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THE SOLUTIONS OF A SYMMETRIC MARKET GAME

L. S. Shapley

P-1392

5 June 1958

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SUMMARY

Solutions are obtained to a symmetrical market game in which the value of a coalition is assumed to be proportional to the number of buyers or sellers participating, whichever is smaller.

THE SOLUTIONS OF A SYMMETRIC MARKET GAME

L. S. Shapley

1. INTRODUCTION

This paper is directed to the problem of determining the full sets of solutions to certain multiperson games that display a rudimentary competitive pattern typical of many economic models. The players are divided into two groups M and N — we may think of them as buyers and sellers of some commodity — and the payoff functions are so constructed that players of opposite types are complementary (i.e., can enter into mutually profitable arrangements) while players of the same type are not; in fact, being perfectly interchangeable in coalitions, they find themselves in relentless competition for the chance to do business with their opposites. The number of players of each type is unrestricted.

Our highly symmetrical characteristic function:

$$(1.1) \qquad v(S) = \min(|S \cap M|, |S \cap N|)$$

(the number of elements in a set X is denoted by $|X|$), emphasizes the basic complementary/substitutability pattern to the exclusion of other features of the market process that might have been included, such as asymmetrical resources, elastic demand functions, indivisible goods, etc.,[1] and the rather exceptional regularity

[1] Characteristic functions embodying a number of these factors are formulated and discussed briefly in [9] (see the bibliography at the end of the paper); we intend to treat them more fully in a future publication.

that we shall observe in the solution sets reflects the structural simplicity of (1.1). Nevertheless, our close analysis of a special class of market games can be expected to point the way for more general types. Also, quite apart from the remote practical significance of the results, the conquest of this class of large-sized games should prove of some theoretical interest, especially since a technical device of considerable generality is developed along the way.[2] As Gillies remarked on a similar occasion: "The intensive study of a particular class of games provides empirical data on the nature of solutions, methods which may be applied to other games, and may suggest or disprove conjectures on solutions in general."[3]

Surprisingly, large market games have been generally neglected by game theorists since the initial work of von Neumann and Morgenstern on the subject.[4] Markets meet the underlying assumptions of complete information, transferable utility, etc., better than most economic phenomena, and ought to provide the material for some good tests of the von Neumann-Morgenstern solution theory. A really decisive confrontation of the theory is not easy to arrange, but in such well-suited applications, as opposed to more artificially-derived examples, one feels that the critics are fully justified in insisting that the

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[2] We refer to the "skew sets," defined in §5 and §7 below.

[3] [3], p. 32[illegible].

[4] [13], §64; see also [9], [10], [12]. Our (1.1) can be obtained from (64.[illegible]) of [13] by specializing the parameters of the latter.

solutions make economic sense: that they satisfy or perhaps extend—but do not contradict—the expectations and intuitions based on observed experience.

The one-parameter sets of imputations that make up our solutions are closely related to the "bargaining curves" that have been observed in many other game solutions.[5] The parameter in the present case can be interpreted as the average net market price of the commodity, or (transformed) as the total profit of the sellers as a group. It varies continuously in each solution from zero profit to the sellers ("cutthroat" pricing) to zero gain for the buyers ("all the traffic will bear"). At any particular parameter value, the solution tells exactly how the individual gains are to be imputed among the players. In other words, if the solution is known and the average price is known, then the outcome, financially speaking, is completely determined.

There is just one symmetrical solution; it corresponds to the "free trade" or "same-price-to-all-comers" standard of behavior. More generally, the solution can be regarded as a description of the institutionalized modes of collusion—premiums, rebates, class discrimination, boycotts, etc.—in terms of their net effect on the outcome. A nonsymmetric solution expresses a stable, self-consistent departure from the "free trade" norm.

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[5] See [3], [5], [7], and [13], §4[illegible].8.2 and §6[illegible]; also [4], p. 209. Bargaining curves are commonly found combined with other point-sets in the solution; the purity of the present occurrences can be ascribed to the very direct "complementary/substitutable" structure of (1.1). There is a close connection between the latter and the (m·n)-person simple game $B^*_m \cdot B^*_n$, with (non-superadditive) characteristic function:

$$v(S) = \min(|S \cap M|, |S \cap N|, 1),$$

whose solutions are all of the bargaining curves connecting the zero faces of the two groups of players (see [illegible], [11]).

Meanwhile the basic function of the actual market institution—that of establishing and maintaining the general price equilibrium—must presumably be carried out by means of "noncollusive" bargaining tactics: bids, prices, concessions, counter-proposals, etc., insofar as they are available among the formally permitted moves of the game. This sets the stage for a rather remarkable division of labor between "cooperative" game theory and its "noncooperative" cousin. We make no attempt in the present paper to solve the noncooperative bargaining problems posed by our cooperative solutions; to do so would require pre-constructing the strategy spaces and payoff functions that underlie (1.1), and would lead us far afield. But the possibility of being able to "solve the solution" is not an unnatural one, when we reflect on how much is left out of the cooperative approach: most of the formal bargaining moves of the extensive-form game are rendered superfluous by the added, free coalition-forming process implicit in the characteristic function; likewise, all of the details of price and money transfer are swallowed up by the hypothesis of unrestricted side-payments. Indeed, the fact that something identifiable as "average price" appears as a parameter in the solutions, after so much has been apparently suppressed, speaks well for the validity of the cooperative solution concept.

