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DIFFERENTIAL GAMES AND OPTIMAL PURSUIT-EVASION STRATEGIES

By

Yu-Chi Ho, A.E. Bryson, Jr. & S. Baron

November 4, 1964

Technical Report No. 457

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ABSTRACT

In this report, we show that variational techniques can be applied to solve games with differential constraints. Conditions for capture and for optimality are derived for a class of optimal pursuit-evasion problems. Results are used to demonstrate that the well known "proportional navigation law" is actually an optimal intercept strategy.

I. INTRODUCTION

The theory of differential games was initiated by Issacs in 1954 [1]. It was later studied in greater detail by Fleming and Berkovitz [2]. Recently, because of the advances in the computational solution of variation problems, renewed interest has developed in this subject.* Stated simply, games are a class of two-sided optimization problem. The simplest games are the type with which we are most familiar. A function J of two discrete variables u and v is given in tabular form. Each particular value of u or v is called a strategy. The problem is the determination of the strategy for u and v such that J is a saddle point (minimax). Because of the discrete nature of the available strategies, it is generally not possible to realize a minimax with a simple choice for u and v. Instead, a mixture of strategies must be employed to realize a minimax on an average basis. The usual game theory devotes considerable effort to the study of the properties of mixed strategies. On the other hand, if u and v were to take on a continuous range of values, then a saddle point could generally be realized by a pair of pure strategies for any reasonably well behaved function $J(u, v)$. The necessary and sufficient conditions for a saddle point are simply,

$$J_u = 0 \qquad J_v = 0 \qquad \text{for all } u, v \tag{1}$$

$$J_{uu} > 0 \qquad J_{vv} < 0 \tag{2}$$

This is in direct analogy to the one sided optimization problem of calculus. A more involved version of the game problem can be stated as that of

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*It is the authors' understanding that Prof. Pontriagin lectured on the subject during the month of October 1964.

determining a saddle point for $J(x, u, v)$ subject to the restriction that $\phi(x, u, v) = 0$. In this case, u and v still have the usual meaning while the variable x is an intermediate variable indirectly determined by the constraint $\phi = 0$. Of course, we can always eliminate x using the constraints and convert the problem into a simple game by writing $J = J^*(g^{-1}(u, v), u, v)$ where $x = g^{-1}(u, v) \Rightarrow \phi = 0$. However, it is usually simpler both theoretically and computationally to solve the problem by the introduction of a Lagrange multiplier again in imitation of the one-sided calculus problem. From this point, it is a straightforward conceptual generalization to consider the differential game problem of determining a saddle point for

$$J = \phi(x(T), T) + \int_0^T L(x, u, v)\, dt . \tag{3}$$

subject to

$$\dot{x} = f(x, u, v, t) ; \quad x(t_o) = x_o \tag{4}$$

where x plays the usual role of the state vector, and u, v the control vectors. Extending our analogy with the one-sided optimization problem, we expect that the techniques of variational calculus should be useful in this context. The purpose of this report is to illustrate that this is indeed so by solving a class of optimal pursuit-evasion problems and deriving conditions for optimality and capture. As an interesting by-product, we shall show that the "proportional navigation" law used in many missile guidance systems actually constitutes a form of optimal pursuit strategy, under the usual simplifying approximations to the equations of motion of the missile and the target.

II. A CLASS OF OPTIMAL PURSUIT-EVASION GAMES

We shall consider two dynamic systems

$$\dot{x}_p = F_p x_p + G_p u \tag{5}$$

$$\dot{x}_e = F_e x_e + G_e v \tag{6}$$

where the subscript p and e stand for pursuer and evader respectively. The pursuing system desires to intercept or rendezvous with the evading system at time T while the latter attempts to do the opposite. Both systems have limited energy sources. Hence, a reasonable criterion would be

$$J = \frac{1}{2} \| x_p(T) - x_e(T) \|_A^2 + \frac{1}{2} \int_0^T [\|u\|_{R_p}^2 - \|v\|_{R_e}^2]\, dt \tag{7}$$

where $A \geq 0$ and R_p and $R_e > 0$. The minus sign in front of the $\|v\|^2$ term comes about due to the fact that we shall attempt to maximize J w.r.t. to the variable v. Following the usual variational procedure, we introduce at this point the multiplier functions $\lambda_p(t)$ and $\lambda_e(t)$ which are used to adjoin (5) and (6) to (7). Now let us take a particular pair of strategies $u(t)$ and $v(t)$ with the associated trajectories $x_p(t)$ and $x_e(t)$ and consider variations $\delta u(t)$ and $\delta v(t)$. The change in J to second order is

