DOC. NO. 61-12-53
COPY NO. REPRO

AD609512

TECHNICAL LIBRARY

HEADQUARTERS
OFFICE OF THE DEPUTY COMMANDER AFSC
FOR AEROSPACE SYSTEMS
UNITED STATES AIR FORCE

VAULT #
61-12-053

BSC 28251

FS-61TN-1013

THE RADAR CROSS SECTION OF AN IONIZED WAKE OF A HYPERSONIC BODY

Prepared by:

C. M. Chu and D. T. Politis

10 November 1961

The work described in the Technical Note was supported by the Department of the Air Force Contract No. AF 04(647)-911

J. Rutkowski, Dept. Mgr.

BENDIX SYSTEMS DIVISION
OF
THE BENDIX CORPORATION
Ann Arbor, Michigan

61 12 05..

I. STATEMENT OF THE PROBLEM

The problem here under investigation can be simply stated as:

"To obtain a parametric estimate of the radar cross section of the ionized wake of a hypersonic body as a function of aspect angle, for a wide range of frequencies and values of the physical parameters involved."

These parameters are:

(a) The radar frequency, f cps

(b) The electron collision frequency, ν $\sec^{-1}$

(c) The electron distribution functions and their parameters.

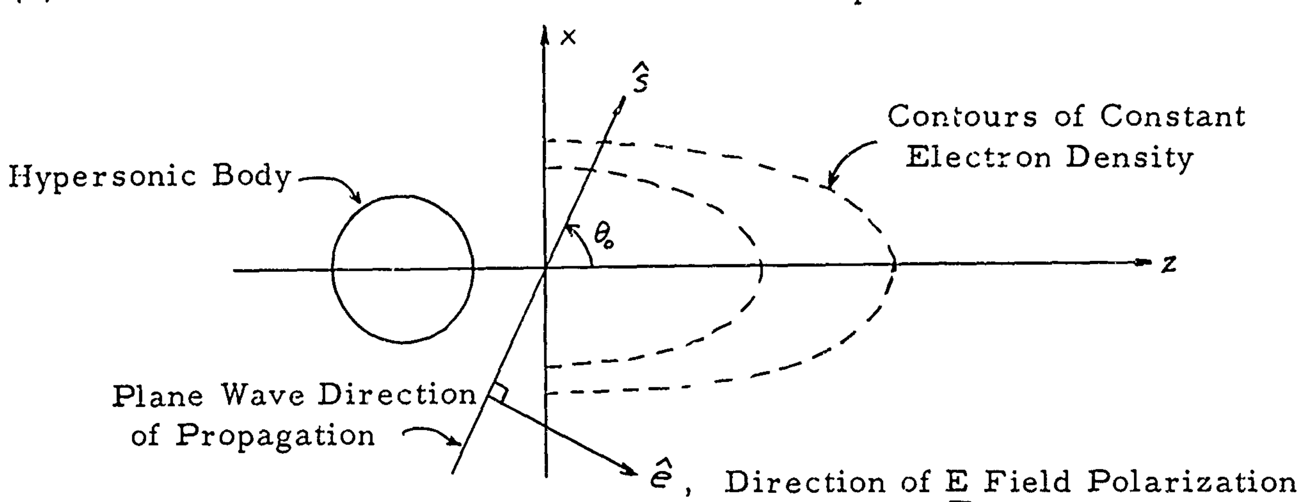


Figure 1 Geometry of the Problem

It will be assumed that the wake is symmetric about the z-axis and therefore at a small distance behind the body (the plane z = o in Figure 1) the electron density N of the ionized wake can be described functionally by

$$N = N(z, \rho, t) \qquad z \geq o$$

The following idealizations will be now imposed on the problem:

(i) Quasi-static approximation, i. e., take $N(z, \rho, t)$ to be a slowly varying function of time, then

$$N(z, \rho, t) \to N(z, \rho)$$

(ii) The space variation of N is assumed to be:

$$N(z, \rho) = N_o f(z) \phi(\rho)$$

and for preliminary investigation, let f and ϕ take the form:

$$f(z) = e^{-\frac{z}{b}} \qquad z \geq o \qquad (1)$$

and

$$\phi(\rho) = e^{-\frac{\rho^2}{a^2}} \qquad \rho \geq o \qquad (2)$$

(iii) The ionization may be replaced by the macroscopic complex dielectric constant, given by:

$$\epsilon_o \epsilon_r (\rho, z) = \epsilon_o \left[1 - \frac{N(\rho, z) q^2}{m\epsilon_o \omega(\omega + i\nu)}\right] = \epsilon_o \left[1 - \frac{\omega_p^2}{\omega(\omega + i\nu)}\right] \qquad (3)$$

wher~

$\epsilon_o = \frac{1}{36\pi} \times 10^{-9}$ farad/meter, is the permitivity of free space

$\epsilon_r(\rho, z) =$ the dielectric constant of the medium

$q = 1.6 \times 10^{-19}$ coulomb, is the charge per electron

$m = 9.1 \times 10^{-31}$ kg, is the mass of an electron

ω = rad/sec is the operating radar frequency

$\omega_p = \sqrt{\frac{N(\rho, z)\, q^2}{m \epsilon_o}}$ rad/sec is the plasma frequency

$\nu = \sec^{-1}$ is the collision frequency

With the above assumptions, the parameters involved for consideration are:

$$N_o, \ a, \ b, \ \omega \text{ and } \nu$$

The selection of Gaussian-Exponentail dependence for the electron density obviously is not the only possible one and the choice was based simply on the fact that at the present state of knowledge it seems to be at least as realistic as any other and that it yields integrals that can be integrated in closed form. However more complicated parametric representation, using functions $f(z)$, or $\phi(\rho)$ other than Equation (2) and (3), will be taken up at a later stage. It must also be noted that due to the variation of density with altitude one should include an altitude dependence of the electron density. But this too will be taken into consideration later.

II. MATHEMATICAL FORMULATION

Mathematically, the solution to the problem stated in (I) is the solution to Maxwell's equations for an inhomogeneous medium. Hence a brief discussion of the problem can be started from Maxwell's Equations:

$$\left.\begin{aligned} \nabla \cdot \epsilon \underline{E} &= 0 \\ \nabla \cdot \underline{H} &= 0 \\ \nabla \times \underline{E} &= i \omega \mu_o \underline{H} \\ \nabla \times \underline{H} &= -i \omega \epsilon_o \epsilon_r \underline{E} \end{aligned}\right\} \qquad \text{II-1}$$

where $\mu_o = 4\pi \times 10^{-7}$ is the permeability of free space

$\epsilon_o = \frac{1}{36\pi} \times 10^{-9}$ is the permittivity of free space

ϵ_r = The dielectric constant of the medium, and the assumptions $\sigma = 0$, $\underline{E} = \underline{E}\ e^{-i\omega t}$ and $\underline{H} = \underline{H}\ e^{-i\omega t}$ have been made.

Elimination of $\underline{E}$ (or $\underline{H}$) in the above equations yields:

$$\left.\begin{aligned} \nabla \times \nabla \times \underline{E} - k^2 \epsilon_r \underline{E} &= 0 \\ \epsilon_r \nabla \times \frac{1}{\epsilon_r} \nabla \times \underline{H} - k^2 \epsilon_r \underline{H} &= 0 \end{aligned}\right\} \qquad \text{II-2}$$

where $k^2 = \omega^2 \mu_o \epsilon_o$.