Our formal results fall short of a complete list of solutions, such as obtained by von Neumann and W. H. Mills on comparable occasions;[6] however, we bring that ultimate goal within reach.

[6] See [13], 65, and [4].

We prove that all solutions are monotonic arcs spanning the simplex of imputations, as discussed above; we also obtain a bound on their location in the simplex and determine explicitly a major, centrally-located subclass of solutions, which in certain cases turns out to be complete (see below, Theorems 3, 2, 1, respectively). As already noted, there is a unique solution possessing the full symmetry of the game.[7]

The main body of this paper is concerned exclusively with the mathematical problem.[8] We have tried to keep the presentation self-contained, and have adopted a mildly expository tone at first, with the idea of easing the way for readers not versed in the intricacies of solution theory.

§1. Preliminaries

Let M and N be the two groups of players, having respectively $m = |M|$ and $n = |N|$ elements. The $(m+n)$-person game to be considered is given by the characteristic function:

$$(1.1) \qquad v(S) = \min(|S \cap M|, \; |S \cap N|) \qquad \text{all } S \text{ in } M \cup N.$$

Define $g = v(M \cup N) = \min(m, n)$. Vectors on $M \cup N$ will be written as

$$x \quad \text{or} \quad (x' : x'') \quad \text{or} \quad (x_1', \ldots, x_m'; \; x_1'', \ldots, x_n'').$$

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[7] This corresponds to Bott's result for the (n, k)-games [1], [3]—another comparable, but incompletely solved, class of large-sized games.

[8] The heuristic account of the game and its solutions, given in [10], still applies for the most part, although the mathematical results therein are now superseded.

Sums over their components will generally be abbreviated

$$x(S) \qquad \text{for} \qquad \sum_{S \cap M} x'_\mu + \sum_{S \cap N} x''_\nu .$$

An imputation is a nonnegative vector on $M \cup N$ such that $x(M \cup N) = g$; the space of all imputations is a simplex of $m + n - 1$ dimensions, denoted by A. The "face" A_S is the $(|S| - 1)$-dimensional set of imputations x such that $x(S) = g$; these are the vectors that impute a total of g to the members of S, and nothing to the other players. The opposing complementary faces A_M and A_N will figure prominently in our analysis.

In general, x is said to dominate y via S provided that

(1.2) $\qquad x - y$ is strictly positive on S, and

(1.3) $\qquad x(S) \leq v(S)$.

In the present instance it will suffice to consider only domination via sets of the form $\{\mu, \nu\}$, $\mu \in M$, $\nu \in N$, since other dominations always imply a domination of this kind. Then (1.2) and (1.3) become simply:

(1.4) $\qquad x'_\mu > y'_\mu \quad \text{and} \quad x''_\nu > y''_\nu$,

(1.5) $\qquad x'_\mu + x''_\nu \leq 1$.

The dominion of a set X of imputations, written dom X, is the (open) set of imputations dominated by elements of X. The set

A − dom A of undominated imputations is called the *core* of the game; it comprises just those imputations x in which all coalitions are "satisfied": $x(S) \geq v(S)$. A *solution* of the game is defined to be any set V of imputations that dominates its complement in A, and nothing else: $V = A - \text{dom } V$. Every solution, being the complement of a dominion, is a closed set and contains the core; no solution contains another. The two properties:

$$V \cap \text{dom } V = 0 \qquad \text{and} \qquad V \cup \text{dom } V = A,$$

that combine to characterize a solution will be referred to sometimes as *internal* and *external stability*, respectively.

LEMMA 1. No solution of (1.1) contains an open set, unless $m = n = 1$.

Proof. For every $x \in A$, there is a pair μ^*, ν^* for which (1.5) holds; otherwise we could sum g inequalities of the form $x'_\mu + x'_\nu > 1$, involving 2g distinct indices, and obtain an absurdity $x(S) > g$. Furthermore, if x is interior to A, every neighborhood of x will contain a y for which (1.4) holds, with respect to μ^*, ν^*, unless $m = n = 1$. Hence every open set in A is internally unstable. But no solution can contain an internally unstable subset. Q.E.D.

§2. The case $m = n$.