$$\begin{aligned}\delta J = {} & [(x_p(T)-x_e(T))^T A^T - \lambda_p^T(T)]\,\delta x_p(T) + [(x_e(T)-x_p(T))^T A^T - \lambda_e^T(T)]\,\delta x_e(T) \\ & + \frac{1}{2}\|\delta x_p(T) - \delta x_e(T)\|_A^2 + \int_0^T [(\dot{\lambda}_p + H_{x_p})\,\delta x_p + (\dot{\lambda}_e + H_{x_e})\,\delta x_e \\ & + H_u\,\delta u + H_v\,\delta v]\, dt + \frac{1}{2}\int_0^T [\delta u \;\; \delta v]^T \begin{bmatrix} R_p & 0 \\ 0 & -R_e \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} dt\end{aligned} \tag{8a}$$

where H is the Hamiltonian

$$H = \frac{1}{2}\left(\|u\|^2_{R_p} - \|v\|^2_{R_e}\right) + \lambda_p^T(F_p x_p + G_p u) + \lambda_e^T(F_e x_e + G_e v) \tag{8b}$$

In order that (8) vanish to first order, we immediately derive the necessary conditions

$$\dot{\lambda}_p^T = -\lambda_p^T F_p \qquad \lambda_p^T(T) = (x_p(T) - x_e(T))^T A^T \tag{9}$$

$$\dot{\lambda}_e^T = -\lambda_e^T F_e \qquad \lambda_e^T(T) = (x_e(T) - x_p(T))^T A^T \tag{10}$$

$$H_u = 0 \quad \Rightarrow \quad u = -R_p^{-1} G_p^T \lambda_p \tag{11}$$

$$H_v = 0 \quad \Rightarrow \quad v = +R_e^{-1} G_e^T \lambda_e \tag{12}$$

Equations (5, 6, 9-12) represent a linear two-point boundary value problem which can be solved in terms of the fundamental matrix solution of the "p" and "e" system, $\Phi_p(t, \tau)$ and $\Phi_e(t, \tau)$. We have,

$$\begin{aligned} \lambda_p(t) &= \Phi_p^T(T, t)\, A\, (x_p(T) - x_e(T)) \\ \lambda_e(t) &= \Phi_e^T(T, t)\, A\, (x_e(T) - x_p(T)) \end{aligned} \tag{13}$$

Substituting (13) into (5-6) and using (11-12), we obtain

$$\begin{aligned} x_p(T) &= \Phi_p(T, t)\, x_p(t) - M_p A\, (x_p(T) - x_e(T)) \\ x_e(T) &= \Phi_e(T, t)\, x_e(t) - M_e A\, (x_p(T) - x_e(T)) \end{aligned} \tag{14}$$

where

$$M_p(T,t) \triangleq \int_t^T \Phi_p(T,\tau)\, G_p\, R_p^{-1}\, G_p^T\, \Phi^T(T,\tau)\, d\tau \tag{15}$$

$$M_e(T,t) \triangleq \int_t^T \Phi_e(T,\tau)\, G_e R_e^{-1} G_e^T \Phi_e^T (T,\tau)\, d\tau \tag{16}$$

Finally, (13) and (14) combine to yield the optimal pursuit and evasive strategies

$$u(t) = -R_p^{-1} G_p^T \Phi_p^T(T,t) A[I+(M_p-M_e)A]^{-1}[\Phi_p(T,t)x_p(t) - \Phi_e(T,t)x_e(t)] \tag{17}$$

$$v(t) = -R_e^{-1} G_e^T \Phi_e^T(T,t) A[I+(M_p-M_e)A]^{-1}[\Phi_p(T,t)x_p(t)-\Phi_e(T,t)x_e(t)] \tag{18}$$

Since $\Phi(T,t)x(t)$ has the interpretation of the predicted terminal state of a dynamic system, the optimal pursuit-evasive strategies are simply linear combinations of the predicted terminal miss – a very reasonable result.

