FDL-TDR-64-133

THEORETICAL STUDIES OF METHODS OF GENERATING RANDOM WAVES FROM HARMONIC SOURCES

TECHNICAL DOCUMENTARY REPORT No. FDL-TDR-64-133

DECEMBER 1964

AIR FORCE FLIGHT DYNAMICS LABORATORY
RESEARCH AND TECHNOLOGY DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

Project No. 4437, Task No. 443705

FOREWORD

The work presented in this report was performed by Dr. J. C. Samuels of the General Technology Corporation, Elgin Illinois, for the Aero-Acoustics Branch, Vehicle Dynamics Division, AF Flight Dynamics Laboratory, Wright-Patterson Air Force Base under Contract Nr. AF 33(657)-10945. This research is part of an effort to simulate complex acoustic waves with an array of harmonic sources. This work was performed under Project Nr. 4437 "High Intensity Sound Environment Simulation" Task Nr. 443705 "Development of Noise Sources." Mr. O. F. Maurer of the Aero-Acoustics Branch, AF Flight Dynamics Laboratory was project engineer. The work was undertaken from May 1963 through June 1964. The group supervisors were Dr. A. C. Eringen and Dr. J. C. Samuels. Contractors report number is General Technology Technical Report Nr. 4-2.

This report has been reviewed and is approved.

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ABSTRACT

This **report** is concerned with the approximation of acoustical noise fields by the use of modulated pure tone sources. Given a specific random signal the methods of orthogonal expansion and Fourier expansion are studied. The integral equation arising from the orthogonal expansion is solved for the normal functions when: a) the power spectrum is given as the ratio of two polynomials, b) Markoff Autocorrelation function, c) stationary band limited Gaussian white noise of mean zero, d) band limited stationary normal white noise on a finite interval, e) white noise. Deterministic and random frequency and amplitude modulation and random switching are examined as ways to broaden spectral components.

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I. Introduction

Because of the many applications, it is found useful to represent a random process in a suitable series of particular functions over some finite or infinite interval. Considerable study has been devoted to the problem of expansions (Ref. 1,2,3). Orthogonal expansions have received particular attention (Ref. 2).

The present report deals with some aspects of the expansion of random functions into orthogonal series. Consideration is also given to how to broaden the spectrum of a single-tone signal by modulation of various sorts. Some attention is also given to optimal approximations of various statistical characteristics of actual random noise fields.

II.   Representation of Acoustic Noise Fields

a)   Orthogonal Expansion of Random Functions

It is desired to represent a randomly varying sound pressure $p(t)$ (see Figure 1) as a series of orthogonal functions with orthogonal random variables as coefficients.
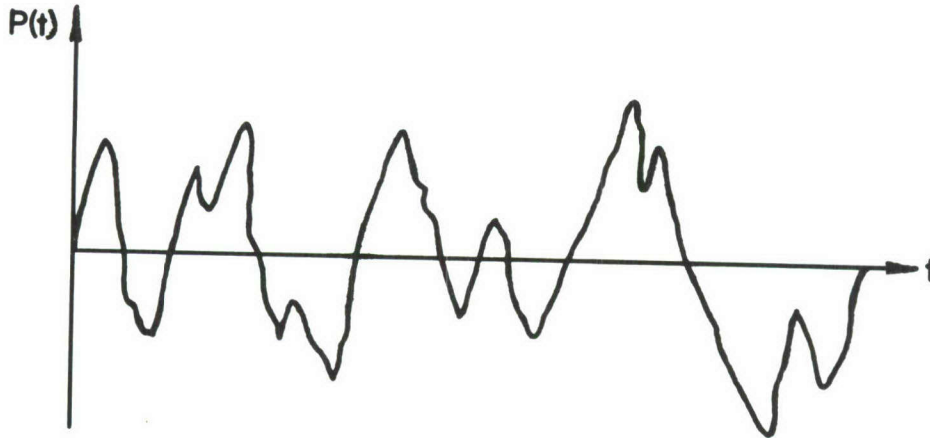
Figure 1.   Typical Recording of Random Sound Field

In particular, we require

$$p(t) = \sum_n |\lambda_n| \, \psi_n(t) \, X_n \tag{1}$$

$$E\{X_n X_k\} = \delta_{nk} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

$$\int_0^T \psi_n(t)\psi_k^*(t)dt = \delta_{nk} \tag{2}$$

To achieve these results, consider

$$p(t)\, p^*(t')\psi_n(t') = \sum_{s,k} |\lambda_s| \, |\lambda_k| \, \psi_s(t)\psi_k^*(t')\psi_n(t') \, X_s X_k$$

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Integrating both sides of this result from $t = 0$ to T and using the second of (2), we find

$$\int_0^T p(t) p^*(t') \psi_n(t') dt' = \sum_{s,k} |\lambda_s| \, |\lambda_k| \delta_{nk} \psi_s(t) \, X_s X_k$$

$$= \sum_s |\lambda_s| \, |\lambda_n| \psi_s(t) \, X_s X_n$$

Taking expection and using the first of (2) gives

$$\int_0^T R(t,t') \psi_n(t') dt' = |\lambda_n|^2 \, \psi_n(t)$$

$$0 \leq t \leq T \qquad (3)$$

$$R(t,t') = E\{p(t) \, p(t')\}$$

Thus we can have such an expansion as (2) provided $\psi_n(t)$ are solutions of (3) and λ_n is the eigenvalue associated with $\psi_n(t)$.

From (1) and (2) it follows that

$$X_n = \frac{1}{|\lambda_n|} \int_0^T \psi_n(t) \, p(t) dt \qquad (4)$$

Therefore if we knew the eigenfunctions $\psi_n(t)$, and the eigenvalues λ_n, and possessed a realized value of $p(t)$, we could calculate the realized values of the orthogonal random variables X_n. All that is needed to find λ_n and $\psi_n(t)$ is the autocorrelation function $R(t,t')$. This is either given or is determined from an actual recording of $p(t)$.

If $p(t)$ is a gaussian random process, the random variables X_k will not only be orthogonal but will be independent random variables with a normal probability distribution. The problem of determining the probability distribution function for $p(t)$ when those of X_k are given is an extremely difficult one and has been solved only in the case of gaussian variates. It is to be noted from (4) that if $p(t)$ is an independent process, X_n will be a normal variate. One can easily prove this by application of the central limit theorem of probability theory.

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b) <u>Fourier Expansion of Random Functions</u>

Because of the difficulty of solving the integral equation (3) and because of simplicity, random functions are expanded into Fourier series in most practical situations. Unfortunately, the random coefficients are not orthogonal on $(0,T)$ except for periodic processes. We can show, however, that the coefficients are asymptotically orthogonal as the length of the recording increases.

Consider a stationary random function $p(t)$ defined on the interval $(0,T)$. The Fourier expansion of this process on $(0,T)$ is

$$p(t) = \sum_{k=0}^{\infty} \{a_k \cos \omega_k t + b_k \sin \omega_k t\}, \quad \omega_k = k\omega_o \qquad (5)$$

where

$$a_o = \frac{1}{T} \int_0^T p(t)dt, \quad T = \frac{2\pi}{\omega_o}$$

$$a_k = \frac{2}{T} \int_0^T p(t) \cos k\omega_o t \, dt, k=1,2,\dots \qquad (6)$$

$$b_k = \frac{2}{T} \int_0^T p(t) \sin k\omega_o t \, dt, k=1,2,\dots$$

Now

$$E\{a_k a_\ell\} = \frac{4}{T^2} E\{\int_0^T \int_0^T p(t)p(t') \cos k\omega_o t \cos \ell\omega_o t' dt dt'\}$$

$$= \frac{4}{T^2} \int_0^T \int_0^T R(t-t') \cos k\omega_o t \cos \ell\omega_o t' dt dt'$$

$$= \frac{4}{T^2} \int_0^T \int_0^T R(\tau) \cos k\omega_o t \cos \ell\omega_o t' dt dt'$$

$$\tau = t-t'$$

Changing variables the double integral becomes

$$E\{a_k a_\ell\} = \frac{4}{T^2} \int_0^T \cos k\omega_o t dt \int_{-t}^{T-t} R(\tau) \cos \ell\omega_o (t+\tau) dt$$