The case where M and N are of equal size can be disposed of quickly. Let V be the set of imputations of the form:

$$(2.1) \qquad z_p = (p, \ldots, p; 1-p, \ldots, 1-p), \qquad 0 \leq p \leq 1.$$

For any $S \in M \cup N$ we have:

$$z_p(S) = p|S \cap M| + (1-p)|S \cap N|$$

$$\geq \min(|S \cap M|, |S \cap N|) = v(S).$$

Thus z_p is undominated, and V is at least contained in the core. On the other hand, any $x \in A$ <u>not</u> in V must have $x'_{\mu^*} + x'_{\nu^*} < 1$ for at least one pair μ^*, ν^*, and hence is dominated via $\{\mu^*, \nu^*\}$ by some $y \in A$. Thus V is precisely the core. What is more, we can take y in the preceding argument to be one of the elements of V, namely z_{p^*} where p^* satisfies

$$x'_{\mu^*} < p^* < 1 - x'_{\nu^*} .$$

We conclude that V dominates all of $A - V$, making V both the core and a solution of the game. Under these circumstances V is necessarily the unique solution. Geometrically, V is the straight line joining the centers of gravity of the opposing faces A_M and A_N.

§3. The case min(m, n) = 1.

The case of monopoly or monopsony,[9] min(m, n) = g = 1, will now be considered. We may assume m = 1, n > 1. Since v takes on only the values 0 and 1 we have what is called a simple game: the "winning" coalitions are those consisting of the single member of M and one or more players from N. This game happens to be factorable into one-person simple games, and a complete description of its solutions is therefore available (see [11]; also footnote 5 above). They turn out to be monotonic curves running from the face A_N to the opposite vertex of the simplex. Stated precisely, a solution is any set of points of the form:

$$(3.1) \qquad z_p = (p; f_1(p), \ldots, f_n(p)), \qquad 0 \leq p \leq 1,$$

where the functions $\{f_i\}$ satisfy $f_i \geq 0$, $\sum f_i(p) = 1-p$ and are continuous and nonincreasing. In contrast with the preceding case, the core, which consists of the single imputation (1; 0, ..., 0), does not dominate even a part of its complement.

The proof that the curves (3.1) are solutions, and that they are the only solutions, is omitted in deference to the more general results proved later on (see Theorem 1 and the second corollary to Theorem 3). Figure 1 illustrates two solutions for the 3-person case[10] m = 1, n = 2; the shaded areas represent the dominions of typical points on the curves, and make it more

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[9]Compare [13], §§4.2.2.

[10]Included, of course, in the complete analysis of general-sum 3-person games in [13], §§0.3 and 04.[illegible].

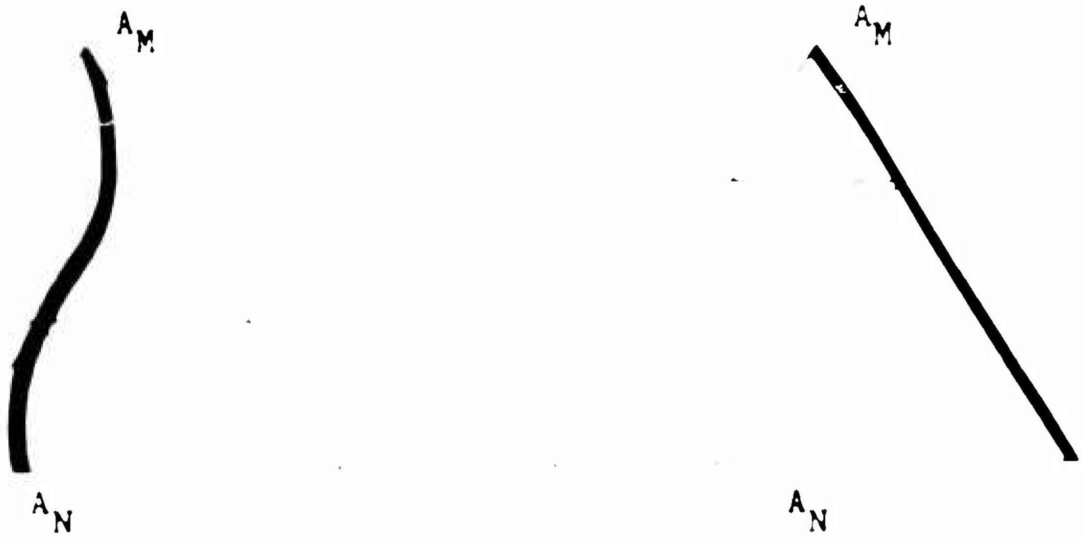

Fig. 1

or less apparent why the functions f_i must be monotonic to avoid internal instability—i.e., self-domination.

§4. The general case.