In the available literature, exact solutions for the above set of equations are given only for simple one-dimensional problems. However, reasonably good approximate solutions can be obtained by use of the following arguments:

(a) Few algebraic steps reduce II-2 into[1]

$$\left.\begin{aligned} \nabla^2 \underline{E} + k^2 \epsilon_r \underline{E} &= -\nabla\left(\frac{\nabla\epsilon_r}{\epsilon_r} \cdot \underline{E}\right) \\ \nabla^2 \underline{H} + k^2 \epsilon_r \underline{H} &= -\frac{\nabla \epsilon_r}{\epsilon_r} \times \nabla \times \underline{H} \end{aligned}\right\} \qquad \text{II-3}$$

Equation II-3 is the basis of geometric optics approximation. The argument is that, if

$$\frac{\nabla\epsilon_r}{\epsilon_r} << k \qquad \text{II-4}$$

(an approximation which improves as ω increases) then the right hand side of Equation II-3 is of small order of magnitude, so that it can be neglected. This reduces the problem into a solution of the homogeneous equation:

$$\nabla^2 \phi + k^2 \epsilon_r \phi = 0$$

where the standard procedure of ray tracing may apply.

It is to be noted that in most applications of geometric optics, $\epsilon_r > 1$, therefore Equation II-4 can be satisfied as long as $\nabla\epsilon_r$ is comparatively small. In the equivalent dielectric constant of ionized media, on the other hand, $\epsilon_r < 1$, so that Equation II-4 can never be satisfied near the region $\epsilon_r \cong o$, even when $\nabla\epsilon_r$ is small. In so far as we know, in general, no

[1] Luneberg, "Mathematical Foundations of Geometric optics," Brown University Lectures, 1944.

satisfactory mathematical discussion exists of the solution of Equation II-3 when $\epsilon_r \cong o$, but the physical approximation of using the concept of critical density sometimes yields useful numerical solutions.

(b) An alternate approach, more amenable to numerical calculations is given by Khizhniak.[2] This is done by rewriting Equations II-2 into the form:

$$\left.\begin{aligned} \nabla \times \nabla \times \underline{E} - k^2 \underline{E} &= k^2 (\epsilon_r - 1) \underline{E} \\ \nabla \times \nabla \times \underline{H} - k^2 \underline{H} &= -i\omega\epsilon_o \nabla \times (\epsilon_r - 1) \underline{E} \end{aligned}\right\} \quad \text{II-5}$$

(From here on for notational simplicity ϵ implies ϵ_r)

By treating the right hand side as a "source" term, the standard Green's function technique should apply. Using the free space Green's function

$$G(\underline{r}, \underline{r}') = \frac{e^{i k(\underline{r} - \underline{r}')}}{4\pi (\underline{r} - \underline{r}')} \qquad \text{II-6}$$

where $\underline{r}$ and $\underline{r}'$ are the coordinates of the source and point of observation respectively, Equations II-5 can be expressed as integral equations:

$$\left.\begin{aligned} \underline{E}(\underline{r}) &= E_o(\underline{r}) + \iiint_V [\epsilon(\underline{r}') - 1][\{\underline{E}(\underline{r}') \cdot \nabla'\}\nabla' G(\underline{r}, \underline{r}') + k^2 \underline{E}(\underline{r}') G(\underline{r}, \underline{r}')] dV' \\ \underline{H}(\underline{r}) &= \underline{H}_o(r) - i\omega\epsilon_o \nabla \times \iiint_V [\epsilon(\underline{r}') - 1] \underline{E}(\underline{r}') (G(\underline{r}, \underline{r}') dV' \end{aligned}\right\} \quad \text{II-7}$$

---

[2] Khizhniak, N. A., "Green's Function for Maxwell's Equations for Inhomogeneous Media." Zhurnal Tekhnicheskoi Fiziki, 7-28, 1958.

where $E_o(\underline{r})$, $H_o(\underline{r})$ are some incident field, and $\underline{E}(\underline{r})$, $\underline{H}(\underline{r})$ are the total field. If we introduce the notation $\underline{E}_s(\underline{r})$ and $\underline{H}_s(\underline{r})$ as the scattered field, then Equations II-7 can be written as integral equations for the scattered field:

$$\underline{E}_s(\underline{r}) = \iiint_V [\epsilon(\underline{r}')-1] \, [\{\underline{E}_s(\underline{r}')+\underline{E}_o(\underline{r}')\}\cdot\nabla']\nabla' G(\underline{r},\underline{r}') + k^2\{\underline{E}_s(\underline{r}')+\underline{E}_o(\underline{r}')\}G(\underline{r},\underline{r}') \, dV'$$

$$\underline{H}_s(r) = -i\omega\,\epsilon_o \iiint_V \nabla' G(\underline{r},\underline{r}')\times\{\underline{E}_s(\underline{r}')+\underline{E}_o(\underline{r}')\}[\epsilon(\underline{r}')-1] \, dV' \qquad \text{II-8}$$

It is noted that equations II-7 or II-8 have a very clear physical interpretation. If we write,

$$\underline{\Pi} = \iiint_V [\epsilon(\underline{r}')-1] \, [\underline{E}_o(\underline{r}')+\underline{E}_s(\underline{r}')] \, G(\underline{r},\underline{r}') \, dV' \qquad \text{II-9}$$

as the Hertzian potential due to induced polarization

$$\epsilon_o[\epsilon(\underline{r})-1] \, [\underline{E}_o(\underline{r})+\underline{E}_s(\underline{r})]$$

then it is obvious that Equations II-8 can be reduced to

$$\underline{E}_s(r) = \nabla\nabla\cdot\underline{\Pi} + k^2\underline{\Pi}$$

$$\underline{H}_s(\underline{r}) = -i\omega\,\epsilon_o\nabla\times\underline{\Pi} \qquad \text{II-10}$$

which agrees with the standard formulation of the radiation problem.

Equations II-8 are integral equations of the Fredholm type for the scattered field $\underline{E}_s(\underline{r})$. Perhaps some detailed investigation is necessary for the nature of the solutions for various forms of $[\epsilon(\underline{r})-1]$.

For immediate numerical answers, we found that this formulation is particularly attractive for the following reasons:

(i) For the region where $|\epsilon(\underline{r}) - 1| < 1$, which is true for frequencies higher than the critical frequency, approximate solutions can be obtained by using $\underline{E}_o(r)$ in place of $\underline{E}(\underline{r})$. This is essentially the Born approximation.

(ii) In principle, the high order corrections to (i) can be obtained by standard Neumann series. But this involves too much numerical work, unless some analytical integration can be done first.

(iii) In conjunction with (i), which actually amounts to a "guess" of $\underline{E}(\underline{r})$ in the integral II-7 or II-9, a reasonable guess of $\underline{E}(\underline{r})$ (or more precisely the induced polarization) should improve the results greatly. A classical example is the Kirchhoff diffraction theory where the induced surface current on a body is guessed from the incident field. A reasonable "guess" of the field inside the ionized region, especially at the region where $\epsilon(\underline{r}) < o$, will be proposed later.

(iv) From the correct guess of $\underline{E}$, the scattering cross section can be obtained by evaluating one integral. This is given by:

$$\sigma(\hat{s}', \hat{s}) = \frac{k^4}{|E_o|^2} \frac{1}{16\pi^2} \left| \hat{s}' \times \iiint_V [\epsilon(\underline{r}')-1] \, \underline{E}(\underline{r}') \, e^{-ik\hat{s}' \cdot \underline{r}'} \, dV' \right|^2 \qquad \text{II-11}$$

where $\hat{s}'$ is the direction of observation and E_o is the amplitude of the incident field.[1]

[1] For derivation of II-11 see Appendix.

