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WENGERT'S NUMERICAL METHOD FOR  
PARTIAL DERIVATIVES, ORBIT  
DETERMINATION, AND QUASILINEARIZATION

R. Bellman, H. Kagiwada and R. Kalaba

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PREFACE AND SUMMARY

In a recent article R. Wengert suggested a technique for machine evaluation of the partial derivatives of a function given in analytical form. In solving nonlinear boundary-value problems using quasilinearization many partial derivatives must be formed analytically and then evaluated numerically. Wengert's method appears very attractive from the programming viewpoint and permits the treatment of large systems of differential equations which might not otherwise be undertaken.

In this Memorandum we show how to apply the technique to some problems of orbit determination, though our ultimate goal is to handle much more complex problems, such as arise in adaptive control.

## I. INTRODUCTION

The method of quasilinearization for the numerical solution of nonlinear multi-point boundary-value problems is discussed in Refs. 1-3. Applications to system identification, cardiology and design and control are given in Refs. 4-6. The determination of orbits is discussed in Ref. 1. The method involves the calculation of many partial derivatives which can be quite onerous and the source of errors. In a recent note<sup>(7)</sup> R. Wengert suggested a procedure for the calculation of partial derivatives which relieves the analyst and the programmer of much routine differentiation. The purpose of this Memorandum is to show the usefulness of the Wengert scheme in carrying out the quasilinearization procedure. Our discussion is built around the orbit determination problem discussed in Ref. 1.

## II. ORBIT DETERMINATION

Consider a heavenly body H moving about the sun in a planar orbit, the x-y plane. The equations of motion are

$$\ddot{x} = \frac{-x}{(x^2 + y^2)^{1.5}} = f(x,y), \quad (1)$$

$$\ddot{y} = \frac{-y}{(x^2 + y^2)^{1.5}} = g(x,y). \quad (2)$$

At various times  $t_i$ ,  $i=1,2,\dots,N$ , the angle between the vector from a fixed observer at (1,0) and the x-axis is measured,  $\tan \theta(t_i) \approx b_i$ , which results in the multi-point boundary conditions

$$b_i \approx \tan \theta(t_i) = y(t_i) / (x(t_i) - 1), \quad i=1,2,\dots,N. \quad (3)$$

We wish to determine a set of conditions on  $x$ ,  $\dot{x}$ ,  $y$  and  $\dot{y}$  at an "initial instant"  $t=t_1$  so that the solution of Equations (1) and (2), subject to these conditions, agrees as well as possible with the observations in the sense of the method of least squares; i.e., we wish to minimize the sum  $S$ , where

$$S = \sum_{i=1}^N \left\{ y(t_i) - b_i [x(t_i) - 1] \right\}^2. \quad (4)$$

### III. QUASILINEARIZATION

(1,2,4)

Our computational formalism proceeds as follows:

We first select an initial approximation to the initial conditions and integrate the Equations (1) and (2) numerically to produce the functions  $x_0(t)$  and  $y_0(t)$  on the interval  $t_1 \leq t \leq t_N$ . Then we proceed recursively. The  $(k+1)^{\text{st}}$  approximation is produced from the  $k^{\text{th}}$  by solving the linear multi-point boundary-value problem

$$\begin{aligned} \ddot{x}_{k+1}(t) = & f(x_k, y_k) + (x_{k+1} - x_k) f_x(x_k, y_k) \\ & + (y_{k+1} - y_k) f_y(x_k, y_k), \end{aligned} \quad (5)$$

$$\begin{aligned} \ddot{y}_{k+1}(t) = & g(x_k, y_k) + (x_{k+1} - x_k) g_x(x_k, y_k) \\ & + (y_{k+1} - y_k) g_y(x_k, y_k), \end{aligned} \quad (6)$$

$$\text{Min} \sum_{i=1}^N \left\{ y_{k+1}(t_i) - b_i \left[ x_{k+1}(t_i) - 1 \right] \right\}^2. \quad (7)$$

where the minimization is over  $x_k(t_1)$ ,  $\dot{x}_k(t_1)$ ,  $y_k(t_1)$ , and  $\dot{y}_k(t_1)$ .