We now drop the restrictions on m and n. By a monotonic arc we shall mean a one-parameter family of imputations of the form:

$$(4.1) \qquad z_p = f_1'(p), \ldots, f_m'(p);\ f_1''(p), \ldots, f_n''(p) \qquad \text{all } p \in R,$$

where R is some real interval, and the functions f_μ', f_ν'' are continuous, nonnegative, and respectively nondecreasing and nonincreasing. Without loss of generality, we can choose the parameter p so that

$$(4.2) \qquad \sum_M f_\mu'(p) = p, \qquad\qquad \sum_N f_\nu''(p) = g - p,$$

thereby making R a subinterval of $[0, g]$.[11]

Let E denote the subset of A delimited by the mn inequalities:

$$(4.3) \qquad x'_\mu + x_\nu \leq 1, \qquad \text{all } \mu \in M,\ \nu \in N$$

Clearly E is nonempty, closed, and convex, and has a nonempty intersection with both A_M and A_N. It is the subset of A in which every pair (μ, ν) is an "effective" coalition, in the sense of (1.3).

> THEOREM 1. Every monotonic arc in E connecting A_M and A_N is a solution of the game.

Remark 1. The theorem is not vacuous, since the straight line joining the centers of gravity of A_M and A_N is a monotonic arc and is contained in E.

Remark 2. If $m = n$ then E is exactly the core (see (2.1)), and if $g = 1$ then $E = A$. Thus, all of the solutions described in §2 and §3 are included in the theorem.

Remark 3. When $m \neq n$, one of the contacts $E \cap A_M$ or $E \cap A_N$ is a single point, namely $(1, \ldots, 1; 0, \ldots, 0)$ or $(0, \ldots, 0; 1, \ldots, 1)$, which pins down one end of the solutions of the theorem. The explanation for this lies in the fact that the point in question is the core, and is necessarily contained in all solutions.

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[11] We can now interpret p as the total profit realized by the sellers' group, M, under the imputation z_p. Adding their costs (assumed constant) and dividing by g gives us the average net market price (see the discussion in the Introduction).

Proof of theorem. Let V be a monotonic arc in E running from A_M to A_N and parametrized according to (4.1), (4.2). To show that V is externally stable, take any $x \in A$, and let p_1 be the greatest $p \in R$ such that

$$(4.4) \qquad f'_\mu(p) \leq x'_\mu, \qquad \text{all } \mu \in M,$$

and p_2 the least $p \in R$ such that

$$(4.5) \qquad f''_\nu(p) \leq x''_\nu, \qquad \text{all } \nu \in N.$$

The existence of these extrema is assured by the fact that V touches A_M and A_N, i.e., that $R = [0, g]$. We distinguish two cases.

Case A: $p_1 < p_2$. Let $p_1 < p^* < p_2$. Then for some μ^*, ν^* we have both

$$f'_{\mu^*}(p^*) > x'_{\mu^*} \qquad \text{and} \qquad f''_{\nu^*}(p^*) > x''_{\nu^*}.$$

Using the fact that $V \subset E$, we see that z_{p^*} dominates x via (μ^*, ν^*).

Case B: $p_1 \geq p_2$. Inserting p_1 in (4.4) and p_2 in (4.5) and summing, we obtain

$$p_1 + g - p_2 \leq x(M \cup N) = g,$$

with the aid of (4.2). This means that p_1 p_2 and that equality holds in all of the $m + n$ inequalities (4.4), (4.5), with p_1 and p_2 inserted. Hence we have:

$$x \quad f'(p_1);\ f'(p_2), \qquad z_{p_1} \in V.$$

This completes the proof of external stability, our two cases having shown that every imputation x is either in V or in dom V. As for internal stability, it is obvious that the contrary direction of the two sets of monotonic functions $\{f'_\mu\}$, $\{f_\nu\}$ rules out the possibility of domination within a monotonic arc. This completes the proof of the theorem.

It is easily verified that the solutions given by Theorem 1 fill up all of E. An example, given below in §6, shows that solutions exist that are not contained in E. We now define a larger subset of A, denoted by F, that contains all solutions, giving us the chain:

$$A \quad F \quad \text{union of all solutions} \quad E \quad \text{core}.$$

In fact, let F be the set of all imputations x such that

$$\max_M x'_\mu + \min_N x_\nu \geq 1, \text{ and}$$

$$\min_M x'_\mu + \max_N x_\nu \leq 1.$$

A comparison with (4.3) shows that F contains E. Note that F may not be convex.

THEOREM 2. Every solution of the game is contained in F.

<u>Proof</u>. Let V be any solution, let x be any element of V, and let μ^* be any element of M such that

$$(4.6) \qquad x'_{\mu^*} + x''_{\nu} > 1 \qquad\qquad \text{all } \nu \in N,$$

supposing that such a μ^* exists. The case $x'_{\mu^*} = 0$ leads at once to the absurdity: $x(N) > n \geq g$. If $x'_{\mu^*} > 0$ we can find $z \in A$ that majorizes x in all components but x'_{μ^*}, and that is so near to x that the strict inequalities of (4.6) remain valid for z. Then any imputation that dominates z dominates x as well. Therefore z is undominated by V, and hence must actually be in V. By the same reasoning, a small neighborhood of z is in V. But this is impossible, by Lemma 1. Hence the existence of μ^* fulfilling (4.6) is refuted, and

$$\max_M x'_\mu + \min_N x''_\nu \leq 1$$

is established. A symmetrical argument completes the proof.

