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STATISTICAL CONTROL OF A BERNOULLI PROCESS

BY

HOWARD M. TAYLOR

TECHNICAL REPORT NO. 79

February 4, 1965

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DEPARTMENT OF STATISTICS  
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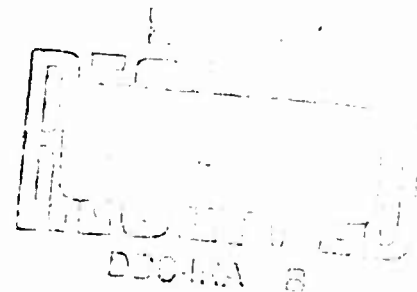
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with the Office of Naval Research

Gerald J. Lieberman, Project Director

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## SECTION 1 - INTRODUCTION

Consider a simple production process which is assumed always to be in one of only two states, a good state and a bad state. Specifically, production begins in the good state and while there a chance event occurs before each item is produced so that the probability of remaining in the good state is  $1 - \pi$  and the probability of a transition to the bad state is  $\pi$ . Once in the bad state, the process remains there until trouble is removed.

Associated with each item produced is a measurable characteristic or quality, denoted  $Y$ , assumed to be a random variable with a distribution depending on the unknown state of the machine. Let  $p_0(\cdot)$  and  $p_1(\cdot)$  be the density function for quality given that the machine is in the good and bad states, respectively.

A statistical control rule is a rule which specifies when the system is to be brought from production to repair, which has the effect of placing the process in the good state. Other than immediately after repair, the true process state is assumed unknown at all times. Hence a control rule must be based on the quality history of produced items. This history is adequately summarized in the posterior probability given the quality history that the next item will come from a machine in the bad state. This probability at time  $t$  is denoted  $X_t$ . Costs are associated with repairing the process and with the quality of each item produced, and the objective is to minimize the average cost per unit time.

This paper studies the 100% inspection case where sampling is by attributes; each item is dichotomized as of good or defective quality. A numerically feasible method is given for calculating the operating characteristics of a rule as a function of the system parameters and the critical value  $\xi^*$  which the rule uses. The operating characteristics which may be computed include the expected time between repairs, the variance of the time between repairs, the overall proportion of defective items produced and the fraction of repairs made to systems still in the good state. The computational method given may be used for the case where quality is any discrete random variable.

The validity of the computational method is checked in two ways. First some inequalities which the optimal solution must satisfy are developed and compared with the computed answer. Second, the discounted cost problem is explored through dynamic programming, and the answers computed using this method are compared with the previously computed answers.

Next some examples of the wide variety of problems which may be attacked through the use of tables of operating characteristics are given.

Lastly the model is checked for its sensitivity to proper choice of system parameters. The Girshick and Rubin equations are used to compute the operating characteristics in the situation where incorrect parameters (transition probability, etc.) are used to compute the posterior probabilities  $X_t$  which the rule uses for its decision making.

## SECTION 2 - A MATRIX FORMULATION

This section develops the matrix formulation of the problem which underlies the computational scheme proposed. The fact that the sequence  $\{X_t, t = 0, 1, 2, \dots\}$  of posterior probabilities forms a discrete parameter Markov chain with a denumerable state space is of fundamental importance. The computational method revolves about the matrix of transition probabilities of this chain.

Let the good system state be denoted by 0 and the bad state by 1 and let the state of the system at time  $t$  be denoted  $\Theta_t$ . It is assumed that if the machine is left undisturbed, these states  $\{\Theta_t, t = 0, 1, 2, \dots\}$  form a Markov chain with transition probabilities

$$\begin{aligned} \Pr[\Theta_{t+1} = 1 | \Theta_t = 0] &= 1 - \Pr[\Theta_{t+1} = 0 | \Theta_t = 0] = \pi \\ \Pr[\Theta_{t+1} = 1 | \Theta_t = 1] &= 1 - \Pr[\Theta_{t+1} = 0 | \Theta_t = 1] = 1. \end{aligned}$$

Rather than restricting the development to the Bernoulli random variables of attributes inspection it is no more difficult to allow the measurable quality characteristic to be any random variable which takes on only a finite number of possible values. Hence, for each  $\theta \in \{0, 1\}$  let  $p(\cdot, \theta)$  be a probability distribution on the elements  $y$  in a finite set  $\mathcal{Y}$ . That is for every  $\theta \in \{0, 1\}$ ,  $\sum_{y \in \mathcal{Y}} p(y, \theta) = 1$  and for all  $y \in \mathcal{Y}$ ,  $p(y, \theta) \geq 0$ .

Let  $\{Y_0, Y_1, Y_2, \dots\}$  be a sequence of random variables whose conditional distribution given  $\{\Theta_0, \Theta_1, \Theta_2, \dots\}$  is as independent

random variables with  $Y_t$  distributed according to  $p(\cdot, \theta_t)$ . The sequence  $\{Y_0, Y_1, Y_2, \dots\}$  represents the measurements taken on the items the machine has produced.

After each item is produced there is the option of continuing production undisturbed or repairing the machine. A repaired machine is in the bad state with probability  $x^0$  which is usually assumed to be zero.

For each  $y \in \underline{Y}$  let  $C(y)$  be the cost incurred by producing an item of quality  $y$  and let  $K_\theta$  be the cost of repairing a machine which is in state  $\theta$  for  $\theta \in \{0, 1\}$ .

The parameters introduced so far serve to specify the model and are assumed known.

If at any time that the machine is in production it were known exactly which state the machine is in one would be able to make the proper decision whether or not to repair. But the true state of the system is unknown, and thus the only relevant information obtainable from the observations at time  $t$  is the posterior probability  $X_t$  that the machine will be in the bad state for the production of the next item assuming no repair is made. Girshick and Rubin [9] and L. Breiman [6] show that any decision rule of interest may be specified by giving a critical value  $\xi^* \in [0, 1]$  with the interpretation that the machine is repaired at time  $t$  if and only if  $X_t \geq \xi^*$ .

Thus the sequence  $\{X_0, X_1, X_2, \dots\}$  is of importance and it is desirable to formalize its structure. Let  $\Xi$  be the unit interval  $[0, 1]$  and generically let  $x$  be an element of  $\Xi$ . For each  $y \in \underline{Y}$ , two operators are defined on  $\Xi$  into  $\Xi$  by:

$$\Lambda_y x = xp(y, 1) / [xp(y, 1) + (1 - x)p(y, 0)] \quad (2.1)$$

and

$$\tau_y x = \Lambda_y x + (1 - \Lambda_y x)\pi \quad (2.2)$$

The first operator is an application of Baye's formula. If  $x$  is a prior probability that the machine is in the bad state and  $Y = y$  is produced then the posterior probability that the item came from a machine in the bad state is  $\Lambda_y x$ . The posterior probability that the next item will come from a machine in the bad state is  $\tau_y x$ .

For any critical value  $\xi^*$  define the transformation from  $\Xi$  into  $\Xi$  by:

$$\tau_y^{\xi^*} x = \begin{cases} \tau_y x & \text{if } x < \xi^* \\ \tau_y x^0 & \text{if } x \geq \xi^* \end{cases} \quad (2.3)$$

Let  $X_0 = x^0$  which assumes the system starts in the repaired state and then define the sequence  $\{X_0, X_1, X_2, \dots\}$  recursively by

$$X_{t+1} = \tau_{Y_t}^{\xi^*} X_t \quad (2.4)$$

THEOREM 2.1

If  $X_0 = x^0$  and  $X_{t+1} = \tau_{Y_t}^{\xi^*} X_t$  for  $t = 0, 1, 2, \dots$  then the sequence  $\{X_t, t = 0, 1, 2, \dots\}$  forms a stationary (time homogenous) Markov chain.



PROOF:

Suppose  $X_s = x_s$  for  $s = 0, 1, 2, \dots, t$  and let  
 $S_\eta = \{y: \tau_y^{\xi^*} \xi_t = \eta\}$  for any  $\eta$ .

Then

$$\begin{aligned}\Pr[X_{t+1} = \eta | X_0 = x^0, \dots, X_t = x_t] &= \Pr[Y_t \in S_\eta | X_0 = x^0, \dots, X_t = x_t] \\ &= x_t \sum_{y \in S_\eta} p(y, 1) + (1 - x_t) \sum_{y \in S_\eta} p(y, 0) \\ &= \Pr[X_{t+1} = \eta | X_t = x_t] \quad \text{Q.E.D.}\end{aligned}$$

Since the space  $X$  was assumed finite, there are only a finite number of possible values for the random variable  $X_t$  for any given  $t$ . The union over  $t = 0, 1, 2, \dots$  of the sets of possible values is at most countable and this set, denoted by  $J$ , is the state space for the Markov chain.