Let $\lambda = t/T$, then

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$$TE\{a_k a_\ell\} = 4 \int_0^1 \cos 2\pi k\lambda d\lambda \int_{-\lambda T}^{T(1-\lambda)} R(\tau) \cos 2\pi\ell\left(\frac{\tau}{T} + \lambda\right) d\tau$$

$$= 4 \int_0^1 \cos 2\pi k\lambda d\lambda \{\cos 2\pi\ell\lambda \int_{-\lambda T}^{T(1-\lambda)} R(\tau) \cos \frac{2\pi\ell\tau}{T} d\tau$$

$$- \sin 2\pi\ell\lambda \int_{-\lambda T}^{T(1-\lambda)} R(\tau) \sin \frac{2\pi\ell\tau}{T} d\tau\}$$

As $T \to \infty$, $\lambda \neq 0$

$$TE\{a_k a_\ell\} \to 4 \int_0^1 \cos 2\pi k\lambda d\lambda \{\cos 2\pi\ell\lambda \cdot \tfrac{1}{2} S(\omega_\ell) +$$

$$+ \sin 2\pi\ell\lambda \cdot (0) \}$$

(7)

$S(\omega_\ell)$ = Spectral density of $p(t)$ and $\omega_\ell = \frac{2\pi\ell}{T}$. $\omega_\ell \to 0$ as $T \to \infty$ unless ℓ is chosen such that $\lim_{\ell, T \to \infty} \omega_\ell \to \omega > 0$.

(7) may be evaluated to give

$$TE\{a_k a_\ell\} \to S(\omega_\ell) \delta_{\ell k}$$

(8)

The normalized $\tilde{a}_k = a_k / \sqrt{\langle a_k^2 \rangle}$, $\tilde{a}_\ell = a_\ell / \sqrt{\langle a_\ell^2 \rangle}$

obey

$$E\{\tilde{a}_k \tilde{a}_\ell\} \xrightarrow[T \to \infty]{} \delta_{\ell k} , \quad k, \ell \geq 1$$

(9)

In like manner one can show that

$$E\{\tilde{a}_k \tilde{b}_\ell\} \to 0$$

$$E\{\tilde{b}_k \tilde{b}_\ell\} \to \delta_{k\ell}$$

$$\quad k, \ell \geq 1 \qquad (10)$$

$$E\{\tilde{a}_0 \tilde{a}_\ell\} \to \delta_{0\ell}$$

$$E\{\tilde{b}_0 \tilde{b}_\ell\} \to \delta_{0\ell}$$

Thus the expansion coefficients are asymptotically orthogonal.

It is of value to compare various Fourier expansions for different lengths of T of recordings of p(t) with the exact orthogonal expansion on (0,T). This would give some idea of the length of record to be taken so that a Fourier expansion would be as good as the exact one.

III. Solution of the Integral Equation Arising in the Orthogonal Expansion of a Random Function.

    a) General Results for Processes with Rational Spectral Density Functions

We confine ourselves to stationary random sound fields whose spectral density functions are rational functions of frequency. In this case the integral equation becomes

$$\int_0^T R(t-t') \, \psi_n(t') dt' = |\lambda_n|^2 \psi_n(t), \quad 0 \le t \le T \tag{11}$$

By the Wiener-Khintchine theorem, we know that

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \, e^{i\tau\omega} d\omega$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) \, e^{-i\tau\omega} d\tau \tag{12}$$

$S(\omega)$ is the power spectral density of the random sound pressure p(t). We now suppose that $S(\omega)$ is a rational function of ω so that it can be written as the ratio of two polynomials in ω, namely,

$$S(\omega) = \frac{N\left((i\omega)^2\right)}{D\left((i\omega)^2\right)} \tag{13}$$

D is at least one degree greater than N. The argument of the N and D polynomials was chosen as a square of ω because of the general requirements placed on $S(\omega)$ as a power spectral density function. We wish to show that when the process has such a spectral density as (13), the functions $\psi_n(t)$ satisfy a linear constant coefficient differential equation.

We start by inserting (13) into (3) to obtain

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$$\frac{1}{2\pi}\int_0^T \int_{-\infty}^{\infty} \frac{N((i\omega)^2)}{D((i\omega)^2)} e^{i(t-t')\omega} \psi_n(t') d\omega dt' = |\lambda_n|^2 \psi_n(t) \tag{14}$$

Let

$$N((i\omega)^2) = \sum_{k=0}^{q/2} b_k \cdot (i\omega)^{2k}$$

$$D((i\omega)^2) = \sum_{k=0}^{p/2} a_k \cdot (i\omega)^{2k} \qquad p > q \tag{15}$$

a_k and b_k are constants. Now operate upon both sides of (14) with the operator

$$D\{\frac{d^2}{dt^2}\} = \sum_{k=0}^{p/2} a_k \frac{d^{2k}}{dt^{2k}} \quad \text{to give}$$

$$\frac{1}{2\pi}\int_0^T \int_{-\infty}^{\infty} \frac{N((i\omega)^2)}{D((i\omega)^2)} D((i\omega)^2) e^{i(t-t')\omega} \psi_n(t') dt' d\omega \tag{16}$$

$$= |\lambda_n|^2 D\{\frac{d^2}{dt^2}\} \psi_n(t)$$

Multiplying by $N((i\omega)^2)$ under the integral $\int_{-\infty}^{\infty}$ is equivalent to operating on the integral with $N\{\frac{d^2}{dt^2}\}$. Thus (16) can be written as

$$\frac{1}{2\pi}\int_0^T [N\{\frac{d^2}{dt^2}\} \int_{-\infty}^{\infty} e^{-i(t-t')\omega} d\omega] \psi_n(t') dt'$$

$$= |\lambda_n|^2 D\{\frac{d^2}{dt^2}\} \psi_n(t) \tag{17}$$

But

$$\int_{-\infty}^{\infty} e^{i(t-t')\omega} d\omega = 2\pi \; \delta(t-t') \tag{18}$$

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where $\delta(t-t')$ is the Dirac delta function. Using (18) in (17), we find

$$N\{\frac{d^2}{dt^2}\} \int_0^T \delta(t-t') \psi_n(t')dt' = |\lambda_n|^2 D\{\frac{d^2}{dt^2}\} \psi_n(t)$$

or

$$N\{\frac{d^2}{dt^2}\} \psi_n(t) = |\lambda_n|^2 D\{\frac{d^2}{dt^2}\} \psi_n(t) \qquad (19)$$

This is the basic differential equation that must be satisfied by the expansion functions $\psi_n(t)$.

The solution of (19) will contain λ_n and p arbitrary constants A_{nk}. When this solution is substituted back in the integral equation (11), it will be seen that the integral equation cannot be satisfied except for certain values of λ_n, the characteristic values, and the constants A_{nk} must satisfy certain conditions. These conditions show that for each n there is only one independent constant. This one constant is to be determined from the normalizing conditions on $\psi_n(t)$, namely,

$$\int_0^T \psi_n(t) \psi_j^*(t)dt = \{ \begin{matrix} 1 & n = j \\ 0 & n \neq j \end{matrix} \qquad (20)$$

The procedure just outlined will now be carried out explicitly. The differential equation (19) becomes

$$\sum_{k=0}^{q/2} b_k \frac{d^{2k}\psi_n(t)}{dt^{2k}} = |\lambda_n|^2 \sum_{k=0}^{p/2} a_k \frac{d^{2k}\psi_n(t)}{dt^{2k}} \qquad (21)$$

Try a solution

$$\psi_n(t) = A_n e^{\Omega_n t} \qquad (22)$$

A_n is a constant and Ω_n is a parameter. Putting (22) in (21) we find that Ω_n must be a root of

$$\sum_{k=0}^{q/2} b_k \Omega_n^{2k} = |\lambda_n|^2 \sum_{k=0}^{p/2} a_k \Omega_n^{2k} \qquad (23)$$

This equation will have, in general (special situations can be taken care of), p roots $\pm \Omega_{nk} = \pm \Omega_{nk}(\lambda_n)$, $k = 1,..,p/2$. Half of the roots are simply the negatives of the other half. A general solution of (21) is then

$$\psi_n(t) = \sum_{k=1}^{p/2} \{A_{nk}^+ e^{\Omega_{nk}t} + A_{nk}^- e^{-\Omega_{nk}t}\} \qquad (24)$$

The autocorrelation function is given by

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N((i\omega)^2)}{D((i\omega)^2)} e^{i\tau\omega} d\omega \qquad (25)$$