COROLLARY. No component of any imputation of any solution exceeds 1.

This is an instance of the general location theorem of Gillies and Milnor,[12] which states that the components of all imputations in a solution must satisfy the inequality:

$$x_i \leq \max_{S \ni i} [v(S) - v(S - \{i\})].$$

A stronger general theorem ([2], p. 21) is available; it would improve on the corollary but not on Theorem 2 itself.

§5. Skewness of solutions.

In this section we shall prove that every solution of the game (1.1) is a monotonic arc connecting A_M and A_N, though not necessarily contained in E. Our main tool will be a property called "skewness", which is a special kind of internal stability. Once we have proved that every solution is skew (Lemma 5) the main result (Theorem 3) follows quickly.

For vectors a and b the notation $a \geq b$ [$a \leq b$] will denote that every component of a − b is nonnegative [nonpositive]. By sup(a, b) [inf(a, b)] we shall mean the least upper [greatest lower] bound of a and b. By med(a, b, c) we shall mean the vector each of whose components is the median of the corresponding components of a, b, c. A vector c will be said to lie between the two vectors a and b if c = med(a, b, c). We observe that the median of any three vectors is between every two of them.

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[12] [[illegible]] and [[illegible]]; or see [[illegible]], p. [illegible].

We call a pair x, y of imputations <u>skew</u> if either

$$\left\{\begin{matrix} x' \geq y' & \text{and} \\ x'' \leq y'' & \end{matrix}\right\}, \quad \text{or} \quad \left\{\begin{matrix} x' \leq y' & \text{and} \\ x'' \geq y'' & \end{matrix}\right\}.$$

A <u>skew set</u> is one in which every pair of elements is skew. A skew set is obviously internally stable, in view of (1.4). Given any three elements of a skew set, one of them will be found to lie between the other two. A monotonic arc is an obvious example of a skew set.

While there is no <u>a priori</u> reason that a solution must be skew, the hypothesis of non-skewness has far-reaching implications that eventually prove contradictory. Our entering wedge is the next lemma; note that it is trivial for skew solutions.

LEMMA 2. If x and y are elements of a solution V, then the two vectors

$$u = \langle \sup(x', y');\ \inf(x'', y'') \rangle \quad \text{and}$$

$$v = \langle \inf(x', y');\ \sup(x'', y'') \rangle$$

are also elements of V.

<u>Proof</u>: If u and v are in A, that is, if

$$(5.1) \qquad u(M \cup N) = v(M \cup N) = g,$$

then it is easily seen that they are in V, since any $z \in V$ that might dominate u (say) would also have to dominate either x or y. Therefore our task is to prove (5.1). Suppose that

$$u(M \cup N) < g.$$

Take a vector w that is strictly greater than u in every component, with $w(M \cup N) = g$. Then w must be in V, since anything dominating it must also dominate either x or y. By the same reasoning, a small neighborhood (in A) of w must likewise be in V which is impossible, by Lemma 1. Hence we have

$$u(M \cup N) \geq g. \tag{5.2}$$

Similarly, we have

$$v(M \cup N) \geq g. \tag{5.3}$$

Adding, and using the identity $u + v = x + y$, we obtain the expression $2g \geq 2g$. This means that we actually have equality in (5.2) and (5.3), as was to be shown.

COROLLARY 1. The median of any three elements of a solution is itself in the solution.

Proof. Apply the lemma repeatedly to the identity:

$$\text{med}(x, y, z) \equiv$$

$$\langle \inf[\sup(x;y'), \sup(y;z'), \sup(z;x')]; \ \sup[\inf(x;y''), \inf(y;z''), \inf(z;x'')] \rangle$$

exploiting the fact that the "med" function can be expanded in terms of "inf" and "sup" in two different, anti-symmetric ways.

COROLLARY 2. If x and y are elements of the same solution V, then $x' \gtrsim y'$ implies $x'' \lesssim y''$. Equivalently, if $x, y \in V$ are not skew, then four indicates $\mu_1, \mu_2 \in M$, $\nu_1, \nu_2 \in N$ can be found such that:

$$x'_{\mu_1} > y'_{\mu_1}, \qquad x'_{\mu_2} < y'_{\mu_2}, \qquad x''_{\nu_1} > y''_{\nu_1}, \qquad x''_{\nu_2} < y''_{\nu_2} .$$

Proof. Suppose $x' \gtrsim y'$ but not $x'' \lesssim y''$. Then $u' = x'$ and $u'' \lesssim x''$, with $u \neq x$ (defining u as in the lemma). Then $u(M \cup N) < x(M \cup N) = g$, contradicting (5.1).

LEMMA 3. If two distinct elements of a solution V are skew, then there is a third, distinct element of V between them.

