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THE EFFECT OF WALL ELASTICITY AND SURFACE TENSION ON THE FORCED OSCILLATIONS OF A LIQUID IN A CYLINDRICAL CONTAINER (PART I ANALYSIS)

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Nomenclature

A_n, B_n	Constants
B_M	Membrane number, see Eq. (16)
B_σ	Bond number, see Eq. (16)
c_n, d_n	Amplitude of the n^{th} sloshing mode
F_1	Potential of the solid-liquid-gas interface
F_3	Potential of the edge load acting on the rim of the tank bottom
$g(t)$	Gravitational acceleration, time dependent, see Eq. (6a)
g_0	Mean local gravitational acceleration
g_1	Amplitude of the imposed axial acceleration
$G(\tau)$	Nondimensional gravitational acceleration, see Eq. (15)
h	Membrane thickness
H	Nondimensional free surface shape, see Eq. (15)
I, I_1, I_2, I_3, I_4	Functionals, see Eqs. (36)
J_0, J_1	Bessel functions of first kind
k_n	n^{th} root of the equation $J_0(k_n) = 0$
ℓ	Depth of liquid
L	Nondimensional depth of liquid, see Eq. (15)
L_1	Pressure energy in nondimensional form, see Eq. (41)
L_2, L_3	Lagrangians in nondimensional form, see Eqs. (43), (44)
M_1, M_2, M_3	Matrix
N_r	Midplane stress resultant
$p_n\omega$	n^{th} sloshing frequency for rigid tank, see Eq. (57)
p	Pressure
P	Nondimensional pressure, see Eq. (15)

r_0	Radius of the tank
(r, θ, z)	Cylindrical coordinates, see Fig. 1
(R, θ, Z)	Nondimensional cylindrical coordinates, see Fig. 1
S_1, S_2, S_3	Surfaces, see Fig. 1
t	Time
U	Modal matrix
w	Transverse deflection of membrane
W	Nondimensional transverse deflection of membrane, see Eq. (15)
z	Vertical coordinate, see Fig. 1
Z	Nondimensional vertical coordinate, see Eq. (15)
α	Nondimensional amplitude of the imposed axial acceleration, see Eq. (15)
γ_n	Constants, see Eq. (59)
Γ_1, Γ_3	Boundary curves of S_1, S_3 respectively
$\delta_1, \delta_2, \delta_3$	Constants, see Eq. (66)
η	Free surface shape
θ	Azimuthal coordinate
λ	Mass ratio, see Eq. (16)
Λ_n	Constants, see Eq. (54a)
μ_n, ν_n	Constants, see Eqs. (51), (52)
ρ	Density of membrane
ρ_0	Density of liquid
σ	Surface tension
τ	Nondimensional time, see Eq. (15)
ϕ	Velocity potential
Φ	Nondimensional velocity potential, see Eq. (15)
ω	Forcing frequency

ABSTRACT

The stability of a fluid contained in a circular cylindrical tank with a flat, flexible bottom under a periodic axial excitation is studied. An analytical difficulty for the solution of the linearized equations in the form of infinite series is discussed. A variational approach is formulated. An approximate solution results in a pair of coupled ordinary differential equations with periodic coefficients. A method of handling the stability of the solutions of such a system of equation is presented. Numerical results will be discussed in a later article.

Ω_1^2, Ω_2^2	Eigenvalues, see Eq. (57)
Ω_M^2	Frequency parameter for the membrane, see Eq. (16)
Ω_T^2	Frequency parameter for surface tension, see Eq. (16)
$\nabla^2, \bar{\nabla}^2$	Two dimensional Laplace operator $\left(\frac{1}{r_0^2}\nabla^2 = \bar{\nabla}^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right.$ in cylindrical coordinates $\left.\right)$
$\nabla, \bar{\nabla}$	Three dimensional gradient operator $\left(\frac{1}{r_0}\nabla = \bar{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z}\right.$ in cylindrical coordinates. For the free surface and the membrane, ignore the $\vec{e}_z \frac{\partial}{\partial z}$ term. $\left.\right)$

1. Introduction

Dynamics of large liquid-fuel rockets naturally involves the motion of a liquid in a flexible container. The symmetric modes of the fluid motion, which influences the pressure at the tank bottom, and thence influencing the pressure in the pump, combustion chamber, and thrust and rocket acceleration, have an important effect on the rocket-structural dynamics. In some instances the longitudinal oscillations were so serious as to affect the safety of the vehicle. For this reason the analysis of the forced oscillations of the liquid container is important.

At ground level perhaps the effects of the flexibility of the tank wall and the surface tension of the free surface are negligible on fuel sloshing. At reduced gravity conditions these effects will become more evident. It is the purpose of this article to evaluate the effects of tank flexibility and surface tension on the stability of liquid motion in the symmetric modes.

Sloshing of liquids has been studied by many authors. Most of them considered rigid containers, see Ref. 4. However, Miles (Ref. 1) considered bending modes of a flexible container. Bleich (Ref. 2) investigated the longitudinal modes approximately. Recently Bhuta and Koval (Ref. 3 and 15) studied the coupled oscillations of a liquid in a tank with a flexible bottom. They defined the normal modes of the system, and treated the orthogonality and expansion theorems. However, the difficulty concerning the convergence of several infinite series, to be explained in Section 5 below, was not considered. In

Refs. 16 and 17, Bhuta and Yeh considered the problem of arbitrarily assigned velocity distribution on the tank bottom; in this case the difficulty referred to above does not appear.

On the other hand, there is substantial literature about the influence of surface tension on sloshing; see Yeh's bibliography, Ref. 18, and papers by Bond and Newton (Ref. 19), and Reynolds (Ref. 20). However, most of these studies are concerned with free oscillations, very little has been done about the influence of surface tension on forced oscillations. No work seems to have been done on the coupling with the flexibility of the tank.

In the present paper a circular tank with a flexible bottom under vertical poriodial excitation is studied. The problem was first formulated in the form of differential equations and then in the form of a variational principle. An approximate solution is presented, which results in a pair of coupled ordinary differential equations with periodic coefficients. The stability of the solutions of these equations is discussed.

2. Statement of the Problem

A circular cylindrical container with rigid side walls and a flat, flexible bottom contains a liquid with a free surface. The tank walls are subjected to an oscillatory axial acceleration, in addition to a constant mean-local-gravitational acceleration which is directed along the axis of the cylinder. A gas with constant pressure exists above the liquid surface. No external force acts underneath the tank bottom. The situation is pictured in Fig. 1. The problem is to determine the motion of the liquid; in particular, its stability.

The fluid properties including the surface tension are assumed to be uniform, constant, incompressible, and inviscid.

The mean free surface of the liquid shall be assumed to be a plane perpendicular to the cylinder axis. In low gravity and finite surface tension one may have to consider a curved mean free surface. The governing criterion is the Bond number defined below. In this paper we shall assume that the Bond number is sufficiently large so that the free surface is approximately a plane. The case of low Bond number will be investigated in a separate article.

As a further simplification we assume that the deviation from the static equilibrium condition is small, so that the deflections of the free surface and of the tank bottom, the fluid velocity, and hence the velocity potential, may be considered as infinitesimal quantities of the first order. Under this assumption all the equations can be linearized, and the mathematical problem is relatively simple. A number of interesting nonlinear problems are ruled out by this assumption. But as an investigation of the initial tendency toward instability, the linearized theory should be adequate.

