ON PROVING THEOREMS IN PLANE GEOMETRY VIA DIGITAL COMPUTER

Richard Bellman

February 1965

P-3068

ON PROVING THEOREMS IN PLANE GEOMETRY
VIA DIGITAL COMPUTER

Richard Bellman*
The RAND Corporation, Santa Monica, California

1. INTRODUCTION

The development of the digital computer has focused attention upon algorithms, logical processes, and decision-making. An offshoot of this has been the set of attempts by various people with varying degrees of success to replicate human thought processes with the aid of a computer. In this connection, let us cite the work in pattern recognition, chess- and checker-playing, and the proving of geometric theorems.

In pursuing these goals, there are two different approaches that can be followed. We can, first of all, imitate what the human mind does. Or, we can accomplish the same task in an entirely different fashion. Since it is generally agreed by knowledgeable people that we possess very little understanding of the workings of the brain, it is clearly rather hazardous to follow the first route. We shall traverse the second path.

We wish to indicate how geometric theorems can first be transformed into algebraic identities (the contribution of Descartes), and then established by the verification of a finite number of cases. The verification will be arithmetic, using a computer.

---

*Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of The RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors. Papers are reproduced by The RAND Corporation as a courtesy to members of its staff.

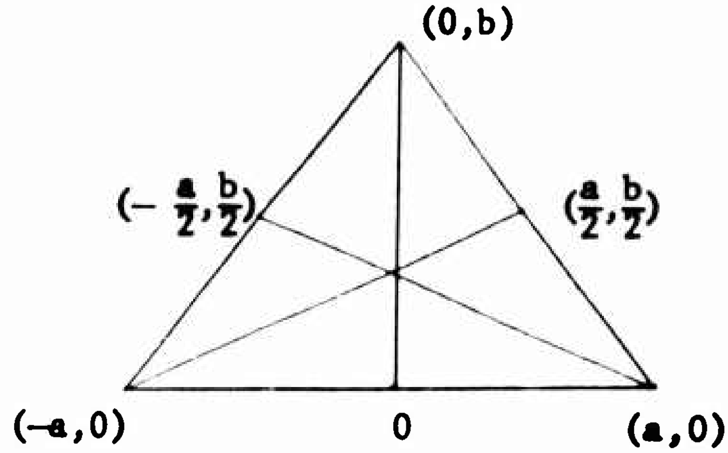
This paper will be submitted for publication in the American Mathematical Monthly.

2. THE MEDIANS OF AN ISOSCELES TRIANGLE

Suppose that we wish to prove that the medians to the equal sides of an isosceles triangle are equal. Considering the figure below, we must prove that

$$(2.1) \qquad [(a - (-\tfrac{a}{2}))^2 + (0 - \tfrac{b}{2})^2]^{1/2}$$
$$= [(\tfrac{a}{2} - (-a))^2 + (\tfrac{b}{2} - 0)^2]^{1/2}.$$

This is obvious upon inspection, but with a digital computer, we are not allowed "inspection."



As far as the computer is concerned, an isosceles triangle is determined by the three vertices (0,b), (a,0), (-a,0), and the lengths of the medians are determined by simple algorithms which provide first the midpoints and then the length according to the distance formula.

We establish the equality of the two sides of (2.1) in a two-step process. We first invoke some general algebraic theorems which assure us that it is sufficient to verify equality for a finite set of values of a and b, and then use the computer to carry out this arithmetic confrontation.

There are several ways tc proceed. Let us sketch one. Since the expressions are homogeneous in a and b (scale is unimportant), it suffices to take $b = 1$. Since equality of the squares implies equality of positive quantities, let us square both sides. It remains to establish the identity of two quadratic polynomials in a. For this, equality at <u>three</u> values of a suffices. Choose three convenient values, e.g., $a = 2,4,6$.

At this point, the reader may justifiably worry about round-off error. After all, computer arithmetic is not ordinary arithmetic. Suppose we had not thought of the artifice of squaring, or, in general, of rationalizing. How would we establish that $M = N$ by comparing the calculated values of $\sqrt{M}$ and $\sqrt{N}$? Does agreement of $\sqrt{M}$ and $\sqrt{N}$ to a sufficiently large number of decimal places assure us that they are equal? The answer is "yes."

Observe that

$$\sqrt{M} - \sqrt{N} = \frac{M - N}{\sqrt{M} + \sqrt{N}}. \tag{2.2}$$

Hence, if M and N are <u>integers</u> and distinct, we must have

$$|\sqrt{M} - \sqrt{N}| \geq \frac{1}{\sqrt{M} + \sqrt{N}}. \tag{2.3}$$

If arithmetic calculations show that

$$|\sqrt{M} - \sqrt{N}| \leq \epsilon < \frac{1}{\sqrt{M} + \sqrt{N}}, \tag{2.4}$$

we can conclude that $M = N$. Starting with the values of M and N, we know how to obtain the accuracy of (2.4).

3. DISCUSSION

We leave it to the reader to investigate the possibility of establishing the existence of the Euler line, the Simpson line, the nine-point circle, and so on. It is clear that we can "generate" theorems of this type in a completely uninspired fashion by tabulating sets of points and lines and testing colinearity, coincidence, etc. Some of this could be pursued in an adaptive fashion, as, for example, the search for the nine-point circle. None of this has any intrinsic interest.

We do feel that experimentation of this type will be very useful to the student learning about the computer and its capabilities. It will force him to study algorithms carefully, and there are many interesting arithmetic and analytic questions associated with the verification process.