Proof. Let x, y be a distinct, skew pair of imputations belonging to V, with $x' \gtrsim y'$, $x'' \lesssim y''$, and let z be any other imputation between them. If z is in V then we are finished;

if not, we can find a $w \in V$ that dominates z via some $\{\mu^*, \nu^*\}$, thus:

$$w'_{\mu^*} > z'_{\mu^*} \geq y'_{\mu^*} ,$$

$$w''_{\nu^*} > z''_{\nu^*} \geq x''_{\nu^*} , \quad \text{and}$$

$$w'_{\mu^*} + w''_{\nu^*} \leq 1 .$$

Since w is not permitted to dominate either x or y, we must also have

$$x'_{\mu^*} \geq w'_{\mu^*} \qquad \text{and} \qquad y''_{\nu^*} \geq w''_{\nu^*} .$$

Let $t = \text{med}(x, y, w)$. By Corollary 1 above, t is in V. Clearly t is between x and y. Inspecting the above inequalities, we see that

$$t'_{\mu^*} = w'_{\mu^*} \neq y'_{\mu^*} \quad \text{and} \quad t''_{\nu^*} = w''_{\nu^*} \neq x''_{\nu^*} .$$

Hence t is distinct from both x and y, as required.

LEMMA 4. Every pair of skew points in a solution V can be connected by a monotonic arc lying entirely within V.

<u>Proof</u>. Define a partial ordering of imputations by the relation:

$$(5.4) \qquad s \overset{\cdot}{>} t \quad \text{if and only if} \quad s' \geq t',\ s'' \leq t'',\ s \neq t.$$

Let x, y ϵ V be skew, with $x \overset{\cdot}{>} y$. By Zorn's Lemma, the set V_{xy} of elements of V that lie between x and y (including x and y) contains at least one maximal chain (maximal linearly-ordered subset). Denote this chain by C. Clearly C contains x and y. Since V_{xy} is a closed set, and the closure of a chain is still a chain, C is a closed set. The continuous function $\phi(s) = s(M)$ maps C into a closed subset $\phi(C)$ of the real interval $I = [y(M), x(M)]$. The mapping is 1:1 as far as it goes, since ϕ is strictly order-preserving, and it covers the endpoints of I. If it failed to cover the interior of I, then $I - \phi(C)$ would contain an open subinterval with endpoints belonging to $\phi(C)$. The inverse images of these endpoints, in C, would be skew to each other. By Lemma 3 there would be a distinct element z ϵ V—C between them. But such a z could be added to the chain C, contradicting the latter's assumed maximality. Therefore we conclude that $\phi(C)$ covers I — i.e., that ϕ is 1:1 onto. The inverse of ϕ, mapping I continuously back onto C, then provides a parametrization of C as a monotonic arc joining x and y, as required.

LEMMA 5. Every solution is a skew set.

<u>Proof</u>. Suppose the contrary, and let $\mathbf{x}$, y be a nonskew pair in a solution V. The associated points $u = \sup(\mathbf{x}', y'); \inf(\mathbf{x}'', y'')$ and $v = \inf(\mathbf{x}', y'); \sup(\mathbf{x}'', y'')$ (see Lemma [illegible]) are then distinct from $\mathbf{x}$ and y, but skew to them both. (In the notation of (?.?), we have $u \dot{>} \mathbf{x} \dot{>} v$ and $u \dot{>} y \dot{>} v$.) Lemma 4 then reveals the existence of four monotonic arcs $u\mathbf{x}$, uy, $\mathbf{x}v$, yv all contained in V. From this configuration we shall derive a contradiction.

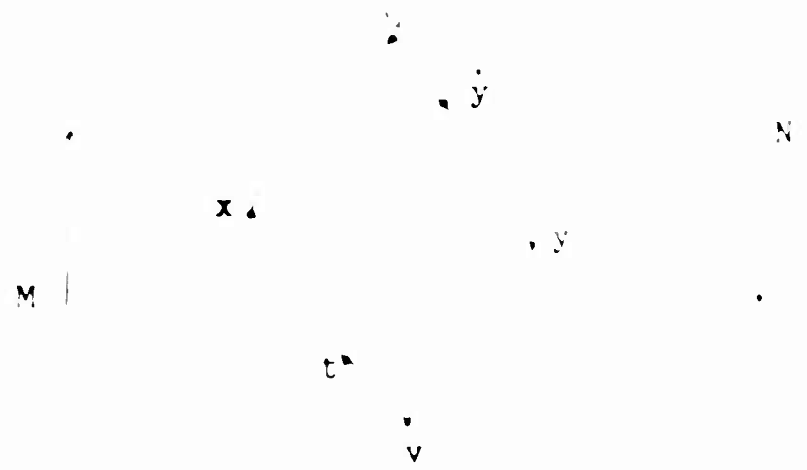

Fig. 2

Define four disjoint sets of indices as follows:

$M_1 = \{\mu \in M \mid \mathbf{x}'_\mu$ [illegible] $y'_\mu\}$

$M_2 = \{\mu \in M \mid \mathbf{x}'_\mu$ [illegible] $y'_\mu\}$

$N_1 = \{\nu \in N \mid \mathbf{x}$ [illegible] y [illegible]$\}$

$N_2 = \{\nu \in N \mid \mathbf{x}''$ [illegible] y [illegible]$\}$.