III. ESTIMATION OF THE RADAR CROSS-SECTION

The evaluation of the radar cross-section (given by Equation II-11) depends on the use of the correct expression for $\underline{E}(r)$. As a first order approximation assume that for $z > o$

$$\underline{E}\,(\underline{r}) \cong \underline{E}_o\,(\underline{r})$$

This approximation enables one to evaluate the integral in Equation (II-11) in closed form.

Refer now to Figure 2; in cylindrical coordinates (ρ, ϕ, z), any point in space is represented by

$$\underline{r}' \sim (\rho \cos \phi, \; \rho \sin \phi, z)$$

and the symbol ~ should be read "whose components are".

The space $z \geq o$ is then replaced by a pseudo dielectric for which

$$\epsilon(\underline{r}') - 1 = -\frac{N_o q^2}{m\epsilon_o \omega(\omega+i\nu)} e^{-\frac{\rho^2}{a^2}} e^{-\frac{z}{b}}$$

$$= -\alpha e^{-\frac{\rho^2}{a^2}} e^{-\frac{z}{b}}$$

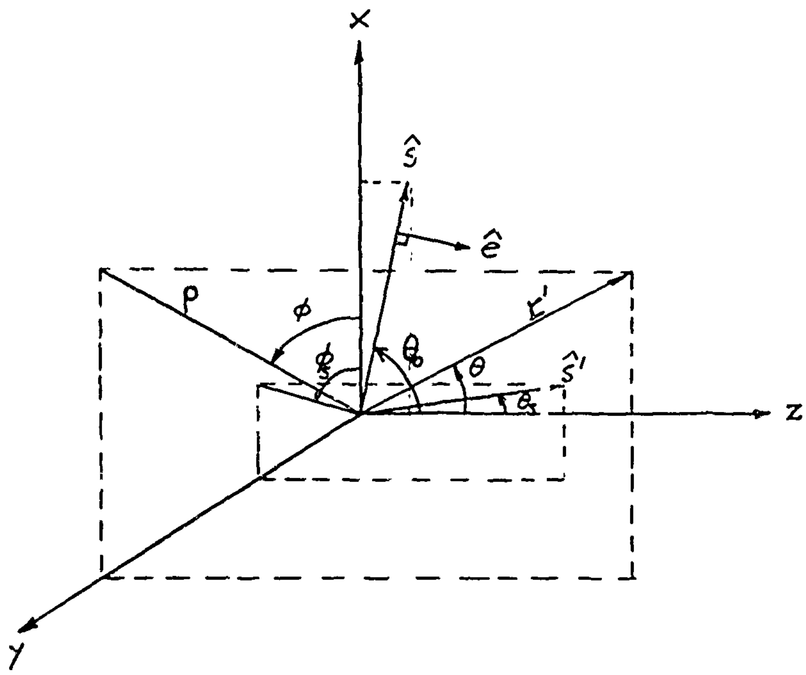
where

$$\alpha = \frac{N_o q^2}{m\epsilon_o \omega(\omega+i\nu)}$$ can be easily calculated.

Because of the axial symmetry the incident plane wave may be taken as

$$\underline{E}_o\,(\underline{r}') = E_o \hat{e}\, e^{ik\underline{r}' \cdot \hat{s}}$$

$$= E_o \hat{e}\, e^{ik(\rho \cos \phi \sin \theta_o + z \cos \theta_o)}$$



$\hat{e} \sim (-\cos\theta_o,\ 0,\ \sin\theta_o)$

$x = \rho\cos\phi$

$\underline{r'} \sim (\rho\cos\phi,\ \rho\sin\phi,\ z)$

$y = \rho\sin\phi$

$\hat{s}' \sim (\sin\theta_s\cos\phi_s,\ \sin\theta_s\sin\phi_s,\ \cos\theta_s)$

$z = z$

For backscattering $\hat{s}' = -\hat{s}$

$\rho = r\sin\theta$

hence:

$\hat{s}$ & $\hat{e}$ are in the x-z plane

$\theta_s = 180^\circ - \theta_o$

$\hat{s} \sim (\sin\theta_o,\ 0,\ \sin\theta_o)$

$\phi_s = 180^\circ$

Figure 2

where the direction of the incident wave

$$\hat{s} \sim (\sin \theta_o, \ 0, \ \cos \theta_o)$$

and its direction of polarization

$$\hat{e} \sim (- \cos \theta_o, \ 0, \ \sin \theta_o)$$

are assumed to be in the x-z plane.

The direction of observation is taken as

$$\hat{s}' \sim (\cos \theta_s \cos \phi_s, \ \sin \theta_s \ \sin \phi_s, \ \cos \theta_s)$$

Using the above expressions, the volume integral in Equation II-11, namely

$$\underline{I} \ (\hat{s}', \hat{s}) = \iiint\limits_{\substack{\text{half space} \\ z \geq 0}} [\epsilon \ (\underline{r}') - 1] \ \underline{E} \ (\underline{r}') \ e^{-ik \hat{s}' \cdot \underline{r}'} \, dv'$$

can be easily carried out in steps. Substituting

$$\epsilon \ (\underline{r}') - 1 = - \alpha \, e^{-\frac{\rho^2}{a^2}} \, e^{-\frac{z}{b}}$$

$$\underline{E} \ (\underline{r}') \cong \underline{E}_o \ (\underline{r}') = E_o \, \hat{e} \, e^{\, i \left[k z \cos \theta_o + k \rho \cos \phi \sin \theta_o \right]}$$

and

$$\hat{s}' \cdot \underline{r}' = \rho \sin \theta_s \cos (\phi - \phi_s) + z \cos \theta_s$$

so that,

$$\left. \begin{aligned} \underline{I} \ (\hat{s}', \hat{s}) = - E_o \, \alpha \, \hat{e} \int_0^{2\pi} d \phi \int_0^{\infty} d z \int_0^{\infty} \rho \, d \rho \, e^{-\frac{\rho^2}{a^2}} \, e^{-\frac{z}{b}} \, e^{ik z \cos\theta_o} \, e^{ik \rho \cos \phi \sin \theta_o} \\ \cdot \left[e^{-ik \rho \sin \theta_s \cos (\phi - \phi_s)} \, e^{-ik z \cos \theta_s} \right] \end{aligned} \right\} \quad \text{III-1}$$

Integrating over z:

$$\int_0^\infty dz\, e^{-\frac{z}{b}}\, e^{ik z \left[\cos\theta_o - \cos\theta_s\right]} = \frac{1}{\frac{1}{b} - ik\,(\cos\theta_o - \cos\theta_s}$$

hence,

$$\underline{I}\,(\hat{s}', \hat{s}) = -\frac{\alpha\,\hat{e}\;E_o}{\frac{1}{b} - ik\,(\cos\theta_o - \cos\theta_s)} \int_0^{2\pi} d\phi \int_0^\infty \rho\, d\rho\, e^{-\frac{\rho^2}{a^2}}\, e^{ik\,\rho\left[\cos\phi\,\sin\theta_o - \sin\theta_s \cos(\phi-\phi_s)\right]}$$