Examination of the second-order terms in (8) shows that the analogous Legendre-Clebsch condition for the saddle point is satisfied.

$$H_{uu} = R_p > 0\,; \qquad H_{vv} = -R_e < 0 \tag{19}$$

Furthermore, the accessory minimax problem is

$$(P)\ \left\{\begin{array}{l} \text{"determine a saddle point for} \\ \delta J = \frac{1}{2}\,\|\delta x_p(T)-\delta x_e(T)\|_A^2 + \frac{1}{2}\int_0^T [\,\|\delta u\|_{R_p}^2 - \|\delta v\|_{R_e}^2\,]\, dt \\ \\ \text{subject to } \delta\dot{x}_p = F_p\,\delta x_p + G_p\,\delta u\,; \quad \delta x_p(0) = 0 \\ \qquad\qquad\quad\ \delta\dot{x}_e = F_e\,\delta x_e + G_e\,\delta v\,; \quad \delta x_e(0) = 0 \end{array}\right.$$

By exactly the same argument leading to the derivation of the conjugate point conditions for the one sided variational problem [3], we find that

$\delta J > 0 \;\; \forall \;\; \delta u \neq 0,\; \delta v$ fixed, and $\delta J < 0 \;\; \forall \;\; \delta v \neq 0,\; \delta u$ fixed, if and only if the matrix solution $X(t)$ is nonsingular during the interval $(0, T)$ where $X(t)$ obeys

$$\begin{bmatrix} \dot{X} \\ \\ \\ \dot{\Lambda} \end{bmatrix} = \begin{bmatrix} F_p & 0 & -G_p R_p^{-1} G_p^T & 0 \\ 0 & F_e & 0 & G_e R_e^{-1} G_e^T \\ 0 & 0 & -F_p^T & 0 \\ 0 & 0 & 0 & -F_e^T \end{bmatrix} \begin{bmatrix} X \\ \\ \\ \Lambda \end{bmatrix} \qquad \begin{array}{l} X(T) = I \\ \\ \Lambda(T) = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \end{array} \tag{20}$$

But from (20),

$$X(t) \quad \begin{bmatrix} \Phi_p(t, T) & 0 \\ 0 & \Phi_e(t, T) \end{bmatrix} \begin{bmatrix} I + M_p A & -M_p A \\ M_e A & I - M_e A \end{bmatrix} \tag{21}$$

Thus, the nonsingularity of $X(t)$ (i.e., nonexistence of conjugate point) is equivalent to the condition

$$\det \begin{bmatrix} I + M_p A & -M_p A \\ M_e A & I - M_e A \end{bmatrix} = \det\,[I + (M_p \cdot M_e)A] \neq 0 \tag{22}$$

which in view of (17) and (18) also guarantees the boundedness of the strategies.

It is also interesting to consider the limiting case where A is positive definite and infinite (e.g., infinite diagonal matrix). This is the situation where u attempts to capture v with minimal energy. Condition (22) becomes

$$\det[I+(M_p - M_e)A] \neq 0 \Longleftrightarrow \det(A^{-1} + (M_p - M_e)) \neq 0$$
$$\Longleftrightarrow \det(M_p - M_e) \neq 0 \qquad (23)$$

In terms of the usual definition of controllability, (23) simply says that the pursuer must be "more controllable" than the evader – an eminently reasonable conclusion. Since both M_p and $M_e > 0$ if the individual systems are controllable, then a sufficient condition for capture is simply $M_p - M_e > 0$. Furthermore, we have the following

Proposition: Let R_p and R_e in (7) be scalar, and the optimal pursuit and evasion energy for $A = \infty$ be $\int_o^T ||u||^2 dt = c_p$, $\int_c^T ||v||^2 dt = c_e$, respectively, then a necessary and sufficient condition for capture of an evader with energy resources c_e by a pursuer with energy resourses c_p is $M_p - M_e > 0$. The proof of this proposition is a direct consequence of the utility interpretation of Lagrange multipliers [4].

III. GUIDANCE LAW FOR TARGET INTERCEPTION

A special case of the class of problems treated in Section II can be formulated as follows: the equations of motion (kinematic) in space for an interceptor and a target are

$$\begin{aligned} \dot{\vec{v}}_p &= \vec{f}_p + \vec{a}_p \\ \dot{\vec{r}}_p &= \vec{v}_p \\ \dot{\vec{v}}_e &= \vec{f}_e + \vec{a}_e \\ \dot{\vec{r}}_e &= \vec{v}_e \end{aligned} \qquad (24)$$

