FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING—IX: VARIATIONAL ANALYSIS, ANALYTIC CONTINUATION, AND IMBEDDING OF OPERATORS

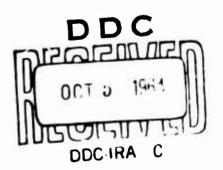Richard Bellman
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SUMMARY

Using the formal identity $\underset{x}{\text{Max}}\left[(x,Ax) - 2(x,y)\right] = - (y,Ay)$ we have in previous papers studied the properties of the Green's function of various functional equations and properties of the resolvent operator. From this latter we were able to deduce properties of the characteristic functions and characteristic values.

In this paper we show how variational techniques can be applied to deduce similar properties for complex operators and for operators which are non—symmetric.

For complex operators we use a min—max variation and analytic continuation, if necessary, while for non—symmetric operators we use an imbedding technique, plus analytic contin- uation if required. A non—symmetric operator is imbedded within a family of symmetric operators associated with a variational problem.

Once the variational problem has been formulated we can apply the functional equation techniques of the theory of dynamic programming.

FUNCTIONAL EQUATIONS IN THE
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Richard Bellman and Sherman Lehman

1. Introduction. Consider the formal identity

$$\text{Max}_{x} \left[(x,Ax) - 2(x,y) \right] = (y,A^{-1}y), \tag{1.1}$$

where x is an element of a space S, and A is a symmetric operator defined over S with the property that (x,Ax) is negative definite. Since the Euler equation associated with this variational problem is $Ax = y$, we see that the element x furnishing the maximum is given in terms of the inverse operator. For the case of ordinary or partial differential operators, these operators are expressible in terms of Green's functions.

On the other hand, several classes of variational problems of this type can be treated by means of the functional equation technique of dynamic programming. Combining the two approaches, we can derive a number of properties of Green's functions. For the case of second order linear differential operators, these techniques were applied in Bellman[1] and Bellman and Lehman[2], while the classical Hadamard variational formula for the Green's function associated with second order partial differential operators, of elliptic type was obtained in this fashion in Bellman and Osborn[3]. Similar results were obtained for linear integral operators and Jacobi matrices in Bellman[4,5].

Since this approach yields results in a quite straightforward fashion, it is of some interest to see whether or not they can be extended to cover cases in which A is not necessarily negative definite, in which A may be symmetric and complex, and in which it need not be symmetric.

In [2] it was shown how analytic continuation could be used to overcome the lack of positive definiteness. Here we shall show how analytic continuation and a min-max variation can be used to treat the case of a complex symmetric operator, with

particular reference to second order linear differential operators, and how analytic continuation and an imbedding technique can be used to handle non-symmetric operators. Specific applications will be given subsequently.

2. Complex Symmetric Operators. Let $A + iB$ be a complex symmetric operator. The equation $(A + iB)(x + iy) = u + iv$ reduces to the two real equations

$$Ax - By = u, \quad Ay + Bx = v. \tag{2.1}$$

If A is negative definite, these can be considered to be the variational equations arising from the problem of determining the maximum over y and the minimum over x of the functional

$$(y, Ay) - (x, Ax) + 2(x, By) - 2(u, x) + 2(v, y). \tag{2.2}$$

If A is not negative definite, we replace A by $A - zI$ where z is a sufficiently large positive scalar, and then employ analytic continuation as in [2].

3. The Second Order Linear Differential Operator. As an illustration of this technique, consider the equation

$$u'' + (g(x) + ih(x))u = p(x) + iq(x) \tag{3.1}$$

over $a \leq x \leq T$. Setting $u = v + iw$, we obtain the equations

$$v'' + g(x)v - h(x)w = p(x),$$
$$w'' + g(x)w + h(x)v = q(x),$$

which are the variational equations connected with the problem of determining the minimum over v and maximum over w of the functional

$$\int_a^T \left[-v'^2 + w'^2 + g(x)v^2 - 2h(x)vw - g(x)w^2 \right. \tag{3.2}$$
$$\left. - 2p(x)v + 2q(x)w \right] dt.$$

Since this functional is convex in w' and concave in v', it is easy to show that $\min_w \max_v = \max_v \min_w$.

With this information, the functional equation technique can be applied as in [2].

If $g(x)$ is not uniformly positive in the interval $[a,T]$, we introduce the function $z + g(x)$ where z is a sufficiently large positive quantity and employ analytic continuation, as indicated in [2].

In a variety of problems in mathematical physics, complex functions occur when energy dissipation is taken into account; cf. Dolph [6], where the min-max formulation is discussed in detail.

4. Non-symmetric Operators.

Let A be a non-symmetric operator, $A \neq A'$. In order to study A^{-1}, and the resolvent operator $(A - \lambda I)^{-1}$, by variational techniques we consider the problem of maximizing the functional

$$(x,Bx) + (y,By) + 2(x,Ay) - 2(u,x) - 2(v,y), \qquad (4.1)$$

where B is a negative definite operator.

The variational equations are

$$\begin{aligned} Bx + Ay &= u, \\ A'x + By &= v. \end{aligned} \qquad (4.2)$$

We have thus imbedded the equation $Ay = u$, not necessarily of variational origin, within a family of variational equations.

In order to carry out the analytic continuation, we replace B by zB and study the analytic character of the symmetric matrix operator

$$M(z) = \begin{pmatrix} zB & A \\ A' & zB \end{pmatrix} \qquad (4.3)$$

and its inverse, as functions of z. Eventually we wish to set $z = 0$ so as to obtain

$$M(0)^{-1} = \begin{pmatrix} 0 & A^{-1} \\ (A')^{-1} & 0 \end{pmatrix}. \qquad (4.4)$$

The analytic details in each case will depend upon the nature of the operators A and B. For the case of linear differential operators, the methods given in ^{2} will yield the desired results.

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