3. Mathematical Formulation

Consider a quantity of inviscid liquid situated in a cylindrical container of radius r_0 as is shown in Fig. 1. The cylindrical polar coordinate system is chosen so that the positive z-direction is directed upward away from the liquid, the zero on this axis being fixed on the mean free surface. On assuming the fluid to be inviscid, incompressible, and the motion to be irrotational, the equation of continuity may be expressed in terms of the velocity potential ϕ :

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0 \tag{1}$$

and the velocity components $\bar{u}, \bar{v}, \bar{w}$ are

$$\bar{u}=\phi_r, \qquad \bar{v}=\frac{1}{r}\phi_r, \qquad \bar{w}=\phi_z. \tag{2}$$

The usual subscript notation is used to denote partial differentiation.

The kinematic conditions at the tank walls and the free surface are

$$\bar{u}=\frac{\partial\phi}{\partial r}=0 \qquad \text{on} \qquad r=r_0 \tag{3}$$

$$\bar{v}=\frac{\partial\phi}{\partial z}=\frac{\partial w}{\partial t} \qquad \text{on} \qquad z=-\ell \tag{4}$$

$$\bar{w}=\frac{\partial\phi}{\partial z}=\frac{\partial\eta}{\partial t} \qquad \text{on} \qquad z=0\ , \tag{5}$$

where w denotes the deflection of the tank bottom, and η denotes the deflection of the free surface, both positive in the positive z-direction, and both are assumed to be infinitesimal.

Since the motion is irrotational, Bernoulli's equation is satisfied throughout the liquid domain. In particular, at the free surface, we have

$$\frac{p}{\rho_0} = -\frac{1}{2}(\bar{\nabla}\phi)^2 - g(t)\,\eta - \phi_t + C(t) \tag{6}$$

$$g(t) = g_0 + g_1 \cos\omega t \tag{6a}$$

where c(t) is an arbitrary function of time, g_0 is the mean local gravitational acceleration, $g_1 \cos\omega t$ is the imposed axial acceleration* both taken as positive if they are directed toward the tank bottom (along the negative z-direction), and p is the pressure just inside of the interface. The pressure p is related to the pressure just outside of the liquid, p_G, by the relation

$$p_G - p = \sigma K \tag{7}$$

where σ is the surface tension, and K is the total curvature of the free surface. In linearized form, under the assumptions that $\eta/r_0 << 1$ and $|\mathrm{grad}\,\eta| < 1$, we have

$$K = \bar{\nabla}^2\eta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\eta}{\partial\theta^2} \tag{8}$$

If the pressure of the gas p_G is a constant, then without loss of generality we may set $p_G = 0$. The function c(t) can be absorbed in ϕ_t. We can also neglect $|\bar{\nabla}\phi|^2$ in Eq. (6) and evaluate ϕ_t on the surface z = 0 under the scheme of linearization. Thus we obtain the linearized free surface condition,

$$\frac{\sigma}{\rho_0}\bar{\nabla}^2\eta = \left(\frac{\partial\phi}{\partial t}\right)_{z=0} + g(t)\,\eta \tag{9}$$

*Here we just write out a special form of imposed axial acceleration. The method developed below can be applied to a general periodic imposed axial acceleration.

Similarly, Bernoulli's equation gives the pressure on top of the bottom wall

$$(p)_{z=-\ell+w} = \rho_0 \left[-(\phi_t)_{z=-\ell} - g(t)(-\ell+w) \right]. \tag{10}$$

No other forces will be assumed to be acting on the tank bottom. If the tank bottom is very thin and is prestressed so that it behaves like a membrane, then the equation of motion of the bottom is

$$N_r \bar{\nabla}^2 w = \rho h \left[\frac{\partial^2 w}{\partial t^2} + g(t) \right] + (p)_{z=-\ell+w} , \tag{11}$$

where N_r is the tensile stress resultant in the tank bottom, and is assumed to be a constant. ρ is the density of the tank bottom material, h is the tank bottom wall thickness, so that ρh is the mass per unit area of the tank bottom.

A combination of (10) and (11) gives the linearized equation of motion of the elastic bottom as a membrane

$$N_r \bar{\nabla}^2 w = \rho h \frac{\partial^2 w}{\partial t^2} - \rho_0 g(t) w - \rho_0 \left(\frac{\partial \phi}{\partial t}\right)_{z=-\ell} + (\rho h - \rho_0 \ell) g(t). \tag{12}$$

In reality, a tank with flat bottom will develop both bending and stretching stresses under fluid pressure. Eq. (12) is a good approximation only if a membrane tension is built-in at the edges by stretching the bottom onto a rigid cylinder before the two are welded together.

It is necessary to specify the boundary conditions for η and w at the edge $r = r_0$. We choose

$$w = 0 \qquad \text{when} \qquad r = r_0 \tag{13}$$

$$\frac{\partial \eta}{\partial r} = 0 \qquad \text{when} \qquad r = r_0 \tag{14}$$

The last condition is a special case of zero capillary-hysteresis. It is consistent with the simplifying assumption that the undisturbed free surface is a plane $z = 0$. In very low gravity condition the mean free surface is curved and Eq. (14) should be replaced by the condition $\partial \eta / \partial r = \gamma \eta$ at the wall, where γ is a physical constant.

These equations define the linear, inviscid problem of sloshing under appropriate initial or periodicity conditions.

4. Dimensionless Equations

Taking the radius of the cylinder r_0 as the characteristic length, the gravitational acceleration g_0 as the characteristic acceleration, and ω as the characteristic frequency, we define the dimensionless variables

$$\left.\begin{aligned} &R = \frac{r}{r_0}, \quad Z = \frac{z}{r_0}, \quad L = \frac{l}{r_0}, \quad \tau = \omega t \\ &\Phi = \frac{\phi}{\omega r_0^2}, \quad H = \frac{\eta}{r_0}, \quad W = \frac{w}{r_0}, \quad \alpha = \frac{g_1}{g_0} \\ &G(\tau) = \frac{g_0 + g_1 \cos \omega t}{g_0}, \quad P = \frac{p}{\rho_0 \omega^2 r_0^2}, \end{aligned}\right\} \quad (15)$$

the dimensionless parameters

$$\left.\begin{aligned} &\text{Bond number} = B_\sigma = \frac{\rho_0 g_0 r_0^2}{\sigma} \\ &\text{Membrane number} = B_M = \frac{\rho_0 g_0 r_0^2}{N_r} \\ &\text{Frequency parameter for surface tension} = \Omega_\sigma^2 = \frac{\rho_0 r_0^3 \omega^2}{\sigma} \\ &\text{Frequency parameter for the membrane} = \Omega_M^2 = \frac{\rho_0 r_0^3 \omega^2}{N_r} \\ &\text{Mass ratio} = \lambda = \frac{\rho h}{\rho_0 r_0} \end{aligned}\right\} \quad (16)$$

and the operator

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} = r_0^2 \bar{\nabla}^2. \quad (17)$$