This integral can be evaluated by contour integration using semicircular arcs and yields

$$R(\tau) = \begin{cases} \sum_{k=1}^{p/2} B_k e^{i\omega_k \tau} & \tau = t-t' > 0 \\ \sum_{k=1}^{p/2} B_k e^{-i\omega_k \tau} & \tau < 0 \end{cases} \qquad (26)$$

ω_k are the roots of

$$D((i\omega)^2) = 0 \qquad (27)$$

lying in the upper half of the complex plane as shown in Figure 2

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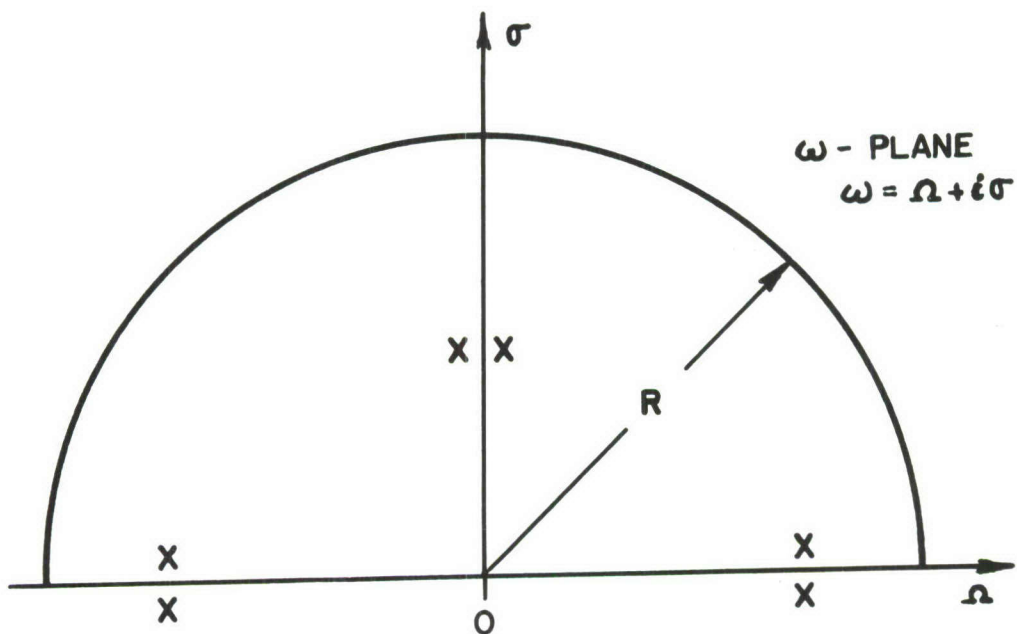
Figure 2. Location of the Roots of $D((i\omega)^2)$

There are p roots of (27) $\pm \omega_k$, $k = 1, \ldots, p/2$. The B_k are given by

$$B_k = \frac{N((i\omega_k)^2)}{\left.\dfrac{dD((i\omega)^2)}{d\omega}\right|_{\omega = \omega_k}} \tag{28}$$

Inserting (24) and (26) into (11), we obtain

$$\int_0^t \sum_{s=1}^{p/2} B_s\, e^{i\omega_s(t-t')} \sum_{k=1}^{p/2} \{A_{nk}^+\, e^{\Omega_{nk}t'} + A_{nk}^-\, e^{-\Omega_{nk}t'}\}\, dt' +$$

$$+ \int_t^T \sum_{s=1}^{p/2} B_s\, e^{-i\omega_s(t-t')} \sum_{k=1}^{p/2} \{A_{nk}^+\, e^{\Omega_{nk}t'} + A_{nk}^-\, e^{-\Omega_{nk}t'}\}\, dt'$$

$$= |\lambda_n|^2 \sum_{k=1}^{p/2} \{A_{nk}^+\, e^{\Omega_{nk}t} + A_{nk}^-\, e^{-\Omega_{nk}t}\}$$

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Carrying out the integrations we obtain

$$\sum_s \sum_k B_s A_{nk}^+ \frac{e^{\Omega_{nk}t}}{\Omega_{nk}-i\omega_s} + \sum_s \sum_k B_s A_{nk}^- \frac{e^{-\Omega_{nk}t}}{-\Omega_{nk}-i\omega_s}$$

$$-\sum_s \sum_k B_s A_{nk}^+ \frac{e^{\Omega_{nk}t}}{\Omega_{nk}+i\omega_s} - \sum_s \sum_k B_s A_{nk}^- \frac{e^{-\Omega_{nk}t}}{-\Omega_{nk}+i\omega_s}$$

$$-\sum_s \sum_k B_s A_{nk}^+ \frac{e^{i\omega_s t}}{\Omega_{nk}-i\omega_s} - \sum_s \sum_k B_s A_{nk}^- \frac{e^{i\omega_s t}}{-\Omega_{nk}-i\omega_s}$$

$$+\sum_s \sum_k B_s A_{nk}^+ \frac{e^{(\Omega_{nk}+i\omega_s)T}}{\Omega_{nk}+i\omega_s} e^{-i\omega_s t}$$

$$+\sum_s \sum_k B_s A_{nk}^- \frac{e^{(-\Omega_{nk}+i\omega_s)T}}{-\Omega_{nk}+i\omega_s} e^{-i\omega_s t} =$$

$$|\lambda_n|^2 \sum_k \{A_{nk}^+ e^{\Omega_{nk}t} + A_{nk}^- e^{-\Omega_{nk}t}\}.$$

This equation can be satisfied identically in t only if the sum of the coefficients of the different functions of time are equated to zero. This gives the system

$$(a) \sum_{k=1}^{p/2} B_s \left\{ \frac{A_{nk}^+}{\Omega_{nk}-i\omega_s} + \frac{A_{nk}^-}{-\Omega_{nk}-i\omega_s} \right\} = 0, \quad s = 1,\ldots,p/2$$

$$(b) \sum_{k=1}^{p/2} B_s \left\{ \frac{A_{nk}^+ e^{(\Omega_{nk}+i\omega_s)T}}{\Omega_{nk}+i\omega_s} + \frac{A_{nk}^- e^{(-\Omega_{nk}+i\omega_s)T}}{-\Omega_{nk}+i\omega_s} \right\} = 0$$

$$s = 1,\ldots,p/2 \qquad (29)$$

$$(c) \sum_{s=1}^{p/2} A_{nk}^+ \left\{ \frac{1}{\Omega_{nk}-i\omega_s} - \frac{1}{\Omega_{nk}+i\omega_s} \right\} B_s = |\lambda_n|^2 A_{nk}^+,$$

$$k = 1,\ldots,p/2$$

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$$\sum_{s=0}^{p/2} A_{nk}^{-} \left\{ \frac{1}{\Omega_{nk} - i\omega_s} - \frac{1}{-\Omega_{nk} + i\omega_s} \right\} B_s = |\lambda_n|^2 A_{nk}^{-}, \quad k = 1, \ldots, p/2$$

Equations (29a) and (29b) are a system of p linear homogeneous equations for the determination of the A_{nk}^{+} and A_{nk}^{-}. For a non-trivial solution, the determinant of the coefficients must vanish, namely,

$$\begin{vmatrix} \frac{1}{\Omega_{n1} - i\omega_1} & \frac{1}{\Omega_{n1} - i\omega} & \cdots & \frac{-1}{\Omega_{n\ell} + i\omega_1} & \frac{-1}{\Omega_{n\ell} + i\omega_1} & \cdots \\ \frac{1}{\Omega_{n1} - i\omega_2} & \cdots & & \frac{-1}{\Omega_{n\ell} + i\omega_1} & \cdots & \\ \vdots & & & & & \\ \frac{1}{\Omega_{n1} - i\omega_{p/2}} & \cdot & \cdot & & \cdot & \\ \frac{e^{(\Omega_{no} + i\omega_o)T}}{\Omega_{n1} + i\omega_1} & \frac{e^{(\Omega_{n1} + i\omega_o)T}}{\Omega_{n1} + i\omega_1} & \cdots & \frac{e^{(-\Omega_{no} + i\omega_o)T}}{-\Omega_{n\ell} + i\omega_1} & \cdots & \end{vmatrix} = 0 \tag{30}$$

From this equation we determine the eigenvalues λ_n, $n = 1, 2, \ldots$ This requires that we know the functions

$$\Omega_{nk} = f_k(\lambda_n) \tag{31}$$

These functions $f_k(\lambda_n)$ are determined by solving for the roots of (23) in terms of λ_n. We can do this analytically in terms of algebraic functions for equations of fourth or lower degree. This would mean that problems could in principle be handled for $D((i\omega)^2)$ of eighth or lower degrees.

