A MULTIMOVE INFINITE GAME WITH LINEAR PAYOFF

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SUMMARY

This paper analyzes a multimove infinite game with linear payoff function. The game had its origin in the consideration of a military problem, but is presented here solely for its mathematical interest. It is symmetric in every respect except that the initial conditions of the two players are different. On each move, each player allocates his resources to tasks that might be described roughly as attacking, defending, and scoring. His resources for the next move are diminished by the amount that his opponent's attack exceeds his own defense, while his score cumulates from move to move. The value of the game and the optimal strategies for the players are rigorously derived in the present paper. It is shown that one player has a pure optimal strategy and the other player must randomize.

A MULTIMOVE INFINITE GAME WITH LINEAR PAYOFF

1. INTRODUCTION

Games can be classified in terms of the number of moves by each player — unimove or multimove — and in terms of the number of choices — finite or infinite — available at each move. The original work of von Neumann [1] on the existence and structure of solutions of games was, in effect, restricted to unimove finite games. Later, Ville [2] proved the existence of optimal strategies for unimove infinite games with continuous payoff function.

Except for games with perfect information, multimove finite games have been analyzed only very recently; and multimove infinite games with an arbitrary number of moves have hardly been touched upon.

In this paper, we analyze a multimove infinite game with a linear payoff function. The game is symmetric in every respect except that the initial conditions of the two players are different. We prove that one player has an optimal pure strategy and that the other player must randomize on the strategies. The optimal strategies and game value are derived.

Although this game had its origin in a military problem [3], and is applicable thereto, it is presented here solely for its mathematical interest.

2. DESCRIPTION OF GAME

We shall analyze the following multimove zero-sum two- person

game. At the n-th move, or stage of the game, Blue has resources given by the state variable p_n and assigns a value to each of two tactical variables under his control, x_n and u_n, subject to the constraints

$$x_n \geq 0, \quad u_n \geq 0, \quad x_n + u_n \leq p_n. \tag{2.1}$$

At the same time, Red has resources given by the state variable q_n and controls the values of the tactical variables y_n and w_n, subject to the constraints

$$y_n \geq 0, \quad w_n \geq 0, \quad y_n + w_n \leq q_n. \tag{2.2}$$

Let us number the moves from the end of the game; i.e., the n-th move means n moves to the end of the game. The state variables at the $(n-1)$-st move are defined by

$$p_{n-1} = \max \left[0, p_n - \max (0, y_n - u_n)\right],$$

$$q_{n-1} = \max \left[0, q_n - \max (0, x_n - w_n)\right]. \tag{2.3}$$

The payoff to Blue is given by

$$\sum_{n=1}^{N} \left|(p_n - x_n - u_n) - (q_n - y_n - w_n)\right|, \tag{2.4}$$

where N is the number of moves in the game.

The play of the game proceeds by first making the N-th move, then the $(N-1)$-st move, ..., the second move, and the first move. An n-th move of the game consists of a choice

by Blue of x_n and u_n satisfying (2.1) and simultaneously a choice by Red of y_n and w_n satisfying (2.2). We assume that each player knows the manner in which the game proceeds from stage to stage; namely, each player has the information expressed by equations (2.3). We also assume that at each stage of the game both players know the state variables and the entire past history of the play; that is, at the n-th move, both players know N, p_N, q_N, and also know x_i, w_i, y_i, w_i for $i = N, N-1, \ldots, n+2, n+1$. It follows that p_i, q_i, for $i = N, N-1, \ldots, n+1, n$, are known at the n-th move.

The strategies of the game in normal form will be defined inductively on the number of moves. First, a strategy for Blue in a one-move game is a point $X_1 = (x_1, u_1)$, where $x_1 \geq 0$, $u_1 \geq 0$, and $x_1 + u_1 \leq p_1$. Similarly a strategy for Red in a one move game is a point $Y_1 = (y_1, w_1)$ where $y_1 \geq 0$, $w_1 \geq 0$, and $y_1 + w_1 \leq q_1$. Now let σ_N be a strategy for Blue in an N-move game. Of course, σ_N is a function of p_N and q_N. Then, in a game of N + 1 moves, at the (N + 1)-st move Blue chooses a point $X_N = (x_{N+1}, u_{n+1})$ in the triangle Δ_{N+1} defined by

$$(2.5) \qquad x_{N+1} \geq 0, \qquad u_{N+1} \geq 0, \qquad x_{N+1} + u_{N+1} \leq p_{N+1},$$

and simultaneously Red chooses a point $Y_{N+1} = (y_{N+1}, w_{N+1})$ in the triangle D_{N+1} defined by

$$(2.6) \qquad y_{N+1} \geq 0, \qquad w_{N+1} \geq 0, \qquad y_{N+1} + w_{N+1} \leq q_{N+1}.$$

These choices yield the state variables p_N and q_N, by equations (2.3). A strategy σ_{N+1} for Blue in the (N + 1) move game is then defined as a choice X_{N+1} in Δ_{N+1} and a function ϕ_N that associates, with each point $(x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1}) \equiv (X_{N+1}, Y_{N+1})$ in the product space $\Delta_{N+1} \times D_{N+1}$ a strategy σ_N in the N-move game. Thus σ_{N+1} can be written as

$$\sigma_{N+1} = (X_{N+1}; \phi_N) = (x_{N+1}, u_{N+1}; \phi_N) ,$$

where ϕ_N assigns the strategy σ_N to the point $(x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1})$.

In a like manner, a strategy τ_{N+1} for Red in the (N + 1)-move game is defined as a choice Y_{N+1} and a function ψ_N that associates, with each (X_{N+1}, Y_{N+1}), a strategy τ_N in the N-move game. Thus we have

$$\tau_{N+1} = (Y_{N+1}; \psi_N) = (y_{N+1}, w_{N+1}; \psi_N) .$$

3. SOLUTION OF GAME

The main result of this paper is the following:

Theorem 1. If N = 1 or 2, the value of the game is given by

$$V_N(p_N, q_N) = N(p_N - q_N).$$

Blue has an optimal pure strategy:

$$\bar{x}_m = \bar{u}_m = 0 \text{ for } m \leq N.$$

Red has an optimal pure strategy:

$$\bar{y}_m = \bar{w}_m = 0 \text{ for } m \leq N.$$

If $N \geq 3$, the value of the game is given by the $(N-2)$-piecewise-linear function.

$$V_N(p_N, q_N) = a_N^i p_N - b_N^i q_N, \quad i = 1, 2, \ldots, N-2,$$

where the constants a_N^i and b_N^i are positive and monotone decreasing in i for fixed N; the value of the superscript i is determined by the ratio p_N/q_N. The optimal strategies for the two players are as follows:

(i) At move $m = 1, 2$ (counting from the end) the players choose

$$\bar{x}_m = \bar{u}_m = \bar{y}_m = \bar{w}_m = 0.$$

(ii) At move $m = 3$, if $p_3 \geq q_3$, then Blue chooses $\bar{x}_3$, $\bar{u}_3$ such that

$$q_3 \leq \bar{x}_3 \leq \min\left(\frac{p_3 + q_3}{2}, \frac{3q_3}{2}\right),$$

$$\bar{u}_3 = \bar{x}_3 - q_3.$$

Red chooses either $\bar{y}_3 = q_3$ or $\bar{w}_3 = q_3$, each with probability $1/2$.

(iii) At the $(m+1)$-st move, where $3 \leq m \leq N-1$, if $p_{m+1} \geq q_{m+1}$, then the ratio p_{m+1}/q_{m+1} determines an integer i, $1 \leq i \leq m-1$, and Blue chooses

$$\bar{x}_{m+1} = \frac{(2m - a_m^i)p_{m+1} - (m - 2b_m^i)q_{m+1}}{m + b_m^i}$$

$$\bar{u}_{m+1} = p_{m+1} - \bar{x}_{m+1}, \quad \text{for } i = 1, 2, \ldots, m-2,$$

and

$$\bar{x}_{m+1} = \left(2 - \frac{1}{b_m^{m-2}}\right) q_{m+1} \quad ,$$

$$\bar{u}_{m+1} = \left(1 - \frac{1}{m}\right) q_{m+1}, \quad \text{for } i = m-1,$$

where the constants a_m^i and b_m^i are those associated with a game of length m and initial condition p_m, q_m. Red chooses either y_{m+1} or $w_{m+1} = q_{m+1}$ with probabilities $\alpha_m^i = b_m^i/(m + b_m^i)$ and $\beta_m^i = m/(m + b_m^i)$, respectively, for $i = 1, 2, \ldots, m-2$; however, if $i = m-1$, Red chooses $y_{m+1} = q_{m+1}$ with probability $\alpha_m^i = 1/m$, or $w_{m+1} = q_{m+1}$ with probability $\beta_m^i = 1/b_m^{m-2}$, or $y_{m+1} = w_{m+1} = 0$ with probability $\gamma_m^i = 1 - 1/m - 1/(b_m^{m-2})$.

The proof of Theorem 1 will be carried out by induction on N, the number of moves of the game. In the course of this argument, recursive definitions will be given for the constants a_N^i and b_N^i. As an illustration of the theorem, Table 1 shows the solutions for games with eight or less moves.

4. A THREE-PART SUFFICIENCY CONDITION WITH MIXED STRATEGIES

From the statement of the theorem, it is seen that mixed strategies will have to be introduced, at least for Red. However, it is sufficient to introduce a restricted class of mixed strategies in order to prove the theorem.

TABLE 1

SOLUTION OF GAME OF N MOVES, FOR $N \leq 8$

Move m	Ratio of State Variables p/q	Associated Superscript i	Value of Game V(p, q)	Blue's Optimal Choice		Red's Optimal Choice		
				$\bar{x}_m^i$	$\bar{u}_m^i$	α_m^i	β_m^i	γ_m^i
1	1.00 to ∞	1	$p - q$	0	0	0	0	1
2	1.00 to ∞	1	$2(p - q)$	0	0	0	0	1
3	1.00 to 2.00	1	$3(p - q)$	q	0	.50	.50	0
	2.00 to ∞	2	$3(p - q)$	1.5q	.5q	.50	.50	0
4	1.00 to 2.33	1	$4.5(p - q)$	.5p + .5q	.5p − .5q	.50	.50	0
	2.33 to ∞	2	$4.00p - 3.33q$	1.67q	.67q	.33	.33	.33
5	1.00 to 1.70	1	$6.35p - 6.35q$	.41p + .59q	.59p − .59q	.53	.47	.0
	1.70 to 2.45	2	$5.82p - 5.45q$	.55p + .36q	.45p − .36q	.45	.55	0
	2.45 to ∞	3	$5.00p - 3.45q$	1.70q	.75q	.25	.30	.45
6	1.00 to 1.44	1	$8.39p - 8.39q$	.32p + .68q	.68p − .68q	.56	.44	0
	1.44 to 1.78	2	$8.00p - 7.83q$	.40p + .56q	.60p − .56q	.52	.48	0
	1.78 to 2.51	3	$7.04p - 6.12q$	.59p + .22q	.41p − .22q	.41	.59	0
	2.51 to ∞	4	$6.00p - 3.51q$	1.71q	.80q	.20	.29	.51
7	1.00 to 1.29	1	$10.50p - 10.50q$	.25p + .75q	.75p − .75q	.58	.42	0
	1.29 to 1.53	2	$10.26p - 10.19q$	.29p + .70q	.71p − .70q	.57	.43	0
	1.53 to 1.84	3	$9.54p - 9.09q$	.41p + .51q	.59p − .51q	.50	.50	0
	1.84 to 2.55	4	$8.21p - 6.64q$	.63p + .11q	.37p − .11q	.37	.63	0
	2.55 to ∞	5	$7.00p - 3.55q$	1.72q	.84q	.17	.28	.55
8	1.00 to 1.25	1	$12.60p - 12.60q$	.20p + .80q	.80p − .80q	.60	.40	0
	1.25 to 1.40	2	$12.48p - 12.45q$	.22p + .78q	.78p − .78q	.59	.41	0
	1.40 to 1.59	3	$12.06p - 11.86q$	.28p + .69q	.72p − .69q	.56	.44	0
	1.59 to 1.88	4	$11.03p - 10.22q$	.42p + .46q	.58p − .46q	.49	.51	0
	1.88 to 2.58	5	$9.36p - 7.07q$	.66p + .01q	.34p − .01q	.34	.66	0
	2.58 to ∞	6	$8.00p - 3.58q$	1.72q	.86q	.14	.28	.58

For a game of one move, a mixed strategy for Red is a probability distribution G_1 over D_1. Now suppose G_N is a mixed strategy for Red in a game of N moves and stats variables p_N and q_N. Then a probability distribution g_{N+1} over D_{N+1} and a function ψ_{N+1} that associates $(x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1})$ with G_N is a mixed strategy G_{N+1} in the $(N + 1)$-move game. Thus we may write the mixed strategy as

$$G_{N+1} = (g_{N+1}, \psi_N).$$

Mixed strategies F_{N+1} for Blue are defined similarly by a distribution function f_{N+1} and a function ϕ_N, and can be written as

$$F_{N+1} = (f_{N+1}, \phi_N).$$

Let $\hat{F}_{N+1}$ denote a mixed strategy for Blue in the $(N + 1)$-move game in which he selects $X_{N+1} = (x_{N+1}, u_{N+1})$ with probability 1 at the $(N + 1)$-st move. Let $\hat{G}_{N+1}$ denote a mixed strategy for Red in which he selects $Y_{N+1} = (y_{N+1}, w_{N+1})$ with probability 1 at the $(N + 1)$-st move.