Then the equations become

$$\left(\frac{\partial^2}{\partial Z^2} + \nabla^2\right)\Phi = 0 \quad (18)$$

$$\nabla^2 H - \Omega_\sigma^2\left(\frac{\partial \Phi}{\partial \tau}\right)_{Z=0} - B_\sigma G H = 0 \quad (19)$$

$$\nabla^2 W - \lambda \Omega_M^2 \frac{\partial^2 W}{\partial \tau^2} + \Omega_M^2 \left(\frac{\partial \Phi}{\partial \tau}\right)_{Z=-L} + B_M G(\tau) W - (\lambda + L) B_M G(\tau) = 0 \tag{20}$$

with the boundary conditions

$$\frac{\partial \Phi}{\partial R} = 0 \qquad \text{on} \qquad R = 1 \tag{21}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{\partial W}{\partial \tau} \qquad \text{on} \qquad Z = -L \tag{22}$$

$$\frac{\partial \Phi}{\partial Z} = \frac{\partial H}{\partial \tau} \qquad \text{on} \qquad Z = 0 \tag{23}$$

$$W = 0 \qquad \text{on} \qquad R = 1 \tag{24}$$

$$\frac{\partial H}{\partial R} = 0 \qquad \text{on} \qquad R = 1 \quad (\text{assuming } \gamma = 0) \tag{25}$$

Eqs. (18) - (25) show that the problem of sloshing depends on the parameters

$$\Omega_\sigma^2,\ B_\sigma,\ \alpha,\ \Omega_M^2,\ B_M,\ \lambda,\ L$$

These dimensionless parameters are not all independent; for

$$\Omega_M^2 = \frac{\sigma}{N_r} \Omega_\sigma^2 \quad , \qquad B_M = \frac{\sigma}{N_r} B_\sigma \tag{26}$$

Hence

$$\frac{B_M}{B_\sigma} = \frac{\Omega_M^2}{\Omega_\sigma^2} \tag{27}$$

However, we retain the symbols $\Omega_\sigma^2, B_\sigma; \Omega_M^2, B_M$ because these two pairs of parameters are not likely to be both important.

$\Omega_M^2, B_M \to 0$ if the tank bottom is rigid

$\Omega_\sigma^2, B_\sigma \to \infty$ if the surface tension has no effect.

5. An Analytical Difficulty

It is not easy to construct an exact solution for the system of equations (18) - (25). To construct a solution in the form of an infinite series, as a direct extension of Benjamin and Ursell's solution for a rigid tank, encounters certain basic difficulties. Consider symmetric modes of motion in which Φ, H, W are independent of the angular coordinates θ . A solution of (18) may be posed as

$$\Phi = \dot{d}_0(\tau) z + \dot{c}_0(\tau) + \sum_{n=1}^{\infty} J_0(k_n R)\left[\dot{c}_n(\tau)\frac{\cosh k_n z}{\sinh k_n L} + \dot{d}_n(\tau)\frac{\sinh k_n z}{\cosh k_n L}\right] \tag{28}$$

Then Eqs. (22) and (23) give

$$H = d_0(\tau) + \sum_{n=0}^{\infty} d_n(\tau)\frac{k_n J_0(k_n R)}{\cosh k_n L} \tag{29}$$

$$W = d_0(\tau) + f(R) + \sum_{n=1}^{\infty} k_n\left[d_n(\tau) - c_n(\tau)\right] J_0(k_n R) \tag{30}$$

Both Eqs. (21) and (25) are satisfied if the k_n's are the roots of the equation

$$J_1(k_n) = 0, \qquad n = 1, 2, 3 \cdots\cdots \tag{31}$$

Eq. (24) is satisfied by taking

$$d_0(\tau) + \sum_{n=1}^{\infty} k_n\left[d_n(\tau) - c_n(\tau)\right] J_0(k_n) = 0 \tag{32}$$

and

$$f(R) = (\lambda + L)\left[1 - \frac{J_0(\sqrt{B_M} R)}{J_0(\sqrt{B_M})}\right]$$

$f(R)$ is the static deflection of the membrane. We assume that if B_M is positive, $\sqrt{B_M}$ is less than the first root of $J_0(x) = 0$, namely, 2.4048. To satisfy Eqs. (19), (20), we substitute Φ , H, W from (28) - (30), collect terms and represent the left hand side as Fourier-Bessel series in $J_0(k_n R)$. Since the series vanish, every coefficient of $J_0(k_n R)$, $n = 1, 2, \cdots$, must vanish. Thus from (19) we obtain the necessary conditions

$$\Omega_\sigma^2 \ddot{c}_0(\tau) + B_\sigma G(\tau)\, d_0(\tau) = 0 \tag{33a}$$

$$\Omega_\sigma^2 \ddot{c}_n(\tau) + k_n \tanh k_n L \left[k_n^2 + B_\sigma G(\tau)\right] d_n(\tau) = 0. \tag{33b}$$

And from (20) and (33a)

$$\Omega_M^2 \ddot{d}_0(\tau) + 2\alpha \sqrt{B_M} \frac{J_1(\sqrt{B_M})}{J_0(\sqrt{B_M})} \cos\tau = 0 \tag{34a}$$

$$\Omega_M^2 (\lambda k_n + \coth k_n L)\, \ddot{c}_n - \Omega_M^2 (\lambda k_n + \tanh k_n L)\, \ddot{d}_n$$
$$+ k_n\left[k_n^2 - B_n G(\tau)\right](c_n - d_n) = \frac{2B_n^{\frac{1}{2}} J_1(\sqrt{B_n})}{J_0(\sqrt{B_n})\, J_0(k_n)} \frac{\alpha}{B_n - k_n^2} \cos\tau. \tag{34b}$$

Thus all differential equations and boundary conditions are satisfied by the assumed form of Φ, H, W provided that c_n, d_n satisfy Eqs. (32), (33), and (34).

A difficulty becomes apparent when one examines the Eqs. (32)-(34). If we truncate the infinite series (28) - (30) to N terms, we see that Eqs. (32) - (34) always impose $N+1$ conditions on N unknowns, which in general have no solution.

The difficulty appears to have risen from the condition $W = 0$ at the edge of the tank bottom. The same difficulty would have appeared in Benjamin and Ursell's rigid tank case if one assumes a "stuck" condition at the edge of the free surface, $H = 0$ when $R = 1$, (physically realizable with certain fluid and wall material). Mathematically, the condition $W = 0$ requires certain discontinuity in the second derivatives of Φ at the edge of the tank bottom. For, we have the boundary conditions

$$\frac{\partial \Phi}{\partial R} = 0 \qquad \text{for} \quad R=1, \ -L \leq Z \leq 0 \qquad \text{(a)}$$

and

$$\frac{\partial \Phi}{\partial Z} = \frac{\partial W}{\partial \tau} \qquad \text{for} \quad 0 \leq R \leq 1, \quad Z=-L \qquad \text{(b)}$$

Hence

$$\frac{\partial}{\partial \tau}\left(\frac{\partial W}{\partial R}\right) = \frac{\partial^2 \Phi}{\partial R \partial Z} = \frac{\partial}{\partial Z}\left(\frac{\partial \Phi}{\partial R}\right) \quad \text{for} \quad 0 \leq R \leq 1, \ Z=-L \qquad \text{(c)}$$

Now if $\frac{\partial^2 \Phi}{\partial R \partial Z}$ were continuous everywhere we must have, from (a), that $\frac{\partial}{\partial Z}\left(\frac{\partial \Phi}{\partial R}\right) = 0$ at $R=1, Z=-L$. Hence (c) yields $\frac{\partial}{\partial \tau}\left(\frac{\partial W}{\partial R}\right) = 0$ at $R=1, Z=-L$ i.e. $\frac{\partial W}{\partial R}$ = const. at $R=1, \ Z=-L$. This is inconsistent with the condition $W=0$ at the same point.* Hence we must conclude that $\frac{\partial^2 \Phi}{\partial R \partial Z}$ must not be continuous at the corner $R=1, \ Z=-L$. This requires that the series (35) and its first and second derivatives be uniformly convergent everywhere, except for $\frac{\partial^2 \Phi}{\partial R \partial Z}$, which loses its analyticity at the corner $R=1, \ Z=-L$. This demand singles out the question of convergence of the series (35) to the foreground. We note, in particular, that any truncated series of (35), (taking the first N terms), will not have the desired property.

