LUNAR-SOLAR PERTURBATIONS OF THE ORBIT OF AN EARTH SATELLITE

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Introduction

Shortly after the launching of Able-3 (1959 delta), Y. Kozai of the Smithsonian Astrophysical Observatory predicted that the lifetime of this highly eccentric satellite would be greatly shortened by the influence of the moon. To confirm this prediction, the following simplified analysis of the lunar-solar perturbations has been made. It is assumed that the Kepler ellipse of the satellite is perturbed only by the sun and moon, ignoring, for the moment, the perturbations of the orbit by the earth's oblateness and atmosphere. It is also assumed that the angular velocities of the sun and moon are small enough compared to the angular velocity of the satellite that we may consider the sun and moon fixed during one revolution of the satellite. This simplifying assumption makes possible the integration of the instantaneous rate of change of the orbital elements over one revolution of the satellite to obtain the change in orbital elements per revolution. The magnitude of the error made by holding the disturbing body fixed is estimated. The results are applied to the orbit of Able-3 (1959 delta).

The Changes in the Orbital Elements

We start with Moulton's equations* giving the rate of change of the orbital elements of the instantaneous osculating ellipse when the perturbing acceleration has a radial component R, a transverse component S in the plane of the instantaneous ellipse, and a component W normal to the plane. The rates of change of the semi-major axis, a, and the eccentricity, e, are affected only by the in-plane components R and S.

* Moulton, An Introduction to Celestial Mechanics, Second Revised Edition, pp.404-405.

The rates are:

$$\frac{da}{dt} = \frac{2e \sin v}{n\sqrt{1-e^2}} R + \frac{2a\sqrt{1-e^2}}{nr} S \tag{1}$$

and

$$\frac{de}{dt} = \frac{\sqrt{1-e^2} \sin v}{na} R + \frac{\sqrt{1-e^2}}{na^2 e} \left[\frac{a^2 (1-e^2)}{r} - r \right] S, \tag{2}$$

where n is the satellite's mean angular motion, r is the radial distance of the satellite from the center of the earth, and v is the true anomaly of the satellite's orbit. From the definition of perigee,

$$q = a\ (1-e),$$

we obtain for the rate of change of perigee,

$$\frac{dq}{dt} = \frac{(1-e)^2}{n\sqrt{1-e^2}} \left[(-\sin v) R + \left(\frac{2 + e - \cos v}{1 + e \cos v} - \cos v \right) S \right] . \tag{3}$$

The orbital inclination i, the node Ω, and the argument of perigee ω (all measured relative to the plane of motion of the disturbing body) change at the rates

$$\frac{di}{dt} = \frac{r \cos(\omega + v)}{na^2 \sqrt{1-e^2}} W , \tag{4}$$

$$\frac{d\Omega}{dt} = \frac{r \sin(\omega + v)}{na^2 \sqrt{1-e^2} \sin i} W , \tag{5}$$

and

$$\frac{d\omega}{dt} = - \cos i \frac{d\Omega}{dt} - \frac{\sqrt{1-e^2}}{nae} \left[R \cos v - S \frac{2 + e \cos v}{1 + e \cos v} \sin v \right] . \quad (6)$$

The components of the disturbing acceleration*, are

$$R = GM_D \left\{ - \frac{r}{\rho^3} + a_D \left[\frac{1}{\rho^3} - \frac{1}{a_D^3} \right] \left[\cos \gamma \cos u + \sin \gamma \sin u \cos i \right] \right\} \quad (7)$$

$$S = GM_D \, a_D \left[\frac{1}{\rho^3} - \frac{1}{a_D^3} \right] \left[- \cos \gamma \sin u + \sin \gamma \cos u \cos i \right] \quad (8)$$

and

$$W = - GM_D \, a_D \sin \gamma \sin i \left[\frac{1}{\rho^3} - \frac{1}{a_D^3} \right] \quad (9)$$

where G = the universal gravitational constant

M_D = the mass of the disturbing body (sun or moon)

u = $\omega + v$

ω = argument of perigee

ρ = distance from the satellite to the disturbing body

a_D = distance between the earth and the disturbing body

i = inclination of the satellite's orbit to the plane of the disturbing body

γ = angle between the line of nodes and a_D

* Moulton, page 340.