By the second corollary to Lemma 2, none of these is empty. Our first task will be to prove:

$$(5.5) \qquad \min_{\mu \in M_2} x'_\mu = \min_{\mu \in M_1} y'_\mu$$

To do this, we start with a fixed $\nu_1 \in N_1$, and select a positive $\epsilon < x'_{\nu_1} - y'_{\nu_1}$. Define $z \in A$ by

$$\begin{cases} z'_\mu = x'_\mu + \epsilon/(m+n-1) & \text{all } \mu \in M \\ z'_\nu = x'_\nu + \epsilon/(m+n-1) & \text{all } \nu \in N - \{\nu_1\} \\ z'_{\nu_1} = x'_{\nu_1} - \epsilon . \end{cases}$$

Suppose first that $z \in V$. Take any $\nu_0 \in N - \{\nu_1\}$ and let μ_0 be such that $z'_{\mu_0} = \min_M z'_\mu$. By Theorem 2 we have $z'_{\mu_0} + z'_{\nu_0} \leq 1$. Hence z dominates x via $\{\mu_0, \nu_0\}$, contradicting the internal stability of V. Therefore $z \notin V$, and there is a $w \in V$ that dominates z via some $\{\mu^*, \nu^*\}$, with

$$w'_{\mu^*} + w'_{\nu^*} \leq 1.$$

But z "almost" majorizes x; to keep w from dominating x as well as z we must have $\nu^* = \nu_1$. Therefore:

$$w'_{\nu^*} = w'_{\nu_1} > z'_{\nu_1} = x'_{\nu_1} - \epsilon > y'_{\nu_1} .$$

But now w threatens to dominate y; to prevent this we must have:

$$y'_{\mu^*} \leq w'_{\mu^*} > z'_{\mu^*} > x'_{\mu^*} ,$$

from which we conclude that $\mu^* \in M_2$. Hence:

$$(5.6) \qquad \min_{M_2} x'_{\mu} \leq x'_{\mu^*} \quad w'_{\mu^*} \leq 1 - w_{1^*}$$

$$1 - x_{1} \quad \cdot \quad \cdot \quad .$$

Now, take a $\mu_1 \in M_1$ so that $y'_{\mu_1} \quad \min_{M_1} y'_{\mu}$, and select a positive $\eta \quad x'_{\mu_1} - y'_{\mu_1}$. There will be a point t on the monotonic arc xv V with $t'_{\mu_1} \quad y'_{\mu_1} + '$. Note that $t_1 \quad x_1 \quad y_1$, so that we are in danger of having t dominate y via $\mu_1, 1$.

To avert this requires $t'_{\mu_1} + t_1 \quad 1$. Thus we have:

$$(5.7) \qquad \min_{M_1} y'_{\mu} \quad y'_{\mu_1} = t'_{\mu_1} -$$

$$1 - t_{1} - \quad 1 - x_{1} - \quad .$$

Since (5.6) and (5.7) are valid for arbitrarily small ε and , we conclude that $\min_{M_1} y'_{\mu} \leq \min_{M_2} x'_{\mu}$. A symmetrical argument establishes the reverse inequality, and (5.5) follows.

To complete the proof of the lemma, choose a point $\dot{y} \in uy$ V far enough from y so that $\dot{y}'_{\mu} \quad y'_{\mu}$ for all $\mu \in M_1$, but not so far that $\dot{y} = u$. (See Fig. 2.) Then x and $\dot{y}$ are nonskew. Defining

$\dot{M}_1$ and $\dot{M}_2$ as above, with respect to the pair x, $\dot{y}$, we see at once that $\dot{M}_1 \subset M_1$ and $\dot{M}_2 \supset M_2$ since $\dot{y}' \geq y'$. Hence

$$\min_{\dot{M}_2} x'_\mu \geq \min_{M_2} x'_\mu, \qquad \text{and}$$

$$\min_{\dot{M}_1} \dot{y}'_\mu < \min_{M_1} \dot{y}'_\mu \leq \min_{M_1} y'_\mu,$$

and (5.5) must fail for one of the pairs x, y or x, $\dot{y}$. This is the desired contradiction.

THEOREM 3. Every solution of the game is a monotonic arc connecting A_M and A_N.

<u>Proof</u>. Let V be a solution, and let $x \in V$ be its "nearest approach" to the face A_M, in the sense that $x(M)$ is maximized. Since V is a skew set (Lemma 5), we have

$$w' \leq x', \qquad w'' \geq x'' \qquad \text{all } w \in V.$$

The imputation $z \in A_M$, given by

$$\begin{cases} z'_\mu = x'_\mu + x(N)/m & \text{all } \mu \in M \\ z''_\nu = 0 & \text{all } \nu \in N \end{cases}$$

is skew to every element of V, and hence is undominated by V.