Suppose that Theorem 1 is valid for games of length $N = n$. Let F_n^* and G_n^* be optimal strategies for Blue and Red, respectively. Let ϕ_n^*, ψ_n^* denote the functions that associate $(x_{n+1}, u_{n+1}, y_{n+1}, w_{n+1})$ with F_n^*, G_n^*, respectively. Suppose, further, that $p_{n+1} \geq q_{n+1}$ (from symmetry, it suffices to consider this case only).

The theorem asserts that at the $(n + 1)$-st move Blue's optimal choice is a point (x_{n+1}, u_{n+1}) that is determined by the

ratio p_{n+1}/q_{n+1}. Denote this point by

$$X^*_{n+1} = (x^*_{n+1}, u^*_{n+1}) = (x^*_{n+1}(p_{n+1}, q_{n+1}), u^*_{n+1}(p_{n+1}, q_{n+1})),$$

and let

$$F^*_{n+1} = (X^*_{n+1}, \phi^*_n).$$

The theorem further asserts that Red chooses (y_{n+1}, w_{n+1}) to be

$$Y^{(1)}_{n+1} \equiv (q, 0), \quad Y^{(2)}_{n+1} \equiv (0, q), \quad Y^{(3)}_{n+1} \equiv (0, 0),$$

with probabilities α_{n+1}, β_{n+1}, and $\gamma_{n+1} = (1 - \alpha_{n+1} - \beta_{n+1})$, respectively, the values of α_{n+1} and β_{n+1} being determined by the ratio p_{n+1}/q_{n+1}. Denote this distribution in Δ_{n+1} by $g^*_{n+1}(p_{n+1}, q_{n+1})$, and set

$$G^*_{n+1} = (g^*_{n+1}(p_{n+1}, q_{n+1}), \psi^*_n).$$

Define

$$L_{n+1}(X_{n+1}, Y_{n+1}) \equiv p_{n+1} - x_{n+1} - u_{n+1} - q_{n+1} + y_{n+1} + w_{n+1}$$

and

$$M_{n+1}(X_{n+1}, Y_{n+1}) \equiv L_{n+1}(X_{n+1}, Y_{n+1}) + v_n(p_n, q_n),$$

where p_n q_n are obtained from p_{n+1}, q_{n+1} by means of (2.3) and the choices x_{n+1}, u_{n+1}, y_{n+1}, w_{n+1}. Let $E_N(F_N, G_N)$ denote the expected payoff of the game of length N if Blue chooses a strategy F_N and Red chooses a strategy G_N. Then

$$E_{n+1}(\hat{F}^*_{n+1}, \hat{G}_{n+1}) = L_{n+1}(X^*_{n+1}, Y_{n+1}) + E_n(F^*_n, G_N)$$

$$\geq M_{n+1}(X^*_{n+1}, Y_{n+1}), \text{ for all } Y_{n+1},$$

where

$$\psi_n(X^*_{n+1}, Y_{n+1}) = G_n.$$

Furthermore, we have

$$E_{n+1}(\hat{F}_{n+1}, G^*_{n+1}) = \alpha_{n+1}\left[L_{n+1}(X_{n+1}, Y^{(1)}_{n+1}) + E_n(F_n, G^*_n\right]$$

$$+ \beta_{n+1}\left[L_{n+1}(X_{n+1}, Y^{(2)}_{n+1}) + E_n(F_n, G^*_n\right]$$

$$+ (1 - \alpha_{n+1} - \beta_{n+1})\left[L_{n+1}(X_{n+1}, Y^{(3)}_{n+1}) + E_n(F_n, G^*_n)\right]$$

$$\leq \alpha_{n+1}M_{n+1}(X_{n+1}, Y^{(1)}_{n+1}) + \beta_{n+1}M_{n+1}(X_{n+1}, Y^{(2)}_{n+1})$$

$$+ (1 - \alpha_{n+1} - \beta_{n+1})M_{n+1}(X_{n+1}, Y^{(3)}_{n+1}),$$

for all X_{n+1}, where

$$\phi_n(X_{n+1}, Y_{n+1}) = F_n.$$

The validity of the following lemma is now apparent.

Lemma 1. Given that Theorem 1 is true for $N = n$, to prove the theorem for $N = n + 1$ with initial conditions $p_{n+1} \geq q_{n+1}$, it suffices to exhibit the X^*_{n+1}, α_{n+1}, and β_{n+1} for which

(4.1) $$E_{n+1}(\hat{F}^*_{n+1}, G^*_{n+1}) = V_{n+1}(p_{n+1}, q_{n+1}),$$

$$(4.2) \qquad M_{n+1}(X^{*}_{n+1}, Y_{n+1}) \geq V_{n+1}(p_{n+1}, q_{n+1})$$

for all Y_{n+1}, and

$$(4.3) \qquad \alpha_{n+1}M_{n+1}(X_{n+1}, Y^{(1)}_{n+1}) + \beta_{n+1}M_{n+1}(X_{n+1}, Y^{(2)}_{n+1}) + (1 - \alpha_{n+1} - \beta_{n+1})M_{n+1}(X_{n+1}, Y^{(3)}_{n+1}) \leq V_{n+1}(p_{n+1}, q_{n+1})$$

for all X_{n+1}

5. SOME SPECIAL CASES

It will also be useful to tabulate the information given by equations (2.3). We may assume that $p_n \geq q_n$, whence $y_n - u_n > p_n$ is impossible and the equations (2.3) can be tabulated as follows, where the subscript n is suppressed:

TABLE 2

DETERMINATION OF VALUES OF STATE VARIABLES p_{n-1} AND q_{n-1}

Region in (X, Y) Space	Region Number	p_{-1}	q_{-1}
$y - u < 0$, $x - w < 0$	I	p	q
$y - u < 0$, $0 \leq x - w \leq q$	II	p	$q - x + w$
$y - u < 0$, $x - w > q$	III	p	0
$0 \leq y - u \leq p$, $x - w < 0$	IV	$p - y + u$	q
$0 \leq y - u \leq p$, $0 \leq x - w \leq q$	V	$p - y + u$	$q - x + w$
$0 \leq y - u \leq p$, $x - w > q$	I	$p - y + u$	0

Games of length N = 1, 2, 3, 4, 5 will now be discussed. From the statement of the theorem, it is clear that separate arguments are needed for N = 1, 2, and for $N \geq 3$. The present discussion is intended to provide insight into the structure of the game and to motivate the general induction step for $N > 3$, which will be presented below. From this discussion, the proof of Theorem 1 for N = 1, 2, 3 will follow. However, not all the work presented here is necessary merely to prove the theorem for N = 3.

For N = 1, an examination of the payoff (2.4) shows that optimal play for Blue is to choose $x_1 = u_1 = 0$, and that optimal play for Red is to choose $y_1 = w_1 = 0$.

As a consequence of Lemma 1, for N = 2 it suffices to consider

$$M_2(X_2, Y_2) = p_2 - x_2 - u_2 - q_2 + y_2 + w_2 + (p_1 - q_1),$$

with $p_2 \geq q_2$. Using Table 2 and dropping the subscript 2, we may write this

$$M(X, Y) = \begin{cases} 2(p - q) - (x + u) + (y + w) & \text{in region I,} \\ 2(p - q) - u + w & \text{in region II,} \\ 2p - q - (x + u) + (y + w) & \text{in region III,} \\ 2(p - q) - x + w & \text{in region IV,} \\ 2(p - q) & \text{in region V,} \\ 2p - q - x + w, & \text{in region VI,} \end{cases}$$

where the region in the (X, Y) space for which each expression on the right is valid is that given in Table 2.

It now follows that the optimal choices at the second move are

$$(\bar{x}, \bar{u}) = (0, 0) \text{ for Blue;} \quad (\bar{y}, \bar{w}) = (0, 0) \text{ for Red;}$$

and that $V_2 = 2(p_2 - q_2)$. Thus the theorem is proved for $N = 2$.

For $N = 3$, it suffices to consider

$$M_3(X_3, Y_3) = p_3 - x_3 - u_3 - q_3 + y_3 + w_3 + 2(p_2 - q_2),$$

where $p_3 \geq q_3$. It follows from Table 2 that, dropping the subscript 3, we may write $M_3(X_3, Y_3)$ as

$$M(X, Y) = \begin{cases} 3(p - q) - (x + u) + (y + w) & \text{in region I,} \\ 3(p - q) + x - u + y - w & \text{in region II,} \\ 3p - q - (x + u) + (y + w) & \text{in region III,} \\ 3(p - q) - x + u - y + w & \text{in region IV,} \\ 3(p - q) + x + u - y - w & \text{in region V,} \\ 3p - q - x + u - y + w & \text{in region VI.} \end{cases}$$

It is now no longer true that each player has an optimal pure strategy at this stage, for

$$(5.1) \qquad \min_Y \max_X M(X, Y) \neq \max_X \min_Y M(X, Y),$$

as will be shown. For each fixed X, a straightforward but tedious computation shows that the function

$$m(x, u) = \min_Y M(X, Y)$$

has the form shown in Fig. 1. Clearly, max m(x, u)

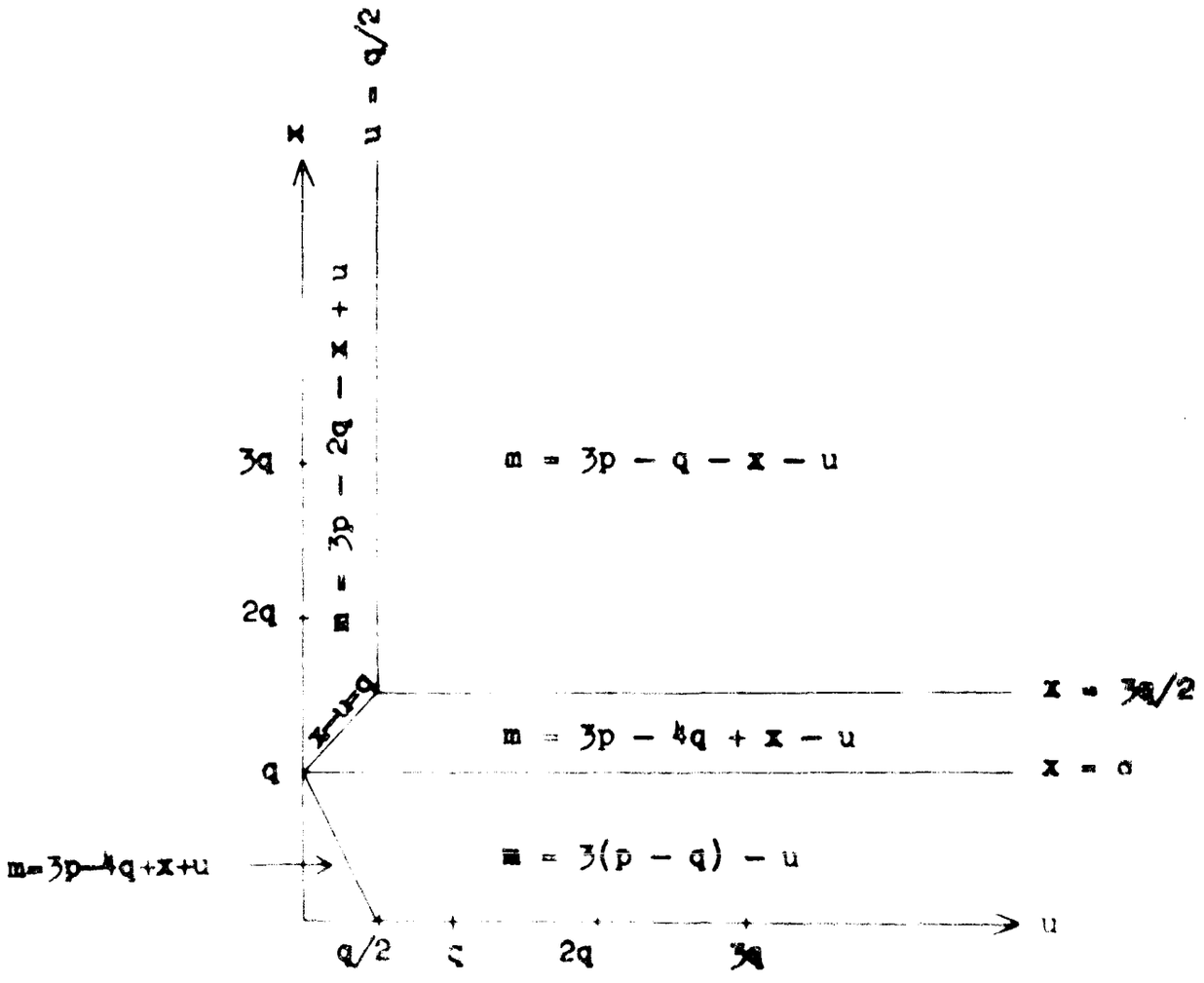

Fig. 1

is attained along the line segment

$$(5.2) \qquad x - u = q, \qquad 0 \leq u \leq \frac{q}{2},$$

subject to the constraint $x + u \leq p$, and the value of the maximum is $3(p - q)$.

