LINEAR PROGRAMMING AND STRUCTURAL DESIGN
I. Limit Analysis

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SUMMARY

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"Limit Analysis" provides the structural engineer with a realistic estimate of the load carrying capacities of structures that are made of ductile materials. From the mathematical point of view, the problem of limit analysis is one of linear programming. The basic concepts are presented and a practical method of solution is discussed. This is essentially the "simplex method with prices," but the various steps are given mechanical interpretations which enable the experienced analyst to capitalize on his intuitive understanding of structural behavior.

LINEAR PROGRAMMING AND STRUCTURAL DESIGN

I. Limit Analysis*

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The title of this lecture is bound to attract a heterogeneous audience, consisting of analysts working in linear programming and engineers interested in structural design, each group having little background in the other's field. Accordingly, the speaker's first task is to establish a common ground from which the expedition may start without risking to lose half of its members at the outset. In the present case, it seems appropriate to choose this ground in the field of structures.

Figure 1 shows a simply supported, horizontal beam with loads P and Q that act in the vertical plane through the axis of the beam. The "fixed support" at A prevents any horizontal or vertical displacement but allows rotation of this end; the "movable support" at B prevents any vertical displacement but allows rotation and horizontal displacement. In the figure, the beam is represented by a line without width or depth; in reality, the beam has finite cross-sectional dimensions, which are, however, supposed to be small compared to its length. Strictly speaking, the line AB in Fig. 1 is the axis of the beam, which is supposed to contain the centroids of all cross sections.

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Consider a generic cross section of the beam, for instance the section I in Fig. 1. The stresses transmitted across this section by the left-hand part of the beam onto the right-hand part are statically equivalent ("equipollent") to a horizontal and a vertical force through the centroid of the cross section I and a couple acting in the vertical plane through the axis of the beam. These "stress resultants" are respectively named "axial force," "shear force," and "bending moment." The strength of a beam can be judged from the bending moments; axial and shear forces play secondary roles in this respect. (A structural element for which this statement does not hold true would not be called a beam.)

The beam in Fig. 1 is "statically determinate," i.e., the bending moment at any cross section can be determined from equilibrium considerations alone. For example, to find the bending moment at the section I, one may first determine the horizontal reaction at A and the vertical reactions at A and B from the three equations establishing the equilibrium between these unknown reactions and the given loads. The bending moment at I is then found by considering only the loads and reactions acting to the left of the section I and summing their moments with respect to its centroid. This static way of computing the bending moment makes it obvious that, for any loadfree segment of the beam, the bending moment varies linearly with the distance measured along the beam. Accordingly, the bending moment cannot assume an extreme value at an interior point of a loadfree segment.

For our purposes, the following kinematic way of finding the required bending moment is of interest. Inserting a hinge into the beam at I and treating as rigid the segments into which this hinge divides the beam, we consider an infinitesimal displacement of the system (indicated by dotted lines in Fig. 2). According to the principle of virtual work, equilibrium requires that the work of the given loads on the displacements of their points of application equal the work of the unknown bending moment on the relative rotation of the two segments joined by the hinge. This kinematic procedure has the advantage of furnishing directly the desired bending moment without requiring the preliminary determination of the reactions at A.

The "design" of our statically determinate beam is straightforward. According to the remark made above, the bending moment assumes its maximum value at one of the points of application of the loads P and Q. We therefore determine the bending moments at these two points and choose the cross section of the beam so that it can support the greater one of these bending moments with the desired margin of safety.

We next consider the "statically indeterminate" beam obtained from that in Fig. 2 by "building in" the end A, i.e., by preventing any rotation there as well as any horizontal or vertical displacement (Fig. 3). This implies that the reactions at A comprise a "clamping couple" in addition to horizontal and vertical forces. Since the vertical reaction at B must be added to these three reactions at A, the three equations of equilibrium do no longer suffice for the determination of the four reactions. The best that purely statical considerations allow us to do, is to express the bending moment at a generic cross section, say I, in terms of the loads and the unknown clamping couple at A. This expression can be obtained kinematically by inserting hinges at A and I and using the principle of virtual work.