Equations (29c) and (29d) will be found to be identically satisfied. All constants A_{nk}^+, $k = 1, \ldots p/2-1$, A_{nk}^-, $k = 1, \ldots$, $p/2$ can be solved for in terms of $A_{np/2}^+$ using equations (29a) and (29b). $A_{np/2}^+$ is determined from the normalizing condition

$$\int_0^T \psi_n(t)\psi_n^*(t)dt = 1 \tag{32}$$

The case where the power spectral density is a rational function of frequency is a very important one and covers a wide range of practical cases.

b) <u>Orthogonal Expansion of Processes with Markoff Autocorrelation Functions</u>

Suppose that the autocorrelation function of a stationary random acoustic noise process $p(t)$ of mean zero is given by

$$E\{p(t)\,p(t+\tau)\} = R(\tau) = A\,e^{-\beta|\tau|}$$

$$\tag{33}$$

$$A = E\{p^2(t)\} = R(o), \quad \beta = \text{constant}$$

This function is shown in Figure 3.
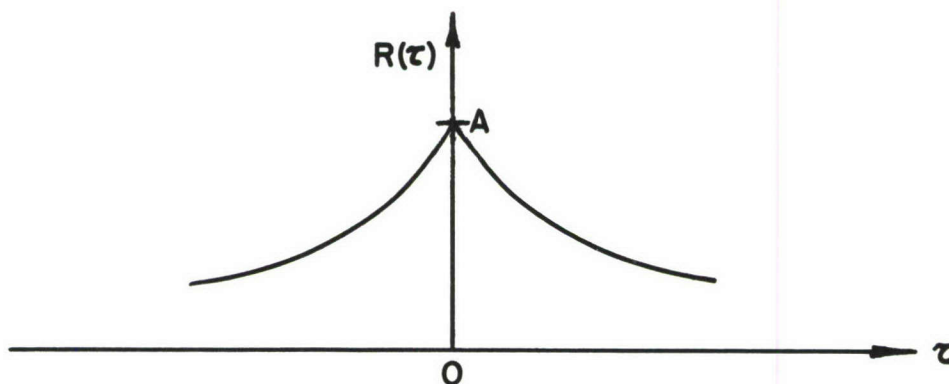
Figure 3. Graph of the Autocorrelation Function $A e^{-\beta|\tau|}$

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The spectral density corresponding to (33) is

$$S(\omega) = \int_{-\infty}^{\infty} A \ e^{-\beta|\tau|} - i\omega\tau \ d\tau = \frac{2A\beta}{-(i\omega)^2 + \beta^2} \qquad (34)$$

This function is plotted in Figure 4.
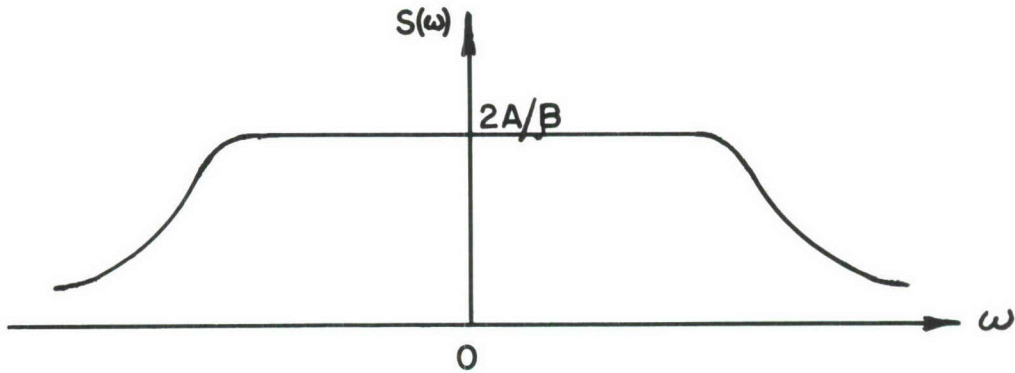
Figure 4. Spectral Density Function of a Process with Markoff
            Autocorrelation Function

From (34), we see that the process under consideration has a
rational spectral density function with

$$N((i\omega)^2) = 2\beta A$$
$$D((i\omega)^2) = -(i\omega)^2 + \beta^2 \qquad (35)$$

The differential equation (19) becomes in this case

$$2\beta A \psi_n(t) = |\lambda_n|^2 [-\frac{d^2\psi_n}{dt^2} + \beta^2 \psi_n] \qquad (36)$$

Let

$$\omega_n^2 = -\beta^2 + \frac{2\beta A}{|\lambda_n|^2} \qquad (37)$$

Then (36) can be written as

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$$\frac{d^2 \psi_n}{dt^2} + \omega_n^2 \psi_n = 0 \tag{38}$$

If $\beta^2 < \frac{2\beta A}{|\lambda_n|^2}$, then (38) has solution

$$\psi_n(t) = a_n e^{i\omega_n t} + b_n e^{-i\omega_n t}, \quad \omega_n > 0. \tag{39}$$

a_n and b_n are arbitrary constants. Substituting (35) into the integral equation (11), we find

$$A\int_0^t e^{-\beta\tau}\psi_n(t')dt' + A\int_t^T e^{\beta\tau}\psi_n(t')dt' = |\lambda_n|^2\psi_n(t), \quad \tau = t-t'$$

Using (39) and carrying out the integrations gives

$$Aa_n \frac{e^{i\omega_n t}-e^{-\beta t}}{\beta + i\omega_n} + Ab_n \frac{e^{-i\omega_n t}-e^{-\beta t}}{\beta - i\omega_n} +$$

$$+ Aa_n \frac{e^{\beta(t-T)+i\omega_n T}}{-\beta + i\omega_n} + Ab_n \frac{e^{\beta(t-T)-i\omega_n T} - e^{-i\omega_n t}}{-\beta - i\omega_n} =$$

$$|\lambda_n|^2 a_n e^{i\omega_n t} + |\lambda_n|^2 b_n e^{-i\omega_n t}$$

This equation is identically satisfied in t if

$$\frac{Aa_n}{\beta + i\omega_n} - \frac{Aa_n}{-\beta + i\omega_n} = |\lambda_n|^2 a_n$$

$$\frac{Ab_n}{\beta - i\omega_n} - \frac{Ab_n}{-\beta - i\omega_n} = |\lambda_n|^2 b_n \tag{40}$$

$$\frac{-Aa_n}{\beta + i\omega_n} - \frac{Ab_n}{\beta - i\omega_n} = 0$$

$$\frac{Aa_n e^{(-\beta + i\omega_n)T}}{-\beta + i\omega_n} - \frac{Ab_n e^{-(\beta + i\omega_n)T}}{\beta + i\omega_n} = 0$$

The first two conditions in (40) are identically satisfied. The second two are two linear homogeneous equations for a_n and b_n. For a nontrivial solution, we require

$$\begin{vmatrix} \dfrac{1}{\beta + i\omega_n} & \dfrac{1}{\beta - i\omega_n} \\[3ex] \dfrac{e^{(-\beta + i\omega_n)T}}{-\beta + i\omega_n} & \dfrac{-e^{-(\beta + i\omega_n)T}}{\beta + i\omega_n} \end{vmatrix} = 0 \qquad (41)$$

This reduces to

$$\omega_n \cos \frac{1}{2} \omega_n T + \beta \sin \frac{1}{2} \omega_n T = 0$$

$$\omega_n = +\sqrt{\frac{2\beta A}{|\lambda_n|^2} - \beta^2} \qquad (42)$$

Equation (42) determines the eigenvalues λ_n. It is clearly a complicated equation and must be solved numerically.

