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ON THE COMPUTATIONAL DETERMINATION OF THE NATURE OF SOLUTIONS OF NONLINEAR SYSTEMS WITH STOCHASTIC INPUTS

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1. Introduction

The advent of large scale digital computers, with their phenomenal speed and accuracy of computation, has created a challenge to mathematicians to use the full capabilities of the machinery to advance knowledge within their field. It is in this spirit that this paper has been written.

Nonlinear dynamics problems prove difficult to investigate analytically or numerically if probabilistic terms are involved in their equations. This paper develops a technique for handling problems of this type that is practical only if a high speed computer is available to perform the attendant computations.

The method is general, but will be applied to the Van der Pol equation:

$$
\begin{aligned}
&\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = r(t), \\
(1)\qquad & \\
&x = u, \quad \dot{x} = v, \quad \text{at} \quad t = t_0.
\end{aligned}
$$

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(A dotting indicates differentiation with respect to t.)

The classical problem considers $r(t) = A \cos wt$. We consider $r(t)$ to be a random function with a known probability distribution.

2. Theory

We first use the phase plane form of the equation:

$$(2) \qquad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\lambda(x^2 - 1)y - x + r(t). \end{aligned}$$

Let $x(t_n) = x_n$, $y(t_n) = y_n$, $r(t_n) = r_n$, $\Delta t = \Delta$ where $x(t_0) = u$, $y(t_0) = v$. The Euler solution procedure for the foregoing equation is:

$$(3) \qquad \begin{cases} x_{n+1} = x_n + \Delta y_n \\ y_{n+1} = y_n + \Delta\left\{-\lambda(x_n^2 - 1)y_n - x_n + r_n\right\}. \end{cases}$$

Let us now define a probability function

$$(4) \qquad P_n = P(C(x_n, y_n);u,v),$$

the probability that at time t_n the integral curve with initial conditions $x_0 = u$, $y_0 = v$, will satisfy the condition $C(x_n, y_n)$.

It is assumed that P_0 is known at all points of the x,y phase plane. Then if r is a fixed variable,

$$P_{n+1} = P(C(x_{n+1}, y_{n+1}); u,v)$$

$$= P(C(x_n + \Delta y_n, y_n + \Delta\{-\lambda(x_n^2 - 1)y_n - x_n + r_n\});u,v)$$

$$= P(C(\overline{x}_n, \overline{y}_n(r));u,v).$$

Since $r(t)$ has a given probability distribution, $G(r)$, we must average the right-hand side over r. This may be expressed as

(5) $$P_{n+1}(x_{n+1}, y_{n+1};u,v) = \int P(\overline{x}_n, \overline{y}_n;u,v)dG(r).$$

Let this formalism now be applied to a question of stability. This requires an investigation of P_n for large n, where $C(x_n,y_n)$ is:

$$\begin{cases} |x_n| < a, \\ |y_n| < b. \end{cases}$$

If a and b are chosen sufficiently large, P_n will converge to 1 uniformly throughout the plane provided that the solution is stable.

If a and b are varied for different runs to determine those values of a, b at which P_n first becomes less than 1 for large n, a region may be set up in which the integral curve must lie. This may be refined to an outer-tangential rectangle by considering $C(x_n,y_n)$ to be

$$x_n \leq a, \quad y_n \leq b.$$

It may be suspected, or desired, that the integral curves lie within a given region. To test this, one assumes the condition $C(x,y)$ to be $f(x,y) < 0$, e.g.,

$$f(x,y) = x^2 + y^2 - R^2, \quad \text{Circular region,}$$

$$f(x,y) = (x^2 - A^2)(y^2 - B^2), \quad \text{Rectangular region,}$$

$$f(x,y) = (x^2 + y^2 - R^2)(x^2 + y^2 - \bar{R}^2), \quad \text{Annulus.}$$

Again, one tests for P_n converging to 1 for large n, uniformly over the plane.

In practice, the infinite (x,y) plane is replaced by a finite cross grid. Suitable adjustments must be made to define the value of the integral in the neighborhood of the boundary. The computational and theoretical errors associated with this procedure are:

1. Truncation error of the Euler process: This can become very bad in the relaxation region and transition regions, for large λ.