Particular and homogeneous solutions of Equations (5) and (6) are readily produced numerically, and the initial conditions are chosen so as to minimize the sum in Equation (7). See Ref. 2.

#### IV. WENGERT'S METHOD

Forming the partial derivatives called for in Equations (5) and (6) is not especially formidable. If other sources of the gravitational field were present, this would not be the case. Let us now see how we can have the computing machine evaluate the needed partial derivatives.

Wengert's method<sup>(7)</sup> is based on the chain rule,

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \frac{dx_2}{dt} + \dots + \frac{\partial u}{\partial x_n} \frac{dx_n}{dt}, \quad (8)$$

where  $u$  is a function of  $x_1, x_2, \dots, x_n$ , and  $x_1, x_2, \dots, x_n$  are functions of  $t$ . By putting  $dx_1/dt = 1$  and  $dx_i/dt = 0$  for  $i=2, \dots, N$ , we find, for example,  $\partial u/\partial x_1 = du/dt$ . In addition, the technique uses the elementary formulas of differentiation. The reader is referred to the paper by Wengert and to that by Wilkins.<sup>(8)</sup> The elementary differentiation subroutines INTEXP, ADD, MGNST, and DIV used in the present orbit determination program have been described in Ref. 8. These are the integer exponentiation, addition, multiplication by a constant, and division routines. In addition to these, we have written subroutine POW for non-integer exponentiation,

$$z = x^c, \quad \dot{z} = c x^{c-1} \dot{x},$$

as here listed.

```

SUBROUTINE POW(X,C,Z)
DIMENSION X(2), Z(2)
Z(1) = X(1)**C           (z = xc)
Z(2) = C*X(1)**(C-1.0)*X(2)  (ż = c xc-1 ẋ)
RETURN
END

```

Let us now combine Wengert's method with quasilinearization and incorporate the resultant scheme into the FORTRAN orbit program. But first, let us write the linear differential equations for the particular and homogeneous solutions of Equations (5) and (6). Let  $p(t)$  be the particular solution corresponding to the variable  $x(t)$ , and let  $q(t)$  be that for the function  $y(t)$ . Let the four homogeneous solutions for  $x(t)$  be represented by  $h^1(t)$ ,  $h^2(t)$ ,  $h^3(t)$ , and  $h^4(t)$ . Similarly for  $y(t)$ , we have the homogeneous solutions  $w^1(t)$ ,  $w^2(t)$ ,  $w^3(t)$ , and  $w^4(t)$ . Then the equations for the particular and homogeneous solutions are

$$\ddot{p} = f(x_k, y_k) + (p - x_k) f_x(x_k, y_k) + (q - y_k) f_y(x_k, y_k) \quad (9)$$

$$\ddot{q} = g(x_k, y_k) + (p - x_k) g_x(x_k, y_k) + (q - y_k) g_y(x_k, y_k) \quad (10)$$

$$\ddot{h}^i = h^i f_x(x_k, y_k) + w^i f_y(x_k, y_k), \quad (11)$$

$$\ddot{w}^i = h^i g_x(x_k, y_k) + w^i g_y(x_k, y_k), \quad (12)$$

$$i = 1, 2, 3, 4.$$

Equations (9), (10), (11), (12) show that the functions  $f$ ,  $f_x$ ,  $f_y$ ,  $g$ ,  $g_x$ ,  $g_y$  must be evaluated at each integration step. We write subroutine FUN1 to evaluate the function  $f(x,y) = -x/(x^2 + y^2)^{1.5}$ , by calling the appropriate elementary differentiation subroutines. Whenever this subroutine is executed, the derivative of  $f$  is simultaneously computed. The following is a listing of FUN1. Beside each FORTRAN statement, we have added the equivalent mathematical expressions.



```

SUBROUTINE FUN1(X,Y,ANS)
DIMENSION X(2), Y(2), ANS(2), Z1(2), Z2(2), Z3(2)
CALL INTEXP(X,2,Z1)            $z_1 = x^2$             $\dot{z}_1 = 2x\dot{x}$ 
CALL INTEXP(Y,2,Z2)            $z_2 = y^2$             $\dot{z}_2 = 2y\dot{y}$ 
CALL ADD(Z1,Z2,Z3)             $z_3 = z_1 + z_2$         $\dot{z}_3 = \dot{z}_1 + \dot{z}_2$ 
CALL POW(Z3,1.5,Z1)            $z_1 = (z_3)^{1.5}$       $\dot{z}_1 = 1.5(z_3)^{0.5} \dot{z}_3$ 
CALL MGNST(-1.0,X,Z2)          $z_2 = -x$             $\dot{z}_2 = -\dot{x}$ 
CALL DIV(Z2,Z1,ANS)            $f = z_2/z_1$           $\dot{f} = (\dot{z}_2 - f\dot{z}_1)/z_1$ 
RETURN
END