From (40)

$$b_n = -\frac{\beta - i\omega_n}{\beta + i\omega_n} a_n \qquad (43)$$

Therefore

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$$\psi_n(t) = a_n \{ e^{i\omega_n t} - \frac{\beta - i\omega_n}{\beta + i\omega_n} e^{-i\omega_n t} \}$$

$$= \frac{2ia_n}{\beta + i\omega_n} \{ \omega_n \cos \omega_n t + \beta \sin \omega_n t \} \tag{44}$$

From the normalizing condition we find

$$|a_n|^2 = \left| \frac{\beta + i\omega}{2} \right|^2 \{ \int_0^T |\omega_n \cos \omega_n t + \beta \sin \omega_n t|^2 dt \}^{-1} \tag{45}$$

The orthogonal expansion of $p(t)$ now takes the form

$$p(t) = \sum_{n=1}^{\infty} \frac{|\lambda_n| a_n}{\beta + i\omega_n} \{ \omega_n \cos \omega_n t + \beta \sin \omega_n t \} X_n \tag{46}$$

The X_n can only be determined when a realized value of $p(t)$ is given. This in principle completes the problem.

c) Other Methods for Solving the Integral Equation

Another method of solving the integral equation (11) much used by physicists is to expand the kernel $R(t-t')$ into a suitable set of orthornormal functions. By this means the problem can be reduced to the problem of finding the non-zero roots of an infinite determinant. By judicious choice of the expansion this process can be made tractable. (Ref. 5.)

d) Other Solutions

1. Stationary Band Limited Gaussian White Noise of Mean Zero (Ref. 4.)

The expansion interval in $-\infty \leq t \leq \infty$. The autocorrelation function for the process is

$$R(\tau) = \langle p^2 \rangle \frac{\sin 2\pi B\tau}{2\pi B\tau} , \quad \tau = t-t' \tag{47}$$

and is shown in Figure 5.

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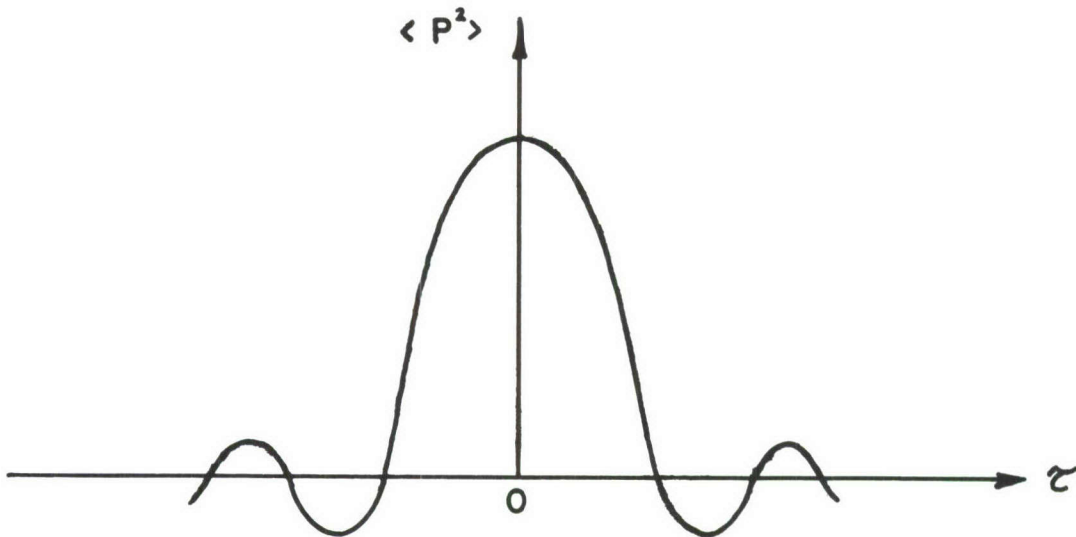
Figure 5. Autocorrelation Function of Band Limited White Noise

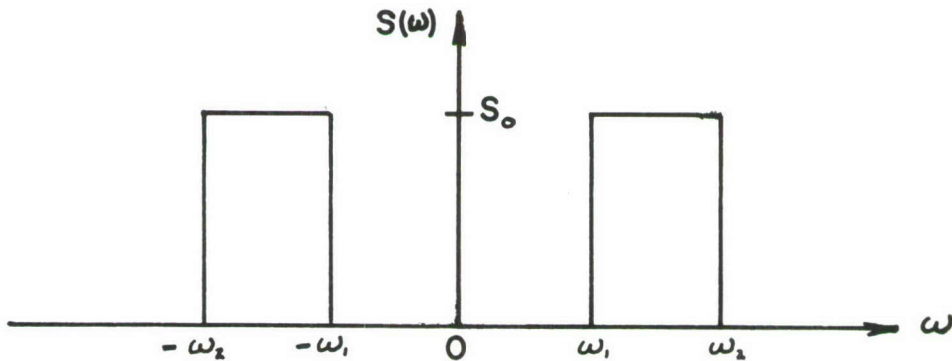$B = \omega_2 - \omega_1$ is the bandwidth of the noise. The spectral density appears as in Figure 6.

Figure 6. Spectral Density of Band Limited White Noise

Note that $\langle p^2 \rangle = S_o B$. The proper functions for the expansion of $p(t)$ on the interval $(-\infty, \infty)$ are found to be

$$\psi_n(t) = \sqrt{2B} \; \frac{\sin 2\pi(t - n/2B)}{2\pi B(t - n/2B)} \tag{48}$$

$\{\psi_n\}$ is an orthonormal set but ψ_n are <u>not</u> solutions of Eq. 11. The expansion takes the form

$$p(t) = (\sqrt{\langle p^2 \rangle})^{-1} \sum_{n=-\infty}^{\infty} \frac{\sin 2\pi B(t - n/2B)}{2\pi B(t - n/2B)} X_n \qquad (49)$$

where

$$X_n = 2B (\sqrt{\langle p^2 \rangle})^{-1} \int_{-\infty}^{\infty} p(t) \frac{\sin 2\pi B(t - n/2B)}{2\pi B(t - n/2B)} dt \qquad (50)$$

2. <u>Band Limited Stationary Normal White Noise on a Finite Interval</u>. (Ref.6.)

The correlation function is still of the form (47), but the expansion is desired on the interval (o,T). The proper functions and eigenvalues are

$$\psi_n(t) = \frac{S_{on}^{(1)}[\pi BT, 2/T(t-1)]}{\sqrt{N_{on}}} \quad , \quad o \leq t \leq T$$

$$\lambda_n = \langle p^2 \rangle T[R_{on}^{(1)}(BT, 1)]^2 \qquad (51)$$

$$N_{on} = 2 \sum_j {}' \frac{(d_j^n)^2}{2j + 1}$$

$R_{n\ell}(c,y)$ and $S_{n\ell}(c, \cos\theta)$ are respectively, the prolate spheroidal Bessel function and prolate spheroidal Legendre function as given by Morse and Feshbach (Ref. 5). These functions have been tabulated. The expansion reads

$$p(t) = \sum \sqrt{\lambda_n} \, \psi_n(t) X_n, \quad o \leq t \leq T$$

$$X_n = \frac{1}{\sqrt{\lambda_n}} \int_o^T p(t) \, \psi_n(t) dt \qquad (52)$$

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X_n are independent normal variates of mean zero.

3. White Noise

Correlation and Spectral density functions are

$$R(\tau) = S_o \delta(\tau)$$

$$S(\omega) = S_o \tag{53}$$

S_o is a constant. The integral equation in this case becomes

$$\int_0^T S_o \delta(t-t') \psi_n(t')dt' = |\lambda_n|^2 \psi_n(t)$$

or

$$S_o \psi_n(t) = |\lambda_n|^2 \psi_n(t)$$

This can be satisfied by any orthonormal set of functions if we take $\lambda_n = \sqrt{S_o}$ for all n. The expansion becomes

$$p(t) = \sqrt{S_o} \sum \psi_n(t) X_n$$

$$X_n = \frac{1}{\sqrt{S_o}} \int_0^T p(t) \psi_n(t)dt \tag{54}$$

The results for white noise are very interesting and potentially very useful. We can make the general statement that very broad band random signals can always be approximately expanded into orthogonal series using any suitable set of orthonormal functions.

IV. Methods of Approximating Acoustic Noise Fields

a) Optimal Approximation Techniques

Once λ_n and $\psi_n(t)$ are determined from solving the integral equation, X_n can be obtained when a typical recording p(t) is available. We get a set of realized values for the X_n. The series $\sum \lambda_n \psi_n(t) = p(t)$ is then an analytic representation of the realized function p(t). The sirens could be used to try to simulate this series or the typical recording.