The period immediately following one repair to and including the next repair will be referred to as a cycle. The next theorem states that for the purpose at hand the process may be studied completely by concentrating on the first cycle only.

#### THEOREM 2.2

Let  $N_K$  be the length of and  $W_K$  the total cost of the  $k^{\text{th}}$  cycle. If  $E[N_K] < +\infty$  then with probability one

$$\lim_m [(W_1 + \dots + W_K) / (N_1 + \dots + N_K)] = E[W_1] / E[N_1]$$



PROOF:

$W_1, W_2, \dots$  are independent, identically distributed random variables with  $E[|R_1|] < +\infty$  and similarly for  $N_1, N_2, \dots$ . The theorem follows immediately from the law of large numbers. Q.E.D.

It is easiest to study the first cycle by considering the Markov chain to be absorbed when  $X_t \geq \xi^*$ . Let  $R$  be the repair region defined by  $R = J \cap [\xi^*, 1]$ , and let  $H$  be the transition probability matrix corresponding to this absorbing chain. The  $i, j^{\text{th}}$  entry of  $H$  is  $\eta_{ij}$  corresponding to a transition from  $x^i \in J$  to  $x^j \in J$ .

$H$  is symbolically partitioned into

$$H = \begin{bmatrix} H_{11} & H_{12} \\ 0 & I \end{bmatrix}$$

where  $H_{11}$  has domain  $J - R \times J - R$ ,  $H_{12}$  has domain  $J - R \times R$  and  $I$  is the  $R \times R$  identity. The partition is symbolic in the sense that each submatrix is usually infinite dimensional.

For the absorbing Markov chain with  $X_0 = x^0$  let  $\psi = (\psi_j)$  be the row vector with  $\psi_j$  equal to the expected number of  $X_t = x^j$  for  $x^j \in R$ . Let  $N$  be the cycle length:

$$N = \inf \{t: X_t \in R\}$$

and let  $A = (A_j)$  be the row vector with

$$A_j = \Pr[X_N = x^j | X_0 = x^0]$$

for  $x^j \in R$ .

For a finite Markov chain the following theorem is well known.

THEOREM 2.3.

Let  $\Delta$  be the row vector  $(1, 0, 0, \dots)$  and  $\mathbf{1}$  the column vector consisting entirely of ones. For the absorbing Markov chain described above,  $\psi$  is the unique non-negative minimal solution to

$$\psi(I - H_{11}) = \Delta \quad (2.5)$$

and further

$$A = \psi H_{12} \quad (2.6)$$

$$E[N] = \psi \mathbf{1} \quad (2.7)$$

For a Markov chain with denumerable state space the development leading to the proof of this theorem may be found in Kemeny, Snell and Knapp [12] in the chapter on transient chains. Although not a rigorous proof, an intuitive justification of Equation 2.5 is easily made. For every column other than the first, the equation may be written

$$\psi_i = \sum_{x^j \notin R} \psi_j \eta_{ji} ; \quad x^i \notin R \quad \text{and} \quad x^i \neq x^0$$

In words, the expected number of times the process  $X_t$  is in state  $x^i$  is equal to the expected number of times the process is in  $x^j$  leading to  $x^i$  weighted by the probability of such a transition, and this is summed for all  $x^j \notin R$ . The first column of the Equation 2.5 is  $\psi_0 = 1$  which states only that a repair occurs once per cycle.

This theorem leads to the computation of the average cost per period for any critical value  $\xi^*$ . Let  $W$  be the total cost of the first cycle and  $\gamma$  the average cost per period.

$$\gamma = E[W]/E[N] \quad (2.8)$$

$W$  has two components of cost:  $W_1$ , the cost due to the quality of items produced and  $W_2$ , the repair cost. For  $X_t = x^i \notin R$  the expected cost of the next item is

$$E_{x^i}[C(Y)] = x^i \sum_{y \in \mathcal{Y}} C(y)p(y, 1) + (1 - x^i) \sum_{y \in \mathcal{Y}} C(y)p(y, 0).$$

$E_{x^i}[C(Y)]$  denotes the expectation of  $C(Y)$  where  $Y$  is distributed according to  $p(\cdot, \Theta)$  and  $\Theta$  is zero with probability  $x^i$  and one with probability  $1 - x^i$ . Thus

$$E[W_1] = \sum_{x^i \notin R} \psi_i E_{x^i}[C(Y)] \quad (2.10)$$

For  $X_N = x^j \in R$ , the probability that repair will be made to a bad machine is  $x^j$ . Hence

$$E[W_2] = \sum_{x^j \in R} A_j [x^j K_1 + (1 - x^j) K_0] \quad (2.11)$$

and  $E[W] = E[W_1] + E[W_2]$ .

Thus by solving the infinite set of linear equations  $\psi(I - H_{11}) = \Delta$  which depend upon a particular choice of a critical value  $\xi^*$  and then computing  $A = \psi H_{12}$  and  $E[N] = \psi 1$  one may compute the average cost per period  $\gamma = E[W]/E[N]$  corresponding to the use of  $\xi^*$  as a critical

value. By doing this for several  $\xi^*$  in a trial and error manner one may compute the minimum cost policy.

It should be noticed that the cost information enters the computations at a late stage. The major computing effort lies in computing the  $\psi$  and  $A$  vectors and it is only after these are computed that cost factors enter. Therefore as a solution to the problem it is proposed to table

$$\text{a. } m_1 = \sum_{x^i \in R} x^i \psi_i = \text{The number of periods per cycle the machine is in the bad state.} \quad (2.12)$$

$$\text{b. } m_0 = \sum_{x^i \in R} (1 - x^i) \psi_i = \text{The number of periods per cycle the machine is in the good state.} \quad (2.13)$$

$$\text{c. } m_0 + m_1 = E[N] = \text{The expected cycle length.} \quad (2.14)$$

$$\text{d. } Y_1 = \sum_{x^j \in R} x^j A_j = \text{The expected fraction of repairs made to bad machines.} \quad (2.15)$$

$$\text{e. } r_0 = 1 - r_1 = \text{The expected fraction of repairs made to good machines.} \quad (2.16)$$

as a function of the system parameters and the critical value  $\xi^*$ . In this notation one has:

$$E[W_1] = m_0 \sum_y C(y) p(y, 0) + m_1 \sum_y C(y) p(y, 1)$$

$$E[W_2] = r_0 K_0 + r_1 K_1$$

Some other easily computed results are of interest. If  $X_t$  is a numerically valued random variable the average outgoing quality level is

$$[m_0 \sum_y yp(y, 0) + m_1 \sum_y yp(y, 1)]/[m_0 + m_1]$$

The repairs per period is  $1/(m_0 + m_1)$ . The second moment of the cycle length may be computed by solving

$$(I - H_{11})\phi = \Delta^T$$

for the column vector  $\phi$  and then

$$E[N^2] = \sum_{x^i \notin R} \phi_i \psi_i .$$

These results may be used for comparison against actual values encountered in an application. Cycle lengths or quality levels differing significantly from predicted values would indicate an erroneous choice of parameters.

The results presented so far may be generalized, at least partially, to the case of a machine with more than two underlying states, say a machine with a mediocre state as well as a bad and a good one. Conceptually the results still hold but the posterior probability distribution  $\bar{\Xi}_t$  which tracks the process must now be multidimensional. This leads to such difficulties both in computing and implementing the policy so as to render this generalization of little interest.

### SECTION 3 -- A NUMERICAL SOLUTION

This section discusses a numerical solution to the set of linear equations

$$\psi(I - H_{11}) = \Delta$$

or

$$\psi_j = \sum_{x^i \notin R} \psi_i \eta_{ij} \quad \text{for } x^j \notin R \text{ and } x^j \neq x^0$$

Usually a reduction of this infinite set of equations to a finite set is the first step in any solution. One method is to consider only those states that the  $\{X_t, t = 0, 1, \dots\}$  process can reach in  $m$  steps, or nearly equivalently, to truncate the quality history and not use the quality of items produced more than  $m$  periods ago in computing the posterior probabilities. For  $m$  sufficiently large the ensuing finite set of equations will closely approximate the infinite set. Unfortunately the number of possible states reached in  $m$  steps of the  $\{X_t, t = 0, 1, \dots\}$  process is astronomically large, even for moderate  $m$ . It has an order of magnitude of the cardinality of  $\underline{Y}$  raised to the  $m - 1$  power.