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* The linearized equation can admit only one boundary condition. The condition $W=0$ is chosen in Eq. (24) on physical basis.

The difficulty considered here seems to be caused by the linearization of the governing equations. If linearization was not made, the kinematic conditions on the free surface and on the bottom are, instead of Eqs. (22) and (23), the following:

$$\frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial R}\frac{\partial \Phi}{\partial R} + \frac{1}{R^2}\frac{\partial H}{\partial \theta}\frac{\partial \Phi}{\partial \theta} = \frac{\partial \Phi}{\partial Z}$$

$$\frac{\partial W}{\partial \tau} + \frac{\partial W}{\partial R}\frac{\partial \Phi}{\partial R} + \frac{1}{R^2}\frac{\partial W}{\partial \theta}\frac{\partial \Phi}{\partial \theta} = \frac{\partial \Phi}{\partial Z}$$

In the nonlinear form the problem is undoubtedly much more difficult, but the particular problem referred to above disappears.

If we relax the demand for an exact solution, we could turn to variational methods. In the following the variational principle and approximate solutions will be discussed.

6. <u>The Variational Principles</u>

If we multiply Eq. (18) by $\delta\Phi$, (19) by $\frac{1}{\Omega_\sigma^2}\delta H$, (20) by $\frac{1}{\Omega_M^2}\delta W$, and integrate over the entire fluid volume V , the free surface S_1 , the bottom surface S_3 , respectively, and transform with appropriate integration by parts, we obtain

$$
\begin{aligned}
&-\delta\int_V \tfrac{1}{2}(\nabla\Phi)^2 dV - \frac{1}{\Omega_\sigma^2}\,\delta\int_{S_1}\left[\frac{(\nabla H)^2}{2} + \Omega_\sigma^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=0} + \frac{B_\sigma G H^2}{2}\right] dS_1 \\
&-\frac{1}{\Omega_M^2}\,\delta\int_{S_3}\Bigg[\frac{(\nabla W)^2}{2} - \frac{\lambda\Omega_M^2}{2}\left(\frac{\partial W}{\partial\tau}\right)^2 - \Omega_M^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=-L} W - B_M\frac{G W^2}{2} \\
&+(\lambda+1)B_M G W\Bigg] dS_3 + \int_{S_1}\left(\frac{\partial\Phi}{\partial z} - \frac{\partial H}{\partial\tau}\right)\delta\Phi\, dS_1 + \int_{S_2}\frac{\partial\Phi}{\partial R}\,\delta\Phi\, dS_2 \\
&+\int_{S_3}\left[-\frac{\partial\Phi}{\partial z} + \frac{\partial W}{\partial\tau}\right]\delta\Phi\, dS_3 + \frac{1}{\Omega_M^2}\int_{\Gamma_3}\frac{\partial W}{\partial R}\,\delta W\, d\ell_3 + \frac{1}{\Omega_\sigma^2}\int_{\Gamma_1}\frac{\partial H}{\partial R}\,\delta H\, d\ell_1 \\
&-\int_{S_3}\left[\lambda\frac{\partial}{\partial\tau}\left(\delta W\frac{\partial W}{\partial\tau}\right) + \frac{\partial}{\partial\tau}(W\delta\Phi)\right] dS_3 + \int_{S_1}\frac{\partial}{\partial\tau}(H\delta\Phi)\, dS_1 = 0
\end{aligned}
\tag{35}
$$

where S_2 is the rigid side-wall surface, Γ_1 is the edge of S_1 , Γ_3 is the edge of S_3 . This suggests the following functional

$$
I[\Phi, H, W; \tau] = I_1 + I_2 + I_3 + I_4 \tag{36}
$$

where

$$
I_1 = \int_V \frac{(\nabla\Phi)^2}{2}\, dV \tag{36a}
$$

$$I_2 = \frac{1}{\Omega_\sigma^2}\int_{S_1}\left[\frac{(\nabla H)^2}{2} + \Omega_\sigma^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=0} H + B_\sigma G\frac{H^2}{2}\right] dS_1 \qquad (36b)$$

$$I_3 = \frac{1}{\Omega_M^2}\int_{S_3}\left[\frac{(\nabla W)^2}{2} - \frac{\lambda\Omega_M^2}{2}\left(\frac{\partial W}{\partial\tau}\right)^2 - \Omega_M^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=-L} W\right.$$

$$\left. - B_M\frac{GW^2}{2} + (\lambda + L)B_M G W - B_M G\frac{L^2}{2}\right] dS_3 \qquad (36c)$$

$$I_4 = \frac{1}{\Omega_\sigma^2}\int_{\Gamma_1} F_1(H;\theta,\tau)\, d\ell_1 + \frac{1}{\Omega_M^2}\int_{\Gamma_3} F_3(W;\theta,\tau)\, d\ell_3 \qquad (36d)$$

It is easy to verify that the variational equation

$$\delta\int_{\tau_1}^{\tau_2} J[\Phi, H, W;\tau]\, d\tau = 0 \qquad (37)$$

under arbitrary variations of Φ, H, W, with the stipulation that $\delta\Phi, \delta H, \delta W$ vanish at $\tau=\tau_1$, and $\tau=\tau_2$, yields

$$\int_{\tau_1}^{\tau_2}\left\{-\int_V \nabla^2\Phi\,\delta\Phi\, dV - \frac{1}{\Omega_\sigma^2}\int_{S_1}\left[\nabla^2 H - \Omega_\sigma^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=0} - B_\sigma G H\right]\delta H\, dS_1\right.$$

$$-\frac{1}{\Omega_M^2}\int_{S_3}\left[\nabla^2 W - \lambda\Omega_M^2\frac{\partial^2 W}{\partial\tau^2} + \Omega_M^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=-L} + B_M G W - B_M G(L+\lambda)\right]\delta W\, dS_3$$

$$+\int_{S_1}\left(\frac{\partial\Phi}{\partial z} - \frac{\partial H}{\partial\tau}\right)\delta\Phi\, dS_1 + \int_{S_2}\frac{\partial\Phi}{\partial R}\delta\Phi\, dS_2 + \int_{S_3}\left(\frac{\partial W}{\partial\tau} - \frac{\partial\Phi}{\partial z}\right)\delta\Phi\, dS_3$$

$$+\frac{1}{\Omega_\sigma^2}\int_{\Gamma_1}\left(\frac{\partial H}{\partial R} + \frac{\partial F_1}{\partial H}\right)\delta H\, d\ell_1 + \frac{1}{\Omega_M^2}\int_{\Gamma_3}\left(\frac{\partial W}{\partial R} + \frac{\partial F_3}{\partial W}\right)\delta W\, d\ell_3 = 0$$