These symbols are further clarified by reference to Figure 1 in which the plane of motion of the disturbing body is taken as the x-y plane. We shall assume that a_D is a constant. In other words, the orbits of the moon and apparent sun are assumed circular. If r is always small compared to a_D, the accelerations can be expanded in powers of $\frac{r}{a_D}$. Then equations (7), (8) and (9) become, to first order,

$$R = K_D \, r \, (1 + 3 \cos 2 \phi), \tag{10}$$

$$S = - 6 \, K_D \, r \cos \phi \left[\cos \gamma \sin u - \sin \gamma \cos u \cos i \right], \tag{11}$$

and

$$W = - 6 \, K_D \, r \cos \phi \sin i \sin \gamma, \tag{12}$$

where ϕ is the angle between r and a_D, and $K_D = \frac{GM_D}{2a_D^{\ 3}}$.

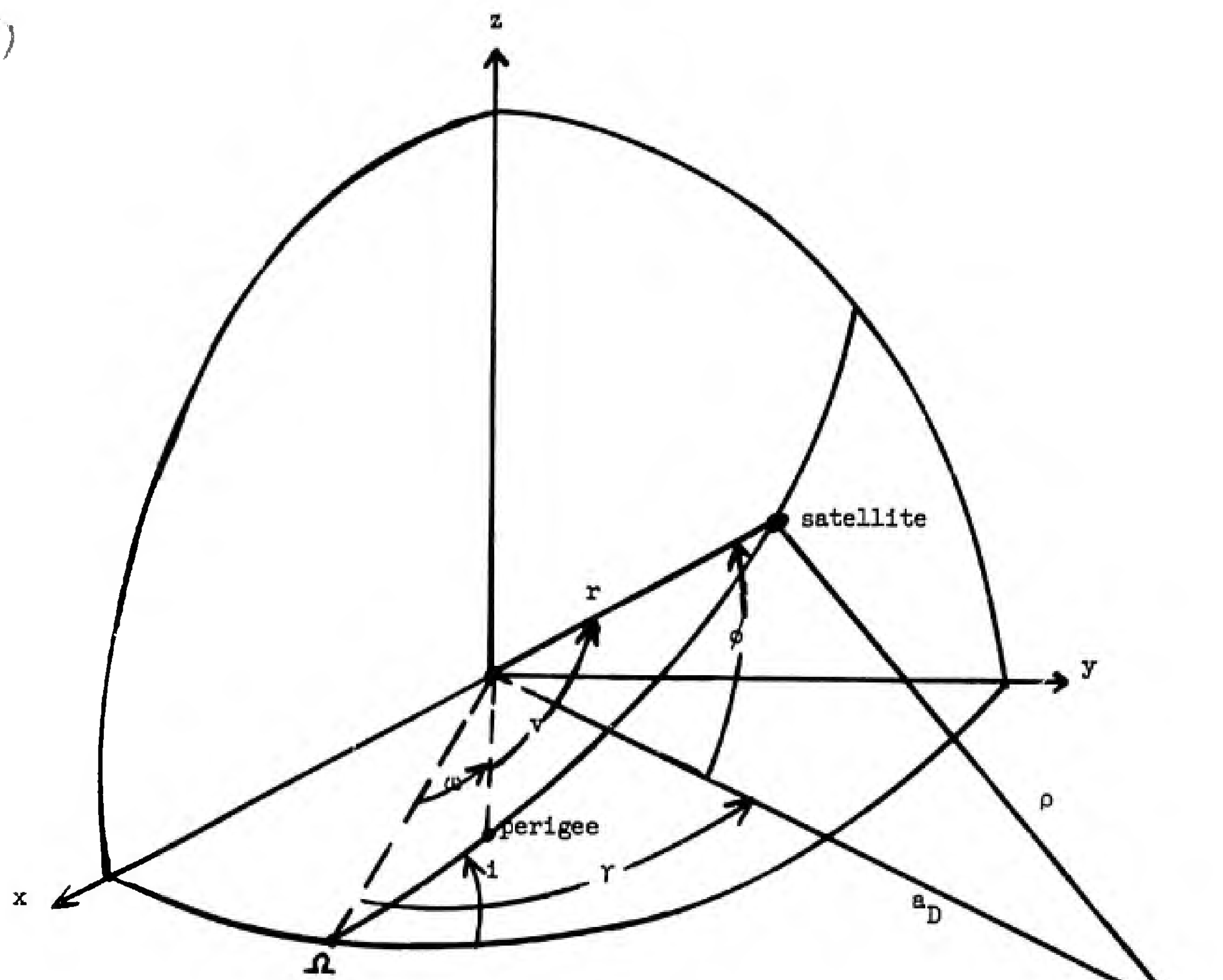

Figure 1

Ω	=	node.
a_D	=	distance from earth to disturbing body, D.
ρ	=	distance from satellite to disturbing body.
r	=	distance from earth to satellite.
ω	=	argument of perigee.
i	=	inclination of satellite's orbital plane to the orbital plane of the disturbing body.
ϕ	=	angle between r and a_D.
γ	=	angular position of the disturbing body, relative to the node.

