# DYNAMIC PROGRAMMING, SUCCESSIVE APPROXIMATIONS AND VARIATIONAL PROBLEMS OF COMBINATORIAL NATURE

Richard Bellman

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SUMMARY

The purpose of this paper is to show that a combination of dynamic programming and the classical method of successive approximations permits a systematic study of various classes of combinatorial problems arising in scheduling theory, communication theory and network theory.

Although the method cannot guarantee convergence to the actual solution, it furnishes a monotonic sequence of approximations by means of approximation in policy space.

An important feature of the method is the use of the solution of sub-problems of considerable magnitude as steps in the approximation procedure. With the aid of digital computers and the techniques of dynamic programming, this is a feasible method.

As examples, we discuss the Hitchcock-Koopmans transportation problem, an allocation problem, and the "travelling salesman" problem.

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## §1. INTRODUCTION

There are a large class of variational problems of combinatorial nature arising from various studies in allocation, scheduling and communication theory which offer formidable analytic and computational difficulties. Although many of these problems are theoretically susceptible to the functional equation technique of dynamic programming, in actuality any direct computational solution is blocked by the large number of state variables occurring.

In this paper we wish to indicate how a combination of dynamic programming and successive approximations, wedded for the purpose of treating nonlinear control processes[1], can be used to provide a systematic, if not complete, procedure for treating various questions of the type mentioned above. As is characteristic of approximation in policy space, we will obtain monotone approximation. Many new problems arise concerning convergence, convergence to local extrema, rapidity of convergence, numerical stability, and so forth.

With the aid of modern digital computers, we employ as a single step in the approximation process the solution of sub-problems which were once difficult in their own right. In this paper we use dynamic programming to accomplish this; in our previous paper[1] we utilized dynamic programming and the solution of systems of linear differential systems. Many examples can be given in which linear programming, and a blend of various techniques can be utilized. The point we wish to emphasize is that the introduction of the digital computer affords vast new scope to that general factotum of analysis, the method of successive approximations.

In place of any abstract general formulation of the approximation procedures that can be used, we will illustrate the basic ideas by means of some examples.

## §2. HITCHCOCK-KOOPMANS TRANSPORTATION PROBLEM

Consider a set of sources, $S_i$, possessing quantities of supplies, $x_i$, and a set of demand points, $T_j$, with requirements $r_j$ such that $\sum_i x_i = \sum_j r_j$. Let the cost of shipping a quantity $x_{ij}$ from $S_i$ to $T_j$ be a given function $g_{ij}(x_{ij})$. If $g_{ij}(x_{ij}) = d_{ij}x_{ij}$, the elegant and powerful techniques of Dantzig[2], and more recently of Ford and Fulkerson[3] and Prager[4] resolve the problem of minimizing the total cost, $\sum_{i,j} g_{ij}(x_{ij})$. If, however, the shipping costs are nonlinear, as, for example, occurs when fixed charges exist, the only general technique available at the moment appears to be that given in [5]. However, the functional technique discussed there, combined with the technique of Lagrange multipliers, is only feasible computationally at the present time if either M or N is small, say less than or equal to 5. On the other hand, if one is small, the other may be quite large, e.g. 1000 or 2000.

Let us now indicate how the method of successive approximations permits us to convert the general transportation problem into a sequence of problems which may be readily resolved via dynamic programming. Since we are using approximation in policy space, we obtain a monotone sequence of decreasing costs.

To begin the approximations, let the supplies at the sources $S_3$, $S_4$, ..., $S_M$ be allocated in an arbitrary fashion to the $N$ demand points. The problem of shipping the supplies from the two remaining sources $S_1$ and $S_2$ to satisfy the remaining demands is now solved by means of functional equations as described in [5]. As indicated there, the solution of this problem involves only sequences of functions of one variable, and is very simply carried out with the aid of a digital computer.

The next step of the approximation procedure consists of using the distribution of supplies from $S_1$, $S_4$, ..., $S_N$ as determined in this way, and reallocating the supplies shipped from $S_2$ and $S_3$ so as to minimize the cost of shipping from these two sources.

The process continues in this way; at each step minimizing over supplies shipped from $S_k$ and $S_{k+1}$, where $S_{N+1} \equiv S_1$.

It is clear that the sequence of costs obtained in this way is monotone decreasing. However, as examples show, these costs can converge to a relative minimum rather than an absolute minimum. Even when there exists only one minimum, an absolute minimum, this process, as an example constructed by O. Gross shows, can converge to a value different from the minimum value. We shall discuss these questions in more detail subsequently.

## §3. GENERAL ALLOCATION PROCESSES

Consider the problem of determining the maximum of

$$R_N(x) = \sum_{i=1}^{N} g_i(x_{i1}, x_{i2}, \ldots, x_{iM}) \tag{1}$$

over all $x_{ij}$ satisfying the constraints

$$\sum_{i=1}^{N} x_{ik} \leq c_k, \quad k = 1, 2, \ldots, M, \quad x_{ij} \geq 0. \tag{2}$$

As we know[6], the problem can be treated by means of recurrence relations involving sequences of functions of $M$ variables. Let us indicate how we can approach the solution by means of sequences of functions of one variable. Consider a determination of the quantities $x_{ik}$, $i = 1, 2, \ldots, N$, $k = 2, \ldots, M$, satisfying (2), and determine the $x_{i1}$ so as to maximize $R_N(x)$. This involves sequences of functions of one variable. Using these values of $x_{i1}$ and the previous values of $x_{ik}$, $k = 3, \ldots, M$, determine the $x_{i2}$ so as to maximize $R_N(x)$. Continuing in this way, we clearly obtain a monotone increasing sequence of values for $R_N(x)$.

## §4. A NETWORK PROBLEM

As a final application of these approximation techniques, consider the problem of tracing a path through $N$ points having minimum "length." In a plane, with the usual metric, this is the travelling salesman problem. Although it is not difficult to resolve this problem by means of functional equations for $N$ of small magnitude, say $N \leq 10$, for $N = 48$, the direct method founders on dimensionality difficulties.

We can, however, employ successive approximations and the feasibility of the solution for small $N$ to obtain an approach to the problem for large $N$. Furthermore, as above, these approximate solutions yield monotone decreasing length.

Let the points be enumerated in some fixed order $P_1, P_2, \ldots, P_N$. Taking the first 10 points, $(P_1, P_2, \ldots, P_{10})$, we determine via functional equations a path of minimum length starting at $P_1$, ending at $P_{10}$, and passing through the intermediate points, $(P_2, P_3, \ldots, P_9)$. Let $(P_1, P_{k2}, P_{k3}, \ldots, P_{10})$ be a sequence of points obtained in this way. Now take the points, $(P_{k2}, P_{k3}, \ldots, P_{10}, P_{11})$, and repeat this procedure. Proceeding in this fashion we obtain a sequence of approximate solutions, each of length less than or equal to that of the preceding.

## §5. DISCUSSION

It is clear that there are a large number of variants of the procedures discussed in the foregoing sections. One that seems particularly interesting is that of using a random sequence of sets instead of the sets (1,2), (2,3), ..., described in §2, and proceeding similarly in a random rather than regular fashion in the processes described in §3 and §4.

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