(38)

Since $\delta\Phi, \delta H, \delta W$ are arbitrary over V, S_1, S_3 respectively, we obtain the differential equations (18), (19), (20), and the following boundary conditions:

$$
\begin{array}{llll}
\text{On } S_1; & \text{Either } \dfrac{\partial\Phi}{\partial Z} - \dfrac{\partial H}{\partial \tau} = 0 & \text{or} & \delta\Phi = 0 \\[2ex]
\text{On } S_2: & \text{Either } \dfrac{\partial\Phi}{\partial R} = 0 & \text{or} & \delta\Phi = 0 \\[2ex]
\text{On } S_3: & \text{Either } \dfrac{\partial\Phi}{\partial Z} - \dfrac{\partial W}{\partial \tau} = 0 & \text{or} & \delta\Phi = 0 \\[2ex]
\text{On } \Gamma_1: & \text{Either } \dfrac{\partial H}{\partial R} + \dfrac{\partial F_1}{\partial H} = 0 & \text{or} & \delta H = 0 \\[2ex]
\text{On } \Gamma_3: & \text{Either } \dfrac{\partial W}{\partial R} + \dfrac{\partial F_3}{\partial W} = 0 & \text{or} & \delta W = 0 .
\end{array}
\qquad (39)
$$

The conditions on the left hand column are the natural boundary conditions; those on the right hand side are the rigid boundary conditions. From physical considerations of our problem we impose the natural boundary conditions over S_1, S_2, S_3 and Γ_1, but the rigid boundary condition $W = 0$ over Γ_3.

The terms in the functional I have the following physical significance. The first term represents the kinetic energy of the fluid (in dimensionless form). The terms

$$\frac{1}{\Omega_\sigma^2}\int_{S_1}\frac{(\nabla H)^2}{2}\,dS_1\,,\quad \frac{1}{\Omega_M^2}\int_{S_3}\frac{(\nabla W)^2}{2}\,dS_3\,,\quad \frac{1}{\Omega_H^2}\int_{S_3}\frac{\lambda\Omega_H^2}{2}\left(\frac{\partial W}{\partial\tau}\right)^2 dS_3$$

represent the change of surface energy of the free surf... ..e change of elastic energy of the bottom, and the kinetic energy of the bottom, respectively.

To interpret the meaning of the other terms, we write the pressure in dimensionless form

$$P(R,\theta,Z,\tau) = -\left\{ \frac{\partial\Phi}{\partial\tau} + \frac{B_M}{\Omega_M^2} G(\tau) Z + \frac{1}{2}(\nabla\Phi)^2 \right\} \tag{40}$$

Then the work done by P through a displacement H is

$$\int_0^H P(R,\theta,Z,\tau)\,dZ$$

On evaluating this integral, and neglecting third order infinitesimals, we see that it is equal to

$$-\frac{1}{\Omega_0^2}\left[\Omega_0^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{Z=0} H + B_0 G(\tau)\frac{H^2}{2}\right]$$

[Note that $\frac{B_0}{\Omega_0^2} = \frac{B_M}{\Omega_M^2}$]. Similarly, the terms

$$-\frac{1}{\Omega_M^2}\left[-\Omega_M^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{Z=-L} W - B_M\frac{G(\tau)W^2}{2} + LB_M G W - B_M G(\tau)\frac{L^2}{2}\right]$$

are equal to the integral of $P(Z,R,\theta,\tau)$ between $-L+W$ and $-L$, with the sign reversed because the pressure P acts on the upper side of the bottom. Finally, $\lambda\frac{B_M}{\Omega_M^2}GW$ represents the work done by the inertia force due to the gravitational acceleration $G(\tau)$ through a displacement W.

F_1, F_3 are the potential of the vertical forces (positive upward) acting on the free surface and the tank bottom, at the edges Γ_1 and Γ_3 respectively. F_1 arises from the capillary surface energy; F_3 normally arises from the reaction of the wall on the membrane, but it can be imposed by an external agency. If F_1 is assumed to be proportional to H^2, then the so-called capillary-hysteresis is obtained. Eq. (14) and Eq. (25) presuppose $F_1 = 0$.

7. An Alternate Form of the Variational Principle

It is known that an appropriate functional from which the equations of fluid mechanics can be derived from the calculus of variations is the "pressure energy"

$$L_1 = -\int P\,dV \tag{41}$$

integrated over the entire volume of the fluid. See Bateman, (Ref. 13) Wang (Ref. 14). With this information, we consider a functional

$$I = L_1 + L_2 + L_3 \tag{42}$$

where L_1 is given by (41), with P expressed in Eq. (40); whereas L_2 and L_3 are the Lagrangian functions of the free surface and the bottom membrane, respectively:

$$L_2 = \frac{1}{\Omega_e^2}\int_{S_1}\frac{1}{2}(\nabla H)^2\,dS_1 + \frac{1}{\Omega_T^2}\int_{\Gamma_1}F_1\,d\ell_1 \tag{43}$$

$$L_3 = \frac{1}{\Omega_M^2}\int_{S_3}\left[\frac{1}{2}(\nabla W)^2 - \frac{\lambda}{2}\Omega_M^2\left(\frac{\partial W}{\partial \tau}\right)^2 + \lambda B_H G W\right]dS_3 + \frac{1}{\Omega_M^2}\int_{\Gamma_3}F_3\,d\ell_3 \tag{44}$$

It can be verified at once that I so obtained from (42) is exactly what is given in (36), except for the term

$$\int_{S_1}\left[\int_{L}^{0}\frac{\partial\Phi}{\partial\tau}\,dz\right]dS_1$$

which make no contribution to the variational equation (37). It is important to note that the volume integral in (41) means

$$\int_0^{2\pi}\int_0^1\int_{-L+W}^{H} P(R,\theta,z,\tau)\,R\,dz\,dR\,d\theta$$

The variable limits for z must be accounted for.

In practical applications, it is common to choose Φ which in Eqs. (36), (36a) as

$$I_1 = \int_{S_1} \left(\Phi \frac{\partial \Phi}{\partial z} \right)_{z=0} dS_1 + \int_{S_2} \left(\Phi \frac{\partial \Phi}{\partial R} \right)_{R=1} dS_2 + \int_{S_3} \left(\Phi \frac{\partial \Phi}{\partial z} \right)_{z=-L} dS_3 \quad (44)$$

Then I contains surface integrals only.