2. Computational Roundoff: This becomes larger for smaller values of Δ, but is not a function of grid size.

3. Intragrid interpolation effects and grid-edge errors.

4. Truncation error associated with the numerical integration.

Neither the effects of these errors, nor their bounds, have been investigated in this paper.

3. Illustrations

To illustrate the method described in this paper, equation (1) was considered, with $r(t)$ assuming the values $+ k$ and $- k$ with equal probability. $C(x_n, y_n)$ was chosen as

$$x_n < a, \quad y_n < b.$$

Two values of λ were considered, $\lambda = +1$, $\lambda = -1$.

It was expected that the nature of the results for small k would not vary too much from $k = 0$.

For $k = 0$, one has the standard Van der Pol equation. For $\lambda = 1$, there exists one periodic solution towards which all solutions converge rapidly(1). The graph of this periodic solution is shown in Figure 1(2). For $a = b = 4$, P_n should converge to 1 for all (x,y).

The value $\lambda = -1$ was selected to exemplify a more varied probability distribution. If in equation (1), the substitution $\tau = -t$ is made, equation (1) as a function of τ would become

$$\ddot{x} - \lambda(x^2 - 1)\dot{x} + x = r(\tau).$$

For negative λ, this equation is identical to the previous case. Hence, the same unique periodic solution exists, but with the y orientation reversed. However, as a periodic solution to the t equation, it is unstable. The singular point at the origin becomes stable.

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(1). Brock, P., "Methods in Nonlinear Vibrations," M. S. Thesis, N.Y.U., 1947.

(2). Andronow, A. A., Chaikin, C. E., "Theory of Oscillations," Princeton, 1949, p. 251.

