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The Moments of the Number of Crossings of a Level by a Stationary Normal Process.

by

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The Moments of the Number of Crossings of a Level by a Stationary Normal Process

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Summary. In this report we consider the number N of upcrossings of a level u by a stationary normal process $\xi(t)$ in $0 \leq t \leq T$. A formula is obtained for the factorial moment $M_k = \mathcal{E}\{N(N-1)\ldots(N-k+1)\}$ of any desired order k. The main condition assumed in the derivation is that $\xi(t)$ have, with probability one, a continuous sample derivative $\xi'(t)$ in the interval $[0,T]$. This condition involves no real restriction since an example shows that even a slight relaxation of it causes all moments of order greater than one to become infinite. The moments of the number of down crossings or total number of crossings can be obtained analogously.

1. Introduction. The problem of obtaining the mean number of crossings, (or equivalently upcrossings) of a given level, by a stationary normal process in a given time, has received a good deal of attention in the literature. In fact, a complete solution to this problem has now been given by Ylvisaker [8]. However, moments of order greater than one of the number of crossings of a level have received less attention. The variance was obtained by Steinberg et al [6], using somewhat heuristic arguments. Rozanov and Volkonski [7] point out in a footnote that the formula given in [6] for the variance is valid under certain precise conditions, of which the main one is that the covariance function of the process have a finite sixth derivative at the origin. Finally in this connection, the variance has been obtained by Leadbetter and Cryer [4] under conditions which assume just a little more than the existence of a second derivative of the covariance function.

There is virtually no literature available in connection with moments of the number of crossings of a level, of higher order than the second. (A partial result is indicated by Ivanov at the end of his paper [3]). It will be our purpose here to obtain explicit expressions for such moments. This will be done for upcrossings, in terms of factorial moments of arbitrary orders and under conditions which are very close to the necessary ones. Corresponding formulae for moments of all orders for the down crossings, or total number of crossings, follow similarly.

2. Moments of the number of upcrossings. We shall, throughout, consider a real valued stationary normal process $\{\xi(t):\ 0 \leq t \leq T\}$ having (for convenience) zero mean, spectrum $F(\lambda)$ possessing an absolutely continuous component, and covariance function $r(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} dF(\lambda)$. We shall further assume that $\xi(t)$ has, with probability one, a continuous sample derivative $\xi'(t)$ on the interval $[0,T]$. Sufficient conditions for this latter property in terms of the behaviour of the covariance function, are well known. Write N for the number of upcrossings of the level u by $\xi(t)$ in $0 \leq t \leq T$; that is N is the number of points t in that interval for which $\xi(t) = u$, $\xi'(t) > 0$. Then the following result holds.

Theorem.

If $\{\xi(t):\ 0 \leq t \leq T\}$ is a normal stationary process, as described, possessing, with probability one, a continuous sample derivative, and k is any positive integer, then

$$M_k = \mathcal{E}\{N(N-1)\ldots(N-k+1)\}$$

$$(1) \qquad = \int_0^T \!\!\ldots\!\int dt_1\ldots dt_k \int_0^\infty \!\!\ldots\!\int y_1\ldots y_k\, p_{\underline{t}}(u,\underline{y})\, dy_1\ldots dy_k$$

in which $p_{\underline{t}}(u,\underline{y}) = p_{\underline{t}}(u,\ldots u,y_1,\ldots y_k)$, $p_{\underline{t}}(x_1,\ldots x_k,y_1\ldots y_k)$ denoting the joint density for the random variables $\xi(t_1)\ldots\xi(t_k)$, $\xi'(t_1)\ldots\xi'(t_k)$. We note here that it follows from the appendix that, when all t_i are different, this is the density corresponding to a non singular joint distribution since $F(\lambda)$ is assumed to have an absolutely continuous component.

Before proceeding to the proof we note that the theorem can easily be modified to refer to "downcrossings" or the total number of crossings of the level u in time T. The discussion will be given here in terms of upcrossings, however.

