n}{(2n+1)!!}\left[t^2 - \frac{k^* y d}{V^* dV^*}\right]^n V^{*n+(1/2)}\left\{J_{n+(1/2)}(V^*) + iY_{n+(1/2)}(V^*)\right\}$$

The binomial power can now be expanded, and the differential operator can be applied.

$$G_{series}(T,y) = i\sqrt{\pi/2}\; k^{*-1}\; X$$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^n}{(2n+1)!!}\;\frac{n!\,(k^* y)^r t^{2(n-r)}}{r!\,(n-r)!}\; V^{*n+r+(1/2)}\; X$$

$$\left\{J_{n+r+(1/2)}(V^*) + iY_{n+r+(1/2)}(V^*)\right\}$$

One expects that this converges if $|t| < 1$. The double sum can be rearranged in several ways. Thus let $m = n + r$. Then

$$G_o(T,y) = G_{series}(T,y) = i\sqrt{\pi/2}\; k^{*-1} X$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m/2} \frac{(-1)^{m-r}(m-r)!\,(k^* y)^r t^{2m-4r}}{(2(m-r)+1)!!\,r!\,(m-2r)!}\; V^{*m+(1/2)} H_{m+(1/2)}(V^*)$$

where $H = J + iY$.

The second type of expansion, which was given in 10.3.2.4, cannot be carried out in a useful way, because the inverse powers of the binomial operator that appear cannot be easily evaluated.

10.6.3.2 General L(p)

The general integral G is made to depend on the special integral G_o for L = 1 by the same procedure as in Section 10.4. Set

$$L(p) = \sum_n b_n \left((p^2 - k^{*2})/k^{*2} \right)^n$$

Then

$$G(T,y) = \sum_{n=0}^{\infty} (-1)^n \, b_n \, (\nabla_{T^*}^2 + 1)^n \, G_o(T,y)$$

But

$$(\nabla_{T^*}^2 + 1)^n \, G_o(T,y) = (-1)^n t^{2n} \, G_o(T,y)$$

$$+ \frac{1}{k^*} \sum_{q=0}^{n-1} (-1)^q \, t^{2(n-q-1)} \, (\nabla_{T^*}^2 + 1)^q \, \frac{\exp iV^*}{iV^*}$$

and

$$(\nabla_{T^*}^2 + 1)^q \, \frac{\exp iV^*}{iV^*} = \sqrt{\frac{2}{\pi}} \left(\nabla_{V^*}^2 + 1 - \left(\frac{k^* yd}{V^* dV^*} \right)^2 \right)^q V^{*-1/2} \times$$

$$\left\{ J_{1/2}(V^*) + iY_{1/2}(V^*) \right\}$$

The two operators $\nabla_{V^*}^2$ and $\left(\frac{d}{V^* dV^*}\right)^2$ do not commute. When applied to an expression of the form

$$V^{*-p} \left(J_p(V^*) + i \, Y_p(V^*) \right),$$

the first increases the index p by 1 and multiplies by 2p; the second simply increases the index p by 2. Thus the non-commutativity does not affect the index, but only the constant factor in front of the expression. Therefore, we can write

$$(\nabla_{T^*}^2 + 1)^q \frac{\exp iV^*}{iV^*} = \sqrt{\frac{2}{\pi}} \sum_{r=0}^{q} c_{q,r} (k^* y)^{2r} V^{*-(q+r+(1/2))} \times$$

$$\left\{ J_{q+r+(1/2)}(V^*) - iY_{q+r+(1/2)}(V^*) \right\} .$$

where the $c_{q,r}$ are numerical constants depending only on q and r. Thus $c_{q0} = (2q-1)!!$ and $c_{qq} = 1$. When these forms are substituted into the expression for G(T,y), the sum $\Sigma\, b_n t^{2n}$ appears, just as in Section 10.4.1. This can be replaced by $L(\lambda)$. A triple sum also appears, which can be rearranged in several ways. The most convenient is apparently

$$G(T,y) = L(\lambda)\; G_o(T,y)$$

$$- \frac{1}{k^* t^2} \sqrt{\frac{2}{\pi}} \sum_{m=0}^{\infty} (-1)^m t^{-2m}\, V^{*-(m+1/2)}\; H_{m+1/2}(V^*) \times$$

$$\sum_{r=0}^{m/2} (-1)^r\, c_{m-r,r} (k^* yt)^{2r}\, a_{m-r},$$

where a_i is the infinite sum defined at the end of Section 10.4.1.

If k^*yt is small, only one term of the summation on r is significant, and the formula above reduces to the result obtained in 10.4.1.

ADDENDUM 1

GREEN'S FUNCTION FOR AN INFINITE ROD

The defining equation for Q(x*,x) is

$$EI\left(\frac{\partial^4}{\partial x^4} - k^4\right)Q(x^*,x) = \delta(x^*-x) \quad .$$

We assume a Fourier integral expansion for Q:

$$Q(x^*,x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} a(p)\, e^{-ipx}\, dp \quad ,$$

and use the result

$$\delta(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ipy}\, dp$$

to obtain

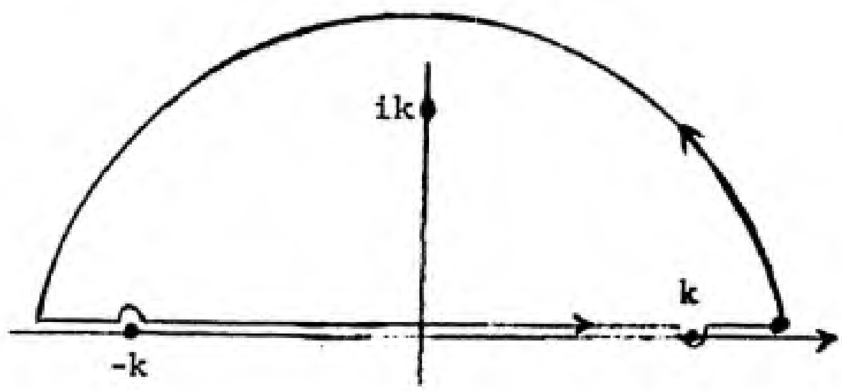
$$a(p) = \frac{1}{\sqrt{2\pi}} \; \frac{e^{ipx^*}}{EI(p^4-k^4} \quad .$$

Thus

$$Q(x^*,x) = \int_{-\infty}^{\infty} \frac{e^{ip(x^*-x)}\, dp}{2\pi EI(p^4-k^4)}$$

There are singularities of the integrand on the real p-axis at p = k and p = -k. The path will be distorted in the complex plane over or under these two poles, in such a way as to ensure no incoming waves. Th appropriate choices depend on the convention adopted for the time dependence, which will be taken as $e^{-i\omega t}$.

Case 1. $x^* > x$. Then the contour is completed by a semicircle above the real p-axis. At p = -k, the path is taken above the pole; and at p = k, below (see the figure). The value of the integral over the semicircle will be zero, in the limit of very large radius. Thus Q(x*,x) will be given by $2\pi i$ X the sum of the residues at the poles within the contour.



The contribution from p = ik is

$$2\pi i \quad \frac{e^{i(ik)(x^*-x)}}{2\pi EI(ik-k)(ik+k)(ik+ik)} = \frac{e^{-k(x^*-x)}}{-4EIk^3}$$

The contribution from p = k is

$$2\pi i \quad \frac{e^{ik(x^*-x)}}{2\pi EI(k+k)(k-ik)(k+ik)} = \frac{ie^{ik(x^*-x)}}{4EIk^3}$$

Thus $Q(x^*,x) = \frac{1}{4EIk^3}\left(ie^{ik(x^*-x)} - e^{-k(x^*-x)}\right), \qquad x^* > x$

If the contour had been closed by a semicircle below the real p-axis, the integral over this portion would not go to zero. If the indentation had been taken <u>under</u> the pole at p = -k, a term with $e^{-ik(x^*-x)}$ as factor would have appeared in Q. Since the source x^* is to the right of x, and the time dependence is $e^{-i\omega t}$, this would be a wave coming in on the left from infinity. If the identation at p= k had been taken above the real axis, so as to exclude the pole, there would not be any traveling wave term in $Q(x^*,x)$.

Case 2. $x > x^*$. In this case, the contour is completed by a semicircle in the lower half-plane, and indentations are made <u>below</u> $p = -k$ and <u>above</u> $p = k$.

The results obtained in the two cases can be combined as

$$Q(x^*-x) = (4EIk^3)^{-1} \left(ie^{ik|x^*-x|} - e^{-k|x^*-x|}\right) .$$

In terms of $y = x^*-x$, the power series expansion is:

$$Q = (4EIk^3)^{-1}\left[i - 1 - (i+1)k^2y^2/2 + k^3|y|^3/3 + (i-1)k^4y^4/4! - (i+1)k^6y^6/6! + \ldots\right]$$

Thus the first and second derivatives of Q are continuous at $y = y^*$, but the third is discontinuous.

ADDENDUM 2
SEVERAL GENERALIZED AXIAL RESONATORS ON A PLATE

There are N resonators of different kinds on a homogeneous plate. They are considered as centered at the points $s_1, \ldots, s_N$. The i-th resonator has the weight-function $w_i(r-s_i)$, where w_i is an arbitrary function of the polar angle θ about s_i and the distance $|r-s_i|$ from a typical point r to s_i. For the moment, we assume reciprocity, so that w_i is appropriate for both sensitivity and response.

At a point r, the i-th resonator exerts the force $F_i w_i(s_i-r)$, where F_i is its strength, when the resonator is excited by a unit <u>weighted</u> <u>displacement</u>. The weighted displacement is defined as:

$$\int dr'' w_i(r''-s_i)\ \eta(r''),$$

where the integration extends over the whole plane. Of course, if $w_i(z) = 0$ for $z > R$, then the domain of integration reduces to a circular disc. No normalization condition will be imposed on w_i.

If there is no water on one side of the plate, the differential operator for plate vibration is $D(\nabla^4 - k^4)$. (cf. Section 3.1.3). In the presence of water there is an additional integral term. In order to include both cases, we shall write the operator simply as O_r. The subscript indicates that it acts only on functions of r.

The Green's function $G(r^*,r)$ for the plate with resonators is by definition the displacement at r due to a concentrated unit force at r^*. Thus it satisfies the following defining equation of motion:

$$O_r(G(r^*,r)) = \delta(r^*-r) + \sum_{n=1}^{N} F_n\left(\int dx\, w_n(x-s_n)\, G(r^*,x)\right) w_n(s_n-r)$$

We shall show that $G(r^*,r)$ can be expressed in terms of the functions $Q_n(x)$, which satisfy the equations:

$$O_x(Q_n(x)) = w_n(x) \qquad n=1,\ldots,N$$

It is clear that $Q_m(x)$ is the response of a plate to which only the m-th resonator is attached, when the excitation of this resonator has unit strength. It will be convenient to set

$$s_o = r^*$$

$$w_o(x) = \delta(x),$$

and to define Q_o by

$$O_x(Q_o(x)) = w_o(x).$$

An integral representation for Q_o (in the presence of water) is given in Section 8.

Before proceding to the evaluation of G, we first demonstrate a reciprocity relation between resonators:

$$\int dx\, w_m(x-s_m)\, Q_n(s_n-x) = \int dx\, Q_m(x-s_m) w_n(s_n-x),$$

which says that the induced excitation of the m-th resonator due to unit excitation of the n-th, is equal to the induced excitation of the n-th due to unit excitation of the m-th. To show this, we observe that the operator O is defined in a way that makes no reference to any point of space, and that

$$O_x(H(x-y)) = O_y(H(x-y))$$

for any function H of the difference between two vectors x and y. Thus in the absence of the water, we have

$$D(\nabla_x^4 - k^4)\ H(x-y) = D(\nabla_y^4 - k^4)\ H(x-y).$$

This is a consequence of the fact that O is an even operator - in particular, it acts on an even function of distance to produce a new even function. The reciprocity relation can now be proved:

$\int dx\ w_m(x-s_m)\ Q_n\ (s_n-x)$

$$= \int dx\ O_x(Q_m(x-s_m)Q_n(s_n-x))$$

$$= \int dx\ O_{s_m}(Q_m(x-s_m))Q_n(s_n-x)$$

$$= O_{s_m}\ (\int dx\ Q_m(x-s_m)Q_n(s_n-x))$$

$$= O_{s_m}\ (\int dy\ Q_m(y)\ Q_n\ (s_n-s_m-y))$$

$$= \int dy\ Q_m(y)\ O_{s_m}(Q_n(s_n-s_m-y))$$

$$= \int dy\ Q_m(y)\ w_n(s_n-s_m-y)$$

$$= \int dx\ Q_m(x-s_m)\ w_n(s_n-x).$$

Note how the linearity of the operator O_{s_m} has been used twice, and how the variable of integration is changed from x to $y = x - s_m$ and back again.