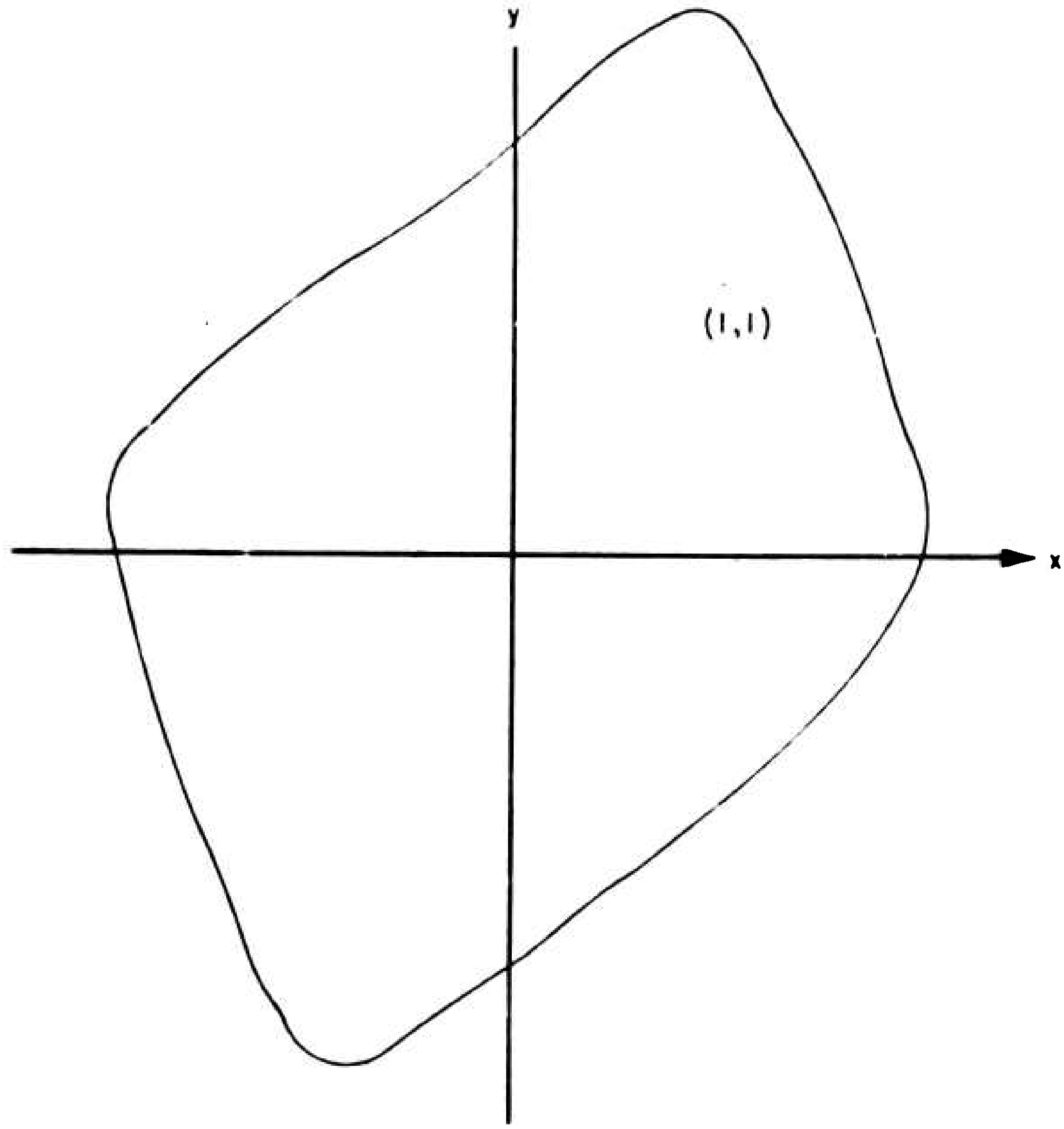


Fig. 1

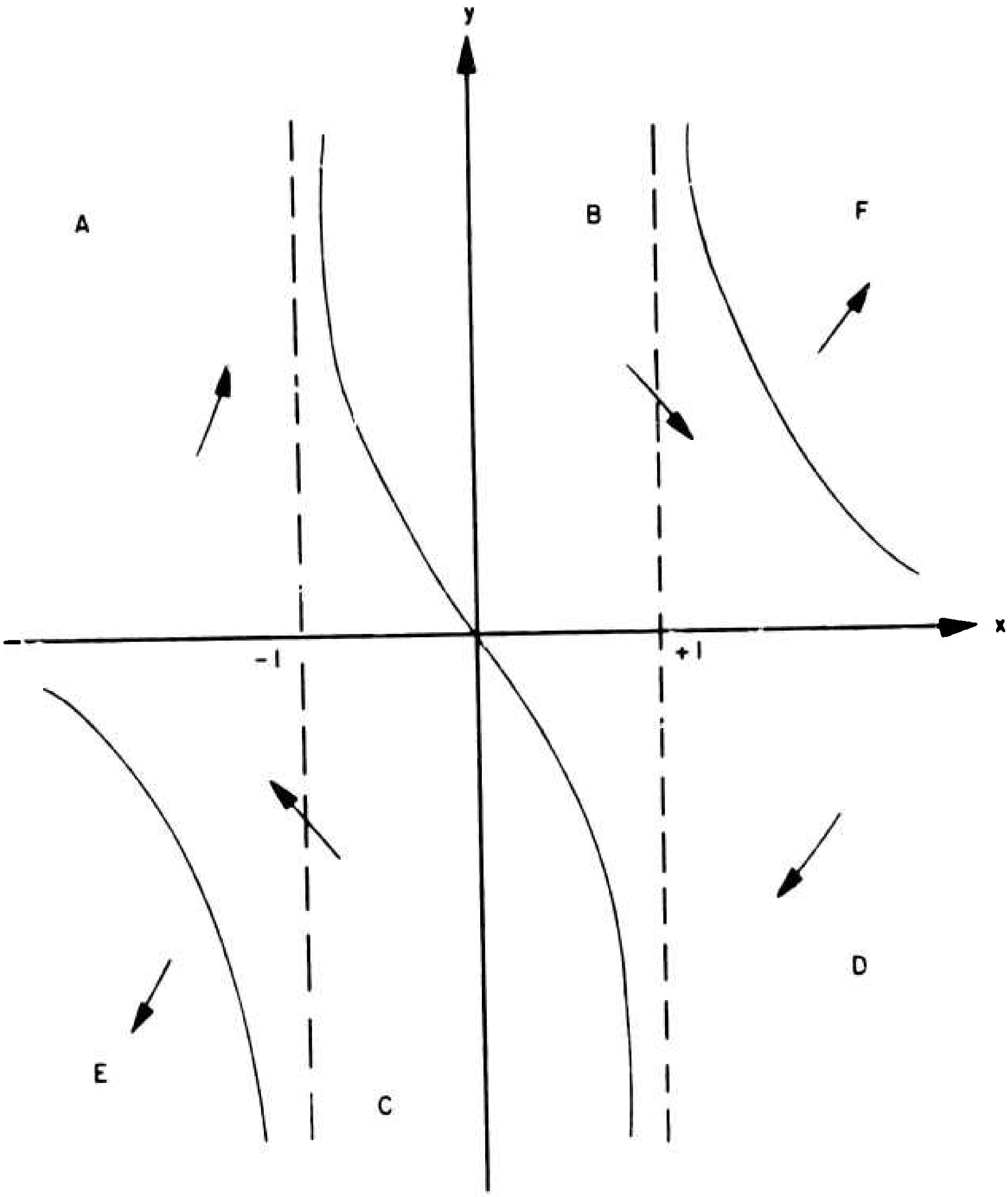


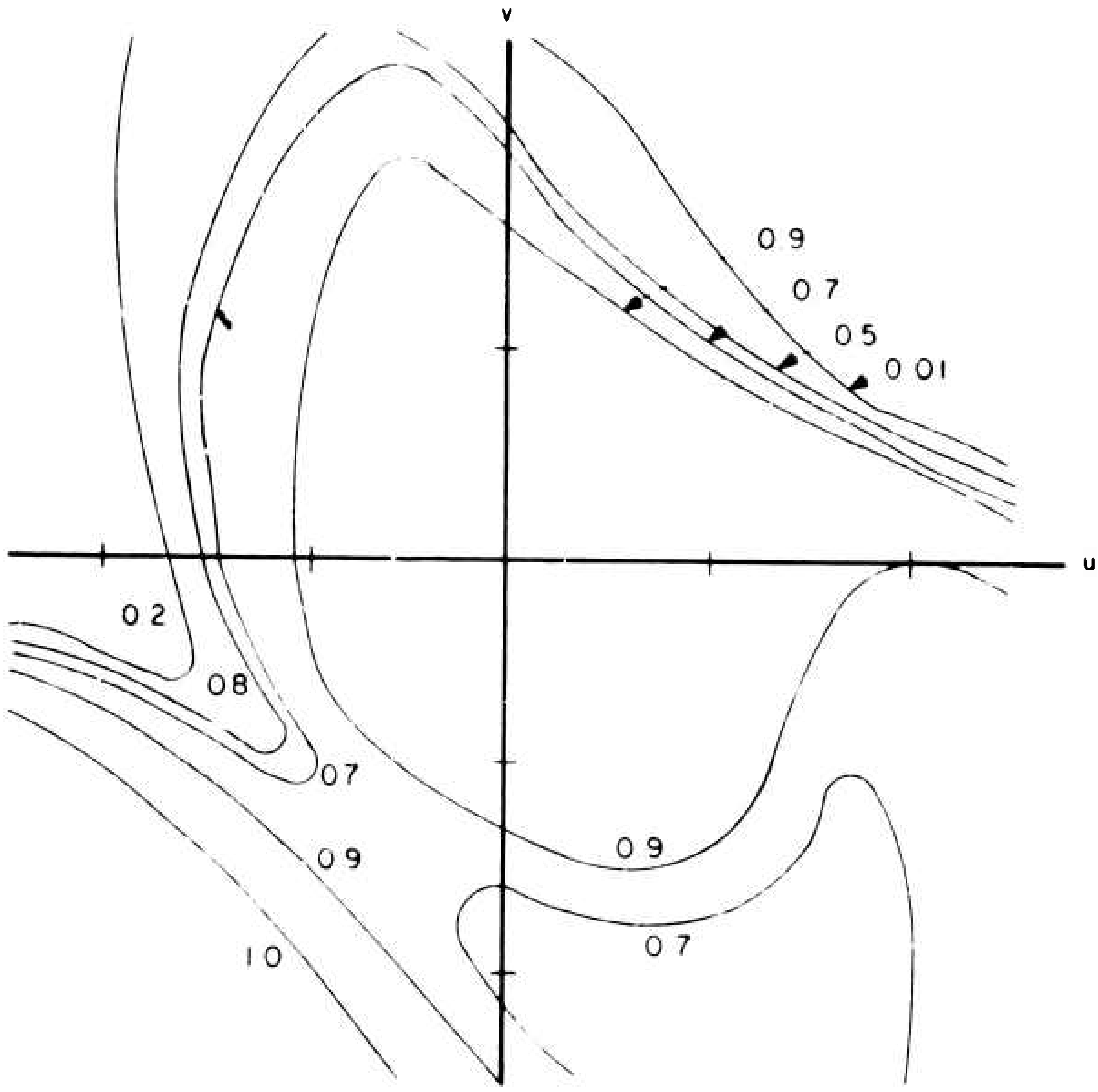
Fig. 2

Figure 2 is a plot

$$(x^2 - 1)y - x = 0.$$

These are isoclines of horizontal slope for the family of integral curves. The direction of slopes in regions A – F are shown in the figure. For (a,b) chosen as $(1,1)$ and large n, it is clear that any curve starting in region F will have $P_n = 0$ while any point curve starting in region E will have a $P_n = 1$. For regions A, B, C, and D, if a curve starts within the limit cycle, it will spiral towards the origin, hence, $P_n = 1$. If it is outside the limit cycle, the integral curve will diverge from the limit cycle and the value of P_n will become 1 or 0 depending on whether the integral curve enters region E or F first.

Figure 3 indicates the isoprobabilistic lines for the case of $k = 1$, $\lambda = -1$. Figure 4 indicates the isoprobabilistic lines for the case $k = 0$, $\lambda = -1$.