But consider the implementation of a policy of the optimal form. A machine operator repairs his machine and starts operation with a probability on the bad state of zero. After inspecting each item produced he updates his posterior probability. When it reaches a critical value, say .60 chance that the machine is in the bad state, he repairs the machine and begins the cycle anew. It would not seem objectionable

if this machinist computed these probabilities only to the nearest .01 decimal place. But this automatically reduces the problem to a finite number, at most 101, of posterior probability states. This simple idea is the keystone in the proposed numerical solution.

As a practical matter it seems best not to have a constant mesh size such as .01 but to have the mesh size vary so as to be sufficiently small to be able to handle the smallest change in posterior probability encountered. For example, if  $X_t = x$  one would want the mesh size at  $x$  to be at least as small as  $\min_y |x - \tau_y x|$ . Unfortunately, this too may be difficult to achieve entirely but it serves as a useful target.

The algorithm for approximating the transition probability matrix will be presented for the case of inspection by attributes where item quality is a Bernoulli random variable. Let  $Y_t = 0$  if the  $t^{\text{th}}$  item is good and  $Y_t = 1$  if it is defective. For this special case let  $p_0$  be the fraction of good items produced when the machine is operating correctly and let  $p_1$  be this fraction when the machine is in the bad state.

$$p_\theta = p(0, \theta) \quad \text{for } \theta \in \{0, 1\} .$$

The algorithm which computes the approximating transition probability matrix is described in the Appendix. Table 3.1 shows such a transition probability matrix, computed on a Burrough's B5000 machine. The states listed are possible values for the posterior probabilities used in controlling the process. The algorithm generating this matrix has two distinct phases. In the first, a finite number of these states are generated, and in the second the transitions among these states are filled in.



Once the finite matrix, denoted  $\hat{H}$ , which approximates the infinite matrix  $H$  has been computed one proceeds to compute operating characteristics as a function of a critical value  $\xi^*$ . It has been shown earlier that the major computing effort lies in solving the equations

$$\psi(I - \hat{H}^{\xi^*}) = \Delta, \quad (3.1)$$

where  $\hat{H}^{\xi^*}$  is the submatrix of  $H$  consisting of states with values less than  $\xi^*$ .

To do this in the most efficient manner it is best to set up the largest possible set of transposed equations,

$$(I - \hat{H})^T \psi^T = \Delta^T, \quad (3.2)$$

and then, using the common Gaussian elimination technique, to solve subsets of these equations for increasing  $\xi^*$ , in an increasing manner. Besides allowing this overlapping of effort, the Gaussian elimination technique recognizes the many zero coefficients in the equations, enabling a faster, more accurate solution. The modified Gaussian elimination procedure used is also presented in the Appendix.

A sample of the results of the program is given in Table II. The process parameters are listed at the top of the page. The operating characteristics corresponding to each choice of critical value appear below this. The entire calculations took 6 seconds of B5000 computer time.

PI = 0.020 FC = 0.220 PF = 0.400 FPSL = 0.007  
 MAX. APPROX. = 0.0201 MATRIX SIZE = 57 ASYM = 0.104 FCHIT = 0.967

THE TRANSITION PROBABILITY MATRIX IS

STATE NO.	STATE (FROM)	STATE NO. (TO)	STATE (TO)	WITH PROB	STATE NO. (TO)	STATE (TO)	WITH PROB
0	0.000	1	0.020	0.970	1	0.020	0.010
1	0.020	2	0.035	0.965	34	0.315	0.014
2	0.035	3	0.049	0.951	37	0.422	0.017
3	0.049	4	0.059	0.941	39	0.504	0.019
4	0.059	5	0.067	0.933	40	0.547	0.021
5	0.067	6	0.074	0.926	41	0.590	0.023
6	0.074	7	0.079	0.921	42	0.633	0.024
7	0.079	8	0.084	0.916	42	0.633	0.025
8	0.084	9	0.087	0.913	43	0.674	0.026
9	0.087	10	0.090	0.910	43	0.674	0.027
10	0.090	11	0.093	0.907	43	0.674	0.027
11	0.093	12	0.095	0.905	43	0.674	0.028
12	0.095	13	0.097	0.903	43	0.674	0.028
13	0.097	14	0.102	0.902	43	0.674	0.028
14	0.102	14	0.102	0.971	44	0.713	0.029
15	0.107	15	0.107	0.970	44	0.713	0.030
16	0.113	15	0.107	0.969	44	0.713	0.031
17	0.114	16	0.113	0.968	44	0.713	0.032
18	0.117	17	0.114	0.968	45	0.749	0.032
19	0.119	18	0.117	0.967	45	0.749	0.033
20	0.122	19	0.119	0.967	45	0.749	0.033
21	0.126	20	0.122	0.966	45	0.749	0.034
22	0.131	21	0.126	0.965	45	0.749	0.035
23	0.136	22	0.131	0.964	45	0.749	0.036
24	0.143	23	0.136	0.963	46	0.782	0.037
25	0.150	24	0.143	0.961	46	0.782	0.039
26	0.159	25	0.150	0.960	46	0.782	0.040
27	0.170	26	0.159	0.958	47	0.812	0.042
28	0.183	27	0.170	0.955	47	0.812	0.045
29	0.198	28	0.183	0.952	48	0.839	0.048
30	0.216	29	0.198	0.949	48	0.839	0.051
31	0.236	30	0.216	0.945	49	0.863	0.055
32	0.259	31	0.236	0.941	50	0.884	0.059
33	0.285	32	0.259	0.936	50	0.884	0.064
34	0.315	33	0.285	0.930	51	0.902	0.070
35	0.348	34	0.315	0.924	52	0.918	0.076
36	0.383	35	0.348	0.917	53	0.931	0.083
37	0.422	36	0.383	0.910	54	0.942	0.090
38	0.462	37	0.422	0.902	54	0.942	0.098
39	0.504	38	0.462	0.894	55	0.952	0.106
40	0.547	39	0.504	0.886	56	0.960	0.114
41	0.590	40	0.547	0.878	57	0.967	0.122
42	0.633	41	0.590	0.870	57	0.967	0.130
43	0.674	42	0.633	0.862	57	0.967	0.138
44	0.713	43	0.674	0.855	57	0.967	0.145
45	0.749	44	0.713	0.848	57	0.967	0.152

Table I (Continued on next page)

46	0.752	45	0.749	0.841	57	0.967	0.159
47	0.812	46	0.782	0.836	999	0.989	0.164
48	0.839	47	0.812	0.831	999	0.991	0.169
49	0.863	48	0.839	0.826	999	0.992	0.174
50	0.884	49	0.863	0.822	999	0.994	0.178
51	0.902	50	0.884	0.819	999	0.995	0.181
52	0.918	51	0.902	0.816	999	0.996	0.184
53	0.931	52	0.918	0.813	999	0.996	0.187
54	0.942	53	0.931	0.811	999	0.997	0.189
55	0.952	54	0.942	0.809	999	0.998	0.191
56	0.960	55	0.952	0.808	999	0.998	0.192
57	0.967	56	0.960	0.806	999	0.998	0.194

Table I An approximate transition probability matrix is given here for the case where a process in the good state has a probability of  $PI = .02$  of moving to the bad state for the production of the next item.  $PO = .99$  and  $P1 = .80$  are the fraction of good items produced when the machine is in the good and bad states respectively. The table has the following interpretation. From any state such as state number 25,  $x = .150$  the posterior probability  $\{X_t, t = 0, 1, \dots\}$  can go to one of two other states. If a good item is observed, which for  $X_t = .150$  occurs with probability .961, the process moves to state number 24 with  $X_{t+1} = .143$ . If a defective item is observed then  $X_{t+1} = .782$ .

$ASYP = .104$  is a value about which the states tend to cluster. An approximation is made here by grouping all states within  $EPSL = .007$  of  $ASYP$  into two classes and assigning a common value to each class. A second approximation occurs when a "TO STATE" cannot be found exactly among the finite number of states listed. Such a "TO STATE" is assigned to its best approximation and  $MAX. APPROX. = .0201$  measures this approximation.