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In general, when we have an analytic expression for $p(t)$, we need a synthesis procedure to approximate $p(t)$ with the sirens. We might try to approximate $p(t)$ itself in the time domain by requiring the mean square difference between the sirens' output $\sum_k p_k(t)$ and $p(t)$ to be a minimum. This requirement is achieved by minimizing

$$\frac{1}{T} \int_0^T [p(t) - \sum_k p_k(t)]^2 dt = \overline{\epsilon^2} \tag{55}$$

with respect to the adjustable parameters of the sirens such as center frequencies, amplitude of outputs, percent modulation etc. Let $\lambda_{i_k}^{(k)}$ be a parameter of the kth siren, then this synthesis procedure requires

$$\frac{\partial}{\partial \lambda_{i_k}^{(k)}} \int_0^T [p(t) - \sum_{k=1}^N p_k(t)]^2 dt = 0 \qquad \begin{array}{l} k = 1,2,\ldots, N \\[6pt] i_k = 1,2,\ldots, L_k \end{array} \tag{56}$$

N is the number of sirens and L_k is the number of adjustable parameters of the kth siren. (56) leads to a system of equations for the optimizing parameters. Unfortunately, the system is nonlinear in the parameters in almost all cases and it is difficult to solve even with digital computers. Here we try to optimally synthesize one realized value of $p(t)$. Since $p(t)$ is a sample function from a random process, it may be better to approximate some statistical characteristics of the process.

Other synthesis criteria might be:

1. Minimize mean square difference between power spectral density of $p(t)$ and that of $\sum_k p_k(t)$.

2. Optimize probability density of $p(t)$.

3. Minimize $\frac{1}{T} \ E\{\int_0^T [p(t) - \sum_{k=1}^N p_k(t)]^2 dt\}$

with respect to the parameters of the sirens. We assume here that some of the parameters of sirens are random in character.

b) <u>Spectral Broadening of a Single-Tone Siren</u>.

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Spectral broadening of a single-tone siren can be achieved
by various forms of modulation or by switching techniques. Let
us consider first spectral broadening by modulation.

1. <u>Amplitude Modulation</u> (Deterministic)

A single-tone siren output may be represented by

$$p(t) = A \cos (\omega_c t + \varphi) \qquad (57)$$

where A is the amplitude of the output, ω_c is the center fre-
quency of the siren, and φ is a phase angle. Such a siren output
as (57) contains only one spectral component. Suppose by some
means we are able to vary A as a function of time. In particular,
let

$$A = A(t) = A_o (1 + \Delta \cos \omega_m t) \qquad (58)$$

A_o, Δ, and ω_m are constants
Then (57) reads

$$p(t) = A_o (1 + \Delta \cos \omega_m t) \cos (\omega_c t + \varphi) \qquad (59)$$

or

$$p(t) = A_o \cos(\omega_c t + \varphi) + \frac{1}{2} A_o \Delta \cos [(\omega_c + \omega_m)t + \varphi] +$$

$$+ \frac{1}{2} A_o \Delta \cos [(\omega_c - \omega_m) t + \varphi] \qquad (60)$$

By this modulation, we see that p(t) now has three spectral
components. It is easy to see that if

$$p(t) = A_o \sum_{k=1}^{N} (1 + \Delta_k \cos \omega_{mk} t) \cos (\omega_c t + \varphi) \qquad (61)$$

p(t) will have $2N + 1$ spectral components. Equation (61) includes
a wide variety of amplitude modulation such as arbitrary periodic
modulation and almost periodic modulation. We note, however, that
as long as A(t) is a deterministic function of time the spectrum
is <u>discrete</u>.

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2. Frequency Modulation (Deterministic)

The instantaneous angular frequency is defined as

$$\omega(t) = \frac{d}{dt}(\omega_c t + \varphi(t)) \tag{62}$$

where the phase angle φ is now considered a function of time. If we vary the frequency by a single tone, the siren output takes the form

$$p(t) = A_o \cos[\omega_c t + \frac{\Delta}{\omega_m} \sin \omega_m t] \tag{63}$$

Δ and ω_m are constants

It is well known that this can be written in the form

$$p(t) = \frac{1}{2} A_o \cos \omega_c t + \frac{1}{2} A_o \sum_{n=1}^{\infty} [1 + (-1)^n] J_n(\frac{\Delta}{\omega_m}) \{\cos (\omega_c t + n\omega_m)t$$
$$+ \cos (\omega_c - n\omega_m)t\} \tag{64}$$

from which we see that there are an infinite number of discrete spectral components. $J_n(\frac{\Delta}{\omega_m})$ is the Bessel function of the first kind of order n. The general appearance of the intensity spectra of $p(t)$ for small, intermediate, and large values of Δ / ω_m are shown in Figure 7 with their envelopes.
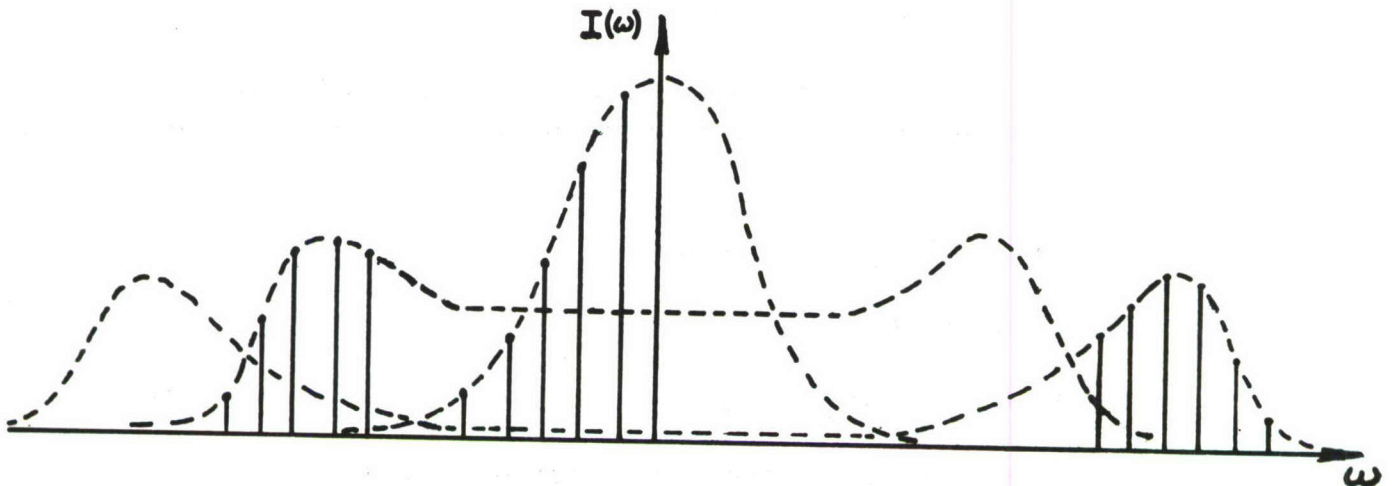

Figure 7. Intensity Spectra of a Single-Tone FM Acoustic Noise Pressure

For arbitrary periodic or almost periodic frequency modulation, the siren output is given by

$$p(t) = A_o \cos \{ \omega_c t + \sum_{k=1}^{N} \frac{A_k}{\omega_{mk}} \sin \omega_{mk} t \} \qquad (65)$$

A_k and ω_{mk} are constants.

The number of sidebands increases tremendously as N increases, but the intensity spectrum is still discrete. Only for an aperiodic modulating wave can we obtain a continuous intensity spectrum or spectral density.

3.) Amplitude Modulation (Random)

Continuous spectral broadening of a single-tone siren can be achieved through random modulation. Consider

$$p(t) = A_o [1 + \Delta P_M(t)] \cos \omega_c t \qquad (66)$$

where $P_M(t)$ is a stationary normal random process and A_o, Δ, and ω_c are constants. The spectral density of $p(t)$ for a modulating noise with spectral density function given by

$$S_M(\omega) = S_o e^{-\omega^2/\omega_b} \qquad (67)$$

$$S_o \text{ and } \omega_b = \text{constants}$$

is shown in Figure 8 for various modulation indices.

It is seen that spectral boradening can be quite considerable. Note that finite energy is in the carrier for all modulation indices shown.