To integrate over ϕ one writes:

$$\cos\phi\sin\theta_o - \sin\theta_s\cos(\phi-\phi_s) = \cos\phi\sin\theta_o - \sin\theta_s\,(\cos\phi\cos\phi_s + \sin\phi\sin\phi_s)$$

$$= \cos\phi\left[\sin\theta_o - \sin\theta_s\cos\phi_s\right] - \sin\phi\left[\sin\theta_s\sin\phi_s\right]$$

$$= d\cos(\phi + \phi_d)$$

where

$$d = \sqrt{(\sin\theta_o - \sin\theta_s\cos\phi_s)^2 + \sin^2\theta_s\sin^2\phi_s}$$

$$= \sqrt{\sin^2\theta_o - 2\sin\theta_o\sin\theta_s\cos\phi_s + \sin^2\theta_s}$$

and

$$\tan\phi_d = \frac{\sin\theta_s\sin\phi_s}{\sin\theta_o - \sin\theta_s\cos\phi_s}$$

From the well known relation

$$\int_0^{2\pi} d\phi\, e^{i(k\rho d)\cos(\phi+\phi_d)} = 2\pi J_o(kd\rho)$$

one obtains:

$$\underline{I}\,(\hat{s}', \hat{s}) = \frac{-\alpha\,\hat{e}\;2\pi\,Eo}{\frac{1}{b} - ik\,(\cos\theta_o - \cos\theta_s} \int_0^\infty \rho d\rho\, e^{-\frac{\rho^2}{a^2}}\, J_o(kd\rho)$$

Finally, integration over ρ yields

$$\underline{I}(\hat{s}', \hat{s}) \frac{-\alpha \hat{e}\, 2\pi E_o}{\frac{1}{b} - ik(\cos\theta_o - \cos\theta_s)} \cdot \frac{a^2}{2}\, e^{-\frac{k^2 d^2 a^2}{4}} \qquad \text{III-2}$$

Making now use of the identity

$$(\hat{s}' \times \hat{e})^2 = 1 - (\hat{s}' \cdot \hat{e})^2 = 1 - [\cos\theta_s \sin\theta_o - \cos\theta_o \sin\theta_s \cos\phi_s]^2$$

the cross section can be written as:

$$\sigma(\hat{s}', \hat{s}) = \frac{\left[\frac{N_o q^2}{m\epsilon_o}\right]^2}{\omega^2(\omega^2 + \nu^2)} \left[\frac{k^4 a^4}{(\frac{1}{b})^2 + k^2(\cos\theta_o - \cos\theta_s)^2}\right] \frac{1}{16}\, e^{-\frac{k^2 d^2 a^2}{2}} \qquad \text{III-3}$$

$$\left\{ 1 - [\cos\theta_s \sin\theta_o - \cos\theta_o \sin\theta_s \cos\phi_s] \right\}$$

Equation III-3 is the expression we were looking for. It is the first order approximation to the radar cross section of an ionized wake viewed from an arbitrary angle θ_s when it is illuminated by an electromagnetic plane wave incident at an arbitrary angle θ_o.

Higher order correction terms to equation III-3 will be the object of future work.

The validity of Equation III-3 is doubtful for all ranges of parameters. It is true if the following conditions hold:

$$\text{(a)} \quad \left| \frac{\frac{N_o q^2}{m\epsilon_o}\, e^{-\frac{\rho^2}{a^2}}\, e^{-\frac{z}{b}}}{\omega(\omega + i\nu)} \right| < 1 \qquad \text{III-4}$$

$$\text{(b)} \quad \nu << \omega \qquad \text{III-5}$$

However, on account of its simplicity a wide range of calculations should be carried out. It can be used at least as the first order estimation of the aspect effect on radar cross-section.

In the case of backscattering, which is of special interest,

$$\theta_s = 180^\circ - \theta_o \quad \text{and} \quad \phi_s = 180^\circ$$

Hence:

$$\cos \phi_s = -\cos \theta_o, \quad \sin \theta_s = \sin \theta_o, \quad \cos \phi_s = -1$$

$$d = 2 \sin \theta_o$$

$$\tan \phi_d = 0$$

and assuming $\nu = 0$, the cross section as a function of aspect angle is given by:

$$\sigma(\theta_o) = \frac{\omega_{p_o}^4}{16\, c^4} \; \frac{a^4}{\frac{1}{b^2} + 4k^2 \cos^2 \theta_o} \; e^{-2a^2 k^2 \sin^2 \theta_o} \qquad \text{III-6}$$

where

ω_{p_o} = rad/sec, the plasma frequency at $z = 0$, $\rho = 0$

c = 3×10^8 m/sec, the velocity of light in vacuum

k = $\frac{\omega}{c}$

and

a & b are scale parameters of the electron distribution.[(1)]

A study of Equation III-6 indicates that as a first order approximation the following conclusions can be drawn:

a. The cross section is proportional to the square to the electron density at the origin.

---

(1) For the case of backscattering at nose-on incidence (i. e. $\theta_o = 0$) and assuming $b >> 1$, the cross section reduces to:

$$\sigma(0) = \frac{1}{64} \; \frac{\omega_{p_o}^4 \, a^4}{c^4 \, k^2} \qquad \text{III-7}$$

This is in exact agreement with the result obtained by S. Altshuler who investigated this special case starting in a different way.

b. The cross section is insensitive to the wake length, except at near broadside view when it varies as the square of the length.

c. The effect of frequency on the cross section is introduced in two ways, by an inverse square relationship and by an exponential to the square of the frequency. Hence the cross section decreases rapidly as frequency increases.

d. For a constant flux of electrons crossing the $z = 0$ plane (see next section) ω_p is inversely proportional to a. Hence the effect of a enters through the exponential term only, and the cross section decreases as a increases.

IV. DETERMINATION OF BACKSCATTERING CROSS-SECTION

For demonstration of the significance of the obtained results, let us substitute values for the parameters in equation III-6 and calculate the backscattering cross section as a function of the aspect angle.

There are four parameters that must be specified before calculations can be made. These are: N_o, f, a, and b.

To obtain a reasonable value for N_o let us take the number of electrons crossing the z = o plane to be 10^{21} electrons/sec, a value typical of wakes of hypersonic bodies, and let its velocity v equal 6,000 m/sec. Then,

$$\int_o^{2\pi} d\theta \int_o^{\infty} v N_o e^{-\frac{\rho^2}{a^2}} \rho \, d\rho = 10^{21} \text{ el/sec}$$

which after integration yields:

$$N_o = \frac{10^{18}}{6\pi a^2} \text{ el/m}^3 \qquad \text{IV-1}$$

At present, flow field studies of wakes indicate that reasonable values for a and b are:

$0.2 \leqslant a \leqslant 1.0$ body diameter

$5 \leqslant b \leqslant 100$ body diameters

and an even greater variation is the value of b, especially at the high end, is possible.

For a one-foot diameter sphere these ranges translate to:

$0.06 \leqslant a \leqslant 0.30$ meter

$1.5 \leqslant b \leqslant 30$ meters

Figures 3, 4, 5 and 6 show the variation of the radar cross section as a function of aspect angle for frequencies in the UHF, L, C and X band respectively, with a as parameter. The values chosen for a are:

$$a = 0.06 \text{ m}$$
and
$$a = 0.15 \text{ m}$$

and represent the range of most probable values of a.

A study of Figures 3 to 8 shows that the cross section has a maximum at $\theta_o = 0^o$, then it decreases as θ_o increases, reaches a minimum somewhere between 45^o and 85^o (the lower the frequency the earlier the minimum occurs) then increases at broadside view. However, due to the exponential relationship to the square of the frequency the cross section falls off so rapidly with aspect angle for frequencies higher than UHF range that it is practically zero for $\theta_o > 10^o$ and although it goes up at $\theta_o = 90^o$ by five or even six orders of magnitude, depending on b, the cross section remains practically zero (e.g. for f = 1.5 KMC/S and a = 0.15 m, $\sigma(85^o) \approx 10^{-16}$ m^2 and $\sigma(90^o) \approx 10^{-11}$ m^2). Hence the curves in Figures 3 to 7 are valid for any of the possible values which b may be expected to take.