The change in any orbital element ϵ after one revolution of the satellite can now be determined by using the relation

$$r^2 \frac{dv}{dt} = h = na^2 \sqrt{1 - e^2} \tag{13}$$

for the angular momentum h per unit mass in a Kepler orbit. Calling $\Delta\epsilon$ the change in any orbital element per revolution, we obtain,

$$\Delta\epsilon = \int_{v=0}^{v=2\pi} \frac{d\epsilon}{dt} dt = \int_0^{2\pi} \frac{d\epsilon}{dt} \frac{r^2}{h} dv. \tag{14}$$

During a single revolution the path of the satellite is taken to be that of the unperturbed Kepler ellipse, so wherever a and e appear, they are treated as constants. Since the integration is to be carried out over the true anomaly v, we use the Kepler equation $r = \frac{a(1 - e^2)}{1 + e \cos v}$ and rewrite equations (10), (11) and (12) as

$$R = 2K_D r \left\{ -1 + 3 \left[\frac{1}{2} \cos i \sin 2\gamma (\sin 2\omega \cos 2v + \sin 2v \cos 2\omega) + \left(\cos^2\gamma - \sin^2\gamma \cos^2 i\right)\left(\cos^2\omega \cos^2 v + \sin^2\omega \sin^2 v - 2 \sin\omega \cos\omega \sin v \cos v\right) + \sin^2\gamma \cos^2 i \right] \right\} \tag{15}$$

$$S = -3 K_D r \left\{ (\cos^2\gamma - \sin^2\gamma \cos^2 i) \left[\sin 2\omega \cos 2v + \sin 2v \cos 2\omega\right] + \sin 2\gamma \cos i \left[1 - 2 \left(\cos^2\omega \cos^2 v + \sin^2\omega \sin^2 v - 2 \sin v \cos v \sin\omega \cos\omega\right)\right] \right\} \tag{16}$$

and

$$W = -6K_D r \sin i \sin\gamma \left\{ \begin{aligned} &\cos\gamma\,(\cos\omega\cos v - \sin\omega\sin v) \\ &+ \sin\gamma\cos i\,(\sin\omega\cos v + \cos\omega\sin v) \end{aligned} \right\} . \qquad (17)$$

Since the angular velocity of the satellite is usually large compared to the angular velocity of the disturbing body, we assume that γ is constant during the time it takes for the satellite to make one revolution about the earth. Then integrals of the type (14) can be evaluated easily. The results are

$$\Delta a = 0 , \qquad (18)$$

$$\Delta q = \frac{15\,K_D\, a\,\pi\, e\,\sqrt{1-e^2}}{n^2} \left\{ \sin 2\gamma \cos 2\omega \cos i - \sin 2\omega \left[\cos^2\gamma - \sin^2\gamma \cos^2 i \right] \right\} \qquad (19)$$

$$\Delta e = -\frac{1}{a}\,\Delta q ,$$

$$\Delta i = \frac{-6\,K_D \pi \sin i \sin\gamma}{n^2\sqrt{1-e^2}} \left[5e^2 \sin\omega\cos\omega\sin\gamma\cos i + \cos\gamma\left[1 + e^2(1-5\cos^2\omega)\right]\right] \qquad (20)$$

$$\Delta\Omega = \frac{-6\,K_D \pi \sin\gamma}{n^2\sqrt{1-e^2}} \left[5e^2 \sin\omega\cos\omega\cos\gamma + \sin\gamma\cos i \left[(1-e^2)\cos^2\omega + (1+4e^2)\sin^2\omega \right] \right] \qquad (21)$$

and

$$\Delta\omega = -\cos i\,\Delta\Omega - \frac{6\,K_D\pi\sqrt{1-e^2}}{n^2} \left[\begin{aligned} &1 - 3\sin^2\gamma\cos^2 i - 10\sin\gamma\cos\gamma\sin\omega\cos\omega\cos i \\ &+ (5\sin^2\omega - 4)(\cos^2\gamma - \sin^2\gamma\cos^2 i) \end{aligned} \right] \qquad (22)$$

If, now, we let γ take values from zero to 2π, we see how the orbital elements change as a function of the time of month or year.