The following proof is divided into two parts A and B. In Part A it is shown that M_k does not exceed the expression on the right hand side of (1), whereas in Part B the reverse inequality is proved. The techniques are straightforward but quite different in each part. It is a perhaps somewhat surprising feature, however, that in both parts use can be made of Fatou's Lemma to give the essential inversions of limiting operations with integrations, in order that inequalities in the desired (opposite) directions may be obtained.

Proof of the Theorem, Part A.

Write $\xi(t) = \xi(t,\omega)$ to exhibit explicit dependence on the "sample point" $\omega \in \Omega$. Let S denote the set of all ω such that the equation $\xi(t) = u$ has at most a finite number of roots t in the interval $I = [0,T]$, while further $\xi(0) \neq u \neq \xi(T)$ and $\xi'(t) \neq 0$ whenever $\xi(t) = u$. According to Bulinskaya [2], Theorem 1 we then have

$$P(S) = 1 \tag{2}$$

Write now $N^{(k)} = N(N-1)\ldots(N-k+1)$ for $k = 1,2\ldots$ and define the functions $\delta_n(x)$, $\sigma(x)$ by

$$\begin{aligned} \delta_n(x) &= n \qquad |x| \leq 1/(2n) \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} \sigma(x) &= x \qquad x > 0 \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Let $D(\epsilon)$ denote the domain in the k-dimensional space R^k with coordinates $t_1\ldots t_k$ defined by the inequalities

$$0 < t_i < T \qquad \text{for } i = 1 \ldots k$$

$$|t_i - t_j| > \epsilon \qquad \text{for } i \neq j .$$

Define also the random variable $J_k(n,\epsilon,\omega)$ by the relation

(3) $$J_k(n,\epsilon,\omega) = \int\!\!\ldots\!\!\int_{D(\epsilon)} \prod_{i=1}^{k} \{\delta_n[\xi(t_i)-u]\,\sigma[\xi'(t_i)]\}\, dt_1 \ldots dt_k \quad .$$

We shall now proceed to prove that

(4) $$\mathcal{E}\{N^{(k)}\} \leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \mathcal{E}\{J_k(n,\epsilon,\omega)\} \quad .$$

In order to prove the validity of (4) we define a subset $S(h)$ of S consisting of all $\omega \in S$ for which the following two conditions are satisfied

(a) The distance between any two upcrossings of $\xi(t)$ with the level u in I is greater than $2h$,

(b) For any zero $t = t_o$ of the derivative $\xi'(t)$ in I, we have $|\xi(t_o) - u| > h$.

According to the definition of S every $\omega \in S$ must also belong to $S(h)$ for some $h > 0$. For it is obvious that property (a) is satisfied for $\xi(t,\omega)$ if h is sufficiently small and if (b) were not satisfied we could find a sequence of points $t_i \in I$ for which

$$\xi'(t_i) = 0 \qquad |\xi(t_i) - u| \leq 1/i \ .$$

But such a sequence $\{t_i\}$ must have a limit point $t_o \in I$ and the continuity of ξ and ξ' show that $\xi'(t_o) = 0$, $\xi(t_o) = u$ contradicting the fact that $\omega \in S$. Hence we thus see that

(5) $$S(h) \uparrow S \qquad \text{as } h \downarrow 0$$

Take now any fixed $\omega \in S(h)$, and let $t = r_1 \ldots r_N$ be all the upcrossings of the corresponding $\xi(t) = \xi(t,\omega)$ in I. Consider the k-dimensional interval I^k in the space R^k, and let $A_{j_1,\ldots j_k}$ denote the point in I^k with coordinates

$$t_1 = r_{j_1}, \ldots t_k = r_{j_k} \quad ,$$

where each j_i may assume the values 1,2...N. Clearly there are N^k different points A, and among these there are exactly $N^{(k)}$ points A' such that no two of the j_i are equal. Since $\omega \in S(h)$, these points A' will all be situated in the domain D(2h), while the remaining $N^k - N^{(k)}$ points A will fall outside D(2h), and even outside $D(\epsilon)$, for any $\epsilon > 0$.