Therefore, it belongs to V, and is in fact the element x. In other words, V actually touches A_M. By a similar argument, V touches A_N. Therefore (Lemma 4) V includes a monotonic arc C connecting the two faces. But C is obviously a maximal skew set; hence $V = C$. This completes the proof.

COROLLARY 1. The only solution possessing the full symmetry of the game is the set of imputations of the form

$$(p, \ldots, p; q, \ldots, q), \quad mp + nq = g,$$

— that is, the line joining the midpoints of A_M and A_N.

Thus, the unique symmetric solution is precisely the set of symmetric imputations.[13]

COROLLARY 2. If $\min(m, n) = g - 1$, then all the solutions of the game are given by Theorem 1. (See §3.)

§6. <u>Two examples</u>.

Theorems 2 and 3, while they narrow the class of candidates to a fairly concise family of sets — the maximal monotonic arcs in P — nevertheless fall short of a complete characterization

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[13] Contrast this with the situation for the familiar three-person simple majority game: the unique symmetric solution consists of three asymetric imputations; the unique symmetric imputation belongs to three asymmetric solutions (see [13], §32, §33).

of the solutions of the game. Maximal monotonic arcs lying entirely in E are certainly solutions, by Theorem 1, but the status of those that enter the region F − E remains in doubt. We give two examples to show that the problem has no simple, all-or-none resolution.

Let $M = \{1, 2\}$, $N = \{3, 4, 5\}$. (Any smaller game would come under the special cases $m = n$ or $g = 1$.) Let V_1 and V_2 be polygonal arcs, with vertices connected in the order listed:

Vertices of V_1	Vertices of V_2
$(1, 1; 0, 0, 0)$	$(1, 1; 0, 0, 0)$
$(3/4, 3/4; 1/6, 1/6, 1/6)$	$(2/3, 2/3; 1/3, 1/6, 1/6)$
$(3/4, 1/4; 1/2, 1/4, 1/4)$	$(2/3, 1/3; 2/3, 1/6, 1/6)$
$(1/4, 1/4; 1/2, 1/2, 1/2)$	$(1/3, 1/3; 2/3, 1/3, 1/3)$
$(0, 0; 2/3, 2/3, 2/3)$	$(0, 0; 2/3, 2/3, 2/3)$

In each case, the third vertex is in F − E, by virtue of $x_1' + x_3'' > 1$. We assert that V_1 is a solution, and that V_2 is not.

To verify the former, we refer to the proof of Theorem 5 and observe that the only statement therein that does not apply to V_1 is the claim of domination via $(\mu^*, \nu^*) = (1, 3)$ for certain values of p^* — a claim that is invalidated by the failure of condition (1.5) in the vicinity of the third vertex. However, the only imputations whose domination depends on this are those with $1/2 \leq x_1' < 3/4$, $1/4 \leq x_3' < 1/2$, and a straightforward argument shows that all such imputations are dominated by V_1 via other pairs (μ, ν), for which (1.5) always holds. We omit the details.

To verify that V_2 is not a solution, we observe simply that the imputation (1/3, 2/3; 1/3, 1/3, 1/3), among others, is undominated by V_2.

§7. General form of the skewness concept

It was remarked in §5 that skewness is a special form of internal stability. It may be of interest to have this relationship made precise in the context of general n-person games, and to suggest why it works so well in the present case.

Following Gillies [2], let us call a coalition S *vital* if there does not exist a nontrivial partition $(S_1, \ldots, S_r)$ of S such that $v(S) = \sum v(S_i)$. (In the game (1.1) the only vital coalitions are the one-element sets and those of the form (μ, ν) with $\mu \in M$, $\nu \in N$.) Let us call two imputations **x** and y *skew* if neither x - y nor y - x is strictly positive on any vital coalition having more than one member.

Our previous definition (§5) is obviously included in the above. It is easy to verify in general that skew sets are internally stable, and that for simple games the solutions are always maximal skew sets. That the converse of the latter is not true may be seen from the example of three-point configurations in the essential zero-sum 3-person game which are maximal skew, but not solutions. An indication of a possible role which skewness might play in the general theory may be gleaned from [13] §30.3 (esp. §30.3.6). However, there are grave difficulties to be overcome in the general approach outlined there, and the best immediate prospect lies in applications to restricted classes of games, especially those with relatively few vital coalitions.

In the present application, the key property is the fact that skew sets are linearly ordered chains in the partial ordering (5.4) defined in the proof of Lemma 4 (but used elsewhere implicitly); that is, skewness of a three-point set implies that one of the points is between the other two.[14] For other classes of games we may hope to be able to discover and exploit other, equally decisive, special properties.

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[14] This is a converse to the general theorem: if z is between x and y, and if {x, y} is skew, then {x, y, z} is skew.

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