8. An Approximate Solution

An approximate solution based on the Rayleigh-Ritz-Galerkin procedure will be given below. We choose Φ, H and W in the following form

$$\Phi = \dot{d}_0(\tau)z + \dot{c}_0(\tau) + J_0(k_nR)\left[\dot{c}_n(\tau)\frac{\cosh k_nz}{\sinh k_nL} + \dot{d}_n(\tau)\frac{\sinh k_nz}{\cosh k_nL}\right] \tag{45a}$$

$$H = d_0(\tau) + \frac{k_n d_n(\tau) J_0(k_nR)}{\cosh k_nL} \tag{45b}$$

$$W = d_0(\tau) - k_n\left[c_n(\tau) - d_n(\tau)\right]J_0(k_nR) + (\lambda + L)\left[1 - \frac{J_0(\sqrt{B_H}R)}{J_0(\sqrt{B_H})}\right] \tag{45c}$$

which satisfy Eqs. (18), (21), (22), (23) and (25). Eq. (24) is satisfied by taking

$$d_0(\tau) = k_n\left[c_n(\tau) - d_n(\tau)\right]J_0(k_n) \tag{46}$$

Eq. (19) is satisfied if we impose

$$\Omega_\sigma^2\,\ddot{c}_0(\tau) + B_\sigma G(\tau)\,d_0(\tau) = 0 \tag{47a}$$

$$\Omega_\sigma^2\,\ddot{c}_n(\tau) + k_n \tanh k_nL\left[k_n^2 + B_\sigma G(\tau)\right]d_n(\tau) = 0 \tag{47b}$$

Eq. (20) is not satisfied, whereas Eq. (38) now becomes

$$\int_{S_3}\left\{\nabla^2 W - \lambda\Omega_H^2\frac{\partial^2 W}{\partial\tau^2} - \Omega_H^2\left(\frac{\partial\Phi}{\partial\tau}\right)_{z=-L} + B_H G W - B_H G(L+\lambda)\right\}\delta W\,dS_3 = 0 \tag{48}$$

Now we shall satisfy (20) in the sense of (48) by choosing

$$\delta W = k_n\left(J_0(k_n) - J_0(k_nR)\right) \tag{49}$$

Then (48) yields

$$\Omega_n^2(\coth k_nL+\lambda k_n)\ddot{c}_n-\Omega_n^2(\tanh k_nL+\lambda k_n)\ddot{d}_n+k_n(k_n^2-B_nG)(c_n-d_n)$$
$$=-(L+\lambda)\left\{\frac{\Omega_n^2\ddot{d}_n}{J_0(k_n)}+\frac{2\alpha\, J_1(\sqrt{B_n})\sqrt{B_n}}{J_0(k_n)\,J_1(\sqrt{B_n})}\left[1+\frac{B_n}{B_n-k_n^2}\right]\cos\tau\right\} \quad (50)$$

A substitution of (46) into (50) gives

$$\Omega_n^2\left(\mu_n\ddot{c}_n(\tau)-\nu_n\ddot{d}_n(\tau)\right)+k_n\left(k_n^2-B_nG(\tau)\right)\left[c_n(\tau)-d_n(\tau)\right] \quad (51)$$
$$=-(L+\lambda)\left\{\frac{2\alpha\, J_1(\sqrt{B_n})\sqrt{B_n}}{J_0(k_n)\,J_0(\sqrt{B_n})}\left(1+\frac{B_n}{B_n-k_n^2}\right)\cos\tau\right\}$$

where

$$\mu_n=\coth k_nL+2\lambda k_n+k_nL$$
$$\nu_n=\tanh k_nL+2\lambda k_n-k_nL \quad (52)$$

The central problem lies in the solution of Eqs. (47b) and (51). These equations can be put in a neater form by letting

$$d_n=x_1\quad,\quad c_n=x_1+x_2 \quad (53)$$

Then we have

$$\ddot{X}+M_1X+\alpha\cos\tau\, M_2X=\alpha\cos\tau\, M_3(\lambda+L) \quad (54)$$

where

$$X=\begin{pmatrix}x_1\\ x_2\end{pmatrix},\qquad M_1=\begin{pmatrix}\frac{\mu_n}{\nu_n}P_n^2 & -\frac{\Lambda_n}{\nu_n}\\ \left(1-\frac{\mu_n}{\nu_n}\right)P_n^2 & +\frac{\Lambda_n}{\nu_n}\end{pmatrix} \quad (54a)$$

$$M_2=\begin{pmatrix}\frac{\mu_n}{\nu_n}k_n\tanh k_nL & +\frac{k_n}{\nu_n}\\ \left(1-\frac{\mu_n}{\nu_n}\right)k_n\tanh k_nL & -\frac{k_n}{\nu_n}\end{pmatrix}\frac{B_n}{\Omega_n^2},\qquad M_3=\frac{2J_1(\sqrt{B_n})\sqrt{B_n}}{\Omega_n^2J_0(k_n)J_0(\sqrt{B_n})}\left(1+\frac{B_n}{B_n-k_n^2}\right)\begin{pmatrix}+1\\ -1\end{pmatrix}$$

$$\Lambda_n=\frac{k_n\left(k_n^2-B_n^2\right)}{\Omega_n^2}$$

If M_1 is positive definite and has two distinct positive eigenvalues Ω_1^2, Ω_2^2 then there exists a nonsingular matrix, U, such that

$$U^{-1} M_1 U = \begin{pmatrix} \Omega_1^2 & 0 \\ 0 & \Omega_1^2 \end{pmatrix} = D. \tag{55}$$

Let $Y = U^{-1} X$ then Eq. (56) assumes the normal form

$$\ddot{Y} + DY + \alpha \cos\tau \, U^{-1} M_2 U \, Y = \alpha \cos\tau \; U^{-1} M_3 (\lambda + L), \tag{56}$$

which will be studied in the following two sections.

9. Free Oscillations

Consider the special case of free oscillation of the system, i.e. the case $\alpha = 0$, or $G = 1$. In this case Eqs. (47b), (51) become two linear differential equations with constant coefficients. By putting

$$c_n(\tau) = C_n e^{i\Omega\tau} \quad , \quad d_n(\tau) = D_n e^{i\Omega\tau}$$

we get

$$-\Omega_\sigma^2 \Omega^2 C_n + k_n \tanh k_n L \, (k_n^2 + B_\sigma) D_n = 0$$

$$\left[-\mu_n \Omega_M^2 \Omega^2 + k_n(k_n^2 - B_M)\right] C_n + \left[\nu_n \Omega_M^2 \Omega^2 - k_n(k_n^2 - B_M)\right] D_n = 0 .$$

The eigenvalues of Ω^2 are

$$\left\{ \begin{matrix} \Omega_1^2 \\ \Omega_2^2 \end{matrix} \right\} = \frac{1}{2} \left[\frac{\Lambda_n}{\nu_n} + \frac{\mu_n}{\nu_n} p_n^2 \mp \sqrt{\left(\frac{\Lambda_n}{\nu_n} + \frac{\mu_n}{\nu_n} p_n^2\right)^2 - 4 \frac{\Lambda_n p_n^2}{\nu_n}} \right] \tag{57}$$

where

$$p_n = \left[\frac{k_n \tanh k_n L \, (k_n^2 + B_\sigma)}{\Omega_\sigma^2} \right]^{1/2}$$

It is recognized that $p_n \omega$ is the circular frequency of sloshing of a liquid in a rigid container. In case $p_n^2 \ll \Lambda_n / \nu_n$, (which is usually the case for a rocket in Earth's gravitational field), we can write in the following form

$$\Omega_1^2 = p_n^2 \left\{ 1 - \frac{\nu_n p_n^2}{\Lambda_n} \left(\frac{\mu_n}{\nu_n} - 1\right) - \left(2 - \frac{\mu_n}{\nu_n}\right)\left(\frac{\mu_n}{\nu_n} - 1\right)\left(\frac{\nu_n p_n^2}{\Lambda_n}\right)^2 + \cdots \right\} \tag{57a}$$

Since $\frac{\mu_n}{\nu_n} > 1$, $|\Omega_1|$ is always less than p_n; which means that the tank flexibility lowers the natural sloshing frequency. Note also that, when either $\Lambda_n \to \infty$, (rigid tank) or $k_n L \to \infty$, (the deep water case, or a high mode) Ω_1 tends to p_n.