Considering, still the same fixed $\omega \in S(h)$ we now take n and ϵ such that

$$0 < n^{-1} < \epsilon < h$$

and consider the integral $J_k(n,\epsilon,\omega)$ defined by (3). The contribution to $J_k(n,\epsilon,\omega)$ arising from small disjoint k dimensional blocks about each point A' is easily seen to be just $N^{(k)}$ for all sufficiently large n (i.e. a unit contribution from each such block). The contribution from the remaining region is zero for all sufficiently large n. (This can be seen clearly from a picture by taking k=2 and writing down the integrals involved). Hence for any fixed $\epsilon < h$, we can always find n_o so large that, for all $n > n_o$ we have

$$J_k(n,\epsilon,\omega) = N^{(k)}$$

and hence also

$$N^{(k)} = \lim_{n \to \infty} J_k(n,\epsilon,\omega)$$

Since this holds for any $\epsilon < h$, while the first member is independent of ϵ, it follows that

(6) $$N^{(k)} = \lim_{\epsilon \to 0} \lim_{n \to \infty} J_k(n,\epsilon,\omega)$$

for any fixed $\omega \in S(h)$. But h can be chosed arbitrarily small and since $S(h) \uparrow S$ as $h \downarrow 0$ it follows that (6) holds for any $\omega \in S$, i.e. with probability one. Finally an application of Fatou's Lemma to the ϵ and n-limits yields the result (4). Thus from (4) we obtain

(7) $$\mathcal{E}\{N^{(k)}\} \leq$$

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int \cdots \int_{D(\epsilon)} dt_1 \ldots dt_k [n^k \int_{u-\frac{1}{2n}}^{u+\frac{1}{2n}} \cdots \int dx_1 \ldots dx_k \int_0^{\infty} \cdots \int y_1 \ldots y_k \; p_{\underline{t}}(x_1 \ldots x_k, y_1 \ldots y_k) dy_1 \ldots dy_k]$$

The entire expression in square brackets on the right hand side of (7) clearly converges to $\int_0^{\infty} \cdots \int y_1 \ldots y_k \; p_{\underline{t}}(u,\underline{y}) \, dy_1 \ldots dy_k$. Further, it can be readily shown that this expression is bounded for all $t_1 \ldots t_k$ in the region $D(\epsilon)$ (using the fact that the determinant of the covariance matrix of $\xi(t_1) \ldots \xi(t_k) \; \xi'(t_1) \ldots \xi'(t_k)$ is bounded away from zero). Hence by dominated convergence

(8) $$\mathcal{E}\{N^{(k)}\} \leq \lim_{\epsilon \to 0} \int \cdots \int_{D(\epsilon)} dt_1 \ldots dt_k \int_0^{\infty} \cdots \int y_1 \ldots y_k \; p_{\underline{t}}(u,\underline{y}) dy_1 \ldots dy_k$$

Finally by monotone convergence it follows that

(9) $$\mathcal{E}\{N^{(k)}\} \leq \int_0^T \cdots \int dt_1 \ldots dt_k \int_0^{\infty} \cdots \int y_1 \ldots y_k \; p_{\underline{t}}(u,\underline{y}) dy_1 \ldots dy_k \quad .$$

Part B.

In order to prove the reverse inequality to (9) we adopt a different procedure (due to Ylvisaker [8]) for counting the number of upcrossings by $\xi(t)$ in $0 \leq t \leq T$. First, however, we note that if X_i, $i = 1,2 \ldots$ are each either zero or one, and $M = \sum_{i=1}^{m} X_i$, then, for any integer $k \leq m$,

$$M^{(k)} = M(M-1)\ldots(M-k+1) = \Sigma' X_{i_1} \ldots X_{i_k} \quad (10)$$

where Σ' denotes summation over all possible ordered sets of distinct integers $i_1 \ldots i_k$. For M is just the number of non zero X_i, and the right hand side of (10) therefore represents the number of ordered sets of distinct integers $i_1 \ldots i_k$ such that each corresponding X_i is non zero, taken out of a total of M possible integers i for which $X_i \neq 0$. But this number is simply $M(M-1)\ldots(M-k+1)$ as required.