The special case n = 0 (the point source at r*) yields

$$\int dx\, w_m(x-s_m)\, Q_o(s_o-x) = \int dx\, Q_m(x-s_m)\, w_o\,(s_o-x)$$

$$= \int dx\, Q_m(x-s_m) \quad \delta(r^*-x)$$

$$= Q_m\,(r^*-s_m).$$

We now assume that the Green's function for the plate with N resonators has the general form

$$G(r^*,r) = Q_o(r^*-r) - \sum_p \sum_q Q_p(r^*-s_p)\, M_{pq} Q_q(s_q-r),$$

where the M_{pq} are independent of r* and r, and shall see that there is only one matrix M_{pq} which leads to satisfaction of the defining equation for G(r*,r). The linearity of O implies

$$O_r(G) = \delta(r^*-r) - \Sigma\,\Sigma\, Q_p(r^*-s_p)\, M_{pq}\, w_q\, (s_q-r)$$

The defining equation will be satisfied if and only if the coefficients of $w_i(s_i-r)$ match, that is:

$$F_n(\int dx\, w_n(x-s_n)\, G(r^*,x) = -\sum_p Q_p\,(r^*-s_p)\, M_{pn}$$

$$n = 1,2,\ldots,N$$

Substitute for G(r*,x), after dividing by F_n:

$$\int dx\, w_n(x-s_n)\, Q_o(r^*-x)$$

$$-\sum_i \sum_j \int dx\, w_n(x-s_n)\, Q_i(r^*-s_i)\, M_{ij}\, Q_j\,(s_j-x)$$

$$= -(1/F_n) \sum_p Q_p(r^*-s_p)\, M_{pn}$$

The first integral is transformed by reciprocity to $Q_n(r^*-s_n)$, as noted above. Then we have

$$Q_n(r^*-s_n) - \sum_i Q_i(r^*-s_i) \sum_j \int dx\, w_n(x-s_n)\, M_{ij}\, Q_j\,(s_j-x)$$

$$= -\,(1/F_n) \sum_p Q_p\,(r^*-s_p)\, M_{pn}$$

This will be satisfied if and only if the total coefficient of each function $Q_m(r^*-s_m)$ is zero:

$$\delta_{mn} - \sum_j M_{mj}\, V_{jn} + (M_{mn}/F_n) = 0$$

where $\quad V_{jn} = \int dx\, w_n(x-s_n)\, Q_j\,(s_j-x)$

This can be written

$$\sum_j (M_{mj}\, V_{jn} - (\delta_{jn}/F_n)) = \delta_{mn}$$

Therefore: if a matrix N is defined by:

$$N_{pq} = \int dx\, w_q\,(x-s_q)\, Q_p\,(s_p-x) - (\delta_{pq}/F_p)$$

then

$$M = N^{-1}$$

V_{pq} will be called the <u>interaction</u> or <u>overlap</u> integral for the p-th and q-th resonators. It is clear that V_{pq} depends on the vector difference $y = s_p - s_q$, and not on s_p and s_q separately. An equation for $V_{pq}\,(y)$ is easily derived by applying the operator O_y to a form of the integral that gives V:

$$O_y(V_{pq}(y)) = O_y \int dx\, w_q(x)\, Q_p\, (y-x)$$

$$= \int dx\, w_q\, (x)\, O_y\, (Q_p(y-x))$$

$$= \int dx\, w_q\, (x)\, w_p(y-x)$$

The integral on the right is to be regarded as a known function, since the w_i are given. In particular, suppose that $w_i(x) = 0$ if $|x| > R_i$, where R_i may be called the radius of the i-th resonator. Then

$$O_y(V_{pq}\, (y)) = 0 \qquad \text{if } |y| > R_p + R_q$$

We now make the assumption of <u>centrosymmetry</u>, which is fulfilled in all our applications:

$$w_i(r-s_i) = \pm\, w_i\, (s_i-r)$$

It is convenient to say that the positive sign holds for even i and the negative for odd i,

$$w_i(-x) = (-1)^i\, w_i(x),$$

but of course this is purely a notational device. It follows that

$$Q_i(-x) = (-1)^i\, Q_i(x)$$

The reciprocity relation then takes the form

$$V_{pq} = \int dx\, w_q\,(x-s_q)\, Q_p\,(s_p-x)$$

$$= \int dx\, Q_p\,(x-s_q)\, w_p\,(s_p-x)$$

$$= (-1)^{p+q} \int dx\, w_p\,(x-s_p)\, Q_q\,(s_q-x)$$

$$= (-1)^{p+q}\, V_{qp}.$$

Suppose now that the weight-function for a resonator is

$$w(|x-s|)\cos n\,\theta$$

where θ is the angle between the prime direction and the vector x-s. We take the Hankel transform of order n of both sides of O (Q) = w. Then application of the operator O to Q corresponds to multiplying the transform of Q by f(p), which was defined in Section 8.1. We thus obtain

$$Q_{(n)}(p) = f^{-1}(p)\, w_{(n)}(p)$$

where $w_{(n)}(p)$ is the Hankel transform of w of order n, and similarly for $Q_{(n)}$. Then inverting the transform,

$$Q(x) = \cos n\,\theta \int_0^{\infty} f^{-1}(p)\, w_{(n)}(p)\, pJ_n(p|x-s|)\, dp.$$

The overlap integral for two resonators centered at s and s', with weight functions w cos n θ and w' cos n' θ, can now be evaluated using Addendum 3:

$$V_{ii'} = \pi\,(-1)^{n}\cos[(n+n')\Omega]\int_{0}^{\infty} f^{-1}(p)w_{(n)}(p)w_{(n')}(p)\;p\;J_{n+n'}(p|s-s'|)dp$$

$$+\;\pi(-1)^{n+n'}\cos[(n-n')\Omega]\int_{0}^{\infty} f^{-1}(p)w_{(n)}(p)w_{(n')}(p)\;p\;J_{n-n'}(p|s-s'|)dp$$

where Ω is the angle from the prime direction to the vector $s-s'$.

If the two resonators are really the same, that is, V is the self-interaction or self-overlap integral, and n is different from zero, then the formula simplifies to

$$V_{ii} = \pi\int_{0}^{\infty} f^{-1}(p)\;w^{2}_{(n)}(p)\;p\;dp$$

The reciprocity restriction will now be removed. The n-th resonator will exert the force

$$F_n w_n(s_n - r)$$

at r, in response to "unit weighted displacement." The weighting function $w^{*}_{n}(x-s)$ for sensitivity is not the same as w_n, but will be written in the form of an operator C_n applied to w_n:

$$w^{*}_{n}\,(x-s) = C_n(w_n(x-s))$$

We shall assume that each C_n commutes with the plate operator O, as will be true if C_n is a linear differential or integral operator (since O has such a form). Furthermore, C_n does not depend on a preferred direction in the plane or on a preferred position. If C_n is regarded as operating on functions of x only, $C_n(w(x-s)) = \pm\, C_n(w(s-x))$.

We define the Green's response function $Q_n(x)$ for the n-th resonator by

$$O(Q_n(x)) = w_n(x)$$

Applying C_n to Q_n, we obtain what may be called the Green's sensitivity function $C_n(Q_n(x))$. Because of the commutation assumption, this satisfies:

$$O(C_n(Q_n(x))) = C_n(w_n(x))$$

We can prove a pseudo-reciprocity relation:

$$\int dx\ C_i(w_n(x-s))Q_m(s_m-x) = \int dx\ Q_n(x-s_n)C_i(w_m(s_m-x))$$

by the same devices used earlier.

The equation of motion, which gives the displacement $G(r^*,r)$ at r due to a unit force at r*, in the presence of N non-reciprocal resonators, is now:

$$O_r(G(r^*,r)) = \delta(r^*-r) + \sum_n F_n[\int dx\ C_n(w_n(x-s_n))G(r^*,x)]\ w_n(s_n-r).$$

Our earlier work suggests a solution in the form

$$G(r^*,r) = Q_o(r^*-r) - \sum_{p,q} C_p(Q_p(r^*-s_p))M_{pq}Q_q(s_q-r)$$

where

$$O(Q_o(x)) = \delta(x)$$

and M is a matrix of constants, to be determined.

We apply O_r to this form for $G(r^*,r)$, equate coefficients of each $w_i(s_i-r)$, and substitute for $G(r^*,x)$ within the integral. We then use the pseudo-reciprocity relation in the special case m=0 and C_o equal to the identity, that is

$$\int dx\ w_n(x-s_n)Q_o(r^*-x) = Q_n(r^*-s_n)$$

The equation of motion will now be satisfied provided that

$$F_n\ \delta_{np} - F_n \sum_q M_{pq} V_{qn} - M_{pn} = 0$$

is satisfied for all n, p = 1, 2,...N. Here V_{qn} is the interaction integral

$$V_{qn} = \int dx\ C_n(w_n\ (x-s_n))\ Q_q(s_q-x)$$

$$= \int dx\ Q_n(x-s_n)\ C_n(w_m(s_m-x)).$$

The equation for the M_{pq} can be written conveniently in matrix notation, as earlier:

$$M = [\ V_{qn} - (\delta_{qn}/F_n)\ ]^{-1}$$

ADDENDUM 3

EVALUATION OF GENERALIZED TWO-DIMENSIONAL CONVOLUTION

The preceding analysis has often led to integrals of the following general nature:

$$I = \int dx\ F^*(x-u)\ G^*(x-v)$$

where the vector variable x ranges over the entire plane, and F^* and G^* have the forms

$$F^* = (a \cos m A + b \sin m A)\ F(|x-u|)$$

$$G^* = (c \cos n B + d \sin n B)\ G(|x-v|) \quad .$$

Here m and n are positive integers or zero, A is the angle between the prime direction and the vector x-u, and B is the angle between the prime direction and x-v. The special case in which F^* and G^* are independent of A and B (that is, $m = n = 0$) has appeared several times.

For convenience in handling the algebra, we will first consider the forms

$$F^* = (\exp\ im\, A)\ F(|x-u|)$$

$$G^* = (\exp\ in\, B)\ G(|x-v|),$$

where m and n can be positive, negative or zero. It will be assumed that F and G can be represented as Hankel transforms, of order m for F and n for G:

$$F(r) = \int_0^\infty F_{(m)}(p)p\ J_m(pr)dp$$

$$G(r) = \int_0^\infty G_{(n)}(q)q\ J_n(qr)dq\ .$$

Then

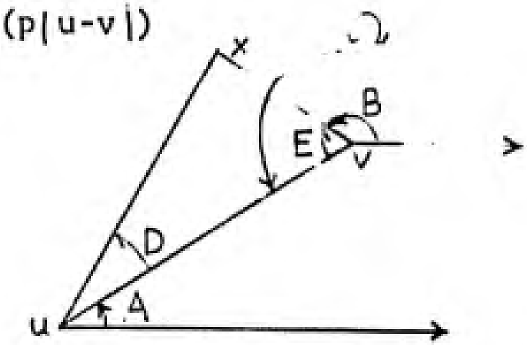
$$I = \int dp \int dq\ p\ F_{(m)}(p)q\ G_{(n)}(q)K$$

where

$$K = \int dx\ \exp\ i(mA+nB)\ J_m(p|x-u|)\ J_n(q|x-v|).$$

We now use an addition theorem for Bessel functions to expand $J_m(p|x-u|)$ around v as center. Let D be the angle between u-v and x-u, and E be angle between x-v and u-v. Both angles are taken as positive and less than π. Then

$$e^{imD} J_m(p|x-u|) = \sum_{h=-\infty}^{\infty} e^{ihE} J_h(p|x-v|) J_{m+h}(p|u-v|)$$



Let Ω be the angle between the prime direction and u-v. It can lie between 0 and 2π. Then from the diagram

$$A = \Omega + D - \pi$$
$$B = \Omega - E.$$

Therefore K becomes

$$K = dx \exp i(m(\Omega+D-\pi) + n(\Omega-E)) \, J_m(p|x-u|) \, J_n(q|x-v|).$$
$$= \exp i(m\Omega - m\pi + n\Omega) L$$

where

$$L = \int dx \exp i(mD - nE) \, J_m(p|x-u|) \, J_n(q|x-v|)$$
$$= \int_{h=-\infty}^{\infty} dx \exp i(hE - nE) \, J_h(p|x-v|) \, J_{m+h}(p|u-v|) \, J_n(q|x-v|)$$

$$= \sum J_{m+h}(p|u-v|) \int dx \exp i(h-n)E \, J_h(p|x-v|) \, J_n(q|x-v|) \; .$$

If we set $r = |x-v|$, the integral over the plane becomes

$$\int_0^{2\pi} \exp i(h-n) E \, dE \int_0^{\infty} J_h(pr) \, J_n(qr) r \, dr$$

$$= 2\pi \, \delta_{hn} \, p^{-1} \, \delta(p-q),$$

by a well-known property of Bessel functions. Therefore

$$L = 2\pi\ p^{-1}\delta(p-q)\ J_{m+n}(p|u-v|)$$

and so

$$I = 2\pi\ \exp\ i(m\Omega - m\pi + n\Omega)\ x$$

$$\int dp \int dq\ pF_{(m)}(p)\ qG_{(n)}(q)\ p^{-1}\delta(p-q)\ J_{m+n}(p|u-v|)$$

$$= (-1)^m\ 2\pi\ \exp\ i(m+n)\Omega \int dp\ F_{(m)}(p)\ G_{(n)}(p)p\ J_{m+n}(p|u-v|).$$

This formula is apparently unsymmetrical between u and v, because of the factor $(-1)^m$. However, this is a consequence of the need to assign a definite sign to u-v in the definition of Ω.

Results when F^* and G^* are real can be easily obtained by writing sines and cosines as sums of imaginary exponentials, and using the following result for Hankel transforms:

$$F_{(-m)} = (-1)^m F_{(n)}$$

$$G_{(-n)} = (-1)^n G_{(n)}\ ,$$

which are derived from

$$J_{-i}(x) = (-1)^i J_i(x)\ .$$

Using the abbreviation

$$S(i) = \int_0^\infty F_{(m)}(p)\ G_{(n)}(p)\ J_i(p|u-v|)\ p\ dp$$

it is found that

$$\int dx\ \cos mA\ \cos nB\ F(|x-u|)\ G(|x-v|)$$
$$= (-1)^m\pi\ S(m+n)\cos(m+n)\Omega + (-1)^{m+n}\pi\ S(m-n)\ \cos\ (m-n)\Omega$$

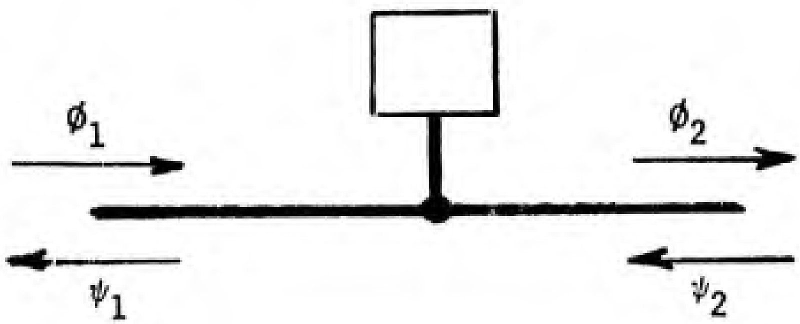
$$\int dx\ \cos mA\ \sin nB\ F(|x-u|)\ G(|x-v|)$$
$$= (-1)^m\pi\ S(m+n)\ \sin(m+n)\Omega - (-1)^{m+n}\pi\ S(m-n)\ \sin(m-n)\Omega$$

$$\int dx\ \sin mA\ \cos nB\ F(|x-u|)\ G(|x-v|)$$
$$= (-1)^m\pi\ S(m+n)\ \sin(m+n)\Omega + (-1)^{m+n}\pi\ S(m-n)\ \sin(m-n)\Omega$$

$$\int dx\ \sin mA\ \sin nB\ F(|x-u|)\ G(|x-v|)$$
$$= -\ (-1)^m\pi\ S(m+n)\ \cos(m+n)\Omega + (-1)^{m+n}\pi\ S(m-n)\ \cos(m-n)\Omega.$$

ADDENDUM 4

TRANSFER MATRIX EXPRESSED IN TERMS OF TRANSMISSION AND REFLECTION MATRICES



Let ϕ_1 and ψ_1 represent the incoming and outgoing (right-directed and left-directed) disturbances on a rod to the left of an obstacle or resonator, and ϕ_2 and ψ_2 the outgoing and incoming (again right-directed and left-directed) disturbances on the right.