On the other hand, the function

$$\mu(y, w) \equiv \max_{X} M(X, Y)$$

has the form shown in Fig. 2 if $p \leq 2q$. The computations are again straightforward and rather long, and so are omitted.

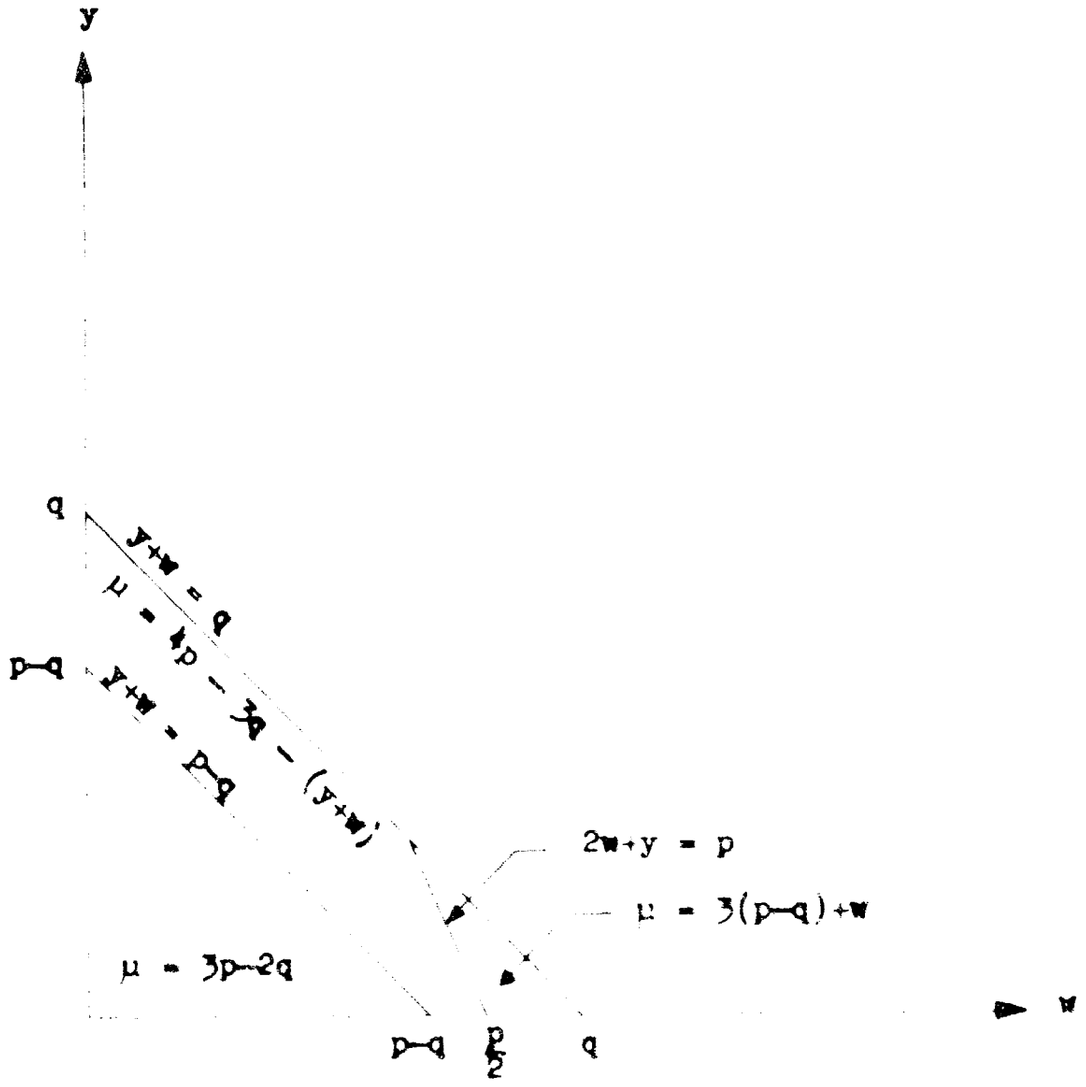

Fig. 2

It therefore follows that min $\mu(y, w)$ is attained along $y + w = q$, $w \leq p - q$, and the value of the minimum is $4(p - q)$. If $p \geq 2q$, then $\mu = 3p - 2q$ for all (y, w).

The validity of (5.1) is now apparent, and thus at least one of the players must randomize. The dictum of the weaker player

randomizes" leads to a computation of

$$E(X, G^*) = \frac{1}{2}\left\{M(X, Y^{(1)}) + M(X, Y^{(2)})\right\},$$

the results of which are shown in Table 3.

TABLE 3

DETERMINATION OF VALUES OF THE FUNCTION E(X, G*)

Region		E
u	x	
$0 \leq u \leq q$	$0 \leq x \leq 2q$	$3(p - q)$
$0 \leq u \leq q$	$2q \leq x$	$3p - q - 2x$
$q \leq u$	$0 \leq x \leq 2q$	$3p - 2q - u$
$q \leq u$	$2q \leq x$	$3p - x - u$

Clearly, we have $E(X, G^*) \leq 3(p - q)$, with the sign of equality holding for all X on the line segment (5.2). Since for each fixed X on this line segment, $M(X, Y) \geq 3(p - 1)$ for all Y, the validity of the theorem for N = 3 follows by induction and the use of Lemma 1.

At N = 4, a new phenomenon manifests itself. Set

$$m_4(x_4, u_4) \equiv \min_{Y_4} M_4(X_4, Y_4)$$

and drop the subscript 4. It can be shown by straightforward, but perhaps tedious, computation that m(x, u) has the form shown in Fig. 3. From the figure, it is evident that:

(a) if $p \leq 7q/3$, then

$$\max_X m(x, u) = \max_X \min_y M(X, y) = \frac{9}{2}(p - q),$$

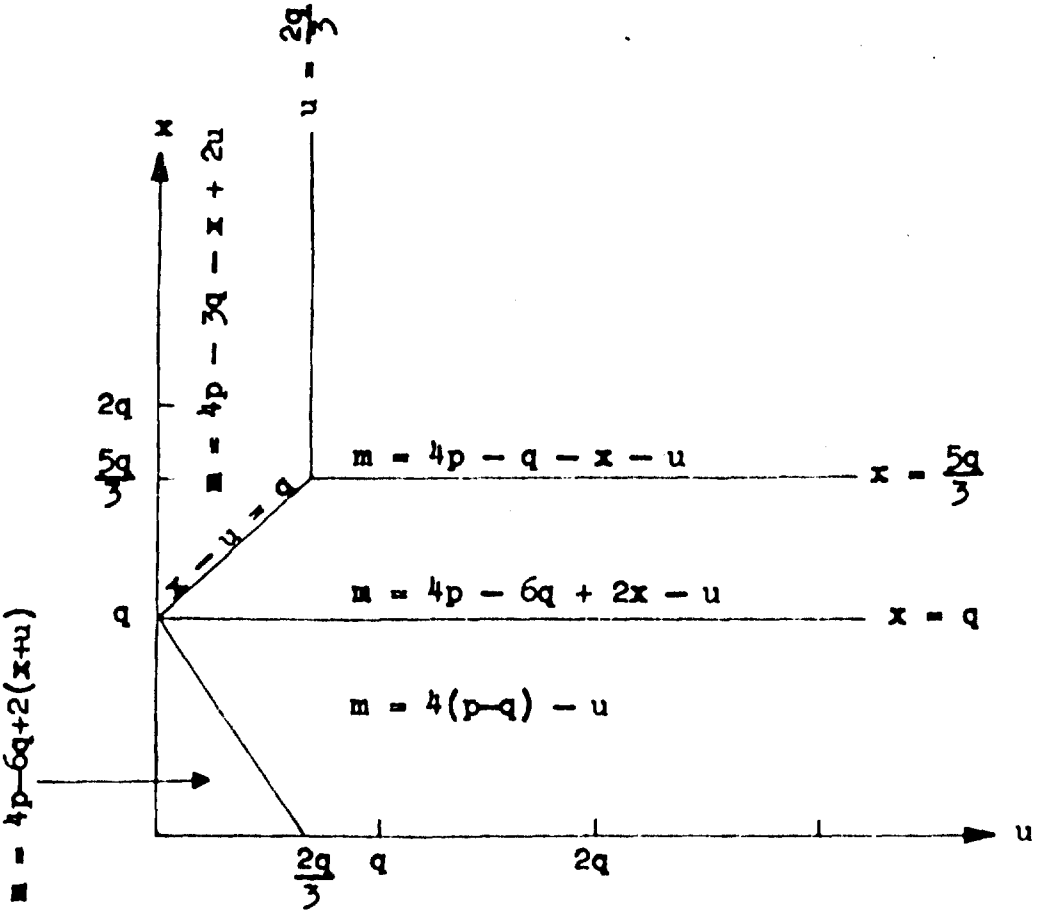

Fig. 3

and this value is attained only at the intersection of $x - u = q$ and $x + u = p$ — that is, for

$$\bar{x} = \frac{p+q}{2} , \quad \bar{u} = \frac{p-q}{2} ;$$

(b) if $p \geq 7q/3$, then

$$\max_X \min_Y M(X,Y) = 4p - \frac{10}{3}q ,$$

and this value is attained only at

$$\bar{x} = \frac{5}{3}q, \qquad \bar{u} = \frac{2}{3}q .$$

It should be observed here that $(\bar{x}, \bar{u})$ lies on the line $\bar{x} + \bar{u} = p$, with $p = 7q/3$. Straightforward calculation shows that these choices constitute optimal play for Blue, while the optimal strategy for Red is to randomize over $Y^{(1)}$ and $Y^{(2)}$ with probabilities $(1/2, 1/2)$ if $p \leq 7q/3$ and to randomize over $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$ with probabilities $(1/3, 1/3, 1/3)$ if $p \geq 7q/3$. The value of the game, of course, is the piecewise-linear function of p and q,

$$V = \begin{cases} \frac{9}{2}(p - q) & \text{if } \frac{p}{q} \leq \frac{7}{3}, \\ 4p - \frac{10}{3}q & \text{if } \frac{p}{q} \geq \frac{7}{3}. \end{cases}$$

All of the characteristics of the game's structure become completely apparent at $N = 5$, and this case will now be studied. Suppose that $p_5 \geq q_5$. As before, we have

$$M_5(X_5, Y_5) = L_5(X_5, Y_5) + \begin{cases} a^*(p_4 - q_4) & \text{if } p_4 \leq \frac{7}{3}q_4, \\ ap_4 - bq_4 & \text{if } p_4 \geq \frac{7}{3}q_4, \end{cases}$$

where

$$a^* = \frac{9}{2}, \quad a = 4, \quad b = \frac{10}{3}.$$

Dropping the subscript 5, we show the values of $M(X, Y)$ in Table 4.

The particular functional forms (i.e., expressions involving A, B) are determined by the regions in the (X, Y) space that appear in Table 2. Each of these regions is then broken into

TABLE 4

DETERMINATION OF VALUES OF THE FUNCTION $M_5(X_5, Y_5)$

Region	Constraint	A	B	M
	$\frac{p}{q} \leq \frac{7}{5}$	$a^* + 1$	$a^* + 1$	
X	$\frac{p}{q} \geq \frac{7}{5}$	$a + 1$	$b + 1$	$Ap - Bq - x - u + y + w$
	$\frac{3}{7}p - q + x \leq w$	$a^* + 1$	$a^* + 1$	
XI	$\frac{3}{7}p - q + x \geq w$	$a + 1$	$b + 1$	$Ap - Bq + (B-2)x - u + y - (B-2)w$
XII	——	$a + 1$	1	$Ap - Bq - x - u + y + w$
XIII	——	$a + 1$	$a^* + 1$	$Ap - Bq - x + (A-2)u - (A-2)y + w$
	$p - \frac{7}{5}q + \frac{7}{5}x + u \leq y + \frac{7}{5}w$	$a^* + 1$	$a^* + 1$	
XIV	$p - \frac{7}{5}q + \frac{7}{5}x + u \geq y + \frac{7}{5}w$	$a + 1$	$b + 1$	$Ap - Bq + (B-2)x + (A-2)u - (A-2)y - (B-2)w$
XV	——	$a + 1$	1	$Ap - Bq - x + (A-2)u - (A-2)y + w$

at most two regions by the conditions $3p_4 \leq 7q_4$, $3p_4 \geq 7q_4$, which determine the constants in $V_4(p_4, q_4)$. It is this further breakdown that determines the particular values assumed by the constants A and B. It will be noted that in some instances an entire region in the (X, Y) plane maps into only one of the regions in the (p_4, q_4) plane.

Let

$$m(x, u) \equiv \min_{Y} M(X, Y)$$

It can be shown by straightforward, but lengthy, computation that m(x, u) has the form shown in Fig. 4.

Since the line $x + u = p$ that passes through the point P_1 has $p = 49/20\ q$, and the one that passes through P_2 has $p = 42/25\ q$, and since $a^* > 2$, $a > 2$, and $b > 2$, the statements indicated in Table 5 hold concerning max m(x, u).