4.) Frequency Modulation (Random)

In this case we have

$$p(t) = A_o \cos [\omega_c t + P_M(t)] \qquad (68)$$

where $P_M(t)$ is a stationary normal random process. A_o and ω_c are constant. The spectrum of $p(t)$ for $P_M(t)$ taken to have spectral density

$$S_M(\omega) = \frac{4 \langle P_M^2 \rangle / B_N}{1 + \omega^2 / B_N^2}$$

$$\frac{4 \langle \tilde{P_M} \rangle B_n}{\frac{B_n}{B_n \gamma}(B_n + \omega^2)} \quad (69)$$

is shown in Figure 9 for various conditions. B_N is the band-width of the noise. $\langle P_M^2 \rangle$ is mean square value of modulating signal.

$$\mu_F = \frac{\langle P_M^2 \rangle}{B_N^2} \qquad [sec\tilde{\ }] \qquad \overline{2500}$$
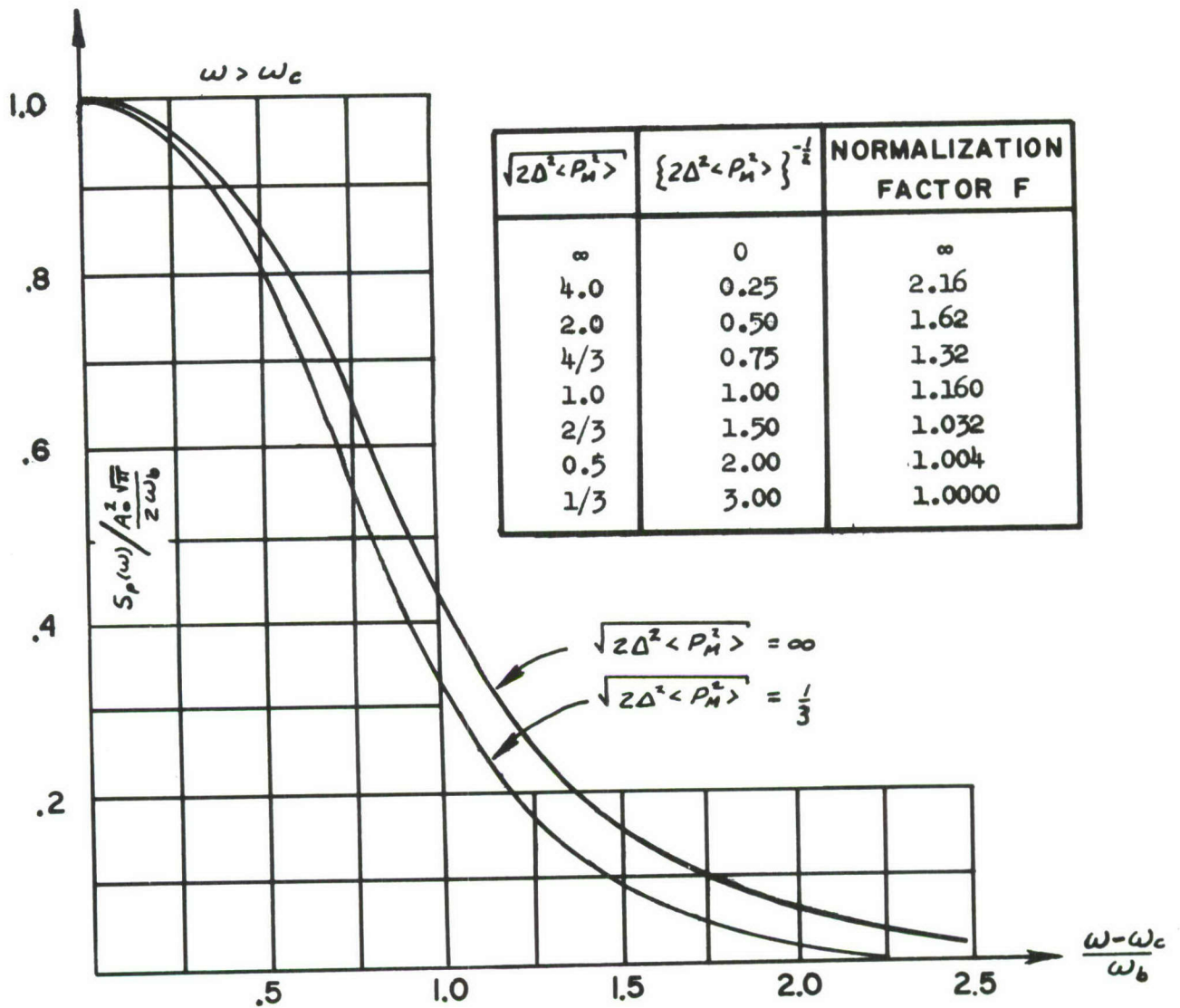
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Figure 8. Spectral Density of Noise Modulated
Single-Tone Siren

The table within the figure:

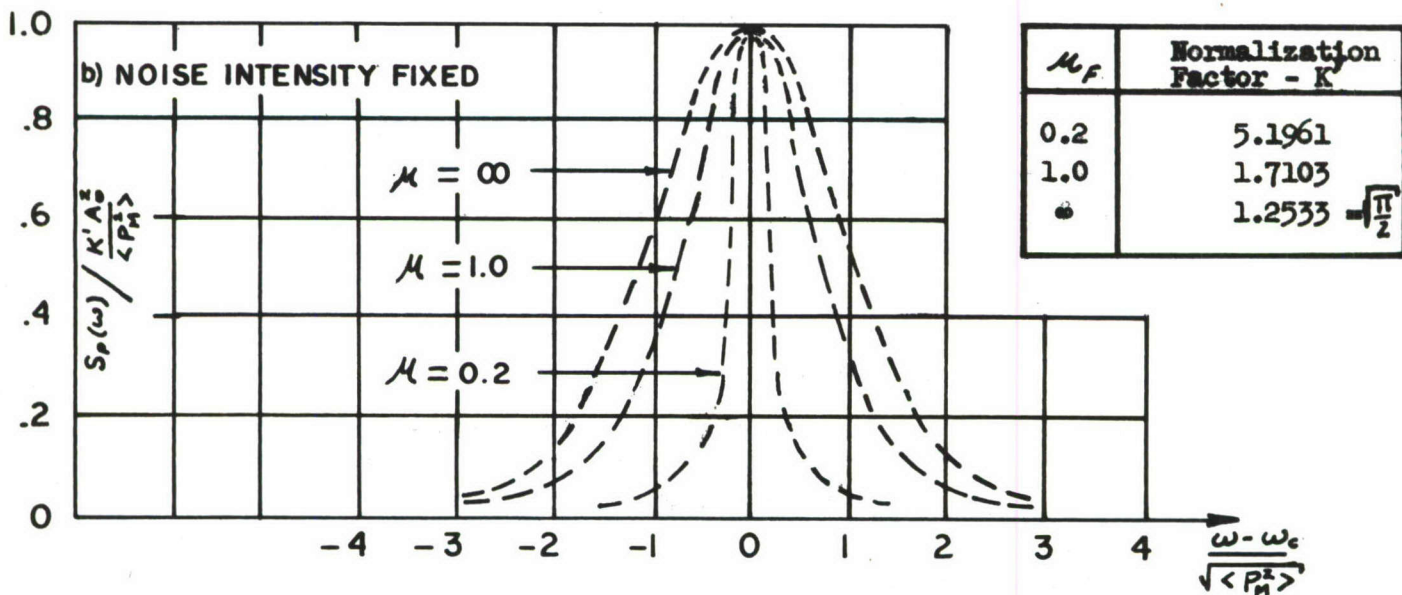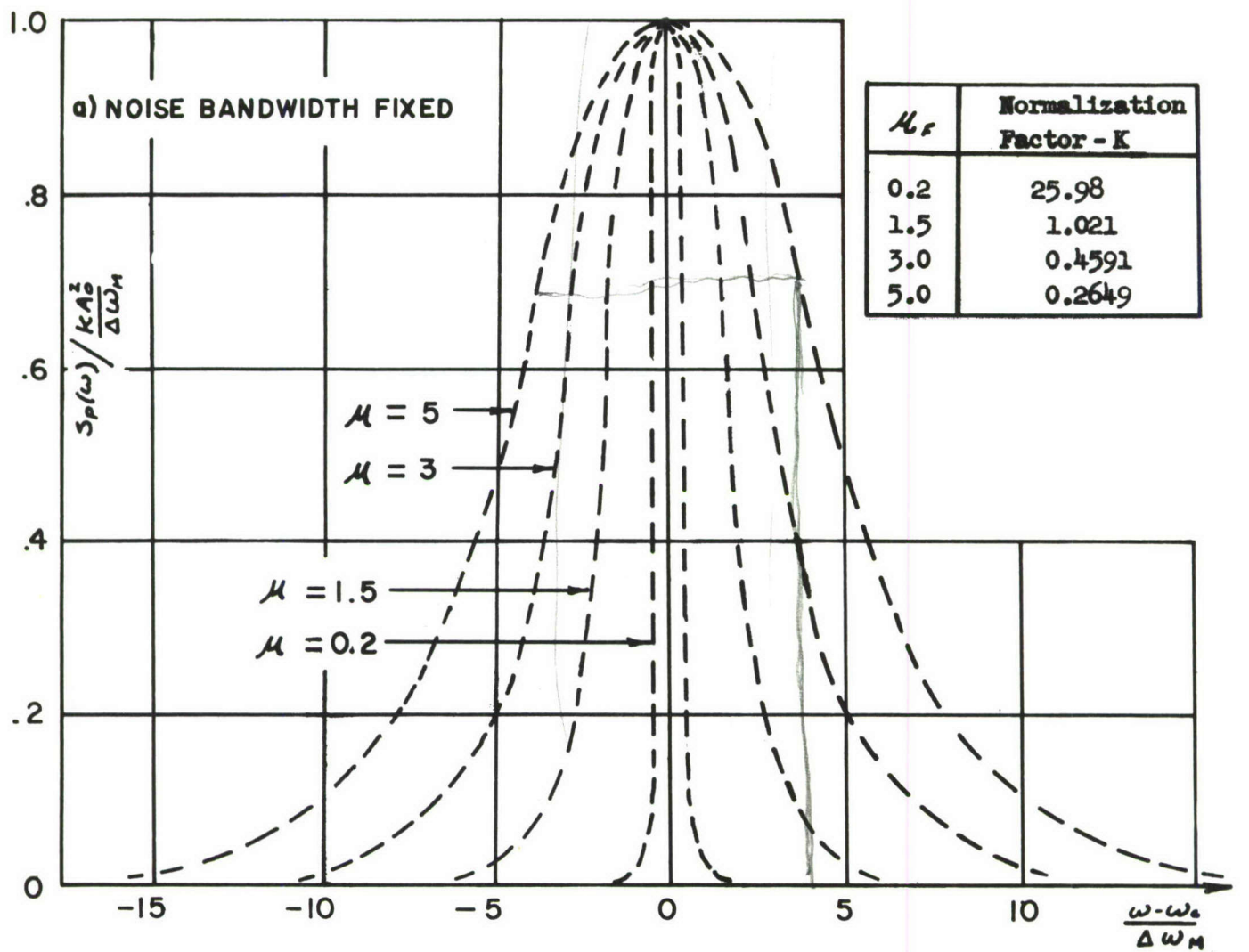| $\sqrt{2\Delta^2 \langle P_M^2 \rangle}$	$\left\{2\Delta^2 \langle P_M^2 \rangle\right\}^{-\frac{1}{2}}$	NORMALIZATION FACTOR F
∞	0	∞
4.0	0.25	2.16
2.0	0.50	1.62
4/3	0.75	1.32
1.0	1.00	1.160
2/3	1.50	1.032
0.5	2.00	1.004
1/3	3.00	1.0000