TRANSITION PROBABILITY IS 0.020  
 FRACTION OF GOOD ITEMS PRODUCED BY A REPAIRED MACHINE IS 0.990  
 FRACTION OF GOOD ITEMS PRODUCED BY A DEFECTIVE MACHINE IS 0.800

CRITICAL VALUE	EXPECTED CYCLE LENGTH	PERIODS FOR WHICH MACH. IS -GOOD- -DEFECT-	FRACTION OF DEFECT. ITEMS	FRACTION OF MACH REPRD WHICH ARE -GOOD- -DEFECT-	REPAIRS PER PERIOD		
0.10	12.49	11.09	0.80	0.0222	0.764	0.236	0.08004
0.15	37.46	34.12	3.34	0.0270	0.316	0.684	0.02669
0.20	37.46	34.12	3.34	0.0270	0.316	0.684	0.02669
0.25	37.46	34.12	3.34	0.0270	0.316	0.684	0.02669
0.30	37.46	34.12	3.34	0.0270	0.316	0.684	0.02669
0.35	37.53	34.43	3.40	0.0271	0.310	0.690	0.02643
0.40	37.63	34.43	3.40	0.0271	0.310	0.690	0.02643
0.45	38.22	34.75	3.46	0.0272	0.303	0.697	0.02617
0.50	38.22	34.75	3.46	0.0272	0.303	0.697	0.02617
0.55	38.99	35.37	3.63	0.0277	0.291	0.709	0.02564
0.60	39.38	35.66	3.72	0.0279	0.285	0.715	0.02540
0.65	40.11	36.20	3.91	0.0285	0.274	0.726	0.02493
0.70	42.15	37.64	4.50	0.0303	0.246	0.754	0.02373
0.75	55.87	46.96	8.91	0.0403	0.062	0.938	0.01790
0.80	56.31	47.23	9.08	0.0406	0.057	0.943	0.01776
0.85	57.05	47.65	9.40	0.0413	0.049	0.951	0.01753
0.90	57.78	48.01	9.78	0.0421	0.042	0.958	0.01731
0.95	59.66	48.67	10.99	0.0450	0.029	0.971	0.01676

Table II For the system whose parameters are listed in the heading, this table gives the operating characteristics of rules defined by a critical value. For example, if the rule "Repair at time  $t$  if and only if  $X_t \geq .65$ " is used then the expected cycle length or number of periods between repairs is 40.11. For 36.20 of these periods the process will be in the good state; for 3.91, the bad. The overall fraction of defective items produced will be .0285. The fraction of machines sent to repair which were actually still in the good state is .274. The remaining fraction represents bad machines. The last column is the reciprocal of the cycle length.

## SECTION 4 - SOME EXAMPLES

Two examples of the use of these tables are given. The first is an illustration of the simplest type of process for which the tables are useful; the second, the most complex.

### Example 1

Suppose the probability that the process moves from the good state to the bad state is  $\pi = .02$ . Let  $p_0 = .99$  and  $p_1 = .80$  be the fraction of good items produced when the system is in the good and bad state, respectively. Each defective item produced costs  $c = .60$  units. A good item has zero cost. Repair is instantaneous and costs 1.00 unit, regardless of from which state repair was made.

For each critical value, the cost per period must be computed. That critical value having the lowest cost then specifies the optimal policy. For this simple example the average cost per period is given by

$$c \times \{\text{Fraction of Defective Items}\} + 1 \times \{\text{Repairs Per Period}\}$$

The fraction of defective items and the repairs per period are obtained from Table II.

This computation yields Table III below.

<u>Critical Value</u>	<u>Expected Cost/Period</u>
0.10	0.09334
0.15	0.04287
0.20	0.04287
0.25	0.04287
0.30	0.04287
0.35	0.04267
0.40	0.04267
0.45	0.04250
0.50	0.04250
0.55	0.04225
0.60	0.04216
0.65	0.04205
0.70	0.04191
0.75	0.04208
0.80	0.04214
0.85	0.04231
0.90	0.04259
0.95	0.04376

Table III This table lists the expected cost per period for each choice of critical value for the problem given as Example 1.

The minimum cost occurs at an optimal critical value of  $\xi^* = .70$  and has a value of .04191 units per item produced. The cost curve is relatively flat. At a critical value of .60 the cost is .04216 and at .80 the cost is .04214. This flatness justifies computing the operating characteristics for critical values in steps as wide apart as .05.

Referring again to Table II one sees that under the optimal policy the expected cycle length is 42.15 periods and for 37.64 of these periods the process operates in the good state. On the average the process operates in the bad state for 4.50 periods before a repair is made. The overall fraction of defective items produced is .0303 and 24.6% of the time an erroneous repair is made to a process still in the good state.

One common criticism of the control approach using posterior probabilities is the difficulty of implementation. This criticism is ill founded in this particular example. By examining the transition matrix in Table I one sees that a posterior probability exceeding .70 can occur in only two ways. The zeroth item, produced immediately after repair is always ignored. If no defective item occurs in items 1 through 14, then repair is made on the occurrence of the first defective. If, however, a defective item is observed in items 1 through 14, then repair is made on the occurrence of the second defective item.

#### Example 2

Again let  $\pi = .02$ ,  $p_0 = .99$  and  $p_1 = .80$ . The direct cost of each item produced is .40 units. The selling price of an item is 1.00 units and each good item produced can be sold. When production is stopped upon suspicion that the process is in the bad state two things occur. First a detailed examination is made to determine if the process has truly broken down. This examination takes 2 periods of time and costs .20 units. If the process is found in the bad state a repair is made which takes 3 periods of time and costs .80 units. The objective is to maximize the expected profit per time period.

The expected cycle length given in Table II is now the expected number of periods in a cycle that the process is in production. To get the true cycle length, an amount representing repair time must be added. One uses the columns giving the fraction of machines repaired which are good and defective. At a critical value of .70 this yields



$$E[N] = 42.15 + 2 + .754 \times (3)$$

$$= 46.41$$

During a cycle 42.15 items on the average will be produced at an item cost of .40 units each to yield

$$\text{Item Cost} = 42.15 \times .40 = 16.86$$

Of these  $(1 - .0303) \times 42.15$  will be good items which may be sold for 1.00 units each to yield

$$\text{Item Return} = .9697 \times 42.15 = 40.87$$

The average cycle repair cost will be

$$\text{Repair Cost} = .20 + .754(.80) = .803$$

This yields, for a critical value of .70 the average profit per period of

$$\gamma = (40.87 - 16.86 - .80)/46.41 = .500$$

Repeating these calculations for several choices of critical values one arrives at Table IV. The optimal critical value is .70. By referring again to Table II one may calculate that the rate of production of good items is

$$\frac{42.15(1 - .0303)}{46.41} = .8807 \text{ items/period .}$$

<u>Critical Value</u>	<u>Total Cycle Length</u>	<u>Profit Per Period</u>
.40	41.90	.4992
.45	42.31	.4995
.50	42.31	.4995
.55	43.12	.4996
.60	43.53	.4998
.65	44.29	.4999
.70	46.41	.5000
.75	50.68	.4997
.80	61.14	.4996

Table IV This table lists the expected profit per period for each choice of critical value for the problem given as Example 2. The maximum profit is seen to occur at a critical value of .70.

## SECTION 5 - VALIDITY OF THE SOLUTION

The validity of the proposed solution is checked in two ways. First several inequalities which the proposed solution must satisfy are developed and the solution is checked against them. Second the problem is solved anew through the method of discounted cost dynamic programming and this answer compared to the original.

Consider the situation depicted earlier in Example 1 where repair is instantaneous and has a cost of one unit irrespective of the state of the process at the time repair is made. Let  $c$  be the cost of a defective item and let  $\pi$ ,  $p_0$ , and  $p_1$  have the same meaning as before. Let  $\underline{\gamma}$  be the average cost per period assuming that the state of the process were known.

$$\begin{aligned}\underline{\gamma} &= \text{Repair Cost} + \text{Item Cost} \\ &= \pi + c(1 - p_0)\end{aligned}\tag{5.1}$$

If  $\gamma$  is the minimum cost assuming the process state unknown then

$$\underline{\gamma} \leq \gamma$$

To get an upper bound on the solution one realizes that the cost under an arbitrary policy must be greater than the cost of the optimal policy. The arbitrary policies considered are of the form:

"Repair every  $m$  periods" for some constant  $m$ .

