ON WEIGHTED PCM AND
MEAN SQUARE DEVIATION

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SUMMARY

In an interesting paper Bedrosian has introduced the concept of weighted pulse-code modulation, WPCM. This differs from normal PCM in that the amplitude of the transmitted pulses representing the binary digits in a pulse-code group are made to depend on the size of the group and on the power of two represented by the individual pulses. In general, the higher the power of two represented by the pulse, the larger the amplitude of the pulse.

In this paper it is shown how the functional equation technique of dynamic programming can be used advantageously in the analysis of WPCM communication systems.

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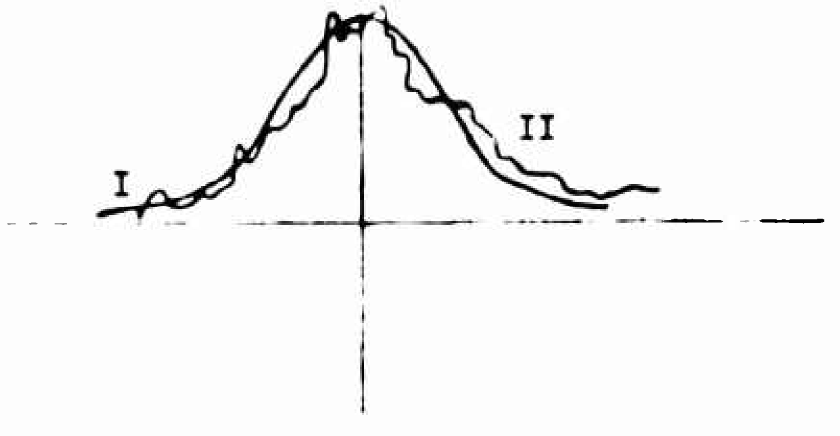
In an interesting paper Bedrosian has introduced the concept of weighted pulse-code modulation, WPCM, [1]. This differs from normal PCM in that the amplitude of the transmitted pulses representing the binary digits in a pulse-code group are made to depend on the size of the group and on the power of two represented by the individual pulses. In general, the higher the power of two represented by the pulse, the larger the amplitude of the pulse. A comparison between PCM and WPCM system performances, indicating the advantages of WCPM, can be found in Bedrosian's paper.

1. The analysis of a WPCM system involves the determination of the optimal power to be allocated to the transmission of each binary digit, under the assumption of a fixed power per pulse-code group, to minimize the mean square deviation of the reconstructed signal at the receiver from the signal sample value presented by the information source. If there are N digits per pulse-code group, then use of the Lagrange multiplier method involves the solution of a system of $N + 1$ simultaneous transcendental equations. Bedrosian accomplishes this under the assumption that the average signal power available is sufficiently great to make the occurrence of more than one error per pulse-code group almost zero. This assumption

leads to a system of $N + 1$ independent equations for the power allocations and the Lagrange multiplier.

In addition, he assumes that the probability of a digit's being received incorrectly as a function of signal strength, and the <u>derivative</u> of this function, are both simply expressible in terms of the error function. This may lead to difficulties in locating the extrema since the smoothed Curve I, in the figure below, may be a reasonable approximation to Curve II, without the slope of I at a point necessarily being a reasonable approximation to the slope of II at the same point.



2. We wish to present an alternative approach to the analysis, based on the functional equation technique of dynamic programming, [2], in which no assumption of smallness is necessary concerning the probability of error; this is signi-

ficant, for a WPCM system may function satisfactorily with signal power so low that several errors per pulse-group occur, the errors being localized to the coefficients of the lower powers of two, which is manifestly not possible with ordinary PCM. In addition, minima are determined without the use of partial derivatives. This permits the probability of error to be given as a function of signal strength either graphically or in tabular form, based upon experimental results, and not necessarily in analytic form. Lastly, the quantization noise is handled in a particularly simple and straightforward fashion.

3. Let us consider that each pulse-code group consists of N binary digits and that the power P is available for sending the entire group. The probability of the correct reception of a digit if the transmitted power is p is defined to be $g(p)$. We assume that transmission and reception of the individual digits are independent. We wish to determine the power p_i to allocate to the transmission of the coefficient of the i-th power of two which results in minimizing the mean square deviation of the received signal from the signal sample value provided by the source of information, where

$$\sum_{i=0}^{N-1} p_i = P, \tag{3.1}$$

$$0 \leq p_i, \quad i = 1, 2, \dots . \tag{3.2}$$

We assume that the transmitter will transmit a 'one' as the coefficient of 2^{N-1} if $a \geq 2^{N-1}$, where a is the signal sample value, and a 'zero' otherwise; call this coefficient c_{N-1}. It transmits a 'one' as the coefficient of 2^{N-2} if $a - c_{N-1}\, 2^{N-1} \geq 2^{N-2}$, and a 'zero' otherwise, and so on.