where

$\vec{v}$ = velocity of a body in three dimensions

$\vec{r}$ = position vector of a body in three dimensions

$\vec{f}$ = external force per unit mass exerted on the body

$\vec{a}$ = control acceleration of a body

We assume that the altitutude difference between the pursuer and the evader is small enough such that $\vec{f}_p = \vec{f}_e$. Hence, if we are only interested in the difference $\vec{r}_p(t) - \vec{r}_e(t)$, the effect of the external forces can be ignored. Now consider the criterion

$$J = \frac{a}{2}(\vec{r}_p(T)-\vec{r}_e(T))(\vec{r}_p(T)-\vec{r}_e(T)) + \frac{1}{2}\int_o^T [c_p^{-1}(\vec{a}_p \cdot \vec{a}_p) - c_e^{-1}(\vec{a}_e \cdot \vec{a}_e)]\, dt \tag{25}$$

where c_p and c_e represent the energy capacity of the pursuer and the evader, respectively. Applying the result of the last section, it can be directly verified that (17) and (18) become in this case

$$\vec{a}_p = \frac{-c_p(T-t)[\vec{r}_p(t) - \vec{r}_e(t) + (\vec{v}_p(t) - \vec{v}_e(t))(T-t)]}{\frac{1}{a} + (c_p - c_e)(T-t)^3/3} \tag{26}$$

$$\vec{a}_e = \frac{-c_e(T-t)[\vec{r}_p(t) - \vec{r}_e(t) + (\vec{v}_p(t) - \vec{v}_e(t))(T-t)]}{\frac{1}{a} + (c_p - c_e)(T-t)^3/3} \tag{27}$$

We note immediately that if

(i) if $c_p > c_e$ (i.e., pursuer has more energy than the evader) then the feedback control gain is always of one sign.

(ii) if $c_p < c_e$ (i.e., pursuer has less energy than the evader) then the feedback gain will change sign at

$$\frac{1}{a} + (c_p - c_e)(T-t)^3/3 = 0 \tag{28}$$

for T sufficiently large.

But (28) is simply the conjugate point condition (22) specialized for this problem. Hence, for case (ii), (26) is no longer optimal for large T. This fact is, of course, obvious to start with, particularly in the case when $a = \infty$. In that case, interception is not possible when $c_p < c_e$ (cf. $M_p < M_e$). Assuming (i) and letting $a = \infty$, the control strategy for the pursuer simplifies to

$$a_p = \frac{-3}{(1 - \frac{c_e}{c_p})(T-t)^2} [\vec{r}_p(t) - \vec{r}_e(t) + (\vec{v}_p(t) - \vec{v}_e(t))(T-t)] \tag{29}$$

Let the pursuer and the target be on a nominal collision course with range R and closing velocity $V_c = R/(T-t)$. Let $x_p - x_e$ represent the lateral deviation from the collision course as shown in Fig. 1

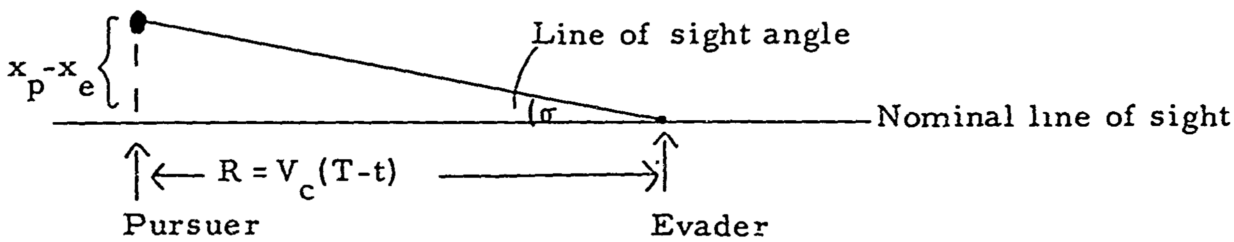

Fig. 1 Geometry of Proportional Navigation.

Then the lateral control acceleration to be applied by the pursuer according to (29) is

$$a\,(\text{lateral}) = \frac{3}{\left(1 - \frac{c_e}{c_p}\right)} V_c \, \dot{\sigma} \tag{30}$$

which is simply proportional navigation with the effective navigation constant $K_e = \frac{3}{1 - \frac{c_e}{c_p}}$. From experience it has been found that the "best" value for K_e ranges between 3 to 5 [5]. In view of (30), we see that the value of 3 corresponds to the case when the target is not maneuverable [6] ($c_e = 0$) ; while the value of 5 corresponds to $\frac{c_e}{c_p} = \frac{2}{5}$.

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