TABLE 5

DETERMINATION OF max m(x, u)

(p, q) Region	max m(x, u)	Point at which max is attained
$1 \leq \frac{p}{q} \leq \frac{42}{25}$	$\frac{108}{17}(p - q)$	$\bar{X}^1 = \begin{cases} \bar{x}^1 = \frac{7p+10q}{17} \\ \bar{u}^1 = \frac{10}{17}(p - q) \end{cases}$
$\frac{42}{25} \leq \frac{p}{q} \leq \frac{49}{20}$	$\frac{4}{11}(16p - 15q)$	$\bar{X}^2 = \begin{cases} \bar{x}^2 = \frac{9p+4q}{11} \\ \bar{u}^2 = \frac{5p-4q}{11} \end{cases}$
$\frac{49}{20} \leq \frac{p}{q}$	$5p - \frac{69}{20}q$	$\bar{X}^3 = \begin{cases} \bar{x}^3 = \frac{17}{10}q \\ \bar{u}^3 = \frac{3}{4}q \end{cases}$

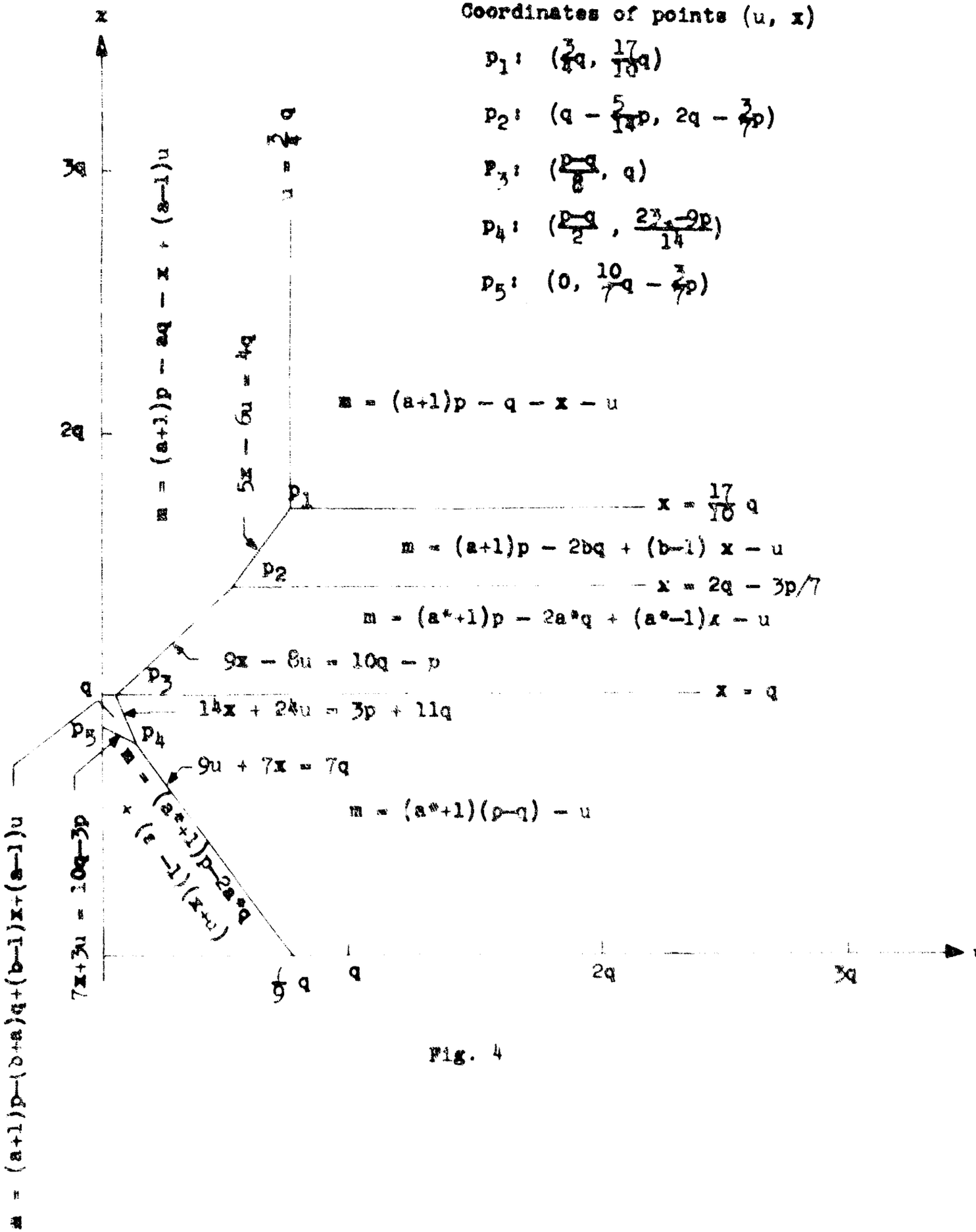
Coordinates of points (u, x)
p_1: $(\frac{3}{4}q, \frac{17}{10}q)$
p_2: $(q - \frac{5}{14}p, 2q - \frac{3}{7}p)$
p_3: $(\frac{p-q}{8}, q)$
p_4: $(\frac{p-q}{2}, \frac{23q-9p}{14})$
p_5: $(0, \frac{10}{7}q - \frac{3}{7}p)$
x
3q
2q
q
$\frac{3}{4}q$
p_1
p_2
p_3
p_4
p_5
$m = (a+1)p - 2aq - x + (a-1)u$
$5x - 6u = 4q$
$m = (a+1)p - q - x - u$
$x = \frac{17}{10}q$
$m = (a+1)p - 2bq + (b-1)x - u$
$x = 2q - 3p/7$
$m = (a^*+1)p - 2a^*q + (a^*-1)x - u$
$9x - 8u = 10q - p$
$x = q$
$14x + 24u = 3p + 11q$
$9u + 7x = 7q$
$m = (a^{**}+1)(p-q) - u$
$m = (a+1)p - (b+a)q + (b-1)x + (a-1)u$
$7x + 3u = 10q - 3p$
$\frac{9}{7}q$
q
2q
3q

Fig. 4

Straightforward calculation shows that

$$M(\bar{X}^1, Y^{(1)}) = M(\bar{X}^1, Y^{(2)}) = \frac{108}{17}(p - q),$$

$$M(\bar{X}^2, Y^{(1)}) = M(\bar{X}^2, Y^{(2)}) = \frac{14}{11}(16p - 15q), \tag{5.3}$$

$$M(\bar{X}^3, Y^{(1)}) = M(\bar{X}^3, Y^{(2)}) = M(\bar{X}^3, Y^{(3)}) = 5p - \frac{69}{20}q.$$

It is thus reasonable to assume that if $20p \leq 49q$ then Red randomizes over $Y^{(1)}$ and $Y^{(2)}$, and that if the inequality is reversed then Red randomizes over $Y^{(1)}$, $Y^{(2)}$, and $Y^{(3)}$. Proceeding on this assumption, we compute $M(X, Y^{(1)})$, $M(X, Y^{(2)})$, and $M(X, Y^{(3)})$, and then seek to determine probabilities α, β, which may depend on (p, q), such that for all X

$$\alpha M(X, Y^{(1)}) + (1-\alpha)M(X, Y^{(1)}) \leq \begin{cases} \frac{108}{17}(p-q) & \text{if } 1 \leq \frac{p}{q} \leq \frac{42}{15}, \\ \frac{8}{11}(16p-15q) & \text{if } \frac{42}{15} \leq \frac{p}{q} \leq \frac{49}{20}, \end{cases}$$

$$\alpha M(X, Y^{(1)}) + \beta M(X, Y^{(2)}) + (1-\alpha-\beta)M(X, Y^{(3)}) \leq 5p - \frac{69}{20}q \quad \text{if } \frac{p}{q} \geq \frac{49}{20}. \tag{5.4}$$

The problem of determining α and β is not as difficult as it may at first appear. For in view of (5.3) it is clear that α and β must have the property that, for a given sector of the (p, q) plane, the sum in (5.4) must either be independent of X in a region of the (x, u) plane containing the appropriate $\bar{X}^i$, $i = 1, 2, 3$, or must be of the form $F(p, q) + C(x + u)$, where F is some function and C is a positive constant, in that region.

Guided by these observations, we compute α and β to be the following:

$$\alpha = \frac{9}{17}, \quad 1 - \alpha = \frac{8}{17} \quad \text{for } 1 \leq \frac{p}{q} \leq \frac{42}{25};$$

$$\alpha = \frac{5}{11}, \quad 1 - \alpha = \frac{6}{11} \quad \text{for } \frac{42}{25} \leq \frac{p}{q} \leq \frac{49}{20};$$

$$\alpha = \frac{1}{4}, \quad \beta = \frac{3}{10}, \ (1 - \alpha - \beta) = \frac{9}{20} \text{ for } \frac{49}{20} \leq \frac{p}{q}.$$

It is then an easy matter to verify that, for these values of α and β, (5.4) is valid. Thus the optimal strategies for both players at the fifth move are determined; namely, Blue chooses X^1 for appropriate i determined by the ratio p/q, and Red chooses the appropriate randomization over $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$, determined also by the ratio of p/q.

6. DEFINITIONS AND PROPERTIES OF CONSTANTS

The first step of the proof of Theorem 1 is to define the sequences $\left\{a_N^1\right\}$, $\left\{b_N^1\right\}$, $\left\{\lambda_N^1\right\}$. To this end, consider the following sequences defined in the manner and order indicated:

$$(6.1) \qquad a_3^1 = 3, \quad a_{n+1}^{n-1} = (a_n^{n-2} + 1); \quad n \geq 3;$$

$$(6.2) \qquad b_3^1 = 3, \quad b_{n+1}^{n-1} = \left(4 - \frac{1}{a_n^{n-2}} - \frac{1}{b_n^{n-2}}\right), \quad n \geq 3;$$

$$(6.3) \qquad b_n^0 = a_n^0 = 0, \ n \geq 3;$$

$$\left.\begin{aligned} (6.4) \qquad a_{n+1}^i &= \frac{a_n^{n-2}(2b_n^i + a_n^i)}{b_n^i + a_n^{n-2}} \\ (6.5) \qquad b_{n+1}^i &= \frac{3a_n^{n-2}b_n^i}{b_n^i + a_n^{n-2}} \end{aligned}\right\} \quad i \geq 1; \ n = i + 2, \ i + 3, \ \ldots;$$

(6.6) $\lambda_n^{n-1} = +\infty$, $n = 3, 4, 5, \ldots$;

(6.7) $\lambda_3^1 = 1$,

$$\lambda_{n+1}^{n-1} = \left(3 - \frac{1}{a_n^{n-2}} - \frac{1}{b_n^{n-2}}\right) = \left(b_{n+1}^{n-1} - 1\right), \quad n = 3, 4, 5, \ldots;$$

(6.8) $\lambda_n^i = \dfrac{b_n^{i-1} - b_n^i}{a_n^{i-1} - a_n^i}$, $n \geq 4$; $\quad i = 1, 2, 3, \ldots, n-3$.

The following properties of the foregoing sequences will be useful in the proof of the theorem; indications of the derivations of the properties are given after the listing:

(6.9) $a_n^{n-2} = n$, $\quad n \geq 3$;

(6.10) $a_n^1 = b_n^1$, $\quad n \geq 3$;

(6.11) $4 > b_{n+1}^{n-1} > b_n^{n-2} \geq 3$, $\quad n \geq 3$;

(6.12) $a_n^i \geq b_n^i > 0$, $\quad i = 1, 2, \ldots, n-2$; $n = 3, 4, 5, \ldots$;

(6.13) $n \leq a_n^i < 2n$, $\quad i = 1, \ldots, n-2$; $n = 3, 4, 5, \ldots$, ;

(6.14) $a_n^i < a_n^{i-1}$, $b_n^i < b_n^{i-1}$, $\quad n \geq 3$; $i = 2, 3, \ldots, n-2$;

(6.15) $\lambda_{n+1}^i = \dfrac{3n\,\lambda_n^i}{(2n-a_n^1)\lambda_n^1 + (n+b_n^1)}$, $\quad n \geq 3$; $i = 1, 2, \ldots, n-2$;

(6.16) $3 > \lambda_{n+1}^{n-1} > \lambda_n^{n-2} \geq 2$, $\quad n \geq 3$;

(6.17) $\lambda_n^{i+1} > \lambda_n^i$, $\quad n \geq 3;\ i = 2, 2, \ldots, n-2;$

(6.18) $\lambda_{n+1}^i \leq \lambda_n^i$, $\quad n \geq 3;\ i = 1, \ldots, n-2.$

Statements (6.9) – (6.11) follow from the definitions and from trivial inductive arguments.

Inequalities (6.12) and (6.13) are proved by induction on n, $n \geq i + 2$, for each fixed i.