The reconstructed received signal will be

$$
\begin{aligned}
s_{N-1} &= 2^{N-1}\, x_{N-1} + 2^{N-2}\, x_{N-2} + \cdots + 2^0\, x_0 \\
&= 2^{N-1}\, x_{N-1} + s_{N-2},
\end{aligned}
\tag{3.3}
$$

where x_k is a random variable assuming only the values zero or one. Since the probability of a correct transmission of the k-th digit with power p_k is $g(p_k)$, we have

$$
p_r\left\{x_k = c_k\right\} = g(p_k). \tag{3.4}
$$

The mean square deviation of the received signal from the signal sample value, a, is

$$
\begin{aligned}
E\left\{(a - s_{N-1})^2\right\} &= E\left\{(a - 2^{N-1}x_{N-1})^2\right\} \\
&\quad - 2\left[a - E\left\{2^{N-1}x_{N-1}\right\}\right] E\left\{s_{N-2}\right\} + E\left\{s_{N-2}{}^2\right\}.
\end{aligned}
\tag{3.5}
$$

By completing the square this becomes

$$
\begin{aligned}
E\left\{(a - s_{N-1})^2\right\} &= E\left\{(a - 2^{N-1}x_{N-1})^2\right\} - \left[a - E\left\{2^{N-1}x_{N-1}\right\}\right]^2 \\
&\quad + E\left\{\left(a - E\left\{2^{N-1}x_{N-1}\right\} - s_{N-2}\right)^2\right\} \\
&= \operatorname{Var}\left\{a - 2^{N-1}x_{N-1}\right\} + E\left\{\left(a - E\left\{2^{N-1}x_{N-1}\right\}\right.\right. \\
&\quad \left.\left. - s_{N-2}\right)^2\right\}.
\end{aligned}
\tag{3.6}
$$

To transform the minimization problem into one involving functional equations we introduce the function

$f_N(a,P)$ = the expected value of the square deviation of the received signal from the source signal, a, using an optimal allocation of the power P to the individual digits, each pulse-code group consisting of N digits.

By definition, then,

$$\min_{\{p_i\}} E\left\{(a - s_{N-1})^2\right\} = f_N(a,P), \tag{3.7}$$

where the minimization is over the set of p_i defined by

$$0 \le p_i, \quad i = 0, 1, \ldots, N-1, \qquad \sum_{i=0}^{N-1} p_i = P. \tag{3.8}$$

If we wish to include the effect of a limitation on peak power, P_{pk}, we may in addition impose the condition that $0 \le p_i \le P_{pk}$. This would be difficult to incorporate in the Lagrange multiplier scheme.

If we now use the principle of optimality, [2], and Eq. (3.6), we obtain

$$f_N(a,P) = \min_{0 \le p_{N-1} \le P} \left\{ \mathrm{Var}\left\{a - 2^{N-1}x_{N-1}\right\} \right.$$

$$\left. + f_{N-1}\left(a - E\left\{2^{N-1}x_{N-1}\right\},\ P - p_{N-1}\right)\right\}, \tag{3.9}$$

$$N = 2,\ 3,\ \ldots\ .$$

If the pulse-code group consists of just one binary digit, all available power is used and we obtain

$$f_1(a,P) = \begin{cases} g(P)a^2 + (1-g(P))(1-a)^2, & 0 \le a < 1, \\ g(P)(1-a)^2 + (1-g(P))a^2, & 1 \le a < \infty. \end{cases} \tag{3.10}$$

Thus the multidimensional optimization problem is converted into a sequence of one-dimensional optimization problems.

4. Using the last two equations we are able to compute recursively, using a high-speed digital computer, the sequence of functions of two variables $\{f_k(a,P)\}$ and the power allocations, p_k, as functions of k, a, and P. This computation is straightforward on a digital computer and is simplified by the fact that if $f_N(a,P)$ is to be determined for a and P in a certain domain $0 \le a \le a_0$, $0 \le P \le P_0$, then all the $f_k(a,P)$ for $k < N$ need only be determined in sub-domains, as we see upon referring to Eq. (3.9).

5. Note that the transmitter is not provided with a feedback link which gives it information concerning whether or not the previous signal was corr ly received, so that the exact state of the system is not known at the transmitter at

each stage. Nonetheless, the transmitter is able to transmit optimally with regard to the least mean square deviation criterion by proceeding as if the signal **a** had been changed purely deterministically to $a - E\{2^{N-1}x_{N-1}\}$.

This observation should find application in other situations involving least mean square error criteria. It means that in some situations we can associate an equivalent deterministic process with a stochastic process.

6. The problems involved in optimizing the performance of null-zone reception systems and null-zone reception systems provided with feedback channels, [3], can be treated similarly, by use of the functional equation technique. It would be of interest to evaluate the performance of communication systems employing combinations of WPCM, null-zero reception, and feedback channels.

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