Let  $\bar{N}$  be the number of periods that the process is in the good state, and let

$$a_n = \Pr[\bar{N} = n] = (1 - \pi)^{n-1} \pi$$

$$b_n = \Pr[\bar{N} \geq n] = \sum_{k=n}^{\infty} a_k = (1 - \pi)^{n-1}$$

Two identities are used.

$$\sum_{i=k}^{\ell} a_i = (1 - \pi)^{k-1} - (1 - \pi)^{\ell} \quad (5.2)$$

$$\begin{aligned} \sum_{n=0}^{m-1} n a_n &= \sum_{k=1}^{m-1} \sum_{i=k}^{m-1} a_i \\ &= \{[1 - (1 - \pi)^{m-1}]/\pi\} - (m - 1)(1 - \pi)^{m-1} \end{aligned} \quad (5.3)$$

If  $\bar{\gamma}^m$  is the average cost per period under a policy of this form then

$$\begin{aligned} \bar{\gamma}^m &= 1/m + c(1 - p_1) + \frac{1}{m} \left\{ \sum_{k=1}^{m-1} c(p_1 - p_0) k a_k + c m (p_1 - p_0) b_m \right\} \\ &= c(1 - p_1) + \frac{1}{m} [c(p_1 - p_0)/\pi + 1] - \frac{c(p_1 - p_0)}{\pi m} (1 - \pi)^m \end{aligned} \quad (5.4)$$

To simplify this expression let  $\beta = c(p_1 - p_0)/\pi$  and use the approximation for large  $m$  and small  $\pi$

$$e^{-m\pi} \approx (1 - \pi)^m.$$

One gets

$$\begin{aligned} \bar{\gamma}^m &\approx c(1 - p_1) + (1 - \beta) \frac{1}{m} + \frac{\beta}{m} e^{-m\pi} \\ &= c(1 - p_0) + (\pi - \frac{1}{m})(\beta - 1) + \frac{\beta}{m} e^{-m\pi} \end{aligned} \quad (5.5)$$

For  $m = \frac{1}{\pi}$  this reduces to

$$\bar{\gamma} = c(1 - p_0) + c(p_0 - p_1)\left(\frac{1}{e}\right) \quad (5.6)$$

Thus

$$\underline{\gamma} \leq \gamma \leq \inf_{m \geq 0} \bar{\gamma}^m \leq \bar{\gamma}$$

For Example 1

$$\begin{aligned} \underline{\gamma} &= .0260 \\ \gamma &= .0419 \\ \inf_{m \geq 0} \bar{\gamma}^m &= .0665 \\ \bar{\gamma} &= .0679 \end{aligned}$$

The discounted cost dynamic programming approach to sequential decision problems is well known.

Let  $\alpha$  be the discount factor such that a unit of value one period hence has a present value of  $\alpha^n$ ;

$K$  be the cost of repairing the machine;

$c$  the cost of a defective item;

$g(\alpha, x)$  the minimum expected total discounted future cost

$$C(x) = c[x(1 - p_1) + (1 - x)(1 - p_0)].$$

If a minimum cost rule exists then one can easily show that

$$g(\alpha, x) = \min \begin{cases} K + C(x^0) + \alpha E_{x^0} g(\alpha, \tau_Y x^0) \\ C(x) + \alpha E_x g(\alpha, \tau_Y x) \end{cases} \quad (5.7)$$

Derman [3] has rigorously shown that such a rule exists and is given by "Repair at time  $t$  if and only if  $X_t \in R_\alpha$ " where

$$R_\alpha = \{x: g(\alpha, x) = K + C(x^0) + \alpha E_x g(\alpha, \tau_Y x^0)\} \quad (5.8)$$

One usefulness of this theory lies in the fact that Equation (5.7) may be solved in a straightforward manner. One defines  $g_0(\alpha, x) = 0$  for all  $x \in [0, 1]$  and then computes recursively:

$$g_{n+1}(\alpha, x) = \min \begin{cases} K + C(x^0) + \alpha E_{x^0} g_n(\alpha, \tau_Y x^0) \\ C(x) + \alpha E_x g_n(\alpha, \tau_Y x) \end{cases} \quad (5.9)$$

It is easily shown that for every  $\alpha \in [0, 1)$  the sequence of functions  $\{g_n(\alpha, \cdot)\}$  converges uniformly and monotonically to a bounded limit function  $g(\alpha, \cdot)$  which is the unique bounded solution to the functional Equation (5.7).

The general theory shows that "The Existence of an Optimal Stationary Non Randomized Rule in the Average Cost Case" shows that

$$\lim_{\alpha \rightarrow 1} g(\alpha, 0)(1 - \alpha) = \gamma$$

and for some subsequence  $\{\alpha_{n_k}\}$

$$\lim_{k \rightarrow \infty} R_{\alpha_{n_k}} = R^*$$

where  $R^*$  is the optimal repair region under the average cost per period criterion and  $\gamma$  is the minimum average cost per period.

Thus a means is provided for checking the proposed solution. For the case considered in Example 1, Equation (5.7) was solved numerically for values of  $\alpha$  of .98, .99, .995, .999.

It should be remarked that convergence was extremely slow and in order to get meaningful results it was necessary to begin the iteration with a function close to the solution. However, the results were very close to those computed earlier as may be seen in Table V, verifying the computational procedure used in the average cost case.

Discount Factor $\alpha$	Crit. Val Defining $R^\alpha$	$(1-\alpha) \times g(\alpha, 0)$	$(1-\alpha) \times g(\alpha, 1)$	$(1-\alpha) \times g(\alpha, \xi^M)$	$\max_{\xi}  g(\alpha, \xi) - [Tg(\alpha, \cdot)](\xi) $	$g(\alpha, 0)$
.98	.77	.03817	.05817	.04130	.002	1.909
.99	.74	.03958	.04958	.04125	.001	3.958
.995	.73	.04041	.04541	.04127	.001	8.082
.999	.72	.04111	.04211	.04129	.001	41.110

Table V Solution of Example 1 by the discounted cost dynamic programming technique. For large  $\alpha$  the critical value should be near the earlier result of .70. The third, fourth and fifth columns should be compared with the average cost per period  $\gamma = .04191$  computed earlier. The relation should be

$$(1 - \alpha)g(\alpha, 0) \leq \gamma \leq (1 - \alpha)g(\alpha, 1) .$$

The modal point of the stationary distribution for  $X_t$  is denoted  $\xi^M$ , and  $(1 - \alpha)g(\alpha, \xi^M)$  should approximate  $\gamma$ . The last two columns are given to evaluate how well the numerical solution satisfies the equation  $g(\alpha, \cdot) = Tg(\alpha, \cdot)$ .



## SECTION 6 - SENSITIVITY ANALYSIS

Before one may apply the optimal control technique proposed in this paper one must estimate the process parameters  $\pi$ ,  $p_0$  and  $p_1$ . Suppose these are estimated incorrectly. A means of detecting incorrect estimates has already been given; compare the actual cycle lengths and fraction defectives produced with the predicted values for these quantities. But still unanswered is the behavior of the process under a control rule based on incorrectly estimated parameters. This section provides a means for answering this question.

It is assumed that the estimates of the parameters are  $\pi$ ,  $p_0$  and  $p_1$  but that the true values are  $\pi'$ ,  $p'_0$ , and  $p'_1$ . The controller has computed an optimal critical value  $\xi^*$  based on his estimates  $\pi$ ,  $p_0$  and  $p_1$  and the process will be controlled as before through the sequence  $\{X_t, t = 0, 1, \dots\}$  defined as before:

$$X_{t+1} = \tau_{Y_t}^{\xi^*} X_t$$

where the estimated values  $\pi$ ,  $p_0$ , and  $p_1$ , are used in defining the transformation  $\tau_y^{\xi^*}$ .

However, the  $\{X_t, t = 0, 1, \dots\}$  process no longer has the Markov property, in general, and its transition probabilities also depend on the unknown parameters  $\pi'$ ,  $p'_0$  and  $p'_1$ .

The technique proposed in the Girshick and Rubin paper affords a means of analysis. One considers the bivariate process  $\{(X_t, \odot_t), t = 0, 1, \dots\}$  which enjoys the Markov property with transition probabilities given in Table VI.

Transition		
From $(X_t, \Theta_t)$	To $(X_{t+1}, \Theta_{t+1})$	Transition Probability
$(x, 0)$	$(\tau_0 x, 0)$	$p'_0(1 - \pi')$
$(x, 0)$	$(\tau_1 x, 0)$	$(1 - p'_0)(1 - \pi')$
$(x, 0)$	$(\tau_0 x, 1)$	$p'_0\pi'$
$(x, 0)$	$(\tau_1 x, 1)$	$(1 - p'_0)\pi'$
$(x, 1)$	$(\tau_0 x, 1)$	$p'_1$
$(x, 1)$	$(\tau_1 x, 1)$	$(1 - p'_1)$

Table VI The possible transitions with corresponding probabilities for the bivariate process  $\{(X_t, \Theta_t), t = 0, 1, \dots\}$ . These results hold only for  $x$  not in the repair region  $R$ .