That the backscattering cross-section should have a relative maximum for $\theta_o = 0$ was expected because of the assumed geometry of the wake, having a plane surface with the highest electron density at the plane $z = 0$.

However what was unexpected is the fact that the cross-section for $0^o \leqslant \theta_o \leqslant 180^o$ is symmetric about the $\theta_o = 90^o$ value. We believe that higher order corrections on the cross-section as given by III-6 will remove this anomaly.

For $\gamma = 0$, as it was taken to be in the backscattering cross section calculations, the validity conditions given by Equations III-4 and III-5 combine into:

$$\left| \frac{\frac{N_o q^2}{m \epsilon_o} e^{-\frac{\rho^2}{a^2}} e^{-\frac{z}{c}}}{\omega^2} \right| < 1 \qquad \text{IV-2}$$

If this inequality is to be satisfied everywhere in the half space $z \geqslant 0$, we must have:

$$\frac{\frac{N_o q^2}{m \epsilon_o}}{\omega^2} = \frac{\omega_{p_o}^2}{\omega^2} < 1 \qquad \text{IV-3}$$

$$\text{or } \omega_{p_o} < \omega \qquad \text{IV-3a}$$

Inserting in equation IV-3 the value calculated for N_o from equation IV-1 and taking $a = 0.15\,m$ one obtains

$$\omega_{p_o} = 8.66 \times 10^{10} \text{ rad/sec}$$

which implies that for these values of N_o and a one must use in equation III-5 frequencies higher than 13.8 KMC/S for the results to be strictly valid. However, conditions III-4 and III-5 are rather stringent and the results are fairly good for

$$\frac{\omega_{p_o}^2}{\omega^2} < 2$$

or even a little higher. This indicates that equation III-6 can be used to obtain radar cross sections at C and X band frequencies with fairly good results.

Currently calculations for bistatic cross-sections are being carried out and will be reported in a subsequent Technical Note.

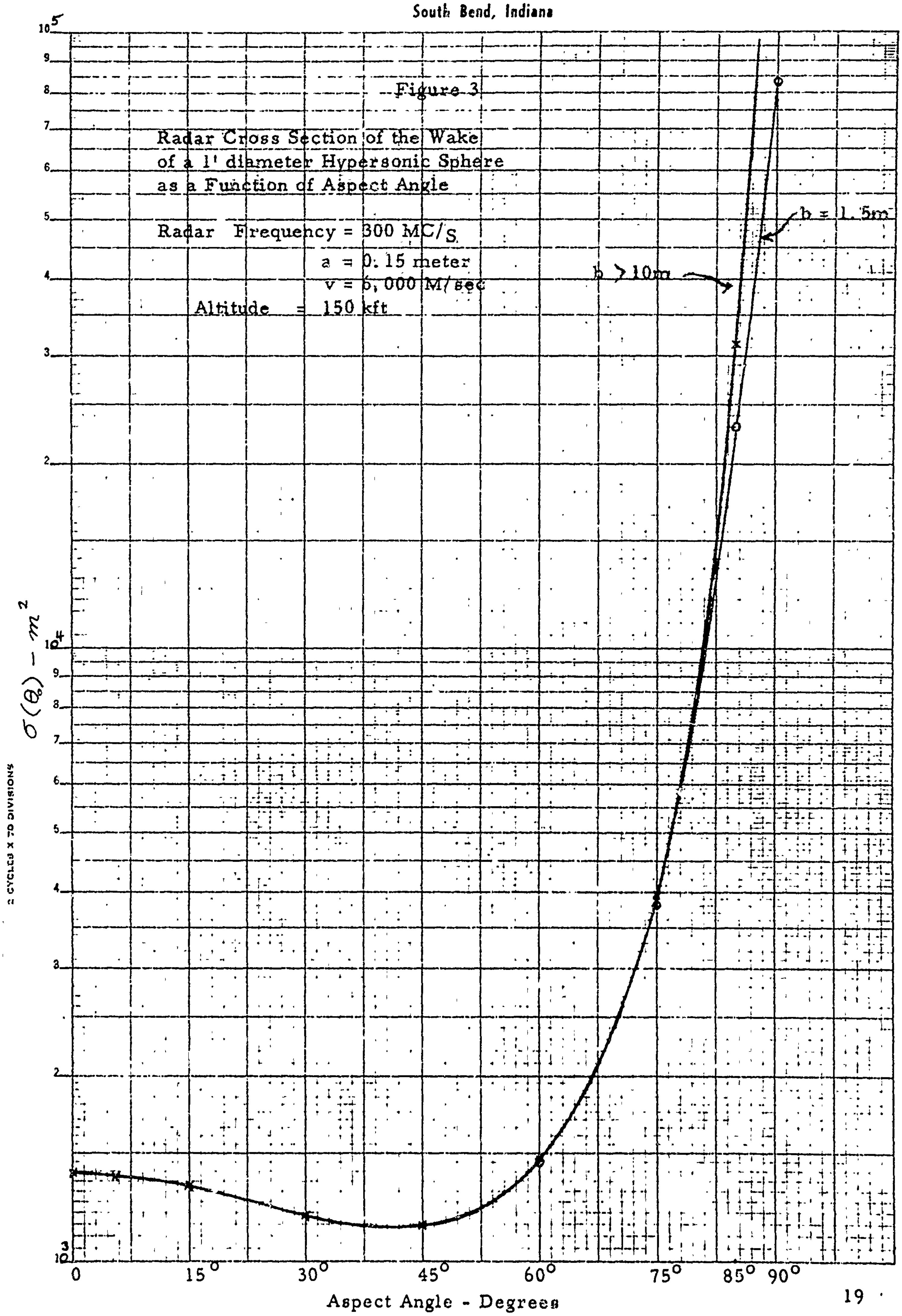
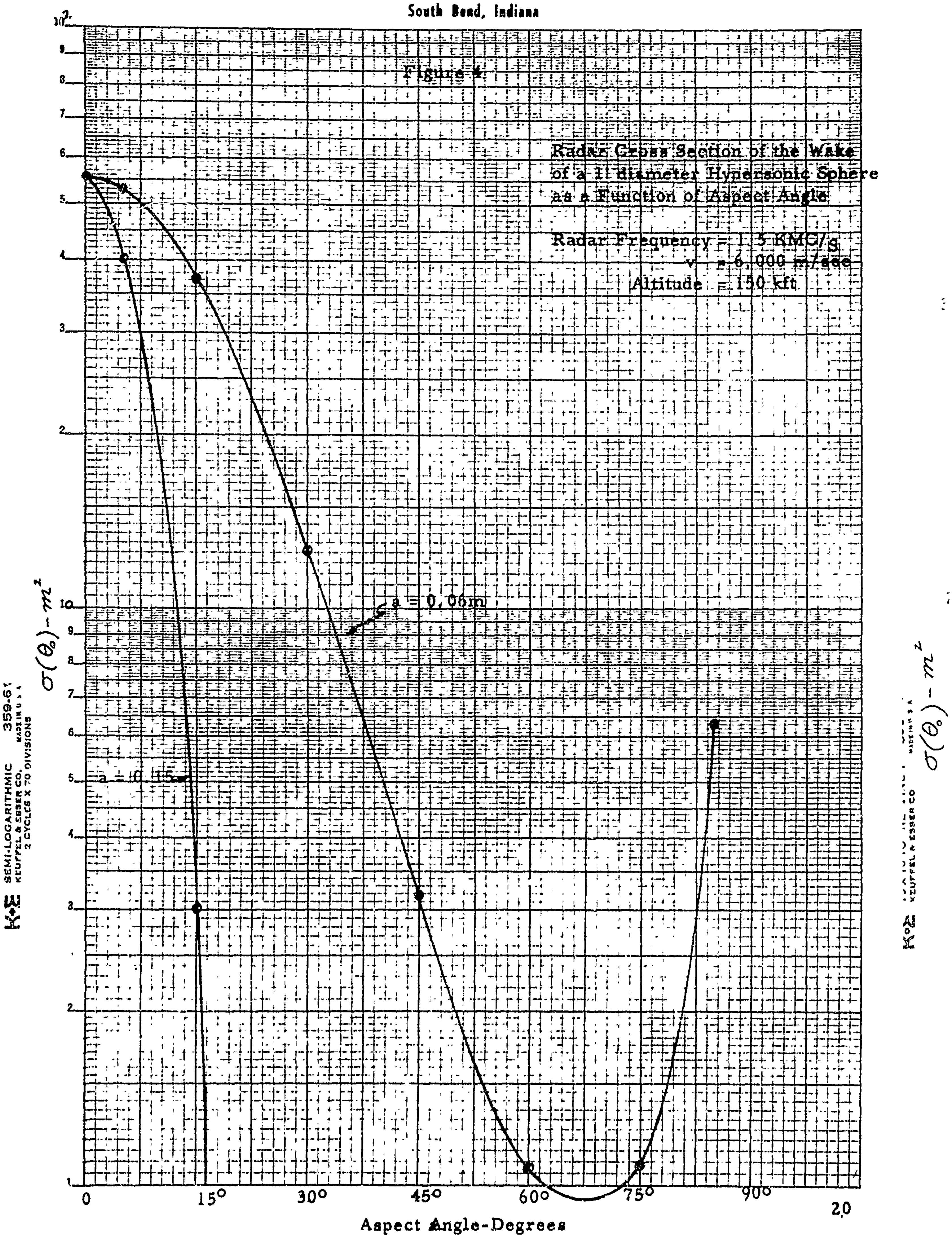