Write now $\xi_i = \xi(Ti/2^n)$, $i = 0, 1 \ldots 2^n$, $n = 1, 2 \ldots$. Let $X_i = 1$ if $\xi_i < u < \xi_{i+1}$, and $X_i = 0$ otherwise. Then if $N_n = \sum_{i=1}^{2^n} X_i$ we have $N_n \uparrow N$ a.s. . (A detailed proof of this latter statement is given by Ylvisaker [8]). Hence by monotone convergence,

$$\mathcal{E}\{N_n^{(k)}\} \longrightarrow \mathcal{E}\{N^{(k)}\} \text{ as } n \longrightarrow \infty . \quad (11)$$

Now from (10) we have with $m = 2^n$,

$$\mathcal{E}\{N_n^{(k)}\} = \sum'_{i_1 \ldots i_k} P\{X_{i_1} = X_{i_2} = \ldots = X_{i_k} = 1\} \quad (12)$$

We note that no terms for which $|i_r - i_s| = 1$ for any r,s appear since we cannot have $X_{i_r} = X_{i_r+1} = 1$. Write $\eta_i = 2^n(\xi_{i+1} - \xi_i)/T$. Then

$$P\{X_{i_1} = X_{i_2} \ldots = X_{i_k} = 1\} = P\{u - 2^{-n}T\eta_{i_r} < \xi_{i_r} < u,\ r = 1 \ldots k\}$$

$$= \int_0^\infty \ldots \int dy_1 \ldots dy_k \int_{u-2^{-n}Ty_1}^{u} \ldots \int_{u-2^{-n}Ty_k}^{u} p_{n,\underline{i}_j}(x_1 \ldots x_k, y_1 \ldots y_k)\, dx_1 \ldots dx_k,$$

where $p_{n,\underline{i}_j}$ is the joint density for the distribution of $\xi_{i_1} \ldots \xi_{i_k}, \eta_{i_1} \ldots \eta_{i_k}$.

(That this distribution is non singular follows from the Appendix.) By a change of the x-variables in this expression we thus obtain from (12)

$$(13)\quad \mathcal{E}\{N_n^{(k)}\} = 2^{-kn}T^k\Sigma'\int_0^\infty\!\!\dots\int dy_1\dots dy_k \int_{-y_1}^{0}\!\!\dots\int_{-y_k}^{0} p_{n,\underline{i}_j}(u+2^{-n}Tx_1,\dots u+2^{-n}Tx_k,y_1,\dots y_k)\,dx_1\dots dx_k$$

Write now $\Psi_{n\,\underline{t}}(x_1\dots x_k,y_1\dots y_k) = p_{n,\underline{i}_j}(x_1\dots x_k,y_1\dots y_k)$ for all $t_1\dots t_k$ such that t_r lies in the interval $(i_rT/2^n,\ (i_r+1)T/2^n)$ for each r. Then (13) may be rewritten as

$$(14)\quad \int_{R_o}\!\!\dots\int dt_1\dots dt_k \int_0^\infty\!\!\dots\int dy_1\dots dy_k \int_{-y_1}^{0}\!\!\dots\int_{-y_k}^{0} \Psi_{n\underline{t}}(u+2^{-n}Tx_1,\dots u+2^{-n}Tx_k,y_1\dots y_k)dx_1\dots dx_k$$

where R_o is the subset of I^k for which no two of $t_1\dots t_k$ are contained in the same or adjacent intervals of the form $(r\,T/2^n,(r+1)\,T/2^n)$. (See the remark following Equation (12)).

Let now $(t_1\dots t_k)$ be a fixed point in R_o. Then $\Psi_{n\underline{t}}$ is a 2k-dimensional normal density function. Suppose that $i_rT/2^n \le t_r < (i_r+1)T/2^n$, $r=1\dots k$. Then corresponding to the point t_r we have the random variables $\xi(i_rT/2^n)$, $\eta(i_rT/2^n)$, yielding the following typical members of the covariance matrix for $\Psi_{n\underline{t}}$ for example

$\text{var}\,(\xi_{i_1}) = r(0)$, writing ξ_i for $\xi(iT/2^n)$

$\text{cov}\,(\xi_{i_1},\xi_{i_2}) = r_p$, writing r_i for $r(iT/2^n)$, $p = i_1-i_2$