Error due to Neglecting the Motion of the Disturbing Body

To estimate the error we have made by taking γ constant, let us consider the special case of co-planar motion of the satellite and the disturbing body ($i = 0$). Let us take the node at perigee, so that $\omega = 0$. Then equation (19) for the change in perigee per revolution becomes simply

$$\Delta q_o = \frac{15 K_D a\pi e \sqrt{1-e^2}}{n^2} \sin 2\gamma_o, \tag{23}$$

where we have used the subscript zero to indicate that the disturbing body is held fixed. Now let us evaluate Δq when the disturbing body is moving by taking

$$\gamma = \gamma_o + n_d t$$

where n_d is the angular velocity of the disturbing body. Equation (14) for Δq will contain integrals of the form:

$$I = \int f(\cos v, \sin v, \cos 2\gamma, \sin 2\gamma)\, dv , \tag{24}$$

where f stands for some function.

We replace cos 2γ and sin 2γ by their Taylor expansions:

$$\cos 2\gamma = \cos 2\gamma_o - 2n_d t \sin 2\gamma_o + \ldots\ldots$$

$$\sin 2\gamma = \sin 2\gamma_o + 2n_d t \cos 2\gamma_o + \ldots\ldots$$

and retain only the first two terms. Our integral now contains the two variables: v, the true anomaly, and t, the time. These may be related through the eccentric anomaly E via the relations

$$t = \frac{E - e \sin E}{n}$$

$$dv = \frac{\sqrt{1-e^2}}{1-e\cos E}\, dE$$

$$\sin v = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E}$$

and

$$\cos v = \frac{\cos E - e}{1-e\cos E} \quad .$$

Our integral is now of the form

$$I = \int F(\gamma_o\ , E, \sin E, \cos E)dE,$$

where F stands for some new function. This type of integral is readily evaluated and yields for Δq

$$\Delta q = \Delta q_o + \frac{K_D \pi a(1-e)^2}{n^2 \sqrt{1-e^2}} \left(\frac{a}{a_D}\right)^{3/2} \left[30\, e\pi \left(\frac{1+e}{1-e}\right)\cos 2\gamma_o + \frac{\sin 2\gamma_o}{\sqrt{1-e^2}}\left(4 + 45e + 58e^2 + 17e^3\right) \right]$$

or

$$\Delta q = \frac{15 K_D \pi a e \sqrt{1-e^2}}{n^2} \sin 2\gamma_o \left[1 + \left(\frac{a}{a_D}\right)^{3/2} \left\{ 2\pi \frac{\cos 2\gamma_o}{\sin 2\gamma_o} + \frac{(1-e)(4 + 45e + 58e^2 + 17e^3)}{15e\ (1+e)\ \sqrt{1-e^2}} \right\} \right]$$

which shows that the additional term, contributed by the motion of the disturbing body, is of the order of $\left(\frac{a}{a_D}\right)^{3/2}$ on the average. Since $\frac{a}{a_D}$ is usually less than 0.1, this term contributes less than 3 per cent error.

An Example

Let us determine the effect of solar-lunar perturbations on the perigee height of 1959 delta during the first month of its lifetime. Assuming that the orbital elements do not change significantly over this period, we may average equation (19) over one month to obtain for the average value of Δq (due to the moon)

$$\overline{\Delta q_m} = - \frac{15 K_m a \pi e \sqrt{1-e^2}}{2 n^2} \sin 2\omega \sin^2 i$$

where the subscript m indicates that the disturbing body is the moon. With a = 15,000 nautical miles, e = .76, i = 41.5, and $\omega = 10.4^\circ$, we obtain

$$\overline{\Delta q}_m = - .061 \text{ nautical miles per revolution.}$$

Since the satellite made about 60 revolutions in the first month, perigee dropped by about 3.7 nautical miles during that time as a result of the moon's perturbing force.

The sun's influence, on the other hand, tends, at first, to raise perigee. Relative to the sun's plane, the argument of perigee started at 2.7°, the orbital inclination was 40.3°, and γ_s was about 48°. The initial change in perigee per revolution, determined from equation (19) is therefore

$$\Delta q_s = +.26 \text{ nautical mile per revolution.}$$

At the end of one month

$$\Delta q_s = +.14 \text{ nautical mile per revolution.}$$

Eventually, the sun will move to such an angular position that Δq_s will be negative. The average effect over a year is a lowering of perigee. This is true of both solar and lunar effects so long as sine $2\omega_s$ and sine $2\omega_m$ are positive.