The monotonicity properties in (6.14) are established as follows. The monotonicity of $\left\{b_{n+1}^i\right\}$, $i = 1, \ldots, n-2$, follows, by induction on n, from the monotonicity of $\left\{b_n^i\right\}$, $i = 1, \ldots, n-2$. To show that $b_{n+1}^{n-2} > b_{n+1}^{n-1}$, it suffices to show that $b_{n+1}^{n-2} \geq 4$. This inequality, however, is obvious from (6.5) and (6.11). When we compute $a_{n+1}^{i-1} - a_{n+1}^i$, $i = 2, 3, \ldots, n-2$, we obtain

$$\frac{n}{(n+b^{i-1})(n+b^i)}\left[b^{i-1}(2n - a^i) - b^i(2n - a^{i-1}) + n(a^{i-1} - a^i)\right]$$

where the subscript n is omitted. From the inductive hypothesis that $\left\{a_n^i\right\}$ is monotone decreasing in i, and from (6.13), the bracketed expression in turn is seen to be larger than the positive quantity

$$b^{i-1}(2n-a^{i-1}) - b^i(2n-a^{i-1}) = (b^{i-1}-b^i)(2n-a^{i-1}) .$$

Furthermore, for $n \geq 3$, by (6.1), (6.4), and (6.11) we have

$$a_{n+1}^{n-2} - a_{n+1}^{n-1} = \frac{b_n^{n-2}(n-1)-1}{b^{n-2}+n} \geq \frac{3(n-1)-n}{b^{n-2}+n} > 0$$

Thus the fact that $\left\{a_{n+1}^{i}\right\}$ is monotone decreasing in i is established.

To prove (6.15), we use (6.4) and (6.5) in the definition (6.8) of λ_{n+1}^{i}.

Inequality (6.16) is obvious, and (6.17) is established inductively as follows. Suppose $\lambda_n^{i+1} > \lambda_n^i$ for $i = 1, \ldots, n-1$. Consider

$$\frac{\lambda_{n+1}^{i+1}}{\lambda_{n+1}^{i}} = \frac{\lambda^{i+1}[(2n-a^i)\lambda^i + (n+b^i)]}{\lambda^i[(2n-a^{i+1})\lambda^{i+1} + (n+b^{i+1})]}, \quad i = 1, 2, \ldots, n-3,$$

where the subscript n is omitted. To prove that this ratio exceeds 1, it therefore suffices to show that

$$n(\lambda^{i+1} - \lambda^i) + \lambda^{i+1}b^i - \lambda^i b^{i+1} > \lambda^i\lambda^{i+1}(a^i - a^{i+1}).$$

Upon replacing λ^{i+1} on the right by its definition (6.8), we see that this last inequality is equivalent to

$$(n + b^i)\,(\lambda^{i+1} - \lambda^i) > 0,$$

the validity of which follows from the inductive hypothesis. The chain

$$\lambda_{n+1}^{n-2} = \frac{3n\lambda^{n-2}}{(\lambda^{n-2}+1)n + b^{n-2}} = \frac{3n\lambda^{n-2}}{b^{n-2}(n+1)} \leq \frac{3\lambda^{n-2}}{b^{n-2}} < \lambda^{n-2} < \lambda_{n+1}^{n-1}$$

completes the proof of (6.17).

It is seen from (6.15) that to verify (6.18) it suffices to show that

$$\lambda_n^i \geq \frac{2n - b_n^i}{2n - a_n^i}, \quad n \geq 3;\ i = 1, 2, \ldots, n-2. \tag{6.19}$$

This inequality is shown to hold by induction on i, as follows. For $i = 1$, equality is obvious. Suppose that the inequality holds for $i = k$. It is then seen to hold for $i = k + 1$, $1 \leq i \leq n - 3$, by writing (omitting the subscript n)

$$\lambda^{k+1}(2n-a^{k+1}) - (2n-b^{k+1}) = \lambda^{k+1}(2n-a^{k}) - (2n-b^{k})$$

$$+ \lambda^{k+1}(a^{k}-a^{k+1}) - (b^{k}-b^{k+1}) >$$

$$\lambda^{k}(2n-a^{k}) - (2n-b^{k}) \geq 0.$$

7. MISCELLANEOUS PREPARATIONS

For $N \leq 3$, Theorem 1 has been proved in Sec. 5. The theorem will now be proved inductively for arbitrary $N > 3$. Suppose then that it has been established for $N = n \geq 3$. It is required to show that it holds for $N = n + 1$.

In order to simplify notation, for the remainder of the proof we shall omit the subscript n. Thus the symbol a_n^i, say, will be written merely as a^i, the symbol λ_{n+1}^{i-1} as λ_1^{i-1}, b_{n-1}^{i+1} as b_{-1}^{i+1}, etc.

From the symmetry of Theorem 1, it is clear that it suffices to consider the case $p_1 \geq q_1$. Define $X_1^* = X_1^*(p_1/q_1)$ as follows:

$$X_1^* = \bar{X}_1^i \equiv (\bar{x}_1^i, \bar{u}_1^i) \quad \text{if } \lambda_1^i \leq \frac{p_1}{q_1} \leq \lambda_1^{i+1}, \; i = 1, \ldots, n-1$$

where

$$\bar{x}_1^i = \frac{(2n-a^i)p_1 - (n-2b^i)q_1}{b^i + n}, \quad \bar{u}_1^i = p_1 - \bar{x}_1^i, \; i = 1, \ldots, n-2,$$

(7.1)

$$\bar{x}_1^{n-1} = (2 - \frac{1}{b^{n-2}})q_1, \qquad \bar{u}_1^{n-1} = (1 - \frac{1}{n})q_1 .$$

Define $\alpha_1 = \alpha_1(p_1/q_1)$ and $\beta_1 = \beta_1(p_1/q_1)$ thus:

a) if $\lambda_1^i \leq \frac{p_1}{q_1} \leq \lambda_1^{i+1}$, $\quad 1 \leq i \leq n-2$,

then

(7.2) $$\alpha_1 = \alpha_1^i = \frac{b^i}{n+b^i}, \quad \beta_1 = \beta_1^i = \frac{n}{n+b^i};$$

b) if $\frac{p_1}{q_1} \geq \lambda_1^{n-1}$,

then

$$\alpha_1^{n-1} = \frac{1}{n}, \qquad \beta_1^{n-1} = \frac{1}{b^{n-2}}.$$

Clearly, $\alpha_1^i > 0$, $\beta_1^i > 0$ for all i satisfying $1 \leq i \leq n-1$; $\alpha^i + \beta^i = 1$ for all i satisfying $1 \leq i \leq n-2$; and $\alpha^i + \beta^i < 1$ for $i = n-1$. Thus α_1 and β_1 are probabilities. Lemma 2 will show that X_1^* is an admissible choice for Blue and will furnish some useful bounds for $\bar{x}_1^i$ and $\bar{u}_1^i$.

Lemma 2. The point X_1^* is an admissible choice of strategic variable for Blue. Furthermore, for all i satisfying $1 \leq i \leq n-1$, we have

(7.3) $$q_1 \leq \bar{x}_1^i \leq 2q_1,$$
$$0 \leq \bar{u}_1^i < q_1.$$

Since X_1^* is defined piecewise, the first step in showing that X_1^* is admissible is to show that the pieces come together — i.e., that X_1^* is well defined. Substitution of $p_1 = \lambda_1^{n-1} q_1$ into the definition (7.1) of $\bar{x}_1^{n-2}$, and application of (6.7), show that

if $p_1 = \lambda_1^{n-1} q_1$, then $\bar{x}_1^{n-2} = \bar{x}_1^{n-1}$. Similarly, it is seen that $\bar{u}_1^{n-2} = \bar{u}_1^{n-1}$ for $p_1 = \lambda_1^{n-1} q_1$. Substitution of $p_1 = \lambda_1^i q_1$, $i = 1, \ldots, n-2$, into the definition (7.1) of $\bar{x}_1^i$, and use of (6.15), show that, for $p_1 = \lambda_1^i q_1$,

$$\bar{x}_1^i = \left(2 - \frac{\lambda_1^i}{\lambda^i}\right) q_1 .$$

Substitution of $p_1 = \lambda_1^{i+1} q_1$, and use of (6.15), show that, for $p_1 = \lambda_1^{i+1} q_1$, $i = 1, \ldots, n-3$,

$$\bar{x}_1^i = \left(2 - \frac{\lambda_1^{i+1}}{\lambda^{i+1}}\right) q_1 .$$

Thus X_1^* is well defined, and for $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $i = 1, 2, \ldots, n-3$,

$$(7.4) \qquad \left(2 - \frac{\lambda_1^i}{\lambda^i}\right) q_1 \leq \bar{x}_1^i \leq \left(2 - \frac{\lambda_1^{i+1}}{\lambda^{i+1}}\right) q_1 ,$$

with equality on the left occurring for $p_1 = \lambda_1^i q_1$, and on the right for $p_1 = \lambda_1^{i+1} q_1$. Similarly, we obtain

$$(7.5) \qquad \left(2 - \frac{\lambda_1^{n-2}}{\lambda^{n-2}}\right) q_1 \leq \bar{x}_1^{n-2} \leq \bar{x}_1^{n-1} .$$

Clearly, (7.3) implies $\bar{x}_1^i \geq 0$, $\bar{u}_1^i \geq 0$ for all i satisfying $1 \leq i \leq n-1$. By definition, $\bar{x}_1^i + \bar{u}_1^i = p_1$ for $1 \leq i \leq n-2$; and for $i = n-1$, we obtain

$$\bar{x}_1^{n-1} + \bar{u}_1^{n-1} = \lambda_1^{n-1} q_1 \leq p_1 .$$

Thus the establishment of the lemma depends on the proof of (7.3). For $i = n-1$, (7.3) is obvious. The inequality (6.18) implies

$\lambda_1^i \leq \lambda^i$, whence it follows that $(2 - \lambda_1^i/\lambda^i)q_1 \geq q_1$ for $i = 1, 2, \ldots, n-2$. Clearly, the inequality $(2 - \lambda_1^i/\lambda^i)q_1 < 2q_1$ holds for $i = 1, \ldots, n-2$. Hence, we obtain

$$q_1 \leq \bar{x}_1^i < 2q_1 \qquad \text{for } i = 1, \ldots, n-2.$$

To verify $0 \leq \bar{u}_1^i < q_1$ for $1 \leq i \leq n-2$, we substitute from the definition of $\bar{x}_1^i$ into the definition $\bar{u}_1^i = p_1 - \bar{x}_1^i$ and obtain

$$\bar{u}_1^i = \frac{(b^i+a^i-n)p_1 + (n-2b^i)q_1}{(b^i+n)}, \qquad 1 \leq i \leq n-2.$$

Hence, showing that $\bar{u}_1^i < q_1$ is equivalent to showing that $p_1/q_1 < 3b^i/(b^i + a^i - n)$. Since $p_1/q_1 \leq \lambda_1^{i+1}$, it clearly suffices to show that $\lambda_1^{i+1} < 3b^i/(b^i + a^i - n)$. From (6.15), it follows that this last inequality is equivalent to

$$b^i - \lambda^i(a^i - n) > 0, \qquad i = 1, 2, \ldots, n-2.$$

Since $a^{n-2} = n$, it follows that for $i = n-2$ the expression on the left of this inequality is equal to b^{n-2}, which is positive. Further, the left hand member is a monotone decreasing function of i, as can be seen by forming the difference of the left-hand side for superscripts $i-1$ and i and getting

$$(\lambda^i - \lambda^{i-1})(a^{i-1} - n) > 0 .$$

This inequality follows from (6.13) and (6.17). Thus it follows that $\bar{u}_1^i < q_1$. The inequality $\bar{u}_1^i \geq 0$, follows from

$$(b^i+a^i-n)p_1 + (n-2b^i)q_1 \geq (a^i-b^i)q_1 \geq 0; \qquad 1 \leq i \leq n-2,$$

and the lemma is proved.

It follows from Lemma 1 and the definitions of X_1^*, α_1, β_1, and from the inductive hypothesis to the effect that Theorem 1 is valid for $N = n$, that the validity of Theorem 1 will be established if (4.1), (4.2), and (4.3) of Lemma 1 are shown to hold for this X_1^*, α_1, β_1. The next sections of the proof will be devoted to the verification of these three statements.

In the course of this verification, it will be necessary to compute

$$(7.6) \qquad M_1(X_1, Y_1) = L_1(X_1, Y_1) + V(p, q)$$

$$= L_1(X_1, Y_1) + a^j p - b^j q, \quad j = 1, \ldots, n - 2,$$

explicitly in terms of p_1, q_1, X_1, Y_1, for certain choices of X_1, Y_1. For any given fixed initial condition (p_1, q_1), an integer $1 \leq i \leq n - 1$ is determined by the inequality $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$. Each choice (X_1, Y_1) by the players falls into one of the six regions enumerated in Table 2* and determines p and q and hence an integer $1 \leq j \leq n - 2$ via the inequality $\lambda^j \leq p/q \leq \lambda^{j+1}$. It is this integer j that appears in (7.6). Clearly, j is a function of p_1, q_1, X_1, and Y_1. In computing $M_1(X_1, Y_1)$ explicitly in terms of the initial conditions and choices X_1, Y_1, it will thus be necessary to take into account the region of the (X_1, Y_1) plane and the superscript j. The statement "(X_1, Y_1) leads to case III^{j_0}" will mean that, for the initial condition being

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*The table is given for passage from n to n − 1, whereas the present situation is for passage from n + 1 to n. The adjustment of subscripts is left to the reader.

considered, the pair (X_1, Y_1) falls into region III of the (X_1, Y_1) plane and the ratio p/q is such that $j = j_0$. At first glance, it appears that there are $6(n-2)$ cases. Actually, not all of these cases are possible; and since some specialization of X_1, Y_1 will occur, not all of the possible cases will be encountered.