$\text{cov}\,(\xi_{i_1},\eta_{i_1}) = 2^n(1-r_1)/T$

$\text{cov}\,(\xi_{i_1},\ \eta_{i_2}) = 2^n(r_{p+1} - r_p)/T$

$\text{var}\,\eta_{i_1} = 2^{2n+1}(1-r_1)/T^2$

$\text{cov}\,(\eta_{i_1},\eta_{i_2}) = -2^{2n}[r_{p+1} - 2r_p + r_{p-1}]/T^2$

For the fixed t_1, t_2 considered i_1, i_2, p depend on n. It is an easy exercise to show that if $\tau = t_2 - t_1$ the above elements converge (in the order given), as $n \longrightarrow \infty$, to

$r(0)$, $r(\tau)$, 0, $r'(\tau)$, $-r''(0)$, $-r''(\tau)$, respectively. Similar conclusions hold for the elements corresponding to any pair t_i, t_j. But this means that the integrand in (14) must converge to $p_{\underline{t}}(u,\underline{y})$ as $n \longrightarrow \infty$ and hence, by Fatou's Lemma

$$\mathcal{E}\{N^{(k)}\} \geq \int_0^T \ldots \int dt_1 \ldots dt_k \int \ldots \int p_{\underline{t}}(u,\underline{y}) dy_1 \ldots dy_k \tag{15}$$

Combining (9) and (15) we obtain the desired equality and hence the truth of the theorem follows.

3. A case when $M_k = +\infty$. Formula 1 was obtained under the condition that $\xi(t)$ have a continuous sample derivative, with probability one. However, this assumption was used in Part A of the proof, but not at all in Part B. Hence if the right hand side of (1) is infinite, the equation is true with both sides infinite. We now give an example of a case where the integral on the right of (1) is infinite, and hence the corresponding moment is infinite.

For this example we take a covariance function of the form

$$r(\tau) = 1 - \lambda_2 \tau^2/2 - \tau^2/\log|\tau| + o(\tau^2/\log|\tau|). \tag{16}$$

That this can be done follows from a result of Pitman [5]. In fact if $H(\lambda) = 1 - F(\lambda) + F(-\lambda)$ for $\lambda > 0$ we can choose $H(\lambda)$ so that

$$H(\lambda) \sim K/(\lambda^2 \log^2 \lambda) \qquad \text{as } \lambda \longrightarrow \infty$$

to give the desired form (16).

Consider now the case $k = 2$, and $u = 0$. Then one can show by some calculation that

$$\iint_0^\infty y_1 y_2 \, p_{\underline{t}}(o,\underline{y}) \, dy_1 dy_2 \sim K|\Lambda|^{\frac{1}{2}}/(1-r^2(\tau))$$

where K again denotes some constant and Λ is the covariance matrix for $\xi(t_1)$, $\xi(t_2)$, $\xi'(t_1)$, $\xi'(t_2)$, $\tau = t_2 - t_1$. But straightforward calculation shows that

$$|\Lambda| \sim \lambda_2 \, \tau^2/\log^2|\tau| \qquad \text{as } \tau \longrightarrow 0$$

and hence

$$\iint_0^{\infty} y_1 y_2 \, p_{\underline{t}}(o,\underline{y}) \, dy_1 dy_2 \sim K/(|\tau| \log |\tau|) \text{ as } \tau \longrightarrow 0.$$

It follows from this that the right hand side of (1) is infinite, in this case.

Finally we note a sufficient condition for $\xi(t)$ to possess a continuous sample derivative, with probability one, is that

$$r(\tau) = 1 - \lambda_2\tau^2/2 + 0\{\tau^2/|\log|\tau||^a\}$$

for some $a > 1$. This follows from the work of Belaev [1]. In our case $r(\tau)$ given by (16) just fails to satisfy this requirement. Hence it appears that the requirements that ξ have a continuous derivative and that the right hand side of (1) be finite, which are <u>sufficient</u> for M_k to be finite and given by (1), are also very close to being <u>necessary</u> for this to be the case.

Appendix.