$k = 1$

$\lambda = -1$

$a = b = 1$

Fig. 3

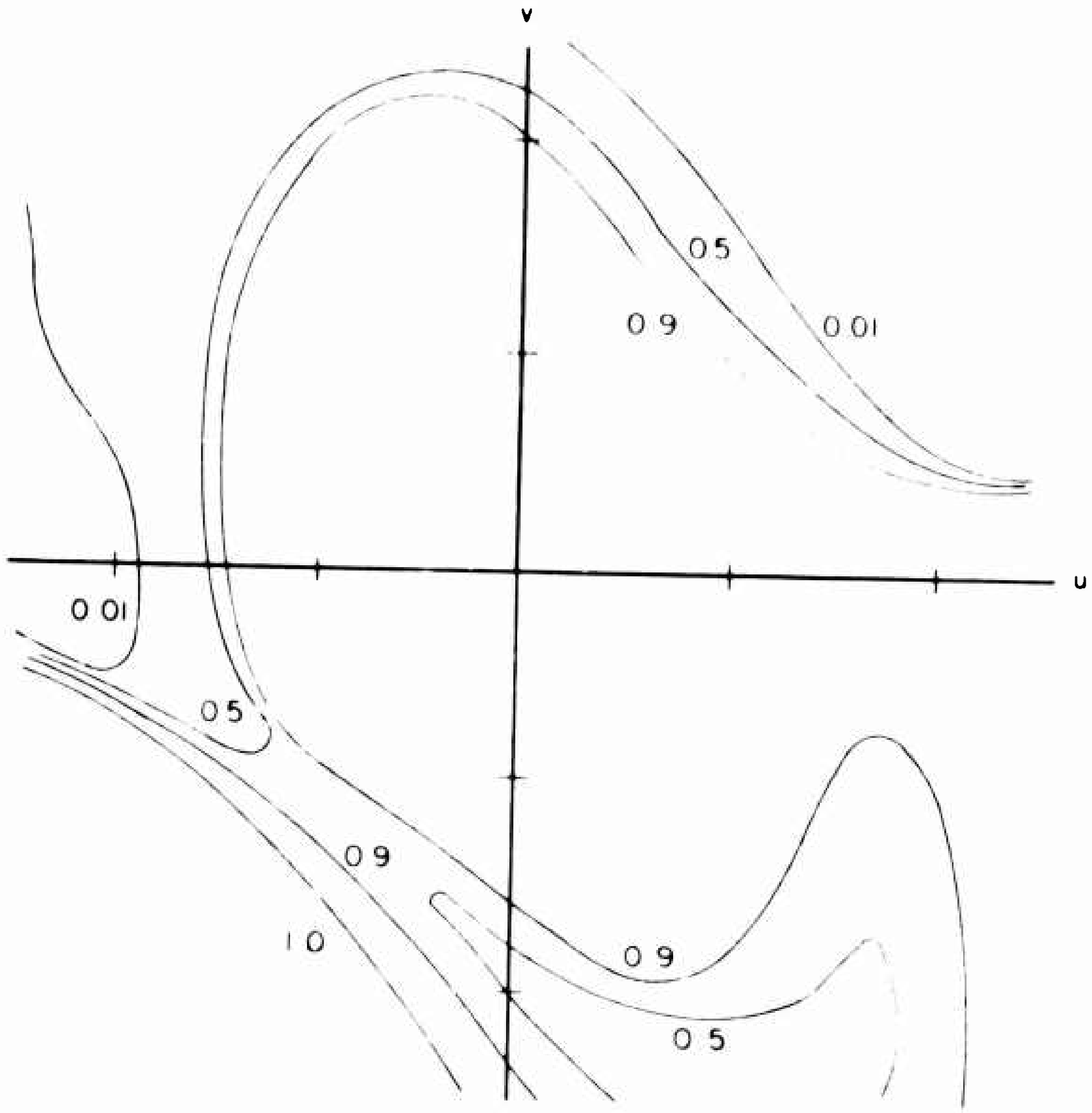


Fig. 4

k = 0

$\lambda = -1$

a = b = 1

4. Computer Techniques

The results illustrated in Section 3, and additional results were obtained on the Datatron Computer at Purdue University.

Since $r(t) = \pm k$ with equal probability, the integral resolved itself into a sum.

$$P_{n+1}(x_{n+1}, y_{n+1}; u,v) = \frac{1}{2}\left[P_n(\overline{x}_n^+, \overline{y}_n^+; u,v) + P_n(\overline{x}_n^-, \overline{y}_n^-; u,v)\right],$$

$$\overline{x}_n^+ = \overline{x}_n^- = x_n + \Delta y_n,$$

(3')

$$\overline{y}_n^+ = y_n + \Delta\left\{-\lambda(x_n^2 - 1)y_n - x_n + k\right\},$$

$$\overline{y}_n^- = y_n + \Delta\left\{-\lambda(x_n^2 - 1)y_n - x_n - k\right\}.$$

λ was chosen as $+1$ or -1 on different runs, k was chosen as 1, .1, 0 on different runs. (a,b) was chosen as $(1,1)$ and $(4,4)$.

A grid of 2601 points (51×51) was selected for ranges of u,v;

$(a,b) = (1,1)$,	$(a,b) = (4,4)$,
$-2.5 \le u \le 2.5$,	$0 \le u \le 5$,
$-2.5 \le v \le 2.5$,	$0 \le v \le 5$,
$\Delta u = \Delta v = .1$,	$\Delta u = \Delta v = .1$,
$\Delta t = .1$,	$\Delta t = .1$.

Initial values of P_0 were selected for all points of the grid; $P_0 = 1$ for all points (x,y) satisfying $x \le a$, $y \le b$; $P_0 = 0$ for all other points.

The arguments for equation (5') were computed in a standard fashion, and probabilities assigned by linear interpolation over the four adjacent corners for points within the grid, and by a like extrapolation for points falling outside the grid.

P_{n+1} values were based upon a full grid of P_n values. When all 2601 P_{n+1} values were calculated, the entire grid function was then replaced. This procedure constituted one cycle.

On the Datatron, one cycle required an average of 70 minutes.

Thirty cycles were calculated for the illustrative figures 3, 4 above.

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