8. VERIFICATION, FIRST PART OF SUFFICIENCY CONDITION

We divide the discussion into two cases.

Case 1: $\quad p_1/q_1 \geq \lambda_1^{n-1}$.

For this initial condition, it is readily seen from Table 2 that $(X_1^*, Y_1^{(1)})$ leads to case VI^{n-2}, $(X_1^*, Y_1^{(2)})$ leads to case II^{n-2}, and $(X^*, Y_1^{(3)})$ leads to case III^{n-2}. It then follows by straightforward computation and the definition of Sec. 6 that

$$(8.1) \qquad M_1(X_1^*, Y_1^{(1)}) = M_1(X_1^*, Y_1^{(2)}) = M_1(X_1^*, Y_1^{(3)}) = V_1(p_1, q_1).$$

The equality $E_1(\hat{F}^*, G_1^*) = V_1(p_1, q_1)$ now follows from

$$\begin{aligned} E_1(\hat{F}_1^*, G_1) = {} & \alpha_1\left[L_1(X_1^*, Y_1^{(1)}) + E(F^*, G^*)\right] + \\ & p_1\left[L_1(X_1^*, Y_1^{(2)}) + E(F^*, G^*)\right] + \\ & (1 - \alpha_1 - \beta_1)\left[L_1(X_1^*, Y_1^{(3)}) + E(F^*, G^*)\right] \\ = {} & \alpha_1 M_1(F_1^*, (X_1^*, Y_1^{(1)})) + \beta_1 M_1(X_1^*, Y_1^{(2)}) + \\ & (1 - \alpha_1 - \beta_1) M_1(X_1^*, Y_1^{(3)}) . \end{aligned}$$

Case 2: $\lambda_1^i \le p_1/q_1 \le \lambda_1^{i+1}$, $i = 1, 2, \ldots, n-2$.

For this initial condition, it is immediately seen from Table 2 that $(X_1^*, Y_1^{(1)})$ leads to case VI^{n-2}, while $(X_1^*, Y_1^{(2)})$ leads to case II^j for appropriate j. To determine the value of j, we first observe that

$$\frac{p}{q} = \frac{p_1}{2q_1 - \bar{x}_1^1} .$$

It follows from (7.4) and (7.5) that $j = 1$. Straightforward computation and use of the definitions in Sec. 6 now show that

$$(8.2) \qquad M_1(X_1^*, Y_1^{(1)}) = M_1(X_1^*, Y_1^{(2)}) = V_1(p_1, q_1),$$

and hence (4.1) follows as before.

9. VERIFICATION, SECOND PART OF SUFFICIENCY CONDITION

Again, as in Sec. 8, we divide the discussion into two cases.

Case 1: $p_1/q_1 \ge \lambda_1^{n-1}$.

Since $\bar{x}_1^{n-2} \ge q_1$ (X_1^*, Y_1) can never fall into region I or IV of the (X_1, Y_1) plane. Since (X_1^*, Y_1) lying in region V implies that

$$y_1 + w_1 \ge \bar{x}_1^{n-2} + \bar{u}_1^{n-2} - q_1 = \left(2 - \frac{1}{b^{n-2}} + \frac{1}{n}\right) q_1 > q_1 ,$$

this event is also impossible.

For Y_1 such that (X_1^*, Y_1) falls in region II, we have

$$\frac{p}{q} = \frac{p_1}{q_1 - \bar{x}_1^{n-2} + w_1} \ge \frac{p_1}{2q_1 - \bar{x}_1^{n-2}} = b^{n-2} \frac{p_1}{q_1} \ge \lambda^{n-2} ,$$

the last inequality following from (6.11) and (6.16). Hence, the superscript associated with region II is n − 2, and

$$M_1(X_1^*, Y_1) = (n+1)p_1 - (b^{n-2}-1)\bar{x}_1^{n-2} - \bar{u}_1^{n-2} - (1+b^{n-2})q_1 + y_1$$

$$- (b^{n-2}-1)w_1 \geq M_1(X_1^*, Y_1^{(2)}) = V_1(p_1, q_1),$$

the last inequality following from (8.1).

The only superscript that can be associated with region III or VI is n − 2. Having noted this, we easily see that for (X_1^*, Y_1) in region III, $\min_{Y_1} M_1(X_1^*, Y_1)$ occurs for $Y_1 = Y_1^{(3)}$, and so (4.2) follows from (8.1). Similarly, for Y_1 such that (X_1^*, Y_1) is in VI,

$$\min_{Y_1} M_1(X_1, Y_1) = M_1(X_1^*, Y_1^{(1)}) = V_1(p_1, q_1).$$

Case 2: $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $i = 1, \ldots, n-2$.

Since $\bar{x}_1^i \geq q_1$, (X_1^*, Y_1) cannot fall in region I or IV. If (X_1^*, Y_1) is in region II, then we have

$$M_1(X_1^*, Y_1) = (a^j+1)p_1 - (b^j+1)q_1 + (b^j-1)\bar{x}_1^i - \bar{u}_1^i$$

$$+ y_1 - (b^j-1)w_1,$$

where j is determined by the ratio

$$\frac{p}{q} = \frac{p_1}{q_1 - \bar{x}_1^i + w_1} .$$

Clearly, j is a nonincreasing function of w_1 alone, $j = j(w_1)$. From (7.4), it follows that $j(q_1) = 1$. For each j, the minimum of $M_1(X_1^*, Y_1)$ is achieved at a point $Y_1 = (y_1, w_1)$, where $y_1 = 0$ and where w_1 is the largest value of w such that (X_1^*, Y_1), $Y_1 = (0, w)$, leads to case II^j. Hence, by the continuity of $M_1(X_1^*, Y_1)$, it follows that the minimum of $M_1(X_1^*, Y_1)$, over all Y_1 such that (X_1^*, Y_1) is in region II, occurs at $Y_1^{(2)}$. Thus, using (8.2), we get

$$M_1(X_1^*, Y_1) \geq M_1(X_1^*, Y_1^{(2)}) = V_1(p_1, q_1).$$

The only superscript possible for (X_1, Y_1) in region III is $n - 2$; thus for Y_1 such that (X_1^*, Y_1) is in III, we have

$$M_1(X_1^*, Y_1) = np_1 - q_1 + (y_1 + w_1).$$

The minimum of this expression over region III is assumed at $Y_1 = (0, 0)$ and is $np_1 - q_1$. Since we are considering the case $p_1/q_1 \leq \lambda_1^{i+1}$, the inequality

$$(9.1) \qquad (b_1^i - 1) - \lambda_1^{i+1}(a_1^i - n) \geq 0$$

implies the inequality $np_1 - q_1 \geq a_1^i p_1 - b_1^i q_1$, and so it suffices to establish (9.1) in order to verify (4.2). With the aid of (6.8) and (6.13), it is easy to see that the left-hand side of (9.1) is a decreasing function of the superscript. Furthermore, for $1 \leq n - 2$ it follows from (6.4), (6.5), and (6.7) that the left-hand side of (9.1) is zero, and so (9.1) is verified.

In the event that Y_1 is such that (X_1^*, Y_1) is in region V, we have

$$(9.2)\quad M_1(X_1^*, Y_1) = (a^j+1)p_1 - (b^j+1)q_1 + (b^j-1)(\bar{x}_1^i - w_1) + (a^j-1)(\bar{u}_1^i - y_1),$$

where j is determined by

$$\lambda^j \leq \frac{p_1 - y_1 + \bar{u}_1^i}{q_1 - \bar{x}_1^i + w_1} \leq \lambda^{j+1}, \quad j = 1, 2, \ldots, n-2.$$

Since X_1^* is fixed, all questions concerned with determining which points in the (X_1^*, Y_1) plane lead to the different cases V^j are thus seen to devolve upon questions concerning point sets in the Y_1 plane. Clearly, the lines L^j,

$$y_1 = -\lambda^j w_1 + \lambda^j(\bar{x}_1^i - q_1) + p_1 + \bar{u}_1^i,$$

in the (y_1, w_1) plane form a finite pencil through the point $y_1 = p_1 + \bar{u}_1^i$, $w_1 = \bar{x}_1^i - q_1$. From the monotonicity properties of the sequence $\{\lambda^i\}$, it follows that for any fixed $y_1 = c$ with $c \leq p_1 + \bar{u}_1^i$ (and so particularly for $y_1 \leq q_1$), as one moves along $y_1 = c$ in the direction of increasing w_1, the lines L^j are encountered in order of decreasing j, with the line L^{n-2} being intercepted at a value of $w_1 > \bar{x}_1^i - q_1$. Thus, the sets in the Y_1 plane giving rise to the various cases V^j are, in general, as indicated by the hatched regions in Fig. 5.

From the fact that $a^j \geq b^j$, it follows that the minimum of $M_1(X_1^*, Y_1)$, over each set V^j of Fig. 5, is assumed at the upper left-hand vertex of V^j. Hence, by the continuity of $M_1(X_1^*, Y_1)$ in Y_1, it follows that the minimum of $M_1(X_1^*, Y_1)$, over all Y_1 such that (X_1^*, Y_1) is in V, is achieved at $w_1 = \bar{x}_1^i - q_1$, $y_1 = 2q_1 - \bar{x}_1^i$. Substituting these values into (9.2) and using $\bar{x}_1^i + \bar{u}_1^i = p_1$,

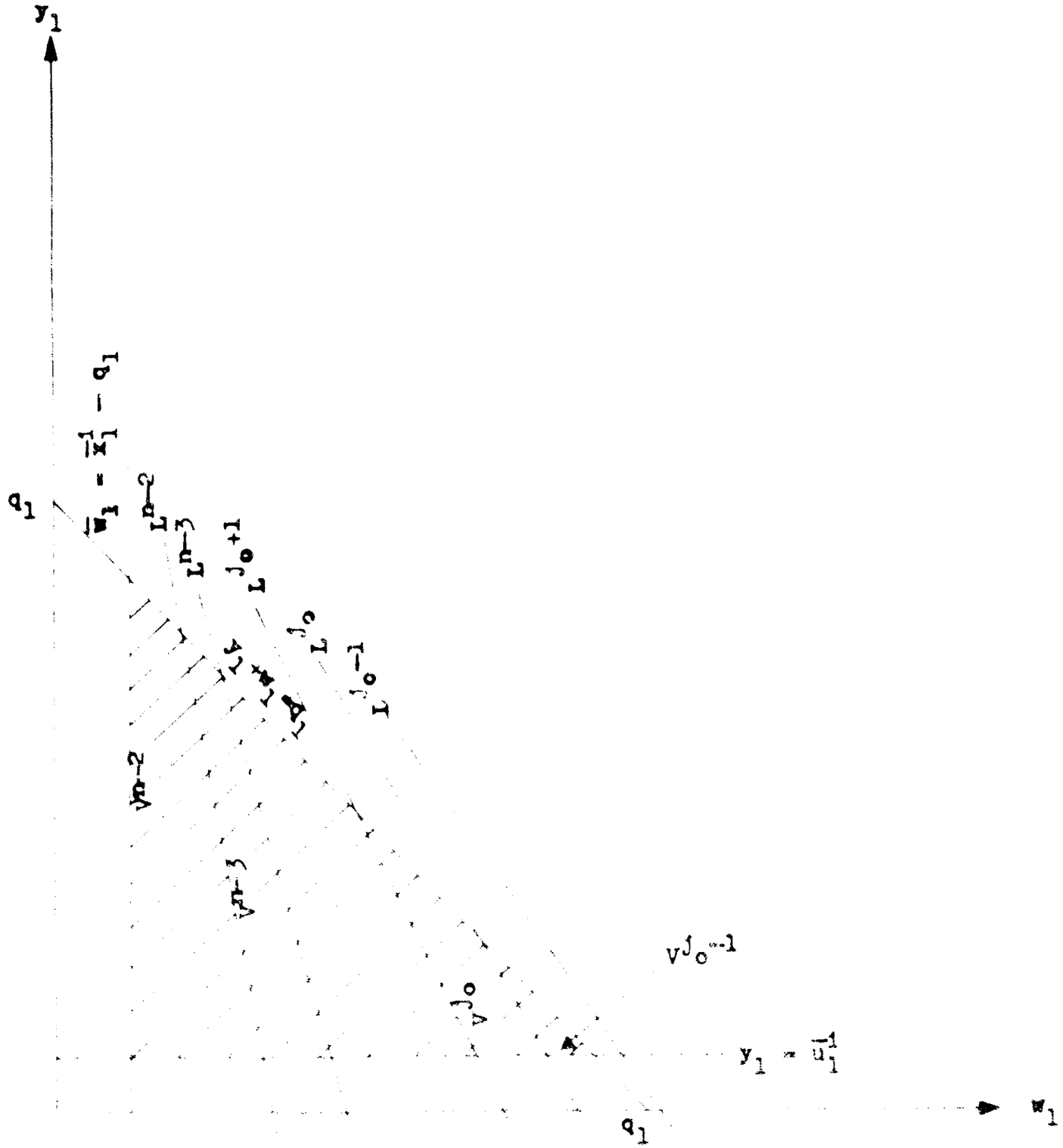
y_1
q_1
$\bar{w}_1 = \bar{x}_1^1 - q_1$
L^{n-2}
L^{n-3}
L^{J_o+1}
L^{J_o}
L^{J_o-1}
v^{n-2}
v^{n-3}
v^{J_o-1}
v^{J_o}
$y_1 = \bar{u}_1^1$
q_1
w_1

Fig. 5

we see that the value of the minimum is $2n(p_1 - q_1)$. It remains to show that

$$2n(p_1 - q_1) \geq a_1^i p_1 - b_1^i q_1 .$$

Since $p_1/q_1 \geq \lambda_1^i$, this inequality is implied by the inequality

$$\lambda_1^i \geq \frac{2n-b_1^i}{2n-a_1^i} ,$$

which is established by induction in exactly the way that (6.19) was established.