It is clear that

$$\phi_2 = T\phi_1 + R^*\psi_2$$

$$\psi_1 = R\phi_1 + T^*\psi_2 \quad ,$$

where

T = transmission operator for disturbance incident from left

T^* = transmission operator for disturbance incident from right

R = reflection operator for disturbance incident from left

R^* = reflection operator for disturbance incident from right.

It is convenient to define also $U = T^{-1}$ and $U^* = T^{*-1}$.

We can solve the first equation for ψ_2:

$$\psi_2 = T^{*-1}(\psi_1 - R\phi_1) = - U^*R\phi_1 + U^*\psi_1 .$$

Substituting this into the second equation, we have also

$$\phi_2 = T\phi_1 + R^*(-U^*R\phi_1 + U^*\psi_1) = (T-R^*U^*R)\phi_1 + R^*U^*\psi_1$$

In terms of the vectors $\begin{pmatrix}\phi_1 \\ \psi_1\end{pmatrix}$ and $\begin{pmatrix}\phi_2 \\ \psi_2\end{pmatrix}$ these can be collected in the single vector equation

$$\begin{pmatrix}\phi_2 \\ \psi_2\end{pmatrix} = \begin{pmatrix} T-R^*U^*R & R^*U^* \\ -U^*R & U^* \end{pmatrix} \begin{pmatrix}\phi_1 \\ \psi_1\end{pmatrix}$$

If the disturbances have been expressed in the wave basis, then the matrix that appears here is clearly the transfer matrix, in the same basis, from a point that is immediately to the left of the obstacle to a point just to the right.

ADDENDUM 5

REGULAR ARRAY OF AXIAL-TRANSVERSE RESONATORS ON A ROD

An infinite series of identical resonators, with axial strength F and transverse strength G, are attached with the spacing s. There is an applied pressure $e^{i(vx - \omega t)}$. After suppressing the factor $e^{-i\omega t}$, the equation of motion is

$$EI\left(\frac{\partial^4}{\partial x^4} - k^4\right)\eta = e^{ivx} + F\eta\sum\delta(x-ns)$$
$$+ G\frac{\partial}{\partial x}\sum\delta(x-ns)\frac{\partial\eta}{\partial x} \quad ,$$

where the sums run from $n = -\infty$ to $n = +\infty$. We now introduce the Fourier expansion for a regular array of δ-functions:

$$\sum\delta(x-ns) = (u/2\pi)\sum e^{inux}$$

where $\quad u = 2\pi/s$

and we assume an expansion for η:

$$\eta = \sum_p a_p\, e^{ipux + ivx} \quad .$$

(All sums on p will also run from $-\infty$ to $+\infty$). We also define the function

$$f\,(w) = EI(w^4 - k^4),$$

and set:

$$F_o = uF/2\pi = F/s$$
$$G_o = uG/2\pi = G/s \quad .$$

Then the equation of motion becomes:

$$\sum_p a_p f(pu + v)\, e^{(pu + v)ix}$$

$$= e^{ivx} + F_o \sum_p a_p \sum_n e^{(nu + pu + v)ix}$$

$$- G_o \sum_p a_p (pu + v)^2 \sum_n e^{(nu + pu + v)ix}$$

$$- G_o \sum_p a_p (pu + v) \sum_n nue^{(nu + pu + v)ix} \quad .$$

The factor e^{ivx} can be divided out. In the sums on the right, we make $m = n + p$ a new summation variable, in place of n, and change the order of summation. Then we rename the summation variable on the left as m. Now the coefficients of corresponding terms e^{imux} can be equated:

$$a_m f(mu+v) = \delta_{mo} + F_o \sum_p a_p - G_o \sum_p a_p (pu+v)^2 - G_o \sum_p a_p (pu+v)(m-p)u$$

$$= \delta_{mo} + F_o \Sigma a_p - G_o \Sigma a_p (mu+v)(pu+v) \qquad m = -\infty, \ldots . +\infty.$$

Now setting

$$A = \sum_p a_p$$

$$P = \sum_p a_p (pu+v) \quad ,$$

we have

$$a_m = f^{-1}(v)\delta_{mo} + f^{-1}(mu+v)(F_o A - G_o (mu+v) P)$$

$$m = -\infty, \ldots , +\infty \ .$$

To determine A, we sum this expression for a_m over m; to determine P, we sum after multiplying by $mu + v$. Then we obtain

$$A = f^{-1}(v) + U_o F_o A - G_o U_1 P$$

$$P = v f^{-1}(v) + U_1 F_o A - G_o U_2 P$$

where we have set

$$U_i = \sum_m f^{-1}(mu+v)(mu+v)^i \qquad i = 0,\ 1,\ 2.$$

Thus we have obtained a system of two simultaneous equations for A and P:

$$(1 - F_o U_o)A + G_o U_1 P = f^{-1}(v)$$

$$- F_o U_1 A + (1 + G_o U_2)P = vf^{-1}(v)$$

If we define

$$T_i = \sum_m f^{-1}(mu+v)m^i,$$

then the determinant Δ of this system can be written in the alternative forms

$$\Delta = 1 - F_o U_o + G_o U_2 + F_o G_o (U_1^2 - U_o U_2)$$

$$= 1 - F_o T_o + G_o(u^2 T_2 + 2uvT_1 + v^2 T_o)$$

$$+ F_o G_o u^2 (T_1^2 - T_o T_2)$$

We then have

$$A = f^{-1}(v)\Delta^{-1}(1 + G_o(u^2 T_2 + uvT_1))$$

$$P = f^{-1}(v)\Delta^{-1}(v + F_o u T_1)$$

Thus the a_m can be found, and then the displacement can be written down. It is exhibited explicitly at the end of Section 7.2.

ADDENDUM 6

SMALL RESONATORS UNIFORMLY INTERSPERSED AMONG REGULAR ARRAY OF LARGE RESONATORS

The large resonators having space s. Between any two of these, there are M - 1 uniformly placed small resonators, of axial strength E. The large resonators have strength E + F in the axial mode.

The equation of motion is

$$EI\left(\frac{\partial^4}{\partial x^4} - k^4\right)\eta = e^{ivx} + F\eta\Sigma\delta(y-ns) + E\eta\Sigma\delta(y-(ns/M))$$

where the summations on n go from $-\infty$ to ∞. We assume the expansion

$$\eta = \sum_p a_p e^{(up+v)ix}$$

where $\quad u = 2\pi/s$

and use the expansion for the regular array of δ-functions. Then after introducing the abbreviations

$$F_o = uF/2\pi$$

$$E_o = NuE/2\pi$$

$$f(w) = EI(w^4-k^4)$$

the equation of motion becomes

$$\sum_p a_p f(pu+v)e^{(pu+v)ix}$$

$$= e^{ivx} + F_o\sum_p a_p\sum_n e^{(nu+pu+v)ix}$$

$$+ E_o\sum_p a_p \sum_n e^{(Mnu+pu+v)ix} \quad .$$

The e^{ivx} factor can be dropped.

In the second term on the right we set m = n+p as new variable, then change the order of summation. Then the term becomes

$$F_o\ A \sum_m e^{muix}$$

where

$$A = \sum_p a_p$$

We now rewrite p, wherever it appears, as $p = j+Mr$, where j goes from 0 to M-1 and r goes from $-\infty$ to $+\infty$. Similarly we set

$$m = j + Mr.$$

Then we have

$$\sum_r \sum_j a_{j+Mr} f((j + Mr)u + v) e^{(j+Mr)uix}$$

$$= e^{ivx} + F_o A \sum_r \sum_j e^{(j+Mr)uix}$$

$$+ E_o \sum_r \sum_j a_{j+Mr} \sum_n e^{(Mn + j + Mr)uix}$$

In the last term, we make $t = n + r$ a new summation variable, in place of n, and take t as the outer summation variable. Then the term can be written

$$E_o \sum_t \sum_j A_j e^{i(j+Mt)uix}$$

where

$$A_j = \sum_r a_{j+Mr} \quad , \qquad j = 0,1,\ldots M - 1 \quad .$$

It is now possible to rename t as r, and then the coefficients of each term $e^{(j+Mr)uix}$ can be equated:

$$a_{j+Mr} f((j+Mr)u+v) = \delta_{jo}\delta_{ro} + F_o A + E_o A_j \quad .$$

We now obtain an equation for each A_j by dividing this equation by f and summing over all r.

Setting

$$T(j) = \sum_{n=-\infty}^{\infty} f^{-1}((j+Mn)u+v),$$

we find

$$A_j = \delta_{jo}\, f^{-1}(v) + F_o T(j) A + E_o T(j) A_j$$

or

$$A_j = \frac{\delta_{jo}\, f^{-1}(v) + F_o T(j) A}{1 - E_o\, T(j)}$$

But of course

$$A = \sum_j A_j$$

and so an equation for A is obtained by summation:

$$A = Z^{-1} f^{-1}(v) \sum_j (1 - E_o T(j))^{-1}$$

where $\quad Z = 1 - F_o \sum T(j)(1 - E_o T(j))^{-1}$

We now retrace the algebra to evaluate A_j, a_p, and finally η.

$$A_j = \frac{\delta_{jo} + F_o T(j) Z^{-1} \sum (1 - E_o T(j))^{-1}}{f(v)(1 - E_o T(j))}$$

$$\begin{aligned} a_{j+Mr} = {} & \delta_{jo}\, \delta_{ro}\, f^{-1}(v) \\ & + \delta_{jo} f^{-1}(v) f^{-1}(Mru+v) E_o (1 - E_o T(0))^{-1} \\ & + f^{-1}(v) f^{-1}((j+Mr)u+v) F_o\, Z^{-1} \sum_i (1 - E_o T(i))^{-1} \\ & \quad X \left\{ 1 + E_o T(j) (1 - E_o T(j))^{-1} \right\} \end{aligned}$$

The final form for η is given in Section 7.4.

ADDENDUM 7

APPROXIMATION FOR THE COMBINED EFFECT OF THE COMPLEX PAIR OF ROOTS OF $P(p^2)$

We shall write one of the complex pair of roots as λ^2 and use a bar to indicate complex conjugate. Then the contribution C of the two roots to the rational part of f(p) has the form

$$C = \frac{\bar{A}\,\lambda^2 + A\,\bar{\lambda}^2 + p^2\,(A + \bar{A})}{p^4 - (\lambda^2 + \bar{\lambda}^2)\,p^2 + \lambda^2\,\bar{\lambda}^2}$$

Using two steps of recursive relation in Section 10.2.2, we obtain the formula:

$$\lambda^2 = -k^2\sqrt{1 \pm ie^* \quad [1 \mp ite^*]^{-1/2}}$$

where

$$t = k^2/(k^2 + k^{*2})$$

$$e^* = e/(k^4(k^2 + k^{*2})^{1/2})$$

Note that

$$e^{*2} = \frac{\rho^{*2}}{\rho^2}\;\frac{1}{h^2(k^2 + k^{*2})}\;,$$

where h is the plate thickness. For many situations of practical interest, e^{*2} is in the range .04 - .08.

The expression for λ^2 is correct to terms in e^{*2}. In explicit power-series form, we then find

$$\lambda^2 + \bar{\lambda}^2 = -\ 2k^2 \quad [\,1 + e*^2(2t + 1)/4\,]$$

$$\lambda^2\ \bar{\lambda}^2 = k^4 \quad [\,1 + e*^2(2t + 2)/4\,]$$

Thus C can be written correct to terms in $e*^2$

$$C = \frac{M\,p^2 + N}{p^4 + 2k^2\,p^2\,(1 + u) + k^4(1 + v)}$$

where u and v are multiples of $e*^2$. We now attempt to approximate C in the form

$$C* = \frac{c}{p^2 + k^2(1 + a\,u + b\,v)} + \frac{c'}{(p^2 + k^2(1 + a'u + b'v))^2}$$

where terms higher than linear in u and v can be discarded. We find that it is necessary to take

$$c = M, \quad c' = M\,k^2 + N$$

Then, in the approximation where terms in $e*^2$ are retained, the error C* - C has the form

$$C* - C = Ju + Kv,$$

where

$$J = \frac{(M\,p^2 + N)\ 2p^2 - M(p^2 + k^2)\ a - (N - M\,k^2)\ 2p^2a'}{(p^2 + k^2)^4}$$

$$K = \frac{(M\,p^2 + N)\ k^2 - M(p^2 + k^2)b - (N - M\,k^2)\ 2p^2b'}{(p^2 + k^2)^4}$$

One can ensure that $K = 0$ by taking

$$b = \frac{(M\,p^2 + N)\,k^2}{M(p^2 + k^2)} \qquad b' = \frac{(M\,p^2 + N)\,k^2}{(N - M\,k^2)\,2p^2}$$

J cannot be made equal to zero, but by adjusting a and a' one can make

$$\max_{p > 0} |J|$$

as small as possible. Set $P^* = p^2/k^2$. Then after taking out constants, the problem has this form: Determine c and d so as to minimize

$$\max_{P^* > 0} \frac{|(P^* - c)\,(P^* - d)|}{(P^* + 1)^4}$$

The values $c = .055$ and $d = .6$ produce the maximum value of .034 at $P^* = 1.8$, and it is not likely that this differs from the minimum over c and d by more than 5 percent.