Figure 3
Radar Cross Section of the Wake
of a 1' diameter Hypersonic Sphere
as a Function of Aspect Angle
Radar Frequency = 300 MC/S
a = 0.15 meter
v = 6,000 M/sec
Altitude = 150 kft
b = 1.5m
b > 10m
$\sigma(\theta_o) - m^2$
10^5
10^4
10^3
0
15°
30°
45°
60°
75°
85°
90°
Aspect Angle - Degrees
2 CYCLES X 70 DIVISIONS

BENDIX ADVANCED DEVELOPMENT LABORATORIES

South Bend, Indiana



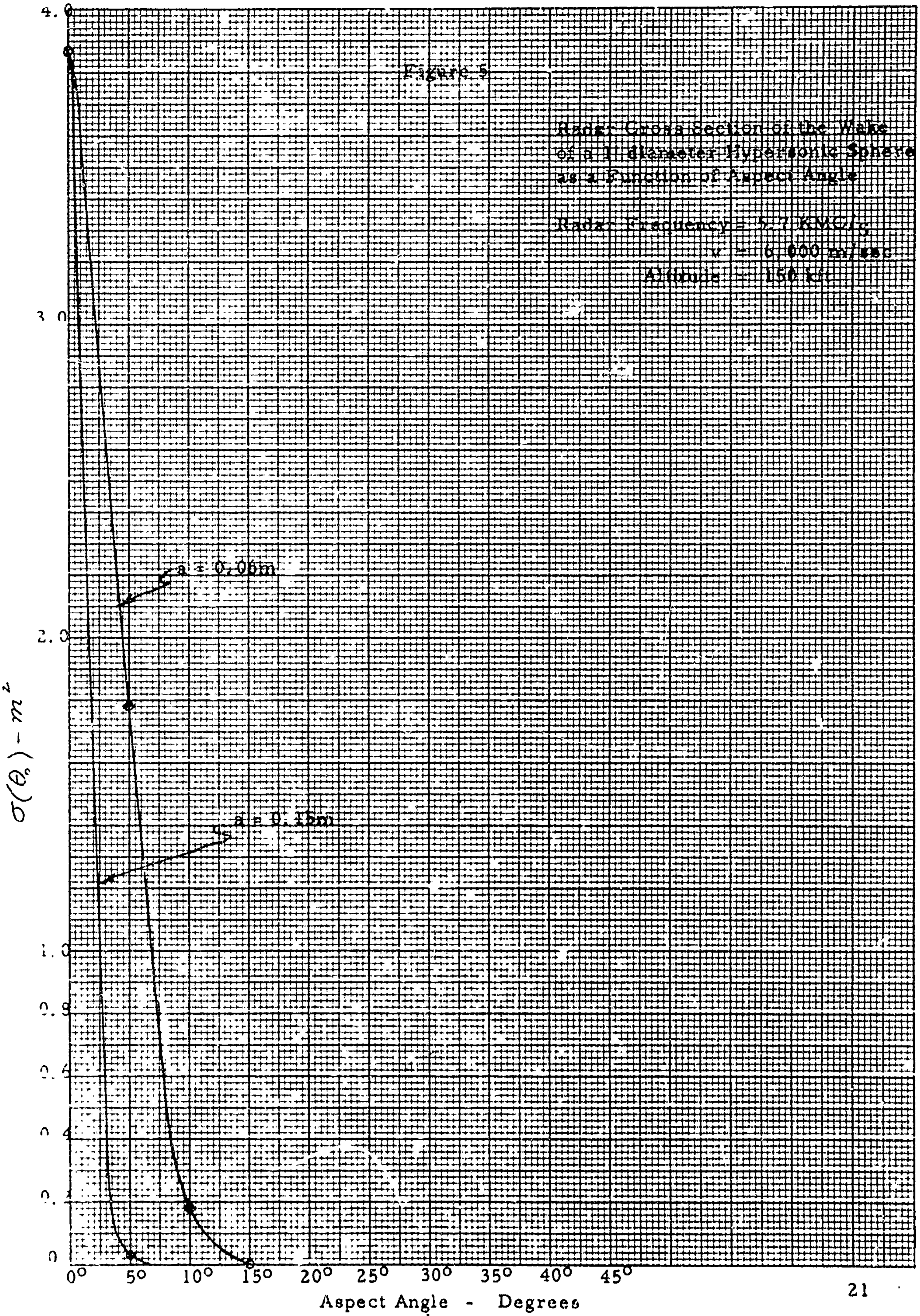


Figure 5
Radar Cross Section of the Wake
of a 1' diameter Hypersonic Sphere
as a Function of Aspect Angle
Radar Frequency = 5.7 KMC/s
v = 6,000 m/sec
Altitude = 150 kft
a = 0.06m
a = 0.15m
$\sigma(\theta_0) - m^2$
4.0
3.0
2.0
1.0
0.8
0.6
0.4
0.2
0
0°
5°
10°
15°
20°
25°
30°
35°
40°
45°
Aspect Angle - Degrees