Let  $\psi^0 = (\psi_j^0)$  be a row vector where  $\psi_j^0$  is the expected number of times in the first cycle that the  $\{(X_t, \Theta_t), t = 0, 1, \dots\}$  process is in state  $(x^j, 0)$  for  $x^j$  not an element of the repair region  $R$ . Let  $H^0 = (\eta_{ij}^0)$  be the transition probability matrix corresponding to transitions of the form  $(x, 0) \rightarrow (\tau_y x, 0)$  for  $x \notin R$  and  $\tau_y x \notin R$ . Then

$$\eta_{ij}^0 = \begin{cases} p'_0(1 - \pi') & \text{if } x^j = \tau_0 x^i \\ (1 - p'_0)(1 - \pi') & \text{if } x^j = \tau_1 x^i \\ 0 & \text{otherwise} \end{cases}$$

By an analysis entirely similar to that used in developing Equation (2.5) one has that  $\psi^0$  is the unique solution to the infinite set of linear equations:

$$\psi_j^0 = \sum_{x^i \notin R} \psi_i^0 \eta_{ij}^0 ; \quad x^j \notin R \quad \text{and} \quad x^j \neq x^0$$

and  $\psi_0^0 = 1$ .

Let  $\psi' = (\psi'_j)$  be a row vector where  $\psi'_j$  is the expected number of times in the first cycle that the  $\{(X_t, \Theta_t), t = 0, 1, \dots\}$  process is in state  $(x^j, 1)$ . In the case where  $\pi = \pi'$ ,  $p_0 = p'_0$  and  $p_1 = p'_1$  one would have  $x^j \psi_j^0 = (1 - x^j) \psi'_j$  and this is the approach Girshick and Rubin suggest. For the case at hand a new system of equations must be developed for the  $\psi'$  vector. Again the expected number of times the process is in any particular state is equated with the sum of the expected number of times the process is in a preceding state weighted by the probability of a transition.

Let  $H' = (\eta'_{ij})$  be the transition probability matrix corresponding to transitions of the form  $(x, 1) \rightarrow (\tau_y x, 1)$  for  $x \notin R$ , and  $\tau_y x \notin R$ .

Then

$$\eta'_{ij} = \begin{cases} p'_1 & \text{if } x^j = \tau_0 x^i \\ (1 - p'_1) & \text{if } x^j = \tau_1 x^i \\ 0 & \text{otherwise} . \end{cases}$$

Let  $H'' = (\eta''_{ij})$  be the transition probability matrix corresponding to transitions of the form  $(x, 0) \rightarrow (\tau_y x, 1)$ . Then

$$\eta''_{ij} = \begin{cases} p'_0 \pi' & \text{if } x^j = \tau_0 x^i \\ (1 - p'_0) \pi' & \text{if } x^j = \tau_1 x^i \\ 0 & \text{otherwise} \end{cases}$$

Assumed		Critical Value	Anticipated			Encountered			
			Cycle Length	Fraction Defect.	Cost per Period	Cycle Length	Fraction Defect.	Cost per Period	
Trans. Prob. $\pi$	Fraction of Good Items		.70	52.41	.0492	.0486	57.36	.0415	.0423
	$p_0$	$p_1$							
.02	.98	.80	.70	52.41	.0492	.0486	57.36	.0415	.0423
.02	.995	.80	.70	45.51	.0229	.0357	38.38	.0275	.0426
.01	.99	.80	.65	98.50	.0264	.0260	56.57	.0407	.0421
.005	.99	.80	.60	174.31	.0192	.0172	57.54	.0417	.0424
.03	.99	.80	.75	31.22	.0368	.0541	40.23	.0290	.0427
.02	.99	.70	.65	38.42	.0290	.0434	39.14	.0281	.0424
.02	.99	.85	.75	58.29	.0375	.0397	54.79	.0396	.0420
.005	.99	.90	.65	179.88	.0186	.0167	57.25	.0416	.0424
.005	.995	.90	.65	197.12	.0132	.0130	54.84	.0396	.0420
.02	.99	.80	.70	42.15	.0303	.0419	42.15	.0303	.0419 ← True

Table VII This table summarizes a sensitivity analysis of the model. The true process has parameters  $\pi = .02$ ,  $p_0 = .99$  and  $p_1 = .80$ . The controller incorrectly estimates these and uses instead the values given in the "Assumed" columns. The critical value is optimal for these assumed numbers. Based on the assumed values, the controller estimates he will encounter the cycle length, etc. given in the "Anticipated" columns. He actually encounters those given in the "Encountered" columns. The last row gives the results if the controller uses the true parameter values. Notice that the other encountered costs are not much greater than achieved when the parameters are estimated correctly.

The vector  $\psi'$  is the unique solution to the infinite system of equations:

$$\psi'_j = \sum_{x^i \notin R} \psi'_i \eta'_{i,j} + \sum_{x^i \notin R} \psi^0_i \eta''_{i,j} \quad (6.2)$$

To investigate the behavior of the process under incorrectly chosen parameters one first finds the critical value  $\xi^*$  defining the repair region that the controller will use, based on his incorrect estimates  $\pi$ ,  $p_0$  and  $p_1$ . This repair region completes the definition of Equations (6.1) and (6.2) which are then solved, using the approximation technique discussed in Section 3, to yield the vectors  $\psi^0$  and  $\psi'$ . Then  $m_\theta$ , the number of periods during the first cycle that the process is in state  $\theta$ , for  $\theta \in \{0, 1\}$  is computed by

$$m_0 = \sum_{x^i \notin R} \psi^0_i$$

$$m_1 = \sum_{x^i \notin R} \psi^1_i$$

The cycle length is  $m_0 + m_1$ . From these three numbers the behavior of the process may be computed for many simple models. The results of such an analysis performed on the case of Example 1 is given in Table VII.

## SECTION 7 - CONCLUSIONS

The simple Bernoulli case, where sampling inspection is by attributes, of the Girshick and Rubin quality control model has been studied in detail. A numerically feasible means of calculating optimal solutions has been demonstrated. The nature of the solution is such that once tables of operating characteristics have been prepared, the solution to a wide variety of problems may be had through slight additional hand calculations from the tables. The solution is optimal in the sense of minimizing the average cost per unit time. It also appears relatively insensitive to errors in the estimates of the model parameters and a means of calculating this sensitivity is provided. A means of checking, over a long period of time, whether or not estimates are correct has also been suggested. It has been shown that the optimal solution is not necessarily difficult to implement.

Of course, further work remains to be done. The entire non 100% inspection case needs to be solved from beginning to end. A means of estimating the model parameters should be found. Some work should be done on simplifying the form of the rule to ease implementation. And, assuming that the model is realistic enough so that there is some demand, tables of operating characteristics should be prepared.

APPENDIX  
A DETAILED DESCRIPTION OF THE NUMERICAL  
SOLUTION METHOD

The proposed numerical solution has two distinct phases. In the first phase a finite transition probability matrix which approximates the natural infinite matrix is computed. In the second phase a set of linear equations is solved for each of several critical values. Each solution determines the operating characteristics of the rule using the corresponding critical value.

The computations were carried out through the facilities of the Stanford Computation Center. Not only did the Center provide free computation time but also many "software" aids. The Gaussian elimination procedure used here is a minor variation of one of their library procedures.

The program was written in Burrough's Extended Algol language. Since this differs from Algol almost solely in the provision for input and output, the description of the numerical solution method will consist of a presentation of the program with annotated comments.

Names of Variables

AA[0: LL, 0: LL]                   , a square array of coefficients in the linear equations.

ASYP                                 , a value about which posterior probabilities tend to cluster



---

BB[0: LL] , the right hand side of the linear equation set.

DMIN , the deviation between an exact "TO STATE" and the state in the finite list which nearest approximates it.

DMAX , the largest of the DMIN's above.

EPS , all states within EPS of the cluster value ASYP are grouped into two classes, those above and those below ASYP. Each class is then assigned a common state to represent all states in the class. EPS, then, is a variable determining the degree of approximation involved.

FS[0: LLL] , the list of states, or "FROM STATES" used in the approximation.

LL, LLL, and LMAX , LMAX is the largest allowable transition matrix size; LL is the matrix size used and  $LLL = LL + 1$ .

MO, M1 , the number of periods per cycle that the machine spends in the good and bad states, respectively.

NFG1[0: LL], NFG2[0, LL] , the state number to which a transition occurs in the event that a good or bad item is observed, respectively.

PI , the transition probability that in a single period a good machine will go bad.

PO, P1 , the fraction of good items that a machine  
 in the good or bad state will produce,  
 respectively.

PMAX , the largest allowable posterior probability  
 state.

PCRIT , the largest posterior probability state  
 encountered.

PCR , a dummy variable running through critical  
 values.

RO , the fraction of repaired machines which are  
 defective.