Figure 9. Spectral Density of a Single-Tone Siren FM by Stationary Normal Noise With Markoff Autocorrelation Function

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If $\mu << 1$, the spectral density is very narrow about ω_c (little broadening). For very large values of μ, the spectral density is broad about ω_c.

5.) Spectral Broadening by Random Switching

Consider a stationary noise pressure of the type shown in Figure 10.
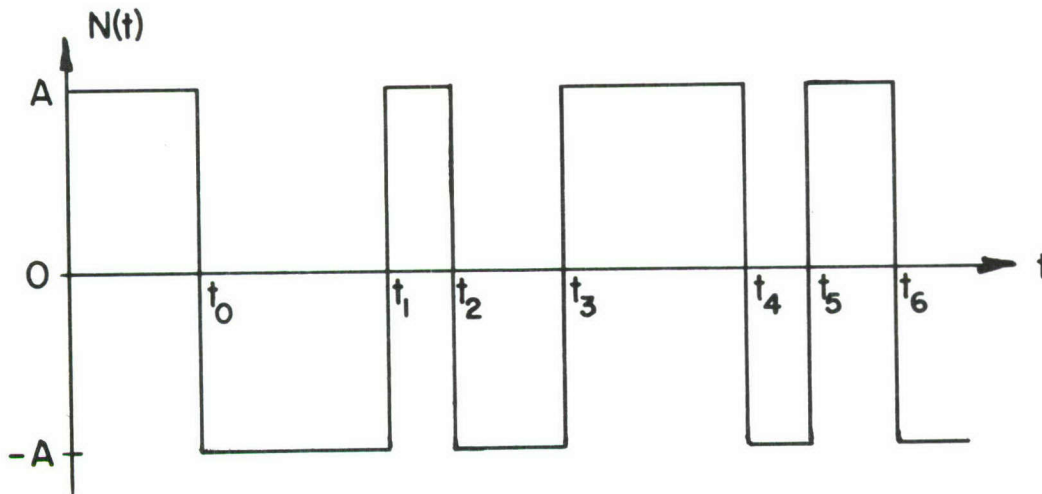

Figure 10. Random Square Wave

The zero crossings occur at purely random instants $t_o, t_1, \ldots$ Such an output would be produced (except for a carrier term $\cos \omega_c t$) by an ideal siren that was being switched on and off at purely random times. The spectral density function of the pressure in Figure 10 is

$$S_N(\omega) = \frac{T}{\pi} \frac{A^2}{1 + \left(\frac{\omega T}{2}\right)^2} \tag{70}$$

when $t_{i+1} - t_i$ is a Poisson distributed random variable. If we use $N(t)$ as an amplitude modulation function of a single-tone siren output, we have the sound pressure

$$P(t) = N(t) \cos[\omega t + \varphi] \tag{71}$$

φ is a random phase uniformly distributed between o and 2π. The autocorrelation function of $p(t)$ is

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$$R_p = E\{N(t) \cos [\omega_c t + \varphi] \cdot N(t + \tau) \cos [\omega_c (t + \tau) + \varphi]\} \quad (72)$$

$$R_p = E\{N(t) N(t + \tau)\} \ \frac{1}{2\pi} \int_0^{2\pi} \cos [\omega_c t + \varphi] \cos [\omega_c (t + \tau) + \varphi] \ d\varphi$$

$$R_p(\tau) = \frac{1}{2} R_N (\tau) \cos \omega_c \tau$$

The spectral density is obtained by use of the Wiéner-Kintchine theorem. We find

$$S_p(\omega) = \int_{-\infty}^{\infty} R_p(\tau) \ e^{-\omega \tau i} d\tau$$

$$S_p(\omega) = \frac{1}{4} \int_{-\infty}^{\infty} \{e^{-(\omega - \omega_c)\tau i} + e^{-(\omega + \omega_c)\tau i}\} R_N(\tau) d\tau \quad (73)$$

$$S_p(\omega) = \frac{1}{4} \{S_N(\omega - \omega_c) + S_N(\omega + \omega_c)\}$$

V. Conclusions and Recommendations for Further Study

The energy of a single-tone siren can be spread over a fairly large frequency band about the center frequency of the siren by various sorts of modulation. The most effective way of spreading the sound energy over the frequency band is by random amplitude or frequency modulation.

A given acoustic noise field may be approximated (in spectral density) by centering the sirens' center frequency at those which constitute the given signal or by dividing the spectral density of the given signal into equal energy bands and assigning an energy band to each siren. The approximation can be improved by amplitude and frequency modulating each siren. Approximation of a given acoustic noise field in some optimal way leads to the solution of a system of non-linear equations.

Further work may be carried out to find other solutions to the integral equation occurring in the Karhunen-Loève expansion of the random noise field. It would be particularly desirable to find some exact solution so that the proper expansions could be compared with a Fourier expansion of the field. Another direction in which a continued investigation might bear fruit is that of trying to devise methods of solving the nonlinear equations arising in trying to optimally approximate some characteristic of the given acoustic noise field with the sirens. Still another area worth looking into further is spectral broadening by randomization of the sirens' parameters.

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