10. Stability of the Solution

To study the stability of the solutions of Eq. (56), (Refs. 11, 12) we shall consider the following more general system of equations*

$$
\begin{aligned}
&\ddot{y}_1 + \sum_{n=-\infty}^{\infty} A_{2n} e^{i2nt} y_1 + \sum_{n=-\infty}^{\infty} A_{2n-1} e^{i2nt} y_2 = 0 \\
&\ddot{y}_2 + \sum_{n=-\infty}^{\infty} B_{2n} e^{i2nt} y_2 + \sum_{n=-\infty}^{\infty} B_{2n+1} e^{i2nt} y_1 = 0
\end{aligned}
\qquad (58)
$$

where $B_0 \neq A_0$, $A_0 > 0$, $B_0 > 0$, $A_{-1} = B_1 \neq 0$, $\sum_{n=-\infty}^{\infty} \beta_n$ (where $\beta_n = |A_n|$ or $|B_n|$) is an absolutely convergent series. Eqs. (58) are invariant when t is changed to $t+\pi$; therefore if $y(t)$ is a solution of Eqs. (58), $y(t+\pi)$ is also a solution. By Floquet theorem (Ref. 9), Eqs. (58) have solutions of the following form

$$y(t) = e^{\xi t}\,\psi(t)$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$\psi(t)$ is periodic function $\bmod(\pi)$. If $Re\,\xi > 0$, $y \to \infty$ as $t \to \infty$ an unbounded solution exists, which is said to be unstable. For a periodic solution $\bmod(\pi)$ to exist, $Im\,\xi$ must be equal to an integer, whereas $Re\,\xi = 0$.

Let us assume a solution of the following form

$$
\begin{aligned}
y_1(t) &= e^{\xi t} \sum_{n=-\infty}^{\infty} \gamma_{2n} e^{i2nt} \\
y_2(t) &= e^{\xi t} \sum_{n=-\infty}^{\infty} \gamma_{2n+1} e^{i2nt}
\end{aligned}
$$

where $\sum_{n=-\infty}^{\infty} n^2 \gamma_{2n}$, $\sum_{n=-\infty}^{\infty} n^2 \gamma_{2n+1}$ are absolutely convergent series.

---

* This is an extension of Hill's method (see Ref. 12, p. 413) to a system of two equations.

Substituting into equations (58), we get

$$\sum_{n=-\infty}^{\infty} \gamma_{2n}(2ni+\xi)^2 e^{(2ni+\xi)t} + \sum_{m=-\infty}^{\infty} A_{2m} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n} e^{(2ni+\xi)t}$$
$$+ \sum_{m=-\infty}^{\infty} A_{2m-1} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n+1} e^{(2ni+\xi)t} = 0$$

$$\sum_{n=-\infty}^{\infty} \gamma_{2n+1}(2ni+\xi)^2 e^{(2ni+\xi)t} + \sum_{m=-\infty}^{\infty} B_{2m} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n+1} e^{(2ni+\xi)t} \qquad (59)$$
$$+ \sum_{m=-\infty}^{\infty} B_{2m+1} e^{2mit} \sum_{n=-\infty}^{\infty} \gamma_{2n} e^{(2ni+\xi)t} = 0$$

On rearranging the terms of the absolute convergent series, and equating the coefficients of $e^{(2ni+\xi)t}$ to zero, we obtain

$$-\gamma_{2n}\frac{(i\xi-2n)^2}{A_0-4n^2} + \sum_{m=-\infty}^{\infty}\frac{A_m}{A_0-4n^2}\gamma_{2n-m} = 0$$
$$-\gamma_{2n+1}\frac{(i\xi-2n)^2}{B_0-4n^2} + \sum_{m=-\infty}^{\infty}\frac{B_m}{B_0-4n^2}\gamma_{2n+1-m} = 0 \qquad (60)$$

provided that $A_0-4n^2 \neq 0$, $B_0-4n^2 \neq 0$. The divisors A_0-4n^2, B_0-4n^2 are introduced in order to make an infinite determinant, which will be formed below, to be convergent.

Equations in (60) are a set of homogeneous equations. For γ_n to have nontrivial solutions the determinant formed by the coefficients of the equations must vanish. Call this determinant $\Delta(i\xi)$; then

$$\Delta(i\xi) = |\alpha_{ij}| = 0 \tag{61}$$

where

$$\alpha_{2m,2m} = \frac{A_0 - (i\xi - 2m)^2}{A_0 - 4m^2}$$

$$\alpha_{2m+1,2m+1} = \frac{B_0 - (i\xi - 2m)^2}{B_0 - 4m^2}$$

$$\alpha_{2m,n} = \frac{A_{2m-n}}{A_0 - 4m^2} \qquad \text{for} \quad 2m - n \neq 0$$

$$\alpha_{2m+1,n} = \frac{B_{2m+1-n}}{B_0 - 4m^2} \qquad \text{for} \quad 2m+1-n \neq 0$$

$$m, n = 0, \pm 1, \pm 2 \cdots$$

We consider another infinite determinant $\Delta_1(i\xi) = |\beta_{ij}|$ where

$$\beta_{m,m} = 1 \tag{62a}$$

$$\beta_{2m,n} = \frac{\alpha_{2m,n}}{\alpha_{2m,2m}} = \frac{A_{2m-n}}{A_0 - (i\xi - 2m)^2}, \qquad 2m - n \neq 0 \tag{62b}$$

$$\beta_{2m+1,n} = \frac{\alpha_{2m+1,n}}{\alpha_{2m+1,2m+1}} = \frac{B_{2m+1-n}}{B_0 - (i\xi - 2m)^2}, \qquad 2m+1-n \neq 0 \tag{62c}$$

Since $\prod_{m=-\infty}^{\infty} \beta_{m,m} = 1$, $\sum_{\substack{m,n=-\infty \\ m \neq n}}^{\infty} |\beta_{m,n}|$ converges provided ξ does not have such a value that one of the denominators of $\beta_{m,n}(i\xi)$ vanishes. Thus the infinite determinant $\Delta_1(i\xi)$ is absolutely convergent. Then (Ref. 7)

$$\Delta(i\xi) = \Delta_1(i\xi) \lim_{M \to \infty} \prod_{n=-M}^{M} \frac{[A_0 - (i\xi - 2n)^2]\,[B_0 - (i\xi - 2n)^2]}{(A_0 - 4n^2)\,(B_0 - 4n^2)}$$

$$= \Delta_1(i\xi)\, \frac{\sin\frac{\pi}{2}(i\xi - \sqrt{A_0}) \sin\frac{\pi}{2}(i\xi + \sqrt{A_0}) \sin\frac{\pi}{2}(i\xi - \sqrt{B_0}) \sin\frac{\pi}{2}(i\xi + \sqrt{B_0})}{\sin^2(\frac{\pi}{2}\sqrt{A_0}) \sin^2(\frac{\pi}{2}\sqrt{B_0})}$$

(63)