TS1[0: LL], TS2[0: LL] , the states to which a transition is made upon  
 the observance of a good or bad item,  
 respectively. The "TO STATES."

TSP1[0: LL], TSP2[0: LL] , the probabilities of the above transitions.

X[0: LL] , the vector of unknowns in the linear equation  
 set.

XO , the overall proportion of defective items  
 produced.

Y , a dummy variable running through posterior  
 probability values.

BEGIN COMMENT THIS PROGRAM COMPUTES THE OPERATING CHARACTERISTICS OF A  
 CONTROL RULE AS A FUNCTION OF THE CRITICAL VALUE OF THE RULE. INPUT IS  
 P1, THE TRANSITION PROBABILITY OF THE MACHINE, AND P0 (P1), THE FRACTION  
 OF GOOD ITEMS A MACHINE IN THE GOOD (BAD) STATE WILL PRODUCE.

```

    REAL P1, P0, P1, Y, EPS, PMAX, ASYP ;
    INTEGER LL, LLL, LMAX;
    LABEL START ;
    LABEL SING ;
    REAL PROCEDURE D(Y);VALUE Y;REAL Y;BEGIN D+(P1-P0)*Y+P0; END D;
    REAL PROCEDURE A(Y);VALUE Y;REAL Y;BEGIN A+((P1-P1*P0)*Y+P1*P0)/D(Y);
    END A;REAL PROCEDURE B(Y);VALUE Y;REAL Y;BEGIN B+(((1-P1)-P1*(1-P0))*Y+
    (1-P0)*P1)/(1-D(Y));END B; REAL PROCEDURE AINV(Y);VALUE Y; REAL Y;
    BEGIN AINV+(Y-P1)*P0/((P0-P1)*Y+P1-P1*P0); END AINV;
    INTEGER PROCEDURE SIZE(EPS);VALUE EPS;REAL EPS;
    BEGIN LABEL OUTA; INTEGER I,K; REAL Y; Y=0; I=0;
    FOR K=0 STEP 1 WHILE I<LMAX DO BEGIN IF Y<ASYP-EPS THEN Y+A(Y) ELSE IF
    Y>ASYP THEN Y+AINV(Y) ELSE BEGIN Y+AINV(ASYP+EPS);I+1;END; IF Y>PMAX
    THEN GO TO OUTA;I+1;END;OUTA:SIZE+IF I<LMAX THEN I ELSE LMAX;END;
    LMAX+75; PMAX+0.97;
  
```

①  
 ②  
 ③

```

    START:READ(P1, P0, P1); ASYP+P1*P0/(P0-P1);
    LL+SIZE(0.02); EPS+0.02; IF LL<LMAX-10 THEN BEGIN LL+SIZE(0.01);EPS+
    0.01;END;IF LL<LMAX-10 THEN BEGIN LL+SIZE(0.007); EPS+0.007;END;
    LLL+LL+1;
    BEGIN
  
```

```

    REAL DMAX, DMIN, PCRIT, PCR;
    REAL M0, M1, X0, R0 ;
    INTEGER I, J, K, M, N1, N2;
    REAL ARRAY FS, IS1, TS2, TSP1, TSP2[0:L];
    INTEGER ARRAY NFG1, NFG2[0:L];
    REAL ARRAY AA[0:L], HB, X[0:L];
    PROCEDURE GAUSSA(M, N1, N2, A, B, X, SING);VALUE M, N1, N2; INTEGER M, N1, N2;
    REAL ARRAY A[0,0], B, X[0]; LABEL SING; BEGIN INTEGER I, J, IMAX, K, L;
    REAL MX, T, QUOT;
    PROCEDURE ELIM(KP1, N, QUOT, A, B);VALUE KP1, N, QUOT; INTEGER N, KP1;
    REAL QUOT; REAL ARRAY A, B[0]; BEGIN INTEGER I; FOR I=KP1 STEP 1 UNTIL N
    DO A[I]+A[I]-QUOT*B[I]; END ELIM;
    M=M-1; N1=N1-1; N2=N2-1; FOR K=N1 STEP 1 UNTIL N2 DO BEGIN
    MX=0; IMAX=K; FOR I=K STEP 1 UNTIL N2 DO
    IF MX<ABS(A[I,K]) THEN BEGIN MX+ABS(A[I,K]); IMAX+I; END;
    IF MX = 0 THEN GO TO SING;
    I=K; IF K ≠ IMAX THEN BEGIN
    J = IMAX; T=B[I]; B[I]+B[J]; B[J]+T; FOR L=0 STEP 1 UNTIL M DO
    BEGIN T+A[I,L]; A[I,L]+A[J,L]; A[J,L]+T; END; END;
    FOR J=K+1 STEP 1 UNTIL M DO BEGIN QUOT+A[J,K]/A[I,K]; IF QUOT≠0 THEN
    BEGIN B[J]+B[J]-B[K]*QUOT; ELIM(K+1, M, QUOT, A[J,+], A[K,+]);END;
    END; END;
    FOR I=N2 STEP -1 UNTIL 0 DO BEGIN
    T=0; FOR L=I+1 STEP 1 UNTIL N2 DO T+T+A[I,L]*X[L];
    X[I]+(B[I]-T)/A[I,I]; END; END OF GAUSSA;
  
```

④

COMMENT THIS GROUP OF INSTRUCTIONS COMPUTES THE TRANSITION PROB-  
 ABILITY MATRIX CORRESPONDING TO THE PROCESS WHICH TRACKS THE POST-  
 ERIOR PROBABILITIES ON THE MACHINE STATE.

## PROGRAM EXPLANATION

1. Several functions are defined.  $Y$  always represents the probability that the machine will be in the bad state for the production of the next item.  $D(Y)$ , then, is the probability that a good item is produced.  $A(Y)$  is the next posterior probability given that a good item is observed.  $B(Y)$  is this probability upon observing a bad item.  $AINV(Y)$  is the inverse function to  $A(Y)$ .
2. The function  $SIZE(EPS)$  gives the approximating matrix size as a function of the approximation constant  $EPS$ . It does this by quickly going through the approximating procedure and counting the size of the resulting matrix.
3. The matrix size for several values of  $EPS$  is computed. The program selects a value for  $EPS$  which results in a matrix size near to but less than  $LMAX$ .
4. The input variables to the  $GAUSSA$  procedure are:  $A$ , an  $M \times M$  matrix of coefficients;  $B$ , an  $M$  vector of constants;  $X$ , an  $M$  vector of unknowns and  $SING$ , a label to which control passes should a singular matrix be found. If this procedure is executed for a value of  $N1 = 1$  and  $N2 = K$  then the values appearing in  $X[I]$  for  $I = 1, 2, \dots, K$  are the solutions to

$$\sum_{J=1}^K A[I, J]X[J] = B[I] ; I = 1, \dots, K$$

If next the procedure is called for a value  $N1 = K + 1$  and  $N2 = L$  then  $X[ ]$  contains the solution to

$$\sum_{J=1}^L A[I, J]X[J] = B[I] ; I = 1, \dots, L$$

And so on, extracting the solution to increasing subsets of equations while solving the largest set.