Clearly, the first term in C^* leads to integrals of the same form as I^* in Section 10.1. The second term produces terms that can be represented in the general form

$$-\frac{1}{k}\,\frac{\partial I^*}{\partial k}\ .$$

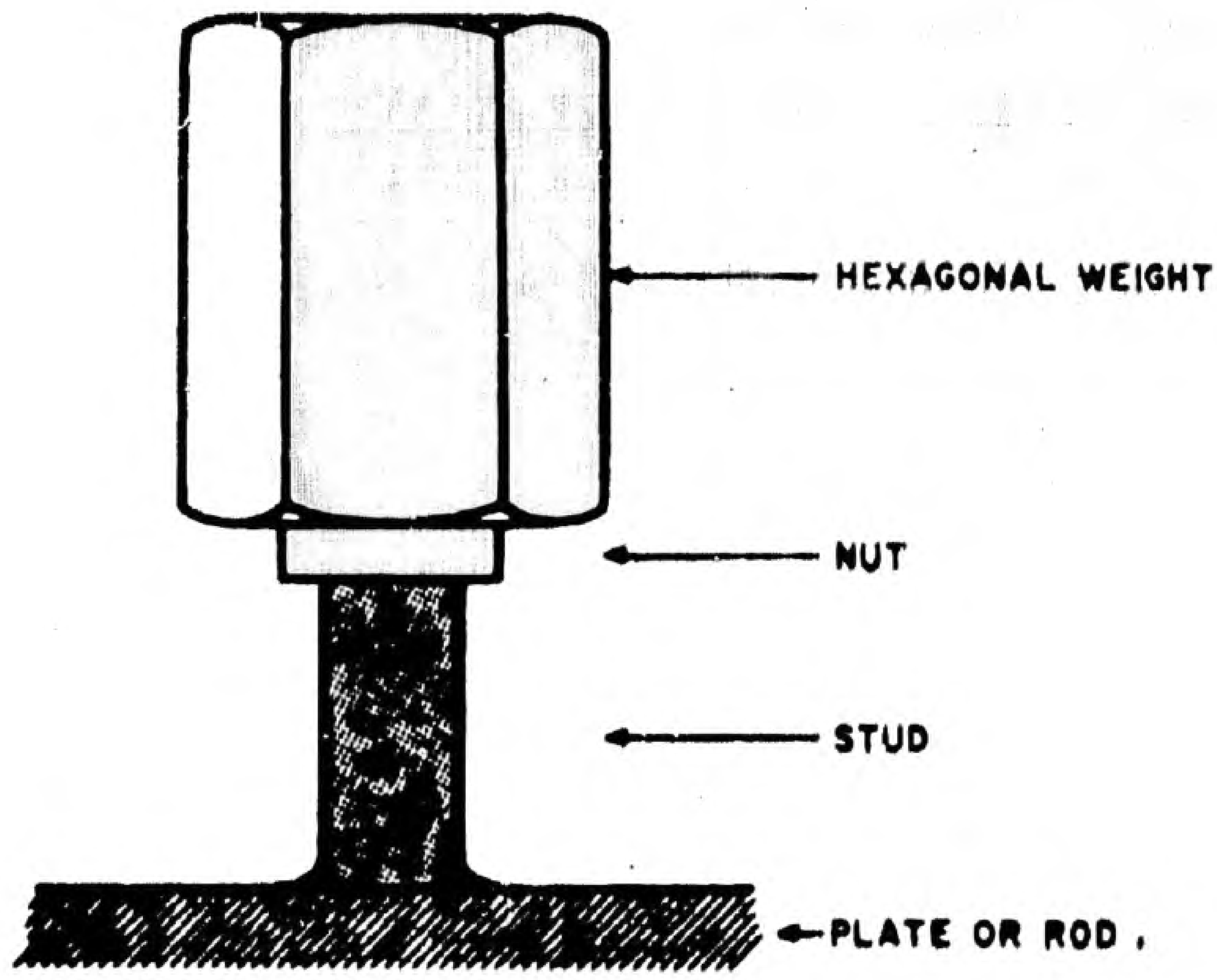


FIGURE 1-1. A STANDARD TRG RESONATOR

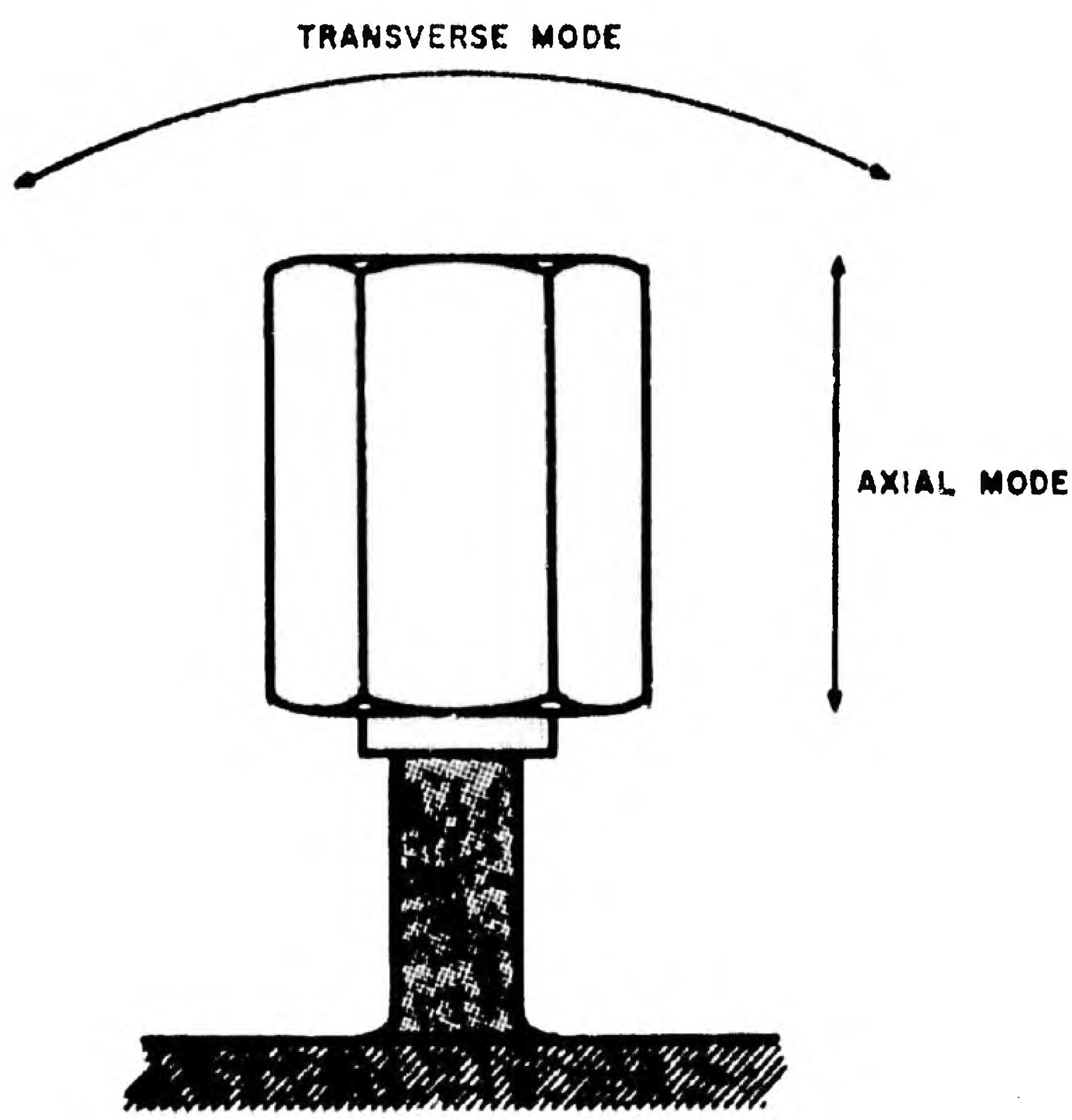


FIGURE 1-2. AXIAL AND TRANSVERSE MODES OF VIBRATION

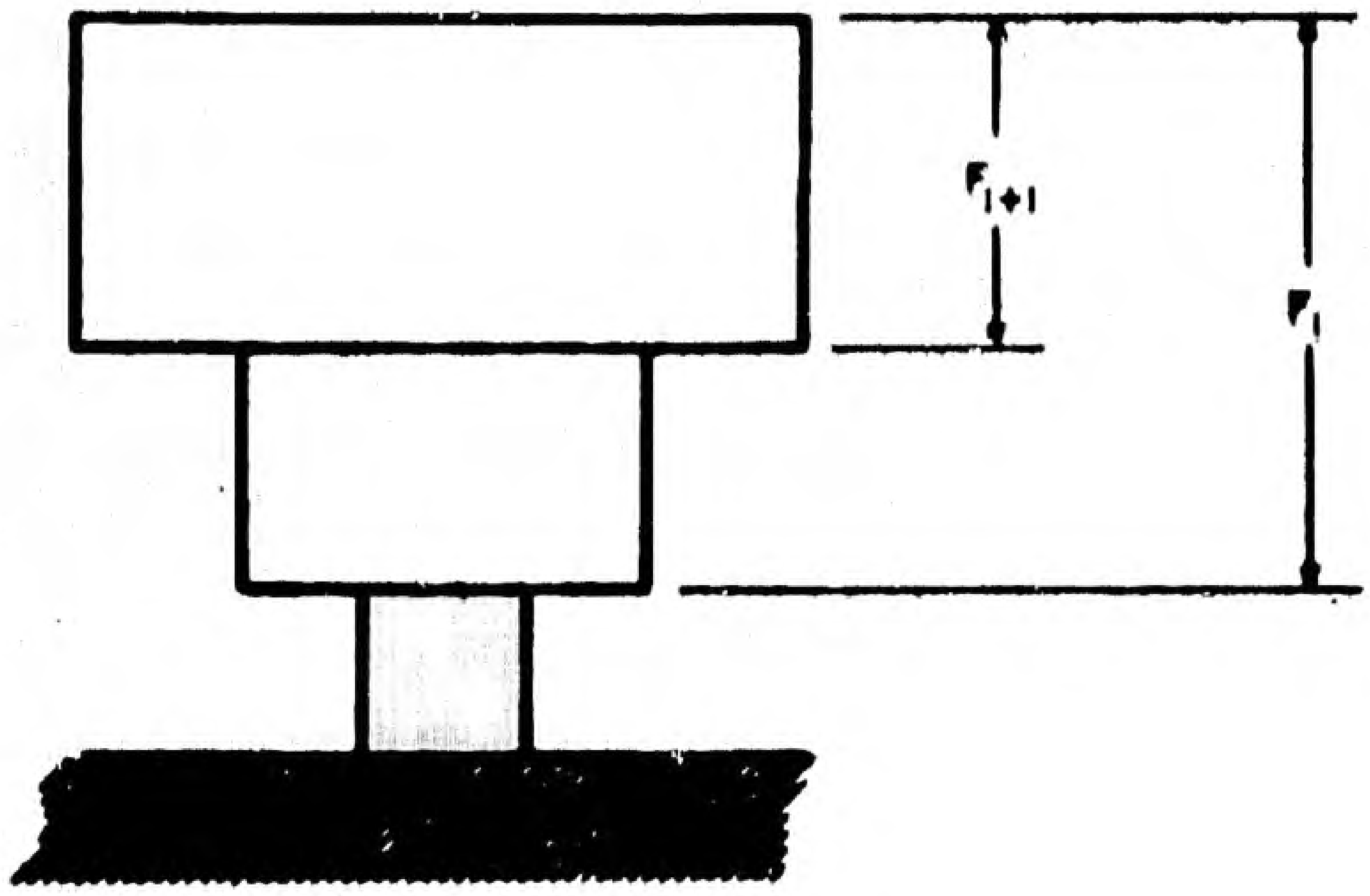


FIGURE 2-1. NOTATION FOR AN n-SEGMENT AXIAL RESONATOR

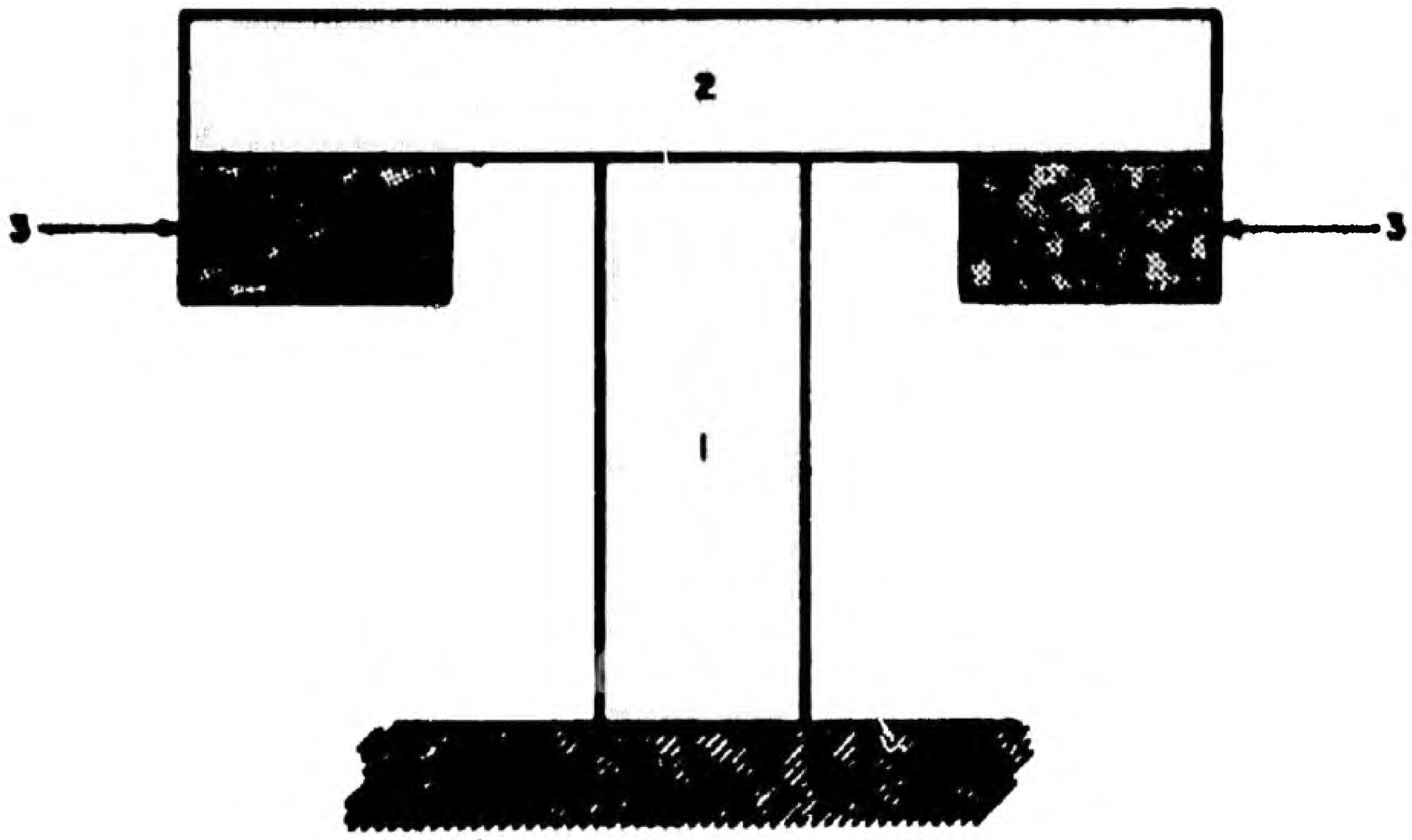


FIGURE 2-2. MORE COMPLICATED TYPE OF RESONATOR

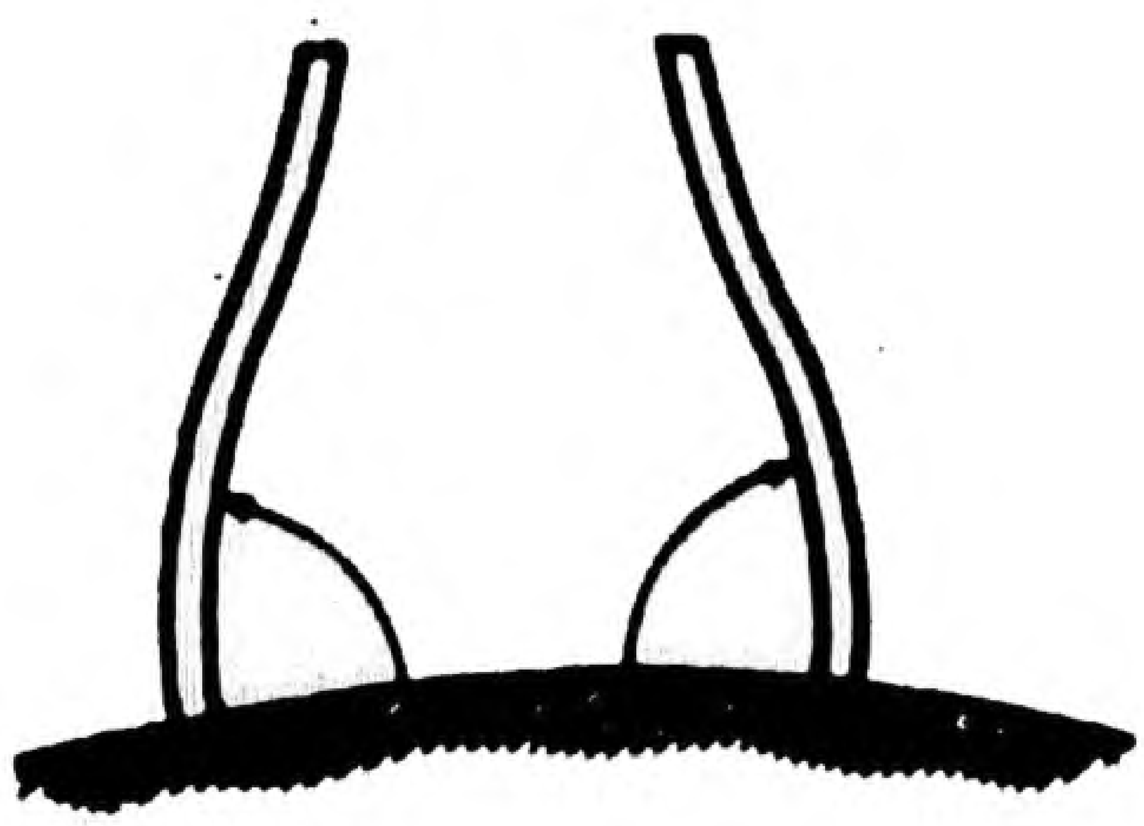


FIGURE 2-3. BREATHING MODE OF A PIPE
(CROSS-SECTION)