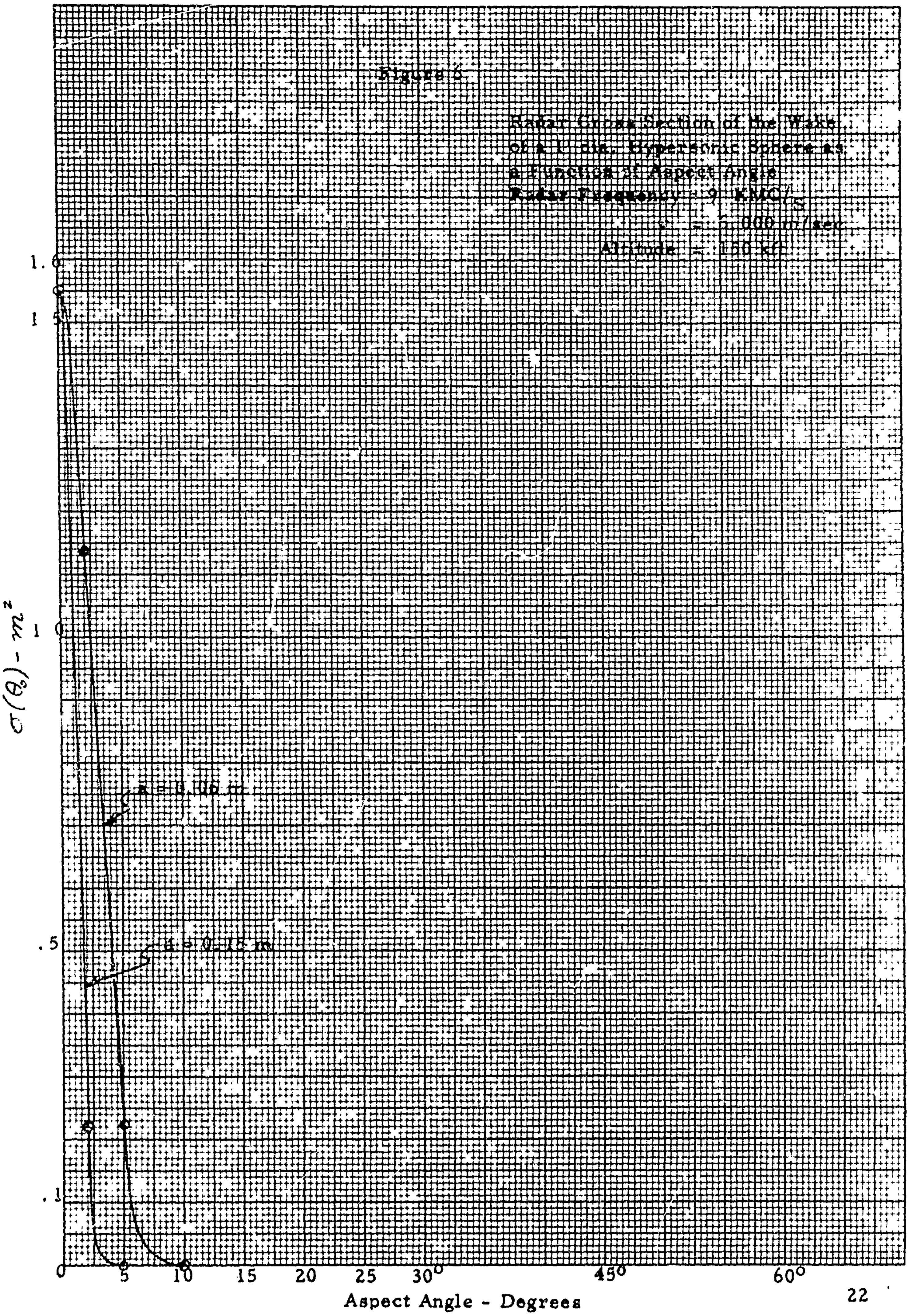


Figure 6
Radar Cross Section of the Wake
of a 1 cm. Hypersonic Sphere as
a Function of Aspect Angle
Radar Frequency = 9 KMC/S
v = 5,000 m/sec
Altitude = 150 kft
$\sigma(\theta_0) - m^2$
1.6
1.5
1.0
.5
.1
0
5
10
15
20
25
30°
45°
60°
Aspect Angle - Degrees

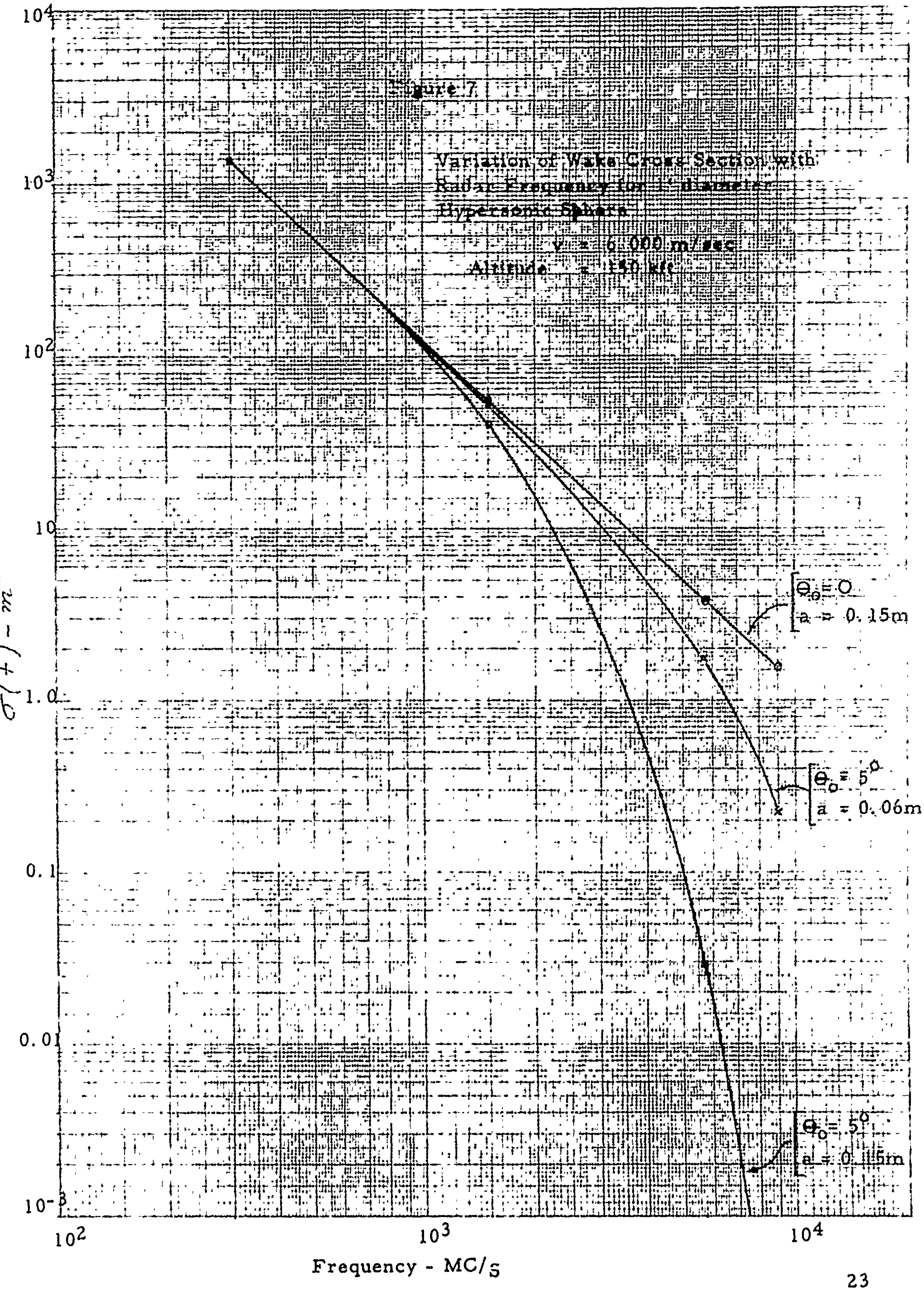


Figure 7
Variation of Wake Cross Section with
Radar Frequency for 1' diameter
Hypersonic Sphere
v = 6,000 m/sec
Altitude = 150 kft
10^4
10^3
10^2
10
1.0
0.1
0.01
10^{-3}
$\sigma(f) - m^2$
$\Theta_o = 0$
a = 0.15m
$\Theta_o = 5^o$
a = 0.06m
$\Theta_o = 5^o$
a = 0.15m
10^2
10^3
10^4
Frequency - MC/S