Finally, the case in which (X_1^*, Y_1) lies in region VI must be considered. Examination of Table 2 shows that the only superscript possible is $j = n - 2$, and so

$$M(X_1^*, Y_1) = (n+1)p_1 - q_1 - \bar{x}_1^i + (n-1)\bar{u}_1^i - (n-1)y_1 + w_1.$$

The minimum of this expression is assumed at $Y_1 = Y_1^{(1)}$. Since $M_1(X_1^*, Y_1^{(1)}) = V_1$, the proof of (4.2) is now concluded.

10. VERIFICATION, THIRD PART OF SUFFICIENCY CONDITION

The proof of (4.3) will clearly involve the computation of

$$(10.1)\quad \pi(X_1) = \alpha_1 M_1(X_1, Y_1^{(1)}) + \beta_1 M_1(X_1 Y_1^{(2)}) + (1-\alpha_1-\beta_1) M_1(X_1, Y_1^{(3)}).$$

Thus, for each X_1 it is necessary to know the case to which we are led by each of the points

$$A: (X_1, Y_1^{(1)}) \quad B: (X_1, Y_1^{(2)} \quad C: (X_1, Y_1^{(3)}).$$

Part of this information is tabulated in Fig. 6. In this tabulation, a symbol such as, say, AV^j in a given region means that, for all X_1 in that region, $(X_1, Y_1^{(1)})$ leads to case V^j. In some instances, the value of j is indicated; in others, the determination of j will be made in the discussions of cases 1 and 2 below.

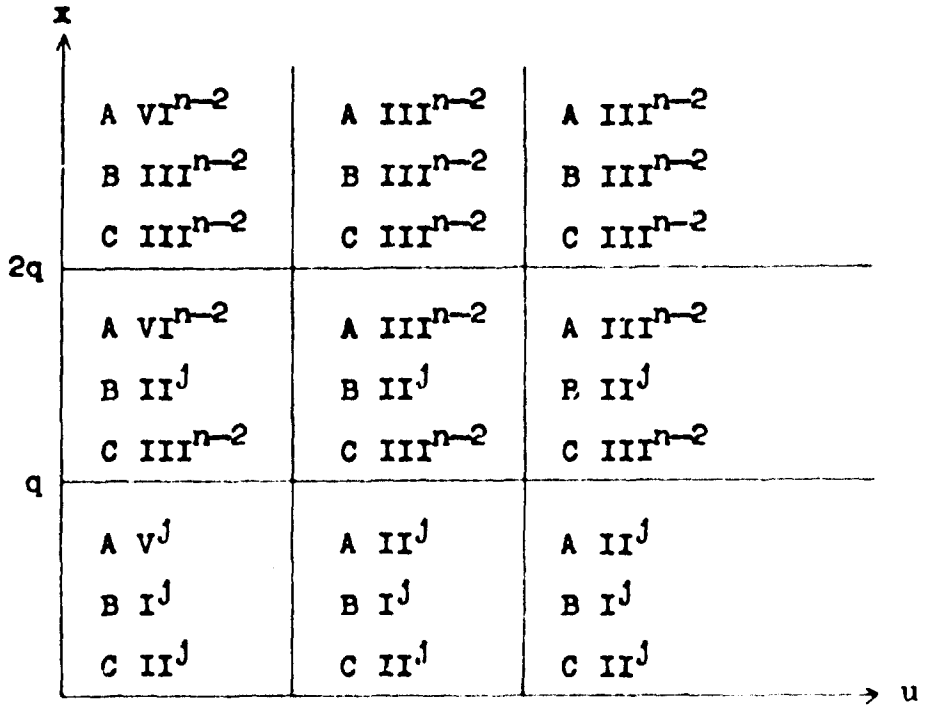

Fig. 6

Case 1: $\lambda_1^{n-1} \leq p_1/q_1$.

First, the values assumed by the superscripts j will be determined. In the case BI^j, Table 2 shows that $p_1/q_1 = p/q$; and since, by (6.16), $\lambda_1^{n-1} \geq \lambda^{n-2}$, it follows that $j = n - 2$. Also, in the case BII^j, we have $j = n - 2$, because

$$\frac{p}{q} = \frac{p_1}{2q_1 - x_1} \geq \frac{p_1}{q_1}, \qquad q_1 \leq x_1 \leq 2q_1.$$

Similarly, in AII^j and CII^j, we have $j = n - 2$, because the relation

$$\frac{p}{q} = \frac{p_1}{q_1 - x_1} \geq \frac{p_1}{q_1}$$

holds there. In the case AV^j, the value of j is determined by the ratio

$$\lambda^j \leq \frac{p}{q} = \frac{p_1 - q_1 + u_1}{q_1 - x_1} \leq \lambda^{j+1}, \qquad j = 1, \ldots, n - 2.$$

The lines

$$\ell^j: \quad \lambda^j x_1 + u_1 = \lambda^j q_1 - (p_1 - q_1)$$

form a finite pencil through the point $x_1 = q_1$, $u_1 = -(p_1 - q_1)$. It follows from the monotonicity of the λ's that if a line $u_1 = c$ with $c \geq -(p_1 - q_1)$ is traversed from $x_1 = q_1$ in the direction of decreasing x_1, then the lines of the pencil are encountered in order of decreasing j, with ℓ^{n-2} being the first line encountered. Thus, the lines ℓ^j divide the square $0 \leq x_1 \leq q_1$, $0 \leq u_1 \leq q_1$ into subregions over each of which a different superscript j is applicable. The number of subregions depends on the ratio p_1/q_1. For sufficiently large values of this ratio, the entire square will have the superscript value $n - 2$ associated with it. The important fact to be noted is that the region with superscript $n - 2$ always exists and contains the line segment $x_1 = q_1$, $0 \leq u_1 \leq q_1$.

Clearly, $\mathcal{M}(X_1)$ is continuous and is of the form

$$\mathcal{M}(x_1) = F(p_1, q_1) + Rx_1 + Su_1 ,$$

where $F(p_1, q_1)$ is a step function on the (x_1, u_1) plane whose values are expressions involving the constants a^j, b^j and the initial conditions p_1, q_1. Its exact form is of no concern here. The coefficients R and S are also step functions on the (x_1, u_1) plane whose values involve the constants a^j, b^j. The information concerning R and S shown in Table 6 is easily obtained from Fig. 6, the preceding discussion, Table 2, and the definitions (7.2) of α_1 and β_1.

TABLE 6

DETERMINATION OF VALUES OF THE COEFFICIENTS R AND S

Region of (x_1, u_1) plane		Region Number	R	S
x_1	u_1			
$2q_1 \leq x_1$	$u_1 \geq q_1$	(1)	-1	-1
$q_1 \leq x_1 \leq 2q_1$	$u_1 \geq q_1$	(2)	0	-1
$0 \leq x_1 \leq q_1$	$u_1 \geq q_1$	(3)	$b^{n-2} - 2$	-1
$2q_1 \leq x_1$	$u_1 \leq q_1$	(4)	-1	0
$q_1 \leq x_1 \leq 2q_1$	$u_1 \leq q_1$	(5)	0	0
$0 \leq x_1 \leq q_1$	$u_1 \leq q_1$	(6)	$\frac{(b^{n-2}-1)b^j}{b^{n-2}} - 1$	$\left(\frac{a^j}{n} - 1\right)$

The superscript j in the entries for region (6) varies as the superscript in AV^j, and assumes the same values as the superscript in AV^j.

For each of the regions (1) through (6) of this table, the

set of points at which the maximum of $m(X_1)$ is achieved on that region is easily determined from the tabulated values of R and S in the region. It then follows from the continuity of $m(X_1)$ that the maximum of $m(X_1)$ is achieved at all points of the square $q_1 \leq x_1 \leq 2q_1$, $0 \leq u_1 \leq q_1$. In particular, it is achieved at $(\bar{x}_1^{n-2}, \bar{u}_1^{n-2})$, since by (7.3) this point is in the square. It now follows from (8.1) that

$$m(X_1) \leq m(X_1^*) = V_1(p_1, q_1) .$$

Case 2: $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $i = 1, 2, \ldots, n-2$.

Under these initial conditions, Fig. 6 is modified from the outset as follows. Point C is eliminated since we have $\alpha_1^i + \beta_1^i = 1$ for $1 \leq i \leq n-2$; and the region $x_1 \geq 2q_1$, $u_1 \geq q_1$ need not be considered since $p_1/q_1 < 3$. In determining the superscript j and the modifications of Fig. 6, it will be convenient to distinguish two cases, namely $p_1/q_1 \geq 2$ and $p_1/q_1 \leq 2$.

Suppose that $p_1/q_1 \geq 2$. The superscript j in BII^j is determined by the ratio.

$$\frac{p}{q} = \frac{p_1}{2q_1 - x_1} , \tag{10.2}$$

where

$$q_1 \leq x_1 \leq \min \begin{cases} 2q_1 \\ \\ p_1 - u_1 \end{cases} .$$

Thus j is a nondecreasing step function of x_1 alone whose value at $x_1 = 2q_1$ is $n-2$, and whose jumps occur at

$$(10.3) \qquad x_1 = 2q_1 - \frac{p_1}{\lambda^j} \ ,$$

where j is such that $2q_1 - p_1/\lambda^j \geq q_1$. At the jump points, j is continuous from the right. Let j_0 denote the lowest value of the superscript j. This is clearly assumed at $x_1 = q_1$, and the defining relation for j_0 can be taken as

$$(10.4) \qquad \lambda^{j_0} \leq \frac{p_1}{q_1} \leq \lambda^{j_0+1} \ .$$

Since, by assumption and (6.18),

$$\frac{p_1}{q_1} \leq \lambda_1^{i+1} < \lambda^{i+1} \ ,$$

it follows that

$$j_0 \leq i \ .$$

It is also necessary to have some information concerning the superscript at $u_1 = q_1$, $x_1 = p_1 - q_1$. Substitution of this value of x_1 into (10.2) gives the quantity $p_1/(3q_1 - p_1)$. It can be shown that

$$\frac{p_1}{3q_1 - p_1} \leq \lambda^{i+1} \ ,$$

and hence it follows that $j \leq i$ at the point $x_1 = p_1 - q_1$, $u_1 = q_1$.

The superscript j in BI^j is determined by the ratio p_1/q_1; and, in view of (10.4), this makes $j = j_0$. In AII^j the superscript is determined by

$$\frac{p}{q} = \frac{p_1}{q_1 - x_1} \ .$$

Thus j is an increasing step function of x_1 alone, having value j_0 at $x_1 = 0$, and $n - 2$ at $x_1 = q_1$. The remarks made under case 1 concerning AV^j are applicable here, too. It is not difficult to see that the lines λ^j split up the square $0 \leq x_1 \leq q_1$, $0 \leq u_1 \leq q_1$ as indicated in Fig. 7, which summarizes the foregoing discussion.

Suppose now that $p_1/q_1 \leq 2$. Most of the remarks concerning the superscript j in BII^j in the case $p_1/q_1 \geq 2$ are also valid here. Now, however, u_1 cannot exceed q_1 when $x_1 \geq q_1$, and so there is no need to discuss the point $u_1 = q_1$, $x_1 = p_1 - q_1$. If j_1 denotes the maximum value of the superscript j, it no longer need be true that $j_1 = n - 2$. However, the relation

$$j_1 \geq i$$

does hold. For, the maximum value of p_1/q_1 is $p_1/(2q_1 - p_1)$, and so the assertion $j_1 \geq i$ is equivalent to

$$\frac{p_1}{2q_1 - p_1} \geq \lambda^i \ .$$

This relation, however, is easily established.