```

Similarly, we write subroutine FUN2(X,Y,ANS) to evaluate the function  $g(x,y) = -y/(x^2 + y^2)^{1.5}$ .

Next, we write the corresponding partial derivative evaluation subroutines. For example, the following subroutine PDRI(X,Y,PD) computes  $f_x$  and  $f_y$  and stores these values in PD(1) and PD(2), respectively.

```

SUBROUTINE PDRI(X,Y,PD)
DIMENSION X(2), PD(2), A(2)
X(2) = 1.0            $\dot{x} = 1$ 
Y(2) = 0.0            $\dot{y} = 0$ 
CALL FUN1(X,Y,A)      $f_x = \dot{f}$ , since  $\dot{x} = 1, \dot{y} = 0$ 
PD(1) = A(2)
X(2) = 0.0            $\dot{x} = 0$ 
Y(2) = 1.0            $\dot{y} = 1$ 
CALL FUN1(X,Y,A)      $f_y = \dot{f}$ , since  $\dot{x} = 0, \dot{y} = 1$ 
PD(2) = A(2)
RETURN
END

```

We write also the partial derivative routine for  $g_x$  and  $g_y$ , subroutine PDR2.

The major steps involved in producing the right-hand sides of the differential Equations (9) to (12), are

- 1) CALL FUN1 to produce  $f(x_k, y_k)$
- 2) CALL PDR1 to produce  $f_x, f_y$
- 3) Evaluate  $\ddot{p}, \ddot{h}^1, \ddot{h}^2, \ddot{h}^3, \ddot{h}^4$
- 4) CALL FUN2 to produce  $g(x_k, y_k)$
- 5) CALL PDR2 to produce  $g_x, g_y$
- 6) Evaluate  $\ddot{q}, \ddot{w}^1, \ddot{w}^2, \ddot{w}^3, \ddot{w}^4$ .

### V. NUMERICAL RESULTS

We repeated one of the numerical experiments mentioned in our earlier paper<sup>(1)</sup>. The 1962 experiment was carried out on the IBM 7090 by means of a FORTRAN II source program. This time, using Wengert's method, we ran the problem on the IBM 7044 with a FORTRAN IV program. The initial approximation to the orbit is a fixed point over all time, this point coinciding with the position of the earth. Seven iterations were carried out, and the results in both runs converged to the correct solution as shown in Table 1.

Table 1

	1962 Experiment	1964 Experiment	True Values
$x(2.5)$	1.19358	1.19358	1.19361
$\dot{x}(2.5)$	-.664268	-.664267	-.664263
$y(2.5)$	1.06063	1.06063	1.06070
$\dot{y}(2.5)$	.247466	.247466	.247499

The computer execution times involved are 1 minute 20 seconds in 1962, and 2 minutes 35 seconds in 1964. It is difficult to estimate the increase in computing time with the use of Wengert's method, since the computing machines and the source languages differed in these two runs. In the intermediate iterations 1 through 5, the predicted values of  $x$ ,  $\dot{x}$ ,  $y$  and  $\dot{y}$  at time 2.5 varied from 6 figures of agreement between runs,

to only 2 figures of agreement. The final results, however, did agree to at least 5 significant figures.

The use of Wengert's method in quasilinearization calculations appears promising, and additional experiments will be carried out.

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