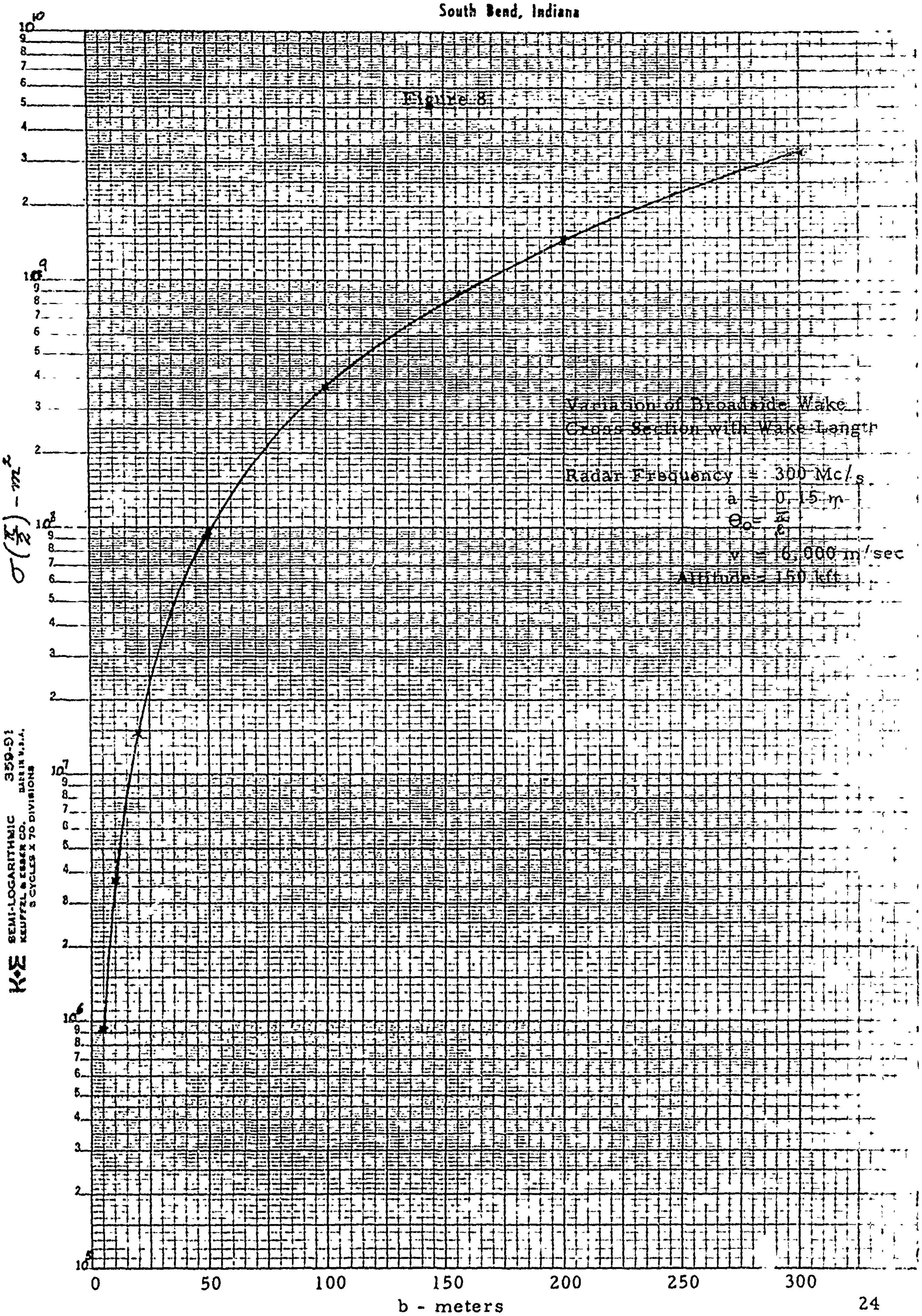


Figure 8
Variation of Broadside Wake Cross Section with Wake Length
Radar Frequency = 300 Mc/s
a = 0.15 m
$\theta_o = \frac{\pi}{6}$
v = 6,000 m/sec
Altitude = 150 kft
$\sigma(\frac{\pi}{2})$ - m^2
b - meters
0
50
100
150
200
250
300
10^{5}
10^{6}
10^{7}
10^{8}
10^{9}
10^{10}
K+E SEMI-LOGARITHMIC 359-91
KEUFFEL & ESSER CO. MADE IN U.S.A.
5 CYCLES X 70 DIVISIONS

APPENDIX

Derivation of Equation II-11.

From Equation II-9,

$$\underline{\Pi} = \iiint\limits_V [\epsilon(\underline{r}') - 1]\, \underline{E}(\underline{r}')\, G(\underline{r}, \underline{r}')\, dV'$$

If the point of observation is at $\hat{s}'\, r$, for $r \to \infty$,

$$G(\underline{r}, \underline{r}') = \frac{e^{ik(\underline{r}-\underline{r}')}}{4\pi(\underline{r}-\underline{r}')} \cong \frac{e^{ikr}}{4\pi r}\; e^{-ik\hat{s}'\cdot\underline{r}'}$$

$$\therefore \underline{\Pi} \cong \frac{e^{ikr}}{4\pi r} \iiint [\epsilon(\underline{r}') - 1]\, \underline{E}(\underline{r}')\, e^{-ik\hat{s}'\cdot\underline{r}'} dV'$$

$$= \frac{e^{ikr}}{4\pi r}\, \underline{I}(\hat{s}', \hat{s})$$

where we formally represent the integral by $\underline{I}(\hat{s}', \hat{s}')$. Now,

$$\underline{H}_s = -i\omega\epsilon_o\, \nabla\times\underline{\Pi} = -\frac{i\omega\epsilon_o}{4\pi}\, \nabla\times\left[\underline{I}(\hat{s}', \hat{s})\, \frac{e^{ikr}}{r}\right]$$

$$= -\frac{i\omega\epsilon_o}{4\pi}\left[\nabla\frac{e^{ikr}}{r}\times\underline{I}(\hat{s}', \hat{s}) + \frac{e^{ikr}}{r}\, \nabla\times\underline{I}(\hat{s}', \hat{s})\right]$$

The terms of order $\frac{1}{r}$ are therefore:

$$\underline{H}_s = \frac{\omega k\epsilon_o}{4\pi}\; \frac{e^{ikr}}{r}\; \hat{s}'\times\underline{I}(\hat{s}', \hat{s}) \qquad \text{II-12}$$

The scattering cross section is therefore,

$$\sigma(\hat{s}', \hat{s}) = \frac{\omega^2 k^2}{|H_o|^2}\; \frac{\epsilon_o^2}{16\pi^2}\; \left|\left[\hat{s}'\times \iiint [\epsilon(\underline{r}') - 1]\, \underline{E}(\underline{r}')\; e^{-ik\hat{s}'\cdot\underline{r}'} dV'\right]\right|^2$$