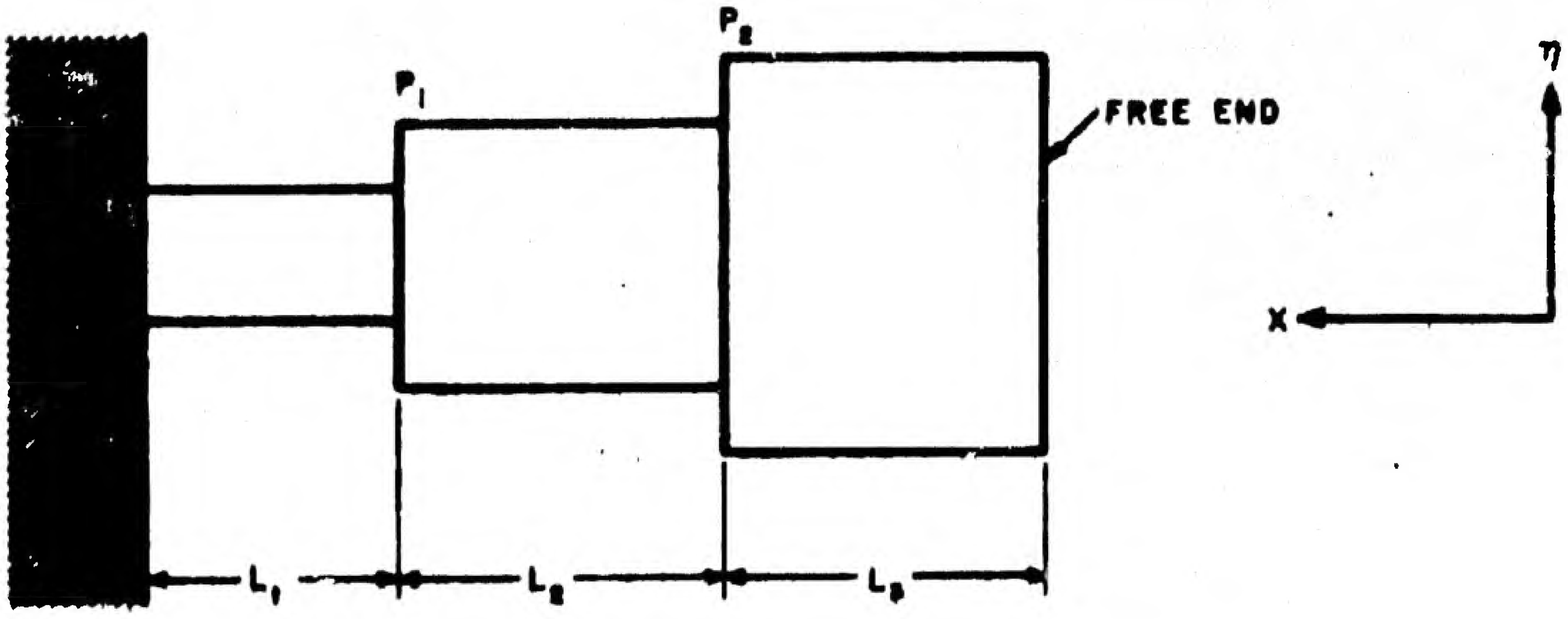


FIGURE 5-1. COMPOUND TRANSVERSE RESONATOR

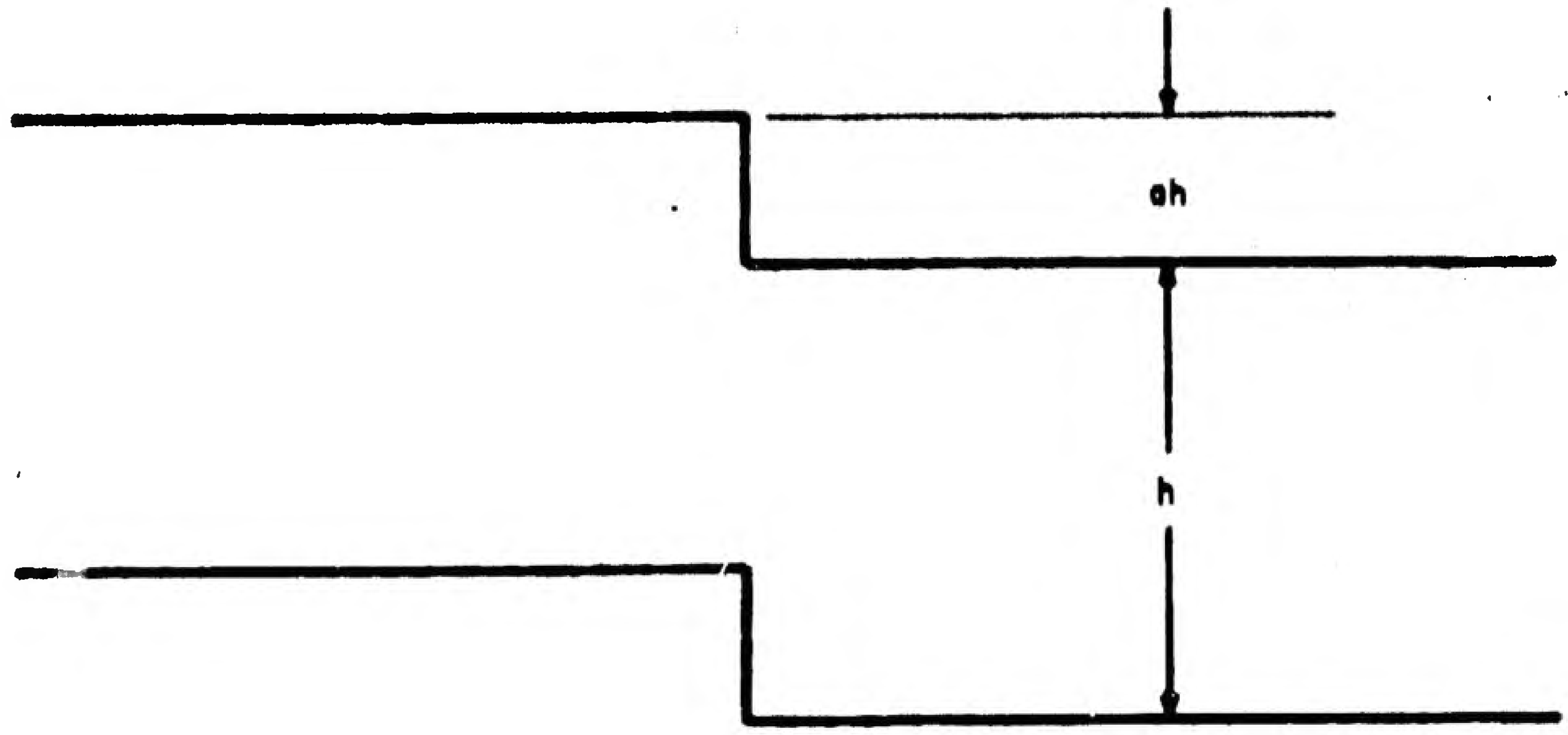


FIGURE 5-2. A STEP IN A ROD

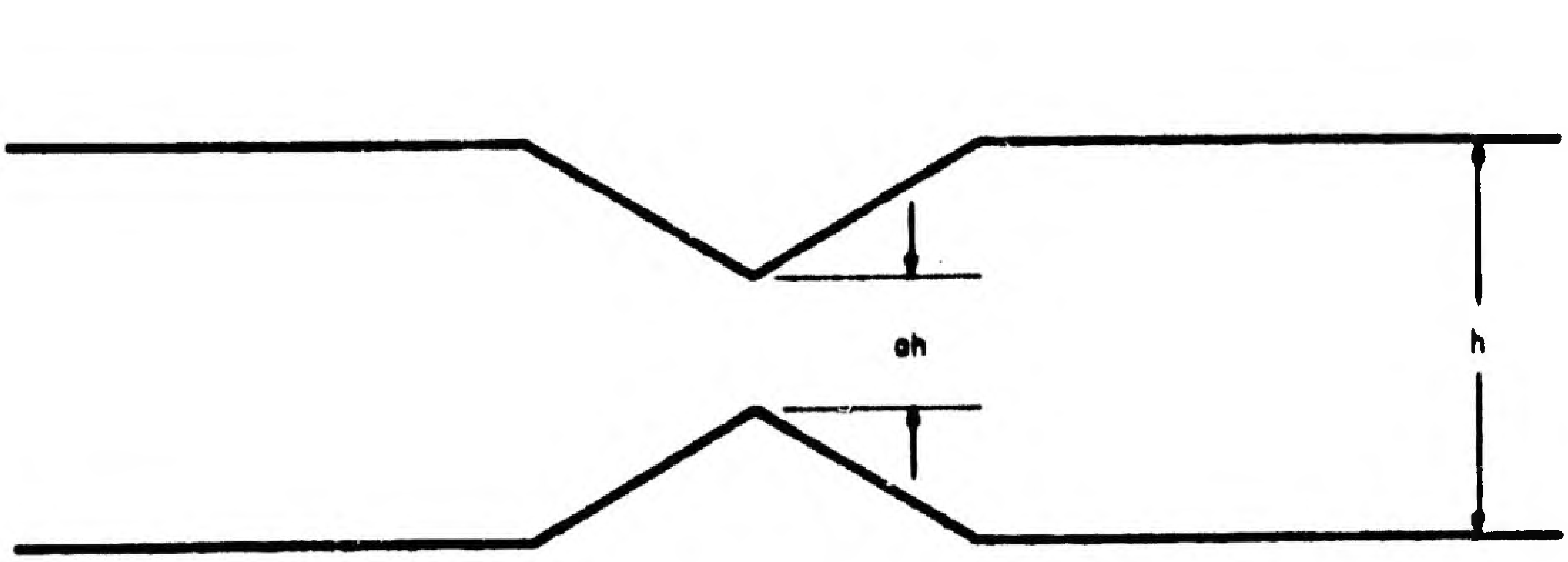


FIGURE 5-3. A PINCH IN A ROD

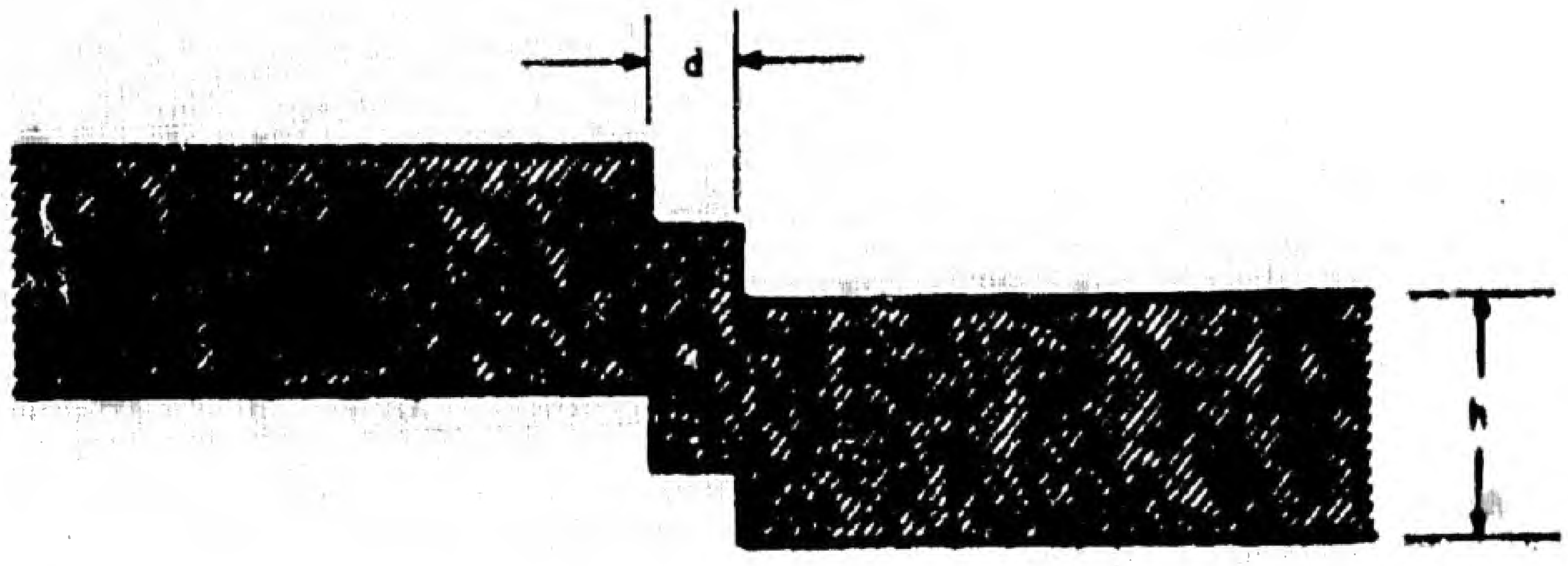


FIGURE 5-4. TWO STEPS CLOSE TOGETHER

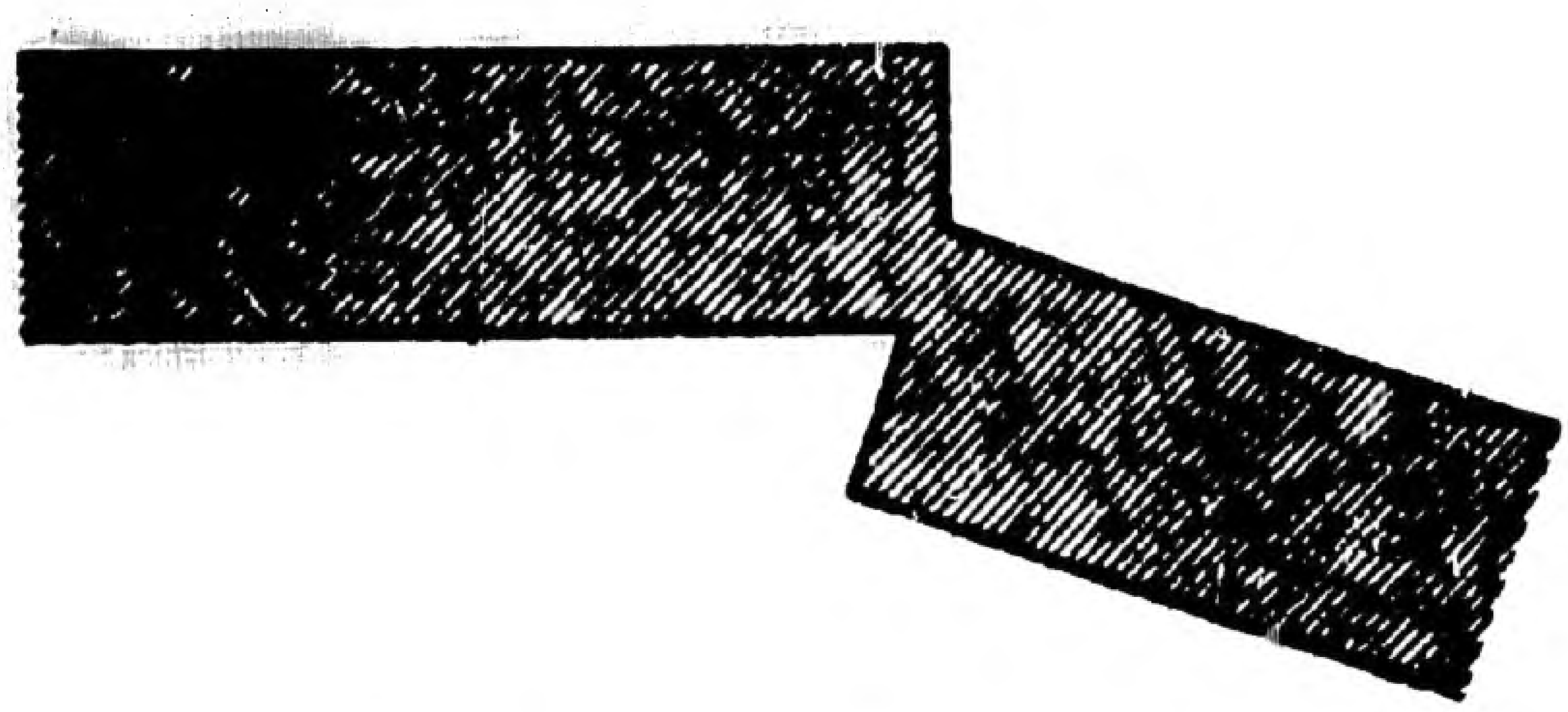


FIGURE 5-5. HINGE EFFECT AT A STEP

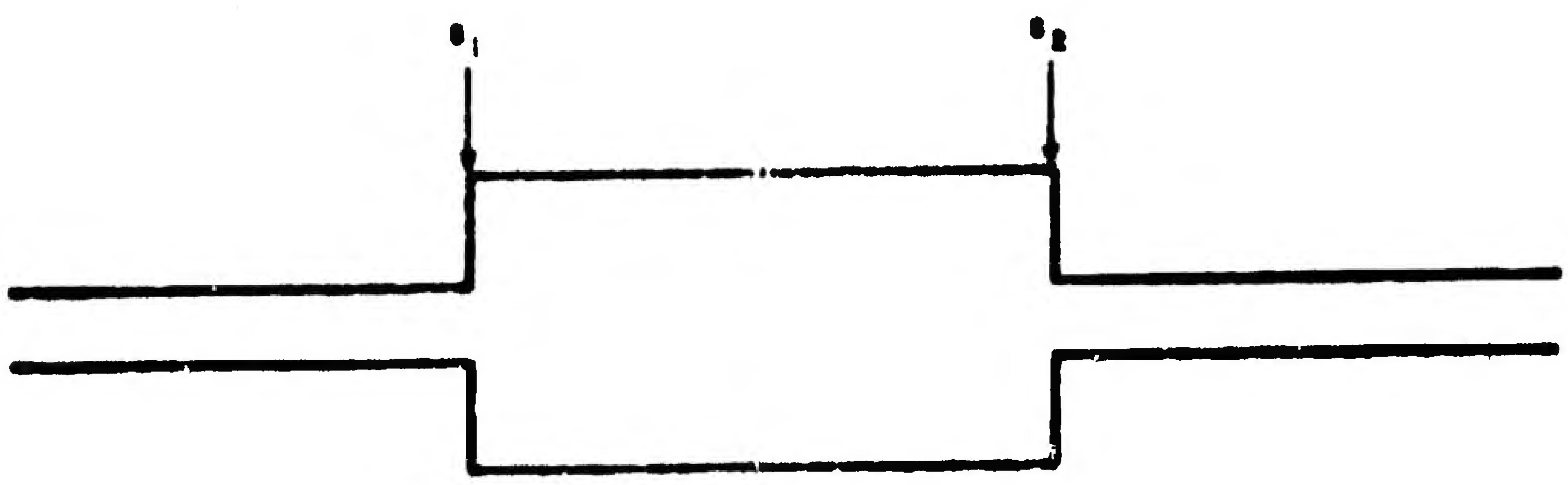


FIGURE 5-6. THICKENING ON A ROD

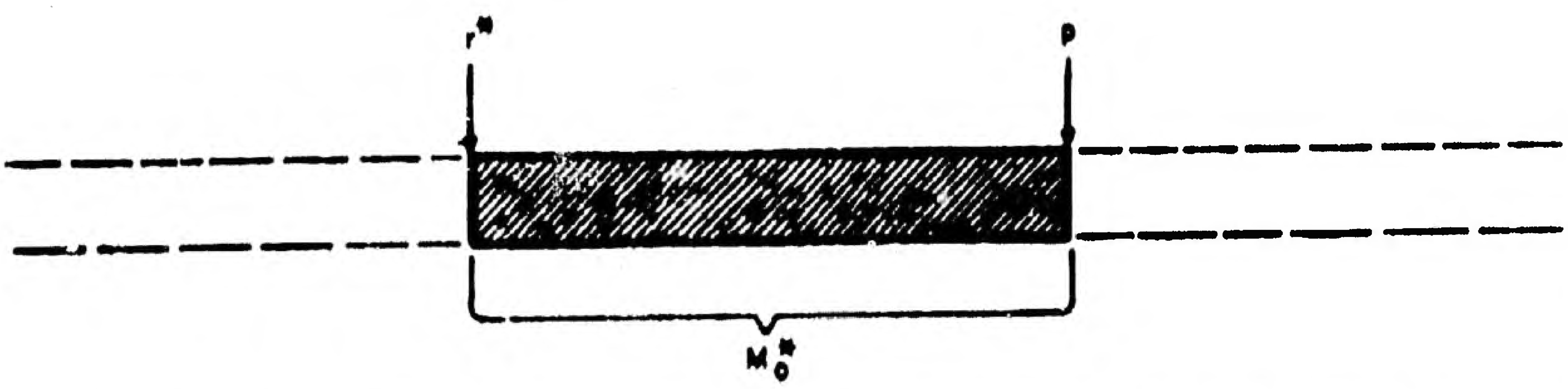


FIGURE 5-7. FORCE AT LEFT END OF ORIGINAL ROD SEGMENT

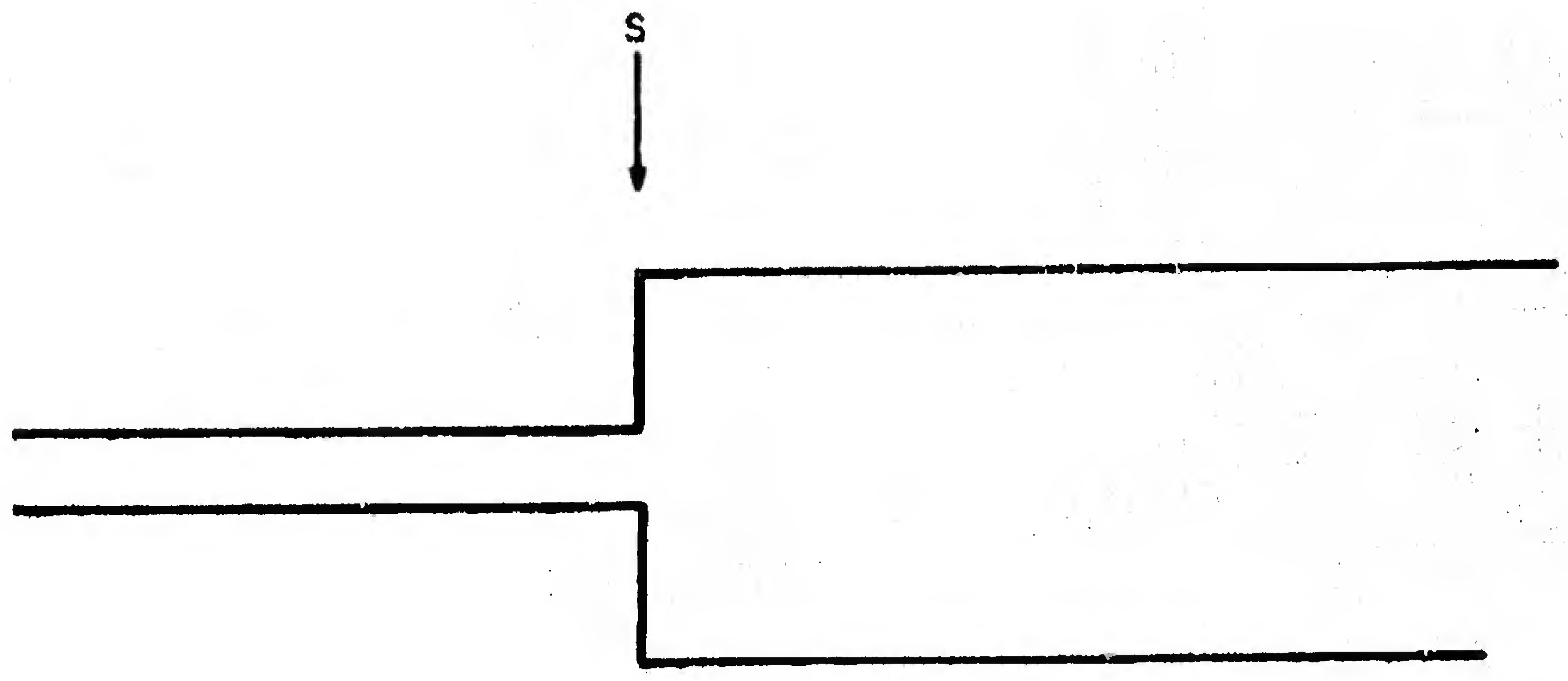


FIGURE 5-8. SYMMETRICAL JUNCTION BETWEEN DISSIMILAR SEGMENTS

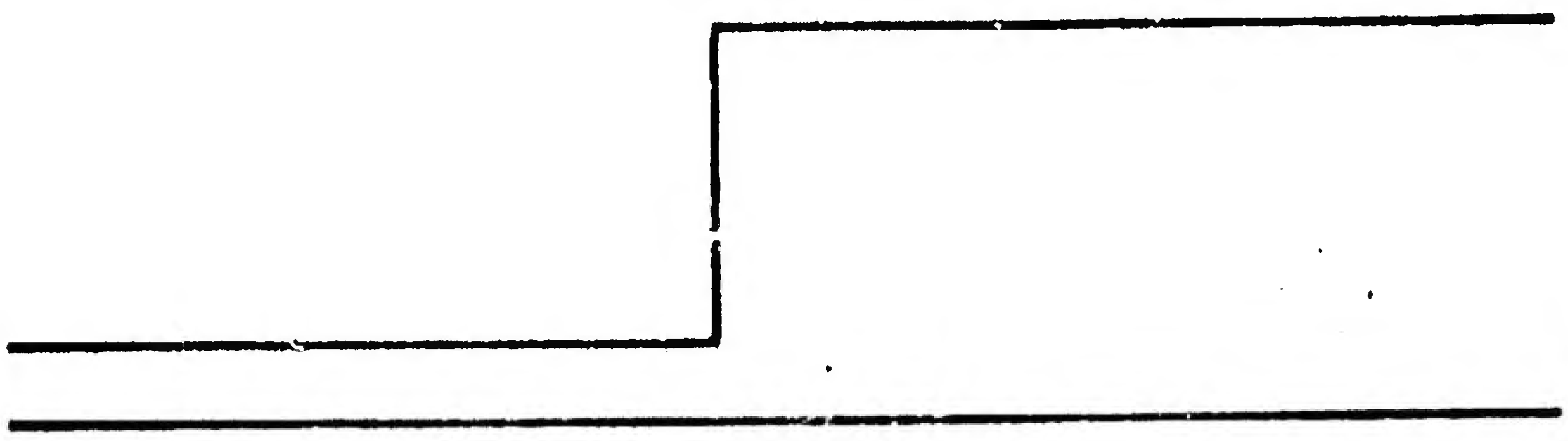


FIGURE 5-9. UNSYMMETRICAL JUNCTION

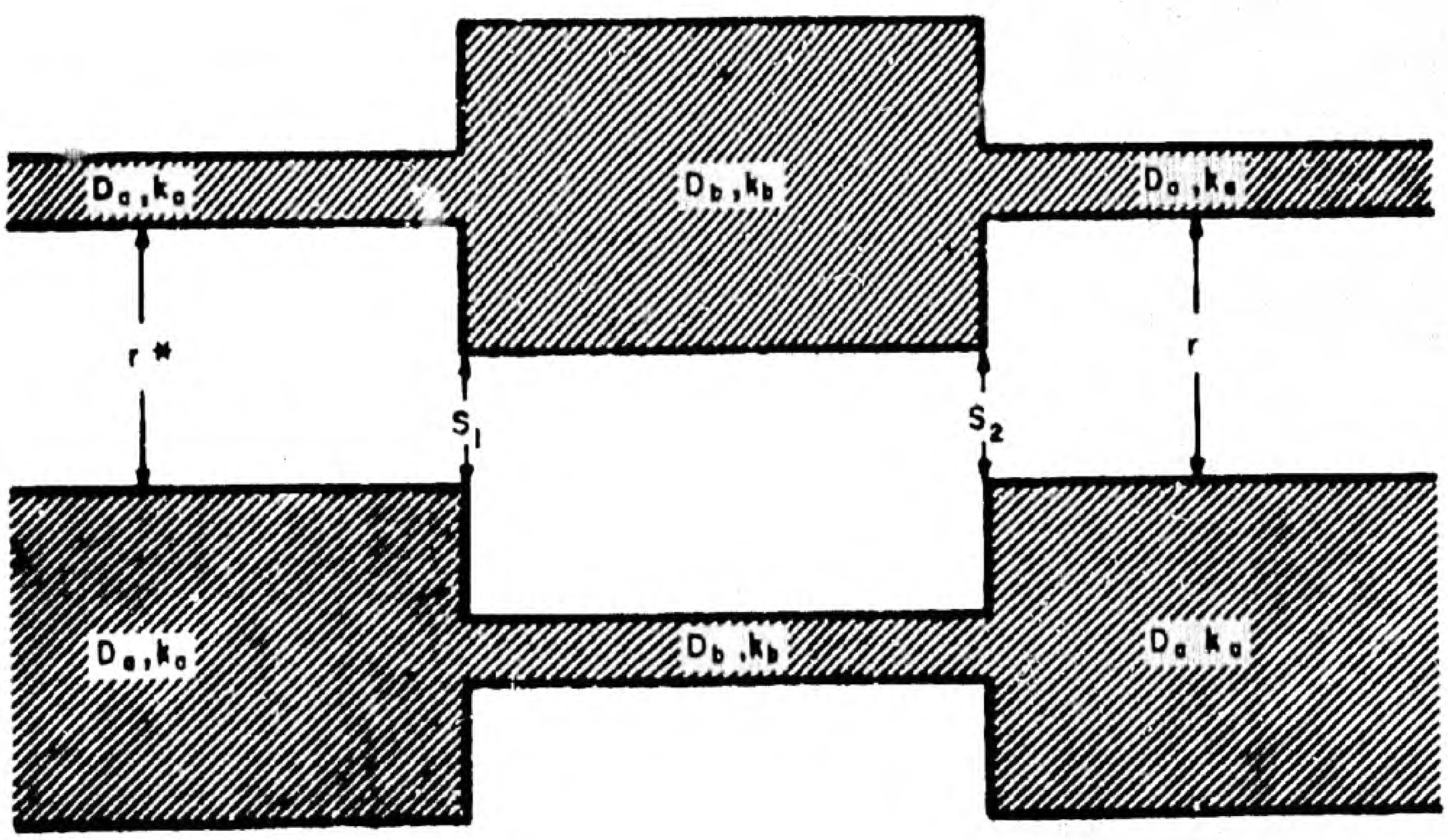


FIGURE 5-10. NOTATION FOR A THICKENING OR THINNING OF A ROD

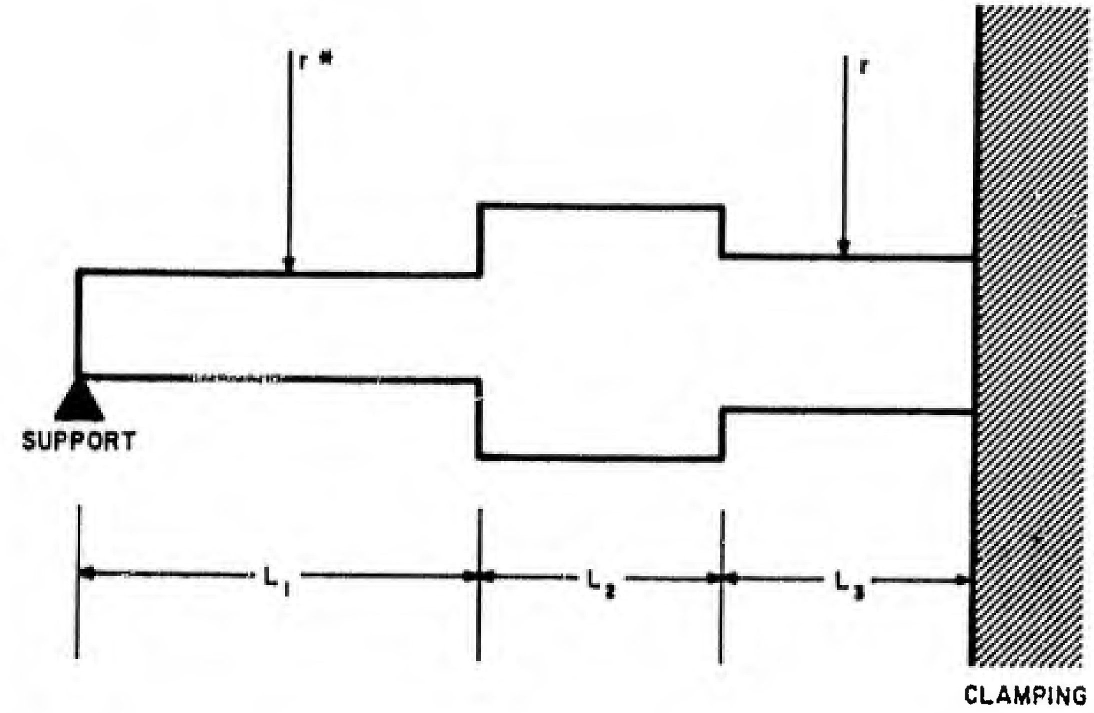


FIGURE 6-1. INHOMOGENEOUS ROD SUPPORTED AT ONE END AND CLAMPED AT THE OTHER