It was stated, in writing down certain density functions that if $t_1 \ldots t_k$ are distinct time points, then

(i) the joint distribution of $\xi(t_1)\ldots\xi(t_k)$ $\xi'(t_1)\ldots\xi'(t_k)$ is non singular, and

(ii) the joint distribution of $\xi(t_1)\ldots\xi(t_k)$ is non singular.

We shall now prove (i), and hence (ii) will also follow.

Let (as assumed throughout) $F(\lambda)$ have an absolutely continuous component and write $\Lambda = [\Lambda_{ij}]$ for the covariance matrix of $\xi(t_1)\ldots\xi(t_k)$, $\xi'(t_1)\ldots\xi'(t_k)$. Let $A = [A_{ij}]$ denote the covariance matrix of $\xi(t_1)\ldots\xi(t_k)$ $B = [B_{ij}]$ that for $\xi'(t_1)\ldots\xi'(t_k)$, and C the matrix of "cross" covariances, $C_{ij} = \mathrm{cov}(\xi(t_i), \xi'(t_j))$. Then

$$\Lambda = \begin{bmatrix} A & C \\ C' & B \end{bmatrix} .$$

Let $\underline{\theta}'$ denote the vector $[\theta_1 \ldots \theta_k, \phi_1 \ldots \phi_k]$, where θ_i, ϕ_i are complex numbers which are not all zero. Then we have

$$A_{j\ell} = \int e^{i(t_j - t_\ell)\lambda} dF(\lambda)$$

$$C_{j\ell} = -\int i\lambda\, e^{i(t_j - t_\ell)\lambda} dF(\lambda)$$

$$B_{j\ell} = \int \lambda^2 e^{i(t_j - t_\ell)\lambda} dF(\lambda)$$

From this we see that

$$\underline{\theta}'\Lambda\underline{\theta} = \int [|\sum_j \theta_j e^{it_j\lambda}|^2 + \lambda^2 |\sum_j \phi_j e^{it_j\lambda}|^2 - i\lambda \sum_j \theta_j e^{i\lambda t_j} \sum_\ell \phi_\ell^* e^{-i\lambda t_\ell} + i\lambda \sum_j \theta_j^* e^{-i\lambda t_j} \sum_\ell \phi_\ell e^{i\lambda t_\ell}] dF(\lambda),$$

in which a * denotes complex conjugate. Thus

$$\underline{\theta}'\Lambda\underline{\theta} = \int | \sum_{j=1}^{k} \theta_j\, e^{it_j\lambda} + i\lambda \sum_{j=1}^{k} \phi_j\, e^{it_j\lambda} |^2 dF(\lambda)$$

Now since the t_j are distinct and θ_j, Φ_j not all zero it follows that $\sum_{j=1}^{k} \theta_j e^{it_j\lambda}$ and $-i\lambda \sum_{j=1}^{k} \Phi_j e^{it_j\lambda}$ are different regular functions of the complex variable λ, and hence cannot be equal for more than a countable number of values of λ. Hence we must have $\underline{\theta}'\Lambda\underline{\theta} > 0$ since $F(\lambda)$ has an absolutely continuous component and the measure it defines is not concentrated on a countable set. Thus Λ is a strictly positive definite matrix and the distribution thus defined is non singular.

Finally we note here that the above argument can be easily generalized to include an arbitrary number of derivatives. That is if $F(\lambda)$ has an absolutely continuous component and is such that $\xi(t)$ has n sample derivatives $\xi(t)$ $\xi'(t)\ldots\xi^{(n)}(t)$, then for any distinct $t_1, t_2, \ldots t_k$, the joint distribution of

$\xi(t_1)\ldots\xi(t_k)\ldots\ldots\xi^{(n)}(t_1)\ldots\xi^{(n)}(t_k)$ is non singular.

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13 ABSTRACT

In this report the number N of upcrossings of a level u by a stationary normal process $\xi(t)$ in $0 \leq t \leq T$, is considered. A formula is obtained for the factorial moment $M_k = \mathcal{E}\{N(N-1)\ldots(N-k+1)\}$ of any order k. The main condition assumed is that $\xi(t)$ possess, with probability one, a continuous sample derivative $\xi'(t)$. An example is given to show that this is very nearly a necessary condition for finiteness of moments of order greater than the first.

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