We note some interesting properties of $\Delta_1(i\xi)$: (1) $\Delta_1(i\xi)$ is a meromorphic function of ξ and tends to 1 as $\mathrm{Re}\,\xi \to \pm\infty$; (2) $\Delta_1(i\xi)$ is a periodic function of ξ with period $2i$. If we form another function

$$
\begin{aligned}
F(\xi) = \Delta_1(i\xi) &- K_1\left[\cot\tfrac{\pi}{2}(i\xi+\sqrt{A_0}) - \cot\tfrac{\pi}{2}(i\xi-\sqrt{A_0})\right] \\
&- K_2\left[\cot\tfrac{\pi}{2}(i\xi+\sqrt{A_0}) + \cot\tfrac{\pi}{2}(i\xi-\sqrt{A_0})\right] \\
&- K_3\left[\cot\tfrac{\pi}{2}(i\xi+\sqrt{B_0}) - \cot\tfrac{\pi}{2}(i\xi-\sqrt{B_0})\right] \\
&- K_4\left[\cot\tfrac{\pi}{2}(i\xi+\sqrt{B_0}) + \cot\tfrac{\pi}{2}(i\xi-\sqrt{B_0})\right],
\end{aligned}
\tag{64}
$$

where K_j's are so chosen that $F(\xi)$ has no poles at $i\xi = \pm\sqrt{A_0}, \pm\sqrt{B_0}$. Since $\Delta_1(i\xi)$ is a periodic function of ξ, it follows that $F(\xi)$ has no poles at

$$
i\xi = 2n \pm\sqrt{A_0}\,,\; 2n\pm\sqrt{B_0}\,, \qquad n = \pm 1, \pm 2 \cdots\cdots
$$

Thus $F(\xi)$ is a meromorphic function with no pole on the entire plane. $F(\xi)$ is certainly bounded, therefore, by Liouville's theorem, $F(\xi)$ must be a constant, say C. As $\mathrm{Re}\,\xi \to \pm\infty$, $\Delta_1(i\xi)=1$. Therefore

$$
\begin{aligned}
C &= 1 + 2(K_2+K_4)i \qquad \text{as } \mathrm{Re}\,\xi \to \infty \\
&= 1 - 2(K_2+K_4)i \qquad \text{as } \mathrm{Re}\,\xi \to -\infty
\end{aligned}
$$

Hence $K_2 + K_4 = 0$, and $F(\xi) = 1$ for all ξ. Using this result and Eqs. (63) and (64) we get

$$
\Delta(i\xi) = \frac{\sin^4\frac{\pi\xi i}{2} - 2\delta_2\sin^2\frac{\pi\xi i}{2} + \delta_1 + \delta_3\sin\pi\xi i}{\sin^2\left(\frac{\pi}{2}\sqrt{A_0}\right)\sin^2\left(\frac{\pi}{2}\sqrt{B_0}\right)}
\tag{65}
$$

where δ_i are some constants relating to K_j's, A_0 and B_0. Put $i\xi = 0, \frac{1}{2}$ and 1 in (65) and we get

$$\delta_1 = \Delta(0) \sin^2\left(\tfrac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\tfrac{\pi}{2}\sqrt{B_0}\right)$$

$$2\delta_2 = 1 + \sin^2\left(\tfrac{\pi}{2}\sqrt{A_0}\right) \sin^2\left(\tfrac{\pi}{2}\sqrt{B_0}\right)\left[\Delta(0) - \Delta(1)\right] \tag{66}$$

$$2\delta_3 = \tfrac{1}{2} + \sin^2\left(\tfrac{\pi}{2}\sqrt{A_0}\right)\sin^2\left(\tfrac{\pi}{2}\sqrt{B_0}\right)\left[2\Delta(\tfrac{1}{2}) - \Delta(0) - \Delta(1)\right]$$

In a special case, if the coefficients of (58) are even functions of t, then (58) is unchanged when we change t to $-t$; we see that if ξ is a solution, then $-\xi$ is also a solution. Therefore, if $\Delta(i\xi) = 0$ when $\sin(i\xi\pi) \neq 0$, then $\delta_2 = 0$. Therefore, when we want to find the roots for $\Delta(i\xi) = 0$, we always have $\delta_2 \sin(\pi\xi) = 0$ and the roots of (65) can be written out in a simple form,

$$\sin^2\left(\frac{\pi\xi_i}{2}\right) = \delta_2 \pm \sqrt{\delta_2^2 - \delta_1} \tag{67}$$

For a bounded solution, i.e. for $Re\, \xi = 0$, we must have

$$1 \geq \delta_2 \pm \sqrt{\delta_2^2 - \delta_1} \geq 0 \tag{68}$$

In Eq. (65) by putting $\Delta(i\xi) = 0$, we can compute ξ, and determine whether this is an unbounded solution or not. Then from (60) we can compute Γ_n, and obtain the complete solution of Eqs. (58). For a periodic solution we must have $\Delta(0) = 0$ or $\Delta(1) = 0$.

If a periodic solution of an inhomogeneous counterpart of Eq. (58) is considered, e.g.

$$\ddot{y} + A(t)\, y = B(t)$$

where $B(t)$ is a column matrix with its elements as periodic function, we can use (59) by putting $\zeta = 0$, whereas Eqs. (60) become inhomogeneous. If $\Delta(0) \neq 0$, we can solve for Y_n uniquely. If $\Delta(0) = 0$ we are on the boundary where Eqs. (58) have an unbounded solution. Therefore for such an inhomogeneous equation, as in our problem in Eq. (57), the zone of instability is determined by the homogeneous solution.

11. Concluding Remarks

In the above the formulation of the problem of stability of a liquid in a cylindrical container is given. The analytical difficulties that occur in this problem are pointed out. Variational approach is favored for approximate solutions. The resulting mathematical analysis is carried out, but numerical results are not included in this report. The trends of fluid stability in low gravity conditions, as influenced by tank flexibility and surface tension, will be presented in a future report.

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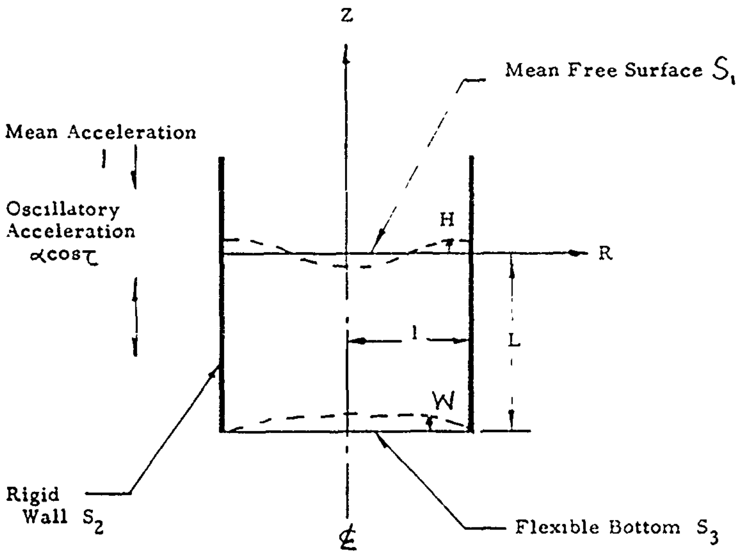

Fig. 1. Geometry of the Problem in Nondimensional Variable.