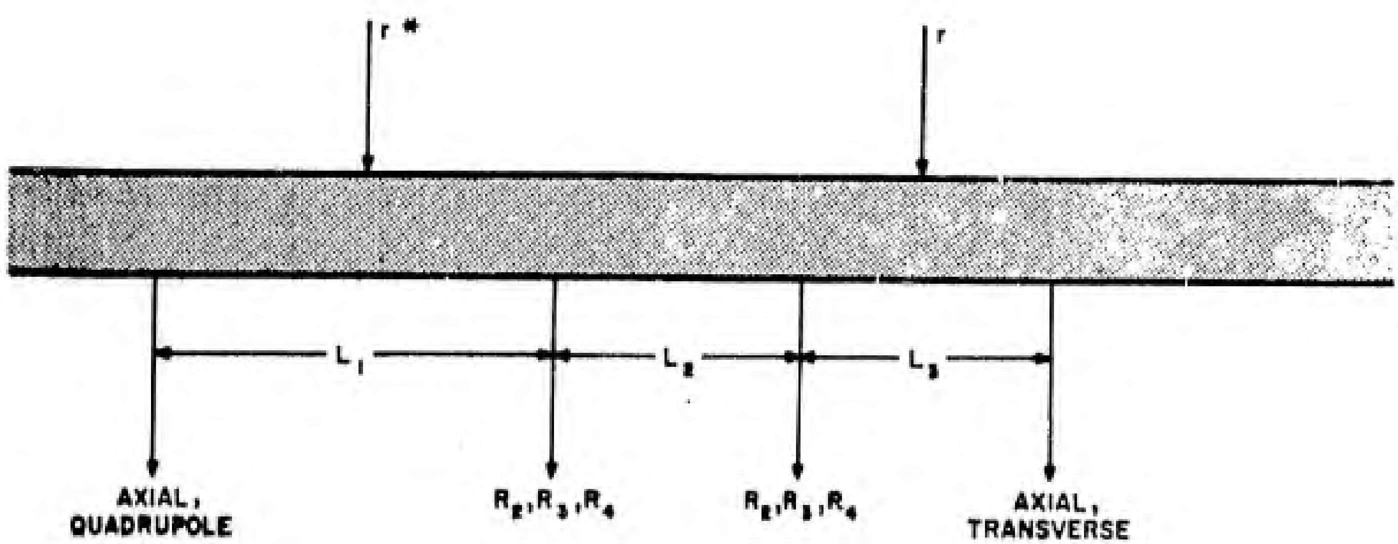


FIGURE 6-2. EQUIVALENT STRUCTURE IN TERMS OF RESONATORS

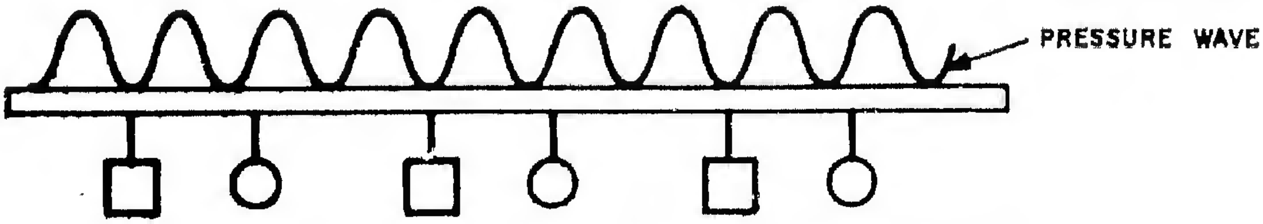


FIGURE 7-1. SEVERAL KINDS OF RESONATORS IN A REPEATING ARRANGEMENT ON A ROD

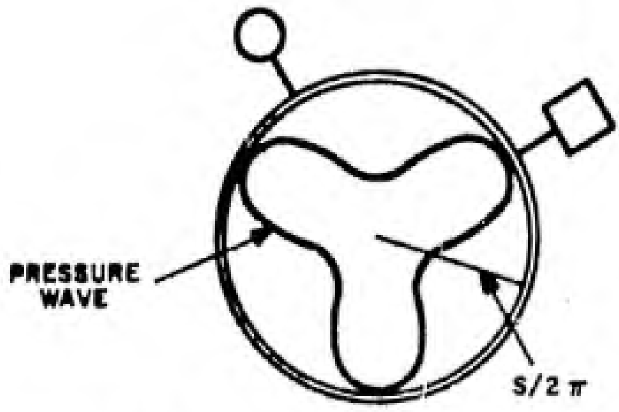


FIGURE 7-2. ROD ROLLED UP

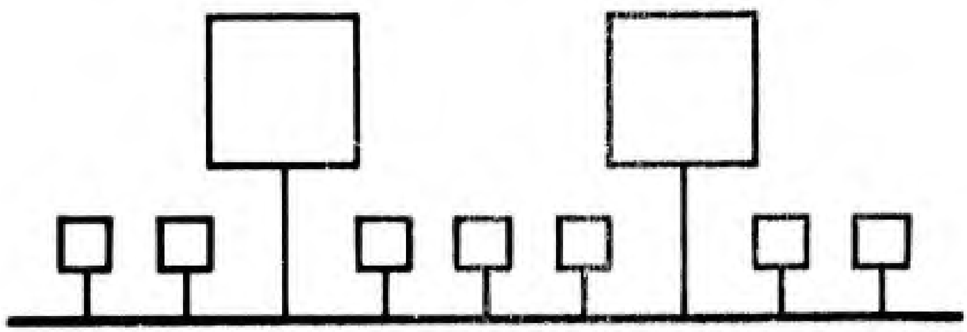


FIGURE 7-3 SMALL RESONATORS EVENLY INTERSPERSED BETWEEN LARGE CNES

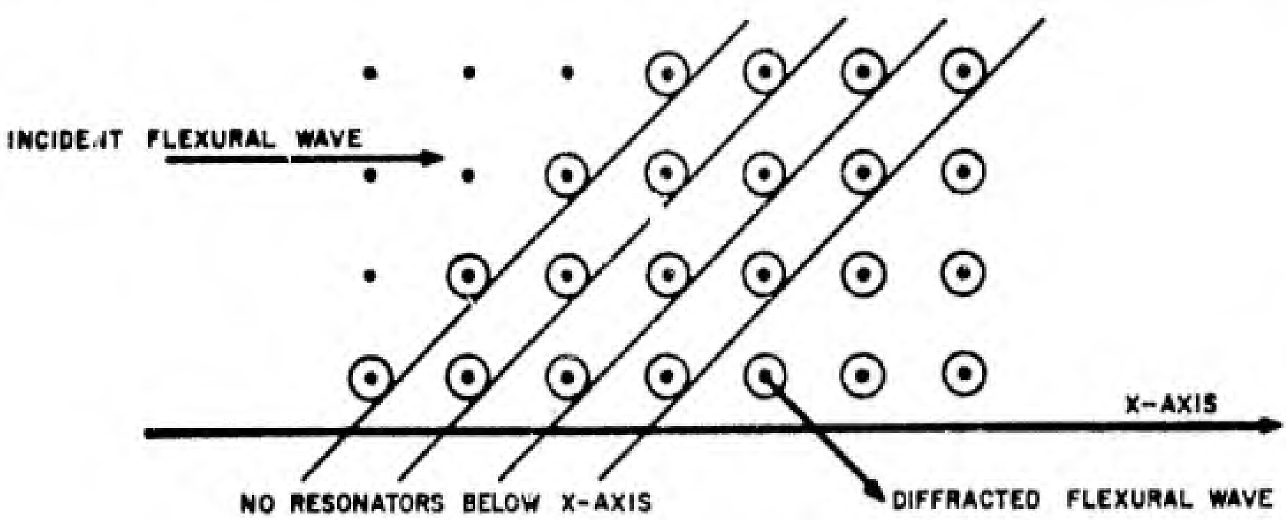


FIGURE 7-4. DIFFRACTION OF A FLEXURAL WAVE STRIKING A REGULAR ARRAY OF RESONATORS

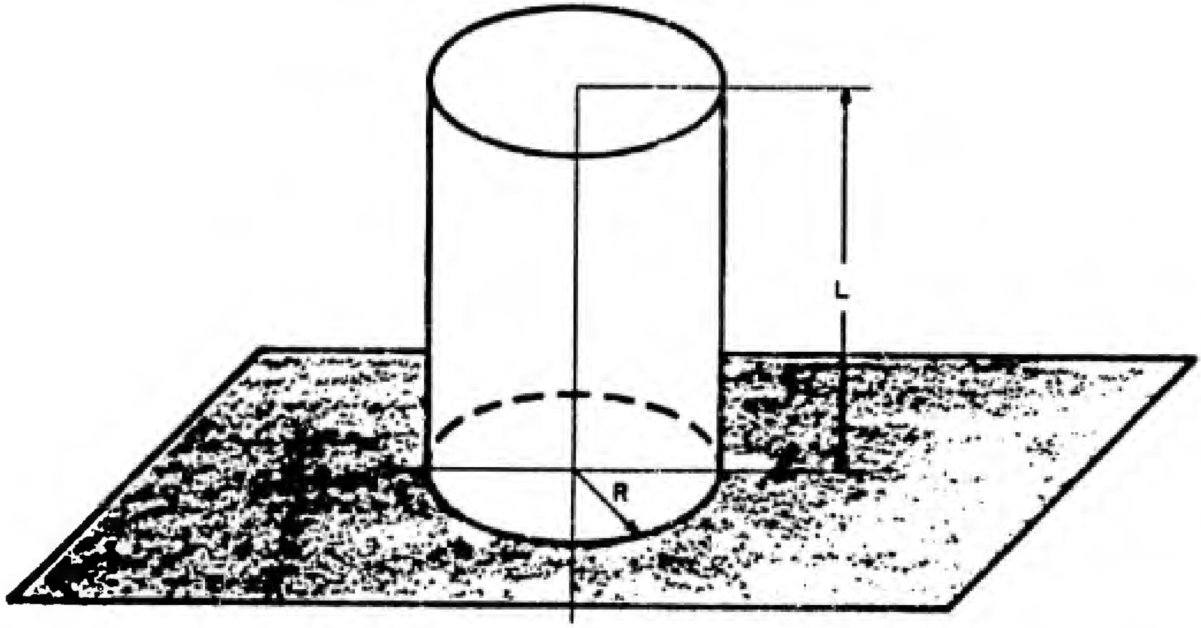


FIGURE 9-1. PLASTIC CYLINDER RESONATOR

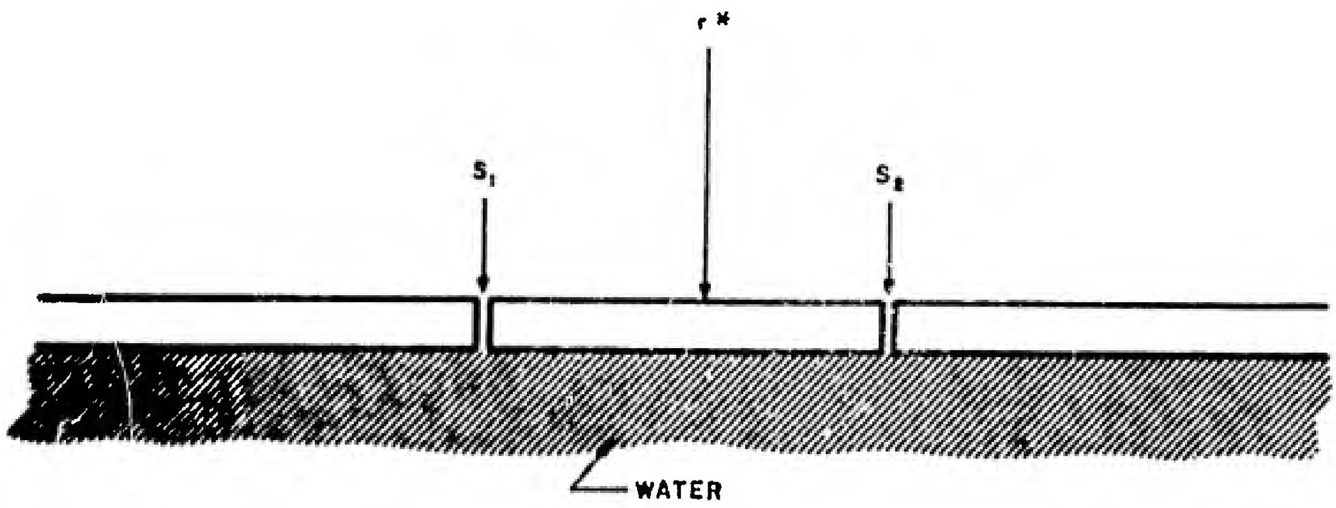


FIGURE 9-2. CROSS-SECTION OF STRIP PISTON

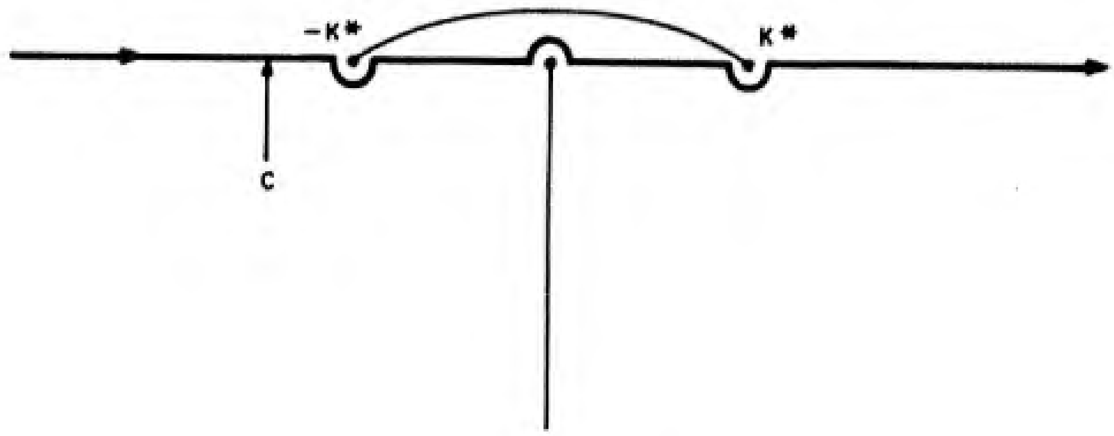


FIGURE 10-1. INTEGRATION PATH C

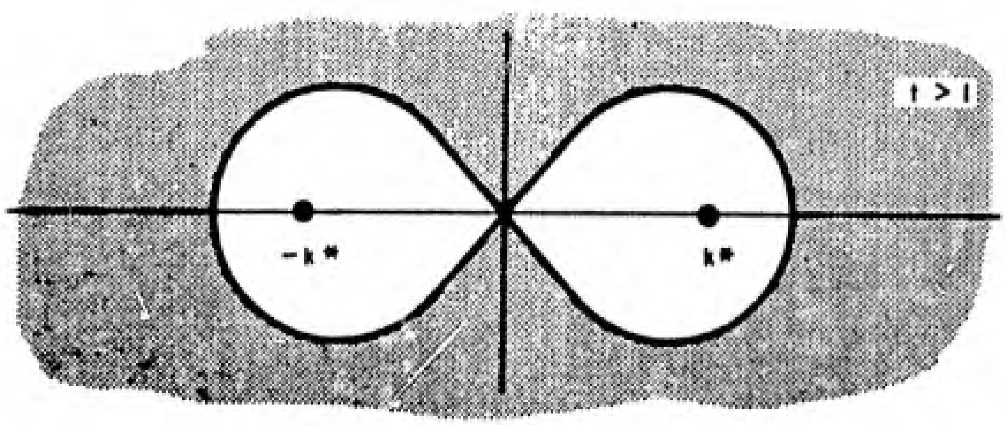


FIGURE 10-2. AREA IN WHICH t > 1 (SHADED)

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13. ABSTRACT

The vibration properties of a plate or a rod with attached mechanical structures, called resonators, are studied by transfer matrix and Green's function techniques. The plate may be in a vacuum, or acoustically coupled to water on one side. The effects of an added resonator at a given frequency depend only on its response strength in each of its modes of vibration. Expressions are derived for the strengths of several useful resonator types. Boundary conditions for the plate, such as clamping or simple support, are satisfied by the mathematical device of attaching resonators of infinite strength. A piston cut out of the plate material is handled similarly. Formulas for the interactions ("influence coefficients") between the modes of different resonators are given. Expansions are developed for the transcendental integrals that arise in the problems treated.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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The vibration properties of a plate or a rod with attached mechanical structures, called resonators, are studied by transfer matrix and Green's function techniques. The plate may be in a vacuum, or acoustically coupled to water on one side. The effects of an added resonator at a given frequency depend only on its response strengths in each of its modes of vibration. Expressions are derived for the strengths of several useful resonator types. Boundary conditions for the plate, such as clamping or simple support, are satisfied by the mathematical device of attaching resonators of infinite strength. A piston cut out of the plate material is handled similarly. Formulas for the interactions ("influence coefficients") between the modes of different resonators are given. Expansions are developed for the transcendental integrals that arise in the problems treated.

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