5

```

COMMENT FIRST THE ALLOWABLE STATES IN THE APPROXIMATION ARE COM-
PUTED AND STORED IN FS( ) ;
Y+0; I+0; FOR K+OSTEP 1 WHILE I<LL DO BEGIN FS(I)+Y;
IF Y<ASYP-EP5 THEN BEGIN TS1(I)+A(Y); NFG1(I)+I+1; TSP1(I)+D(Y);
TSP2(I)+1-D(Y); Y+A(Y); END
ELSE IF Y<ASYP THEN BEGIN TS1(I)+FS(I-1); NFG1(I)+I-1; TSP1(I)+D(Y);
TSP2(I)+1-D(Y); Y+A(INV(Y)); END
ELSE BEGIN TS1(I-1)+FS(I)+Y+ASYP-0.33*EP5; TS1(I)+Y; NFG1(I)+I;
TSP1(I)+D(Y); TSP2(I)+1-D(Y);
I+I+1; Y+ASYP+0.33*EP5; FS(I)+TS1(I)+Y; NFG1(I)+I; TSP1(I)+D(Y);
TSP2(I)+1-D(Y); Y+A(INV(ASYP+EP5)); END;
I+I+1; END;
COMMENT NEXT THE MATRIX IS FILLED IN WITH A "0" STATE BEING
APPROXIMATED BY ITS NEAREST NEIGHBOR ;
DMAX+0.0; PCRIT+FS(LL); FS(LL)+0.99999;
FOR I+OSTEP 1 UNTIL LL DO BEGIN DMIN+0.99; Y+B(FS(I));
IF Y>PCRIT+DMAX THEN BEGIN TS2(I)+Y; NFG2(I)+999; END
ELSE BEGIN FOR J+1 STEP 1 UNTIL LL DO IF ABS(Y-FS(J))<DMIN THEN
BEGIN DMIN+ABS(Y-FS(J)); K+J; END;
TS2(I)+FS(K); NFG2(I)+K; IF DMIN>DMAX THEN DMAX+DMIN; END; END;
COMMENT THE TRANSITION PROBABILITY MATRIX IS WRITTEN OUT ;
BEGIN FORMAT FMT1( X4, "PI = ", F5.3, X2,
"PO = ", F5.3, X2, "P1 = ", F5.3,
X4, "EPS_ = ", F5.3/X4, "MAX.APPRDX. = ", F6.4, X3,
"MATRIX SIZE 0:", I3,
X4, "ASYP = ", F5.3, X4, "PCRIT = ", F5.3 // ),
FMT2( X20, "THE TRANSITION PROBABILITY MATRIX IS" /
X2, "STATE STATE STATE STATE WITH " /
"STATE STATE WITH " /
X2, " NO. (FROM) NO.(TO) (TO) PROB NO.(TO) (TO)",
" PROB " / ),
FMT3( X2, I5, X4, F5.3, X5, I5, X4, F5.3, X2, F5.3, X4,
I5, X4, F5.3, X2, F5.3 ),
FMT4( I6, X2, 4(F5.3, X2), F6.4, X2, I4, 2(X2, F6.4));
LIST OUT1( PI, PO, P1, EPS, DMAX, LL, ASYP, PCRIT),
OUT3 ( I, FS(I), NFG1(I), TS1(I), TSP1(I), NFG2(I), TS2(I),
TSP2(I) ) ;
WRITE(FMT1, OUT1); WRITE(FMT2); FOR I+OSTEP 1 UNTIL LL DO WRITE(FMT3,
OUT3); WRITE((PAGE1)); END;
BEGIN FORMAT TMT1(X10, "TRANSITION PROBABILITY IS ", F5.3 /
X10, "FRACTION OF GOOD ITEMS PRODUCED BY A REPAIRED MACHINE IS", F6.3 /
X10, "FRACTION OF GOOD ITEMS PRODUCED BY A DEFECTIVE MACHINE IS", F5.3 //
X5, "CRITICAL EXPECTED PERIODS FOR FRACTION FRACTION OF MACH",
" REPAIRS"/
X5, " VALUE CYCLE WHICH MACH. IS OF DEFECT. REPRD WHCH ARE ",
" PER "/
X5, " LENGTH -GOOD- -DEFECT- ITEMS -GOOD- -DEFECT-",
" PERIOD" //);
LIST OUT1(PI, PO, P1); WRITE(TMT1, OUT1); END;

COMMENT THIS GROUP OF INSTRUCTIONS SETS UP THE EQUATIONS WHICH WILL
BE USED TO SOLVE FOR THE LONG RUN BEHAVIOR ;
FOR J+OSTEP 1 UNTIL LL DO BEGIN AA(J, J)+1; FOR I+J+1 STEP 1 UNTIL LL DO
AA(I, J)+AA(J, I)+0; END; FOR J+OSTEP 1 UNTIL LL DO BEGIN BB(J)+0;
I+NFG1(J); IF ISLL THEN AA(I, J)+AA(I, J)-TSP1(J);

```

6

5. The relations among the transition matrix variables generated by the algorithm is summarized as:

- (1) If  $Y = FS[I] < ASYP - EPS$  then
  - a.  $FS[I] = A(FS[I-1]), I \neq 0; FS[0] = 0;$
  - b.  $NFG1[I] = I + 1;$
  - c.  $TS1[I] = A(FS[I]);$
  - d.  $TSP1[I] = D(Y);$
  - e.  $TS2[I]$  is the state in the list nearest to  $B(Y);$
  - f.  $TSP2[I] = 1 - D(Y);$
- (2) If  $Y = FS[I] > ASYP + EPS$  then
  - a.  $FS[I] = A(FS[I + 1]);$
  - b.  $NFG1[I] = I - 1;$
  - c.  $TS1[I] = A(FS[I]);$
  - d.  $TSP1[I] = D(Y);$
  - e.  $TS2[I]$  is the state in the list nearest to  $B(Y);$
  - f.  $TSP2[I] = 1 - D(Y);$
- (3) If  $ASYP - EPS \leq Y = FS[I] \leq ASYP + EPS$  then
  - a.  $FS[I] = FS[I - 1] \leq FS[I + 1]$  according as  $Y < ASYP$  or  $Y > ASYP;$
  - b.  $NFG1[I] = I;$
  - c.  $TS1[I] = FS[I] = Y;$
  - d.  $TSP1[I] = D(Y);$
  - e.  $TS2[I]$  is the state in the list nearest to  $B(Y);$
  - f.  $TSP2[I] = 1 - D(Y);$

6. Equation 3.2

$$(I - \hat{H})^T \Psi^T = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is set up as  $AA \cdot X = BB.$

7. Using the modified Gaussian elimination procedure, the subsets of Equation 3.2 are solved. At each step, the operating characteristics are computed according to the finite counterparts of Equations 2.12 to 2.16.

```

I=MSG2(J); IF ISLL THEN AA(I,J)+AA(I,J)-TSP2(J); END;
BR(I)+1;

COMMENT THE LONG RUN BEHAVIOR OF THE SYSTEM IS FOUND FOR EACH POSS-
IBLE CRITICAL VALUE PCR. ;
N2 + -1;
FOR PCR=0.10 STEP 0.05 UNTIL PCR=1 DO BEGIN
FOR I=0 STEP 1 WHILE FS(I) < PCR DO M=I;
N1=N2+1; N2=M; GAUSSA(LL+1,N1+1,N2+1,AA,BB,X,SING) ;
M1=M0+R0+0.0; FOR I=0 STEP 1 UNTIL M DO BEGIN
M1=M1+FS(I)*X(I); M0=M0+(1-FS(I))*X(I);
IF TS1(I) > PCR THEN R0=R0+X(I)*TSP1(I)*TS1(I);
IF TS2(I) > PCR THEN R0=R0+X(I)*TSP2(I)*TS2(I); END;
X0=(M0*X0+M1*X1)/(M0+M1);
BEGIN FORMAT FMT1( X7, F4.2, X3, F8.2, F9.2, F8.2, X2, F7.4, X3, 2(F8.3),
F10.5); LIST OUT1(PCR,M0+M1,M0,M1,1-X0,1-R0,R0,1/(M0+M1));
WRITE(FMT1,OUT1); END; END;
WRITE([PAGE]); GO TO START; END;
SING: BEGIN FORMAT F("SINGULAR MATRIX"); WRITE(F); END; GO TO START;
END.

```

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DOCUMENT CONTROL DATA - R&D		
<i>(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)</i>		
1 ORIGINATING ACTIVITY (Corporate author) Stanford University Department of Statistics Stanford, California		2a REPORT SECURITY CLASSIFICATION Unclassified
		2b GROUP
3 REPORT TITLE  Statistical Control of a Bernoulli Process		
4 DESCRIPTIVE NOTE. (Type of report and inclusive dates) Technical Report, February 4, 1965		
5 AUTHOR(S) (Last name, first name, initial)  Taylor, Howard M.		
6 REPORT DATE February 4, 1965	7a TOTAL NO. OF PAGES 43	7b NO. OF REFS 12
8a CONTRACT OR GRANT NO. Nonr-225(53)(FBM)	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 79	
b PROJECT NO. NR-042-002		
c. Nonr-225(53)(FBM)	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
d		
10. AVAILABILITY/LIMITATION NOTICES  Releaseable without limitations on dissemination		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch Office of Naval Research Washington, D. C. 20360	
13. ABSTRACT  The optimal statistical control of a simple production process which has only two underlying states, good and bad, is studied. The produced items are classified as good or defective, a cost being associated with each defective item produced. A cost is also charged for repairing the process, which has the effect of returning the process to the good state. Other than immediately after repair, the process state is assumed unknown. One seeks a statistical control rule which, based on the quality history of produced items, tells when to repair the machine.  A feasible means of computing the control rule which minimizes the average cost per unit time is given. The nature of the solution is such that once tables of operating characteristics have been prepared, the solutions to a wide variety of problems may be had through slight additional hand calculations. The optimal rule is shown to be relatively insensitive to errors in estimating the model parameters. A means of checking, over a long period of time, whether or not estimates are correct has also been suggested.		

DD FORM 1473  
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