As before, the superscript in BI^j takes on the value j_0. In AII^j, it is readily seen that the superscript j is equal to j_0 at $x_1 = 0$ and increases to the maximum value of j_1. In AV^j, the remarks made in the discussion of $p_1/q_1 \geq 2$ still hold,

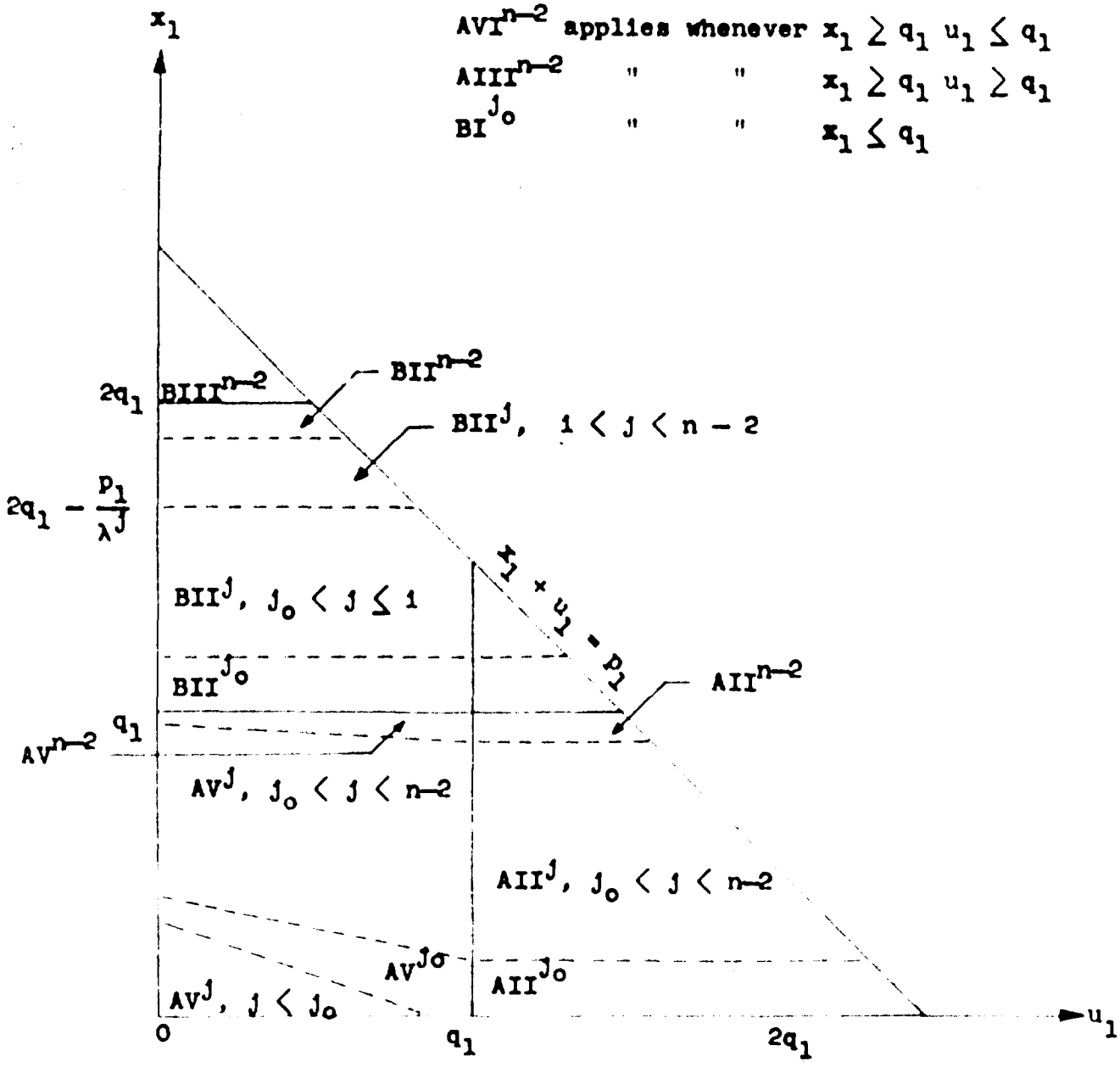
x_1
AVI^{n-2} applies whenever $x_1 \geq q_1$ $u_1 \leq q_1$
$AIII^{n-2}$ " " $x_1 \geq q_1$ $u_1 \geq q_1$
BI^{j_o} " " $x_1 \leq q_1$
$2q_1$
$BIII^{n-2}$
BII^{n-2}
BII^j, $1 < j < n-2$
$2q_1 - \frac{p_1}{\lambda^j}$
BII^j, $j_o < j \leq 1$
$x_1 + u_1 = 2q_1$
BII^{j_o}
AII^{n-2}
AV^{n-2}
q_1
AV^j, $j_o < j < n-2$
AII^j, $j_o < j < n-2$
AV^{j_o}
AII^{j_o}
AV^j, $j < j_o$
0
q_1
$2q_1$
u_1

Fig. 7

except that for $x_1 \geq p_1 - q_1$ the regions are truncated by the line $x_1 + u_1 = p_1$. Furthermore, the smallest superscript involved in a truncated region is clearly j_1, and so is greater than 1. This information is summarized in Fig. 8, below.

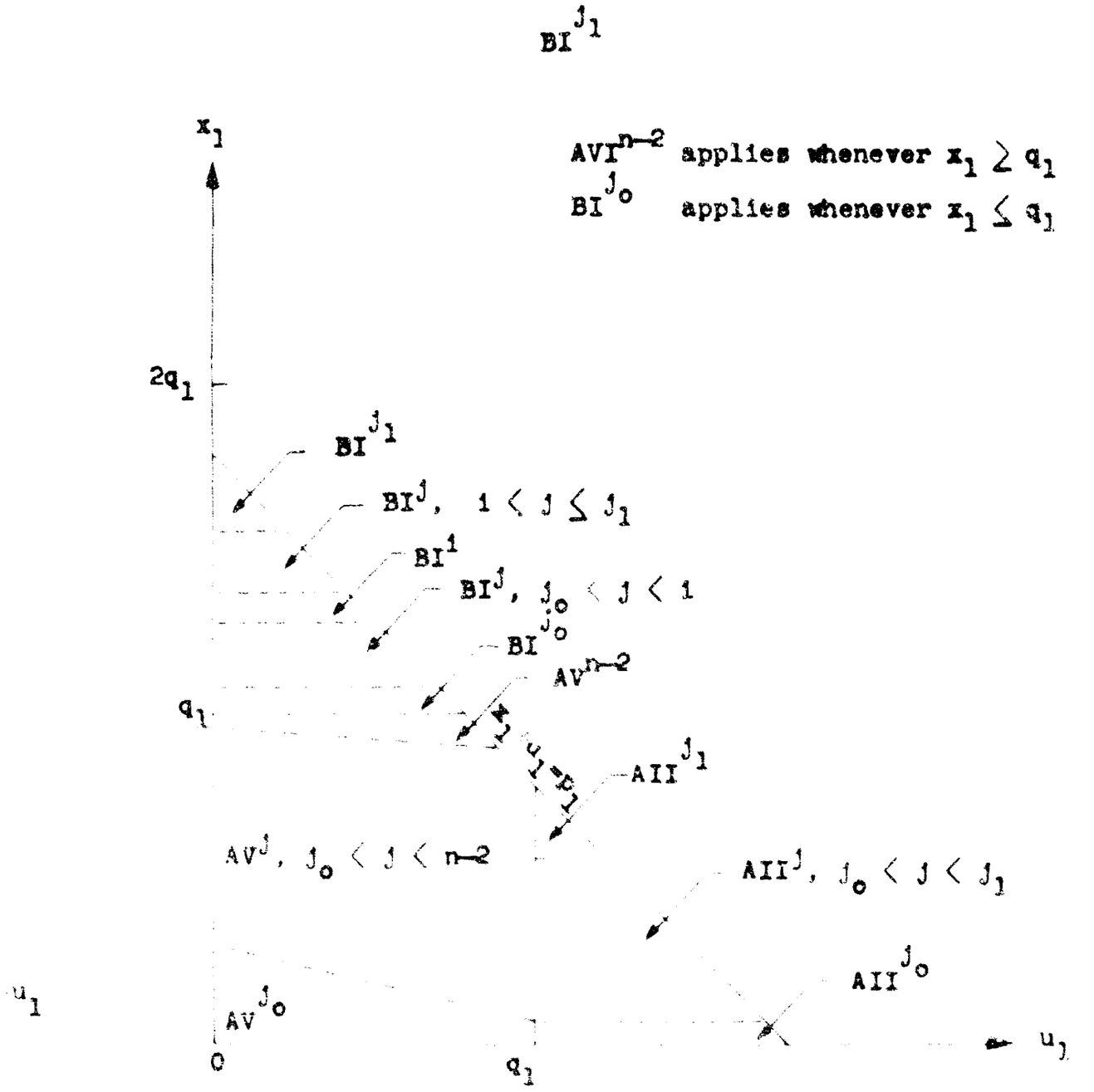

Fig. 8

Regardless of whether $p_1 \geq 2q_1$, or $p_1 \leq 2q_1$, we may write

$$\mathcal{M}(X_1) = H(p_1, q_1) + Tx_1 + Uu_1,$$

where H, T, and U are step functions having values that depend on the choice of X_1, but do not involve the variables x_1 or u_1. The functional values of H, T, and U do involve the constraints a^i, b^i, a^j, b^j, and those of H involve p_1 and q_1 in addition. The superscripts j, of course, are determined by X_1. The values of T and U are shown in Table 7, and the regions of constancy are indicated. Clearly, the regions of constancy of F coincide with those of T and U.

TABLE 7

DETERMINATION OF VALUES OF THE COEFFICIENTS T AND U

Region of (x,u) Plane	Region Number	T	U	Remarks
$2q_1 \leq x_1$ $u_1 \leq q_1$	(1)	-1	$\frac{nb^i}{b^i+n} - 1$	Only applies if $p_1 \geq 2q_1$
$q_1 \leq x_1 \leq 2q_1$ $q_1 \leq u_1$	(2)	$\frac{nb^j}{b^i+n} - 1$	-1	Only applies if $p_1 \geq 2q_1$; exponent j varies as superscript in BII^j
$q_1 \leq x_1 \leq 2q_1$ $u_1 \leq q_1$	(3)	$\frac{nb^j}{b^i+n} - 1$	$\frac{nb^i}{b^i+n} - 1$	Exponent j varies as superscript in BII^j
$0 \leq x_1 \leq q_1$ $q_1 \leq u_1$	(4)	$\frac{b^i b^j}{b^i+n} - 1$	-1	Exponent j varies as superscript in AII^j
$0 \leq x_1 \leq q_1$ $u_1 \leq q_1$	(5)	$\frac{b^i b^j}{b^i+n} - 1$	$\frac{b^i a^j}{b^i+n} - 1$	Exponent j varies as superscript in AV^j

Suppose now that $p_1 \geq 2q_1$. Since $nb^1 > n + b^1$, the maximum of $m(X_1)$ over region (1) of Table 7 occurs at $x_1 = 2q_1$, $u_1 = p_1 - 2q_1$. Then the following four facts, (a) $nb^1 > b^j + n$, (b) the b^j's are decreasing in j, (c) the point $x_1 = p_1$, $u_1 = q_1$ lies in a set for which the superscript j in BII^j does not exceed i, and (d) the continuity of $m(X_1)$, have the following implications: (a) the maximum of $m(X_1)$ over region (2) and that part of region (3) lying below $x_1 = p_1 - q_1$ is attained at $x_1 = p_1 - q_1$, $u_1 = q_1$; (b) the maximum of $m(X_1)$ over that part of region (2) lying above $x_1 = p_1 - q_1$ is attained at all points of the line $x_1 + u_1 = p_1$ that lie in the strip for which the superscript in BII^j assumes the value i. Denote this set of points by $\mathcal{E}$. Again appealing to the continuity of $m(X_1)$, we see that the maximum of $m(X_1)$ over all admissible X_1 for which $x_1 \geq q_1$ is achieved on $\mathcal{E}$.

It is now asserted that X_1^* lies in $\mathcal{E}$. In view of (10.3), this is equivalent to showing that we have

$$2q_1 - \frac{p_1}{\lambda_1^i} \leq \bar{x}_1^i \leq 2q_1 - \frac{p_1}{\lambda_1^{i+1}}, \quad i = 1, \ldots, n-2.$$

The right-hand inequality follows immediately from (7.4) and the initial conditions. The left-hand inequality follows from the definition (7.1) of $\bar{x}_1^i$, (6.15), and the initial conditions.

Thus, it has been proved that

$$(10.6) \qquad m(X_1^*) \leq m(X_1)$$

for all X_1 such that $x_1 \geq q_1$. Since $m(X_1^*) = V_1(p_1, q_1)$, to complete the proof of (4.3) in the case $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$,

$p_1/q_1 \geq 2$, it is sufficient to show that (10.6) holds for all X_1 such that $x_1 \leq q_1$. From the form of S in region (5) of Table 7, it is clear that $\mathcal{M}(X_1)$ attains its maximum along the line $u_1 = q_1$ whenever $x_1 \leq q_1$. From the form of R in this region, it is clear that if $b^i b^{n-2} \geq b^i + n$, then the maximum of $\mathcal{M}(X_1)$ is attained at $(x_1, u_1) = (q_1, q_1)$. Hence, (10.6) follows for all X_1 in this event. On the other hand, if $b^i b^{n-2} < b^i + n$, then the maximum of $\mathcal{M}(X_1)$ will be attained at one of the points

$$x_1 = q_1 - \frac{p_1}{\lambda^j} \geq 0. \quad \text{or} \quad x_1 = 0$$

of the line $u_1 = q_1$. In this event, it can be shown by lengthy computation that (10.6) holds for such X_1. Thus (4.3) is established for $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $p_1 \geq 2q_1$. By similar methods, which will not be carried out here, (4.3) can be established for $p_1 \leq 2q_1$. Thus the validity (4.3), and hence that of the theorem, is established.

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