Figure 2, showing Δq_s and Δq_m for the first month, agrees with the results of STL's numerical integration program for satellite orbits. By summing equation (19) over the proper number of revolutions, (taking into account the change in ω and i), we have determined the behavior of perigee height as a function of time during the four months following launch. The decreasing trend of the curve and the oscillations around this trend show the proper behavior upon comparison with a similar curve obtained from the numerical integration program.

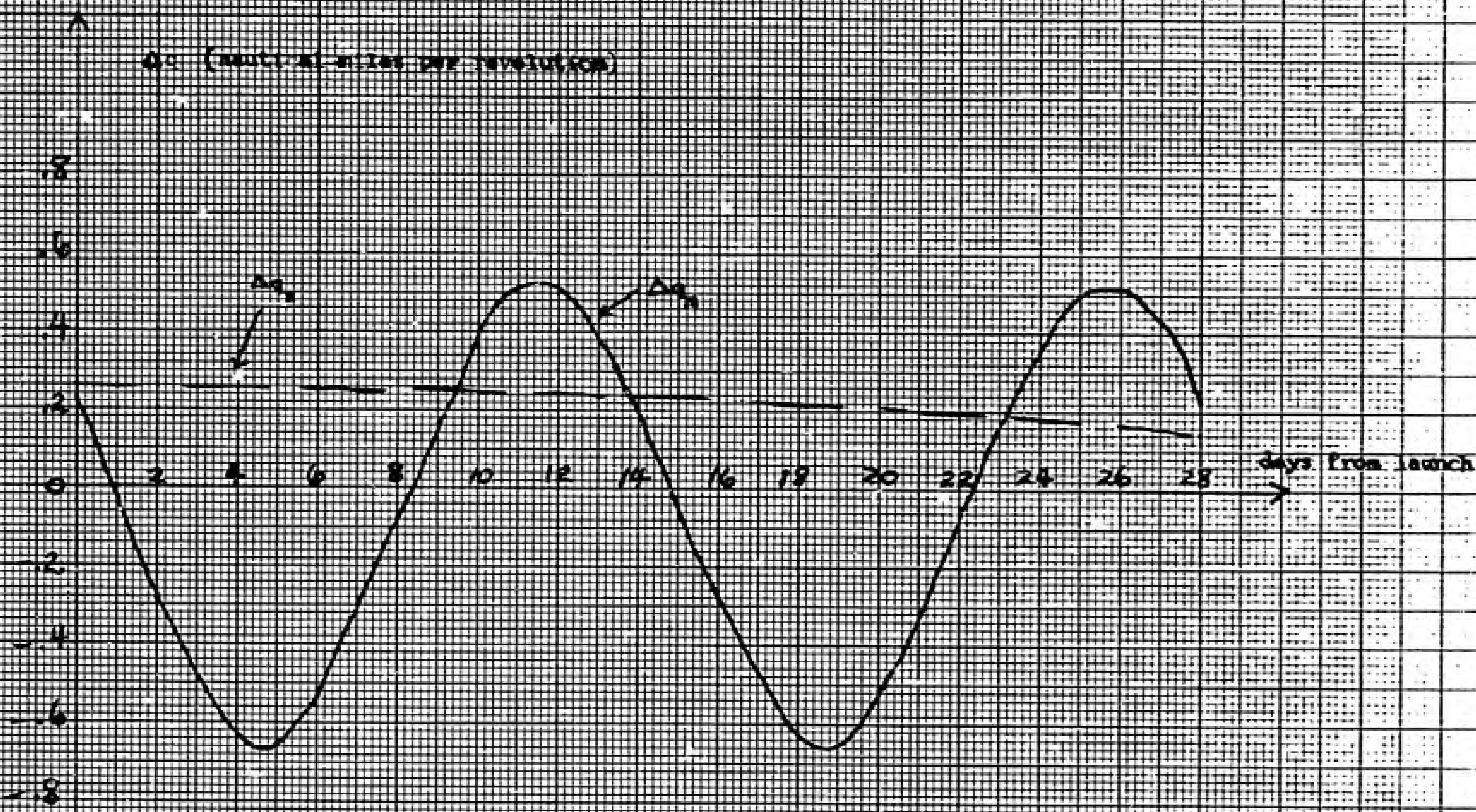

Fig. 2 SOLAR-LUNAR EFFECTS ON THE PERIGEE OF 1959 DELTA

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