To resolve this indeterminacy, the deflections of the beam (shown out of scale in Fig. 3) must be considered, and the condition must be used that the deformed center line of the beam has a horizontal tangent at the clamped end A. The usual "elastic" analysis of beam deflections is based on the assumption that the curvature k of the deformed center line at a generic cross section is proportional to the bending moment M there. For a typical structural beam of mild steel, however, the diagram of bending moment versus curvature has more nearly the shape indicated in Fig. 4a, the essential feature being the existence of a critical value M_0 of $|M|$; when this value is reached, a large increase in $|k|$ requires but an insignificant increase in $|M|$.

An analysis based on the diagram in Fig. 4a would be extremely cumbersome; we therefore idealize this diagram as shown in Fig. 4b. Described in terms of the behavior of a beam element, this diagram stipulates rigidity for

$|M| < M_0$ and unrestricted bending for $|M| = M_0$, while it rules out bending moments that exceed M_0 in absolute value. As a rule, the critical absolute value of the bending moment is only reached at discrete cross sections. A typical deformation of the beam is therefore due to localized bending at these sections, which are called "plastic hinges," and the beam segments between adjacent plastic hinges remain straight.

To illustrate the simplification of analysis that is made possible by this idealization of the actual beam behavior, let us consider the beam shown in Fig. 5a. The bending moment at the support B vanishes, and the bending moments M_A, M_C, and M_D at the sections A, C, and D satisfy

$$-2\,M_A + 3\,M_C \qquad = 4\,Pa, \qquad (1)$$

$$2\,M_A - 7\,M_C + 8\,M_D = 0. \qquad (2)$$

Equation (1) is obtained by inserting hinges into the beam at A and C and applying the principle of virtual work to the infinitesimal deformation indicated in Fig. 5b. Equation (2) is obtained in a similar manner, except that three hinges are now used, the displacements of C and D being chosen so that the combined work of the loads vanishes. Equations (1) and (2) are equations of equilibrium for the bending moments at A, C, and D. There cannot be any equation of equilibrium for these moments that is linearly independent of (1) and (2), because three linearly independent equations of equilibrium would furnish these bending moments in contradiction to the fact that the beam is statically indeterminate.

Assuming the beam to have a uniform cross section of the given "fully plastic moment" M_0, we now propose to determine its "load carrying capacity" for the considered type of loading. In mathematical terms, we propose to determine M_A, M_C, and M_D so as to maximize P as given by (1) subject to the condition (2) and the inequalities

$$-M_0 \leq M_A \leq M_0, \tag{3}$$

$$-M_0 \leq M_C \leq M_0, \tag{4}$$

$$-M_0 \leq M_D \leq M_0. \tag{5}$$

This is a problem in linear programming. To discuss its solution geometrically, we use M_A/M_0 and M_C/M_0 as rectangular Cartesian coordinates in a "stress plane" (Fig. 6). As (2) specifies M_D in terms of M_A and M_C, a generic point of the stress plane ("stress point") represents a set of values M_A, M_C, and M_D, and hence a bending moment distribution in our beam. For a fixed value of Pa/M_0, Equation (1) specifies a straight line (e.g., the dashed line in Fig. 6) containing all stress points that are compatible with this value of Pa/M_0. As P increases, this line undergoes a translation towards the left.

Each of the inequalities (3) and (4) restricts the stress point to a strip that is centered at the origin, and the same remark applies to the inequality obtained by substituting M_D from (2) into (5). The hexagon common to these three strips is shown in Fig. 6. The vertex V of this hexagon obviously corresponds to the largest possible value of P. Since this vertex is the intersection of the sides corresponding to

$$M_A = -M_O \tag{6}$$

and

$$M_D = M_O, \tag{7}$$

the load carrying capacity of the beam is reached when plastic hinges form at A and D. To compute the load carrying capacity, we substitute (6) and (7) into (1) and (2), and eliminate M_C. Thus,

$$\max P = 8\, M_O/(7a). \tag{8}$$

The preceding geometrical solution of our problem relied heavily on the fact that the bending moment distribution could be specified by two independent bending moments (M_A and M_B). Whenever this is the case, a two-dimensional "stress space" can be used. The stress point is then restricted to the region common to several strips which all contain the origin and therefore have a convex polygon as their "intersection." For the maximum load intensity, the stress point falls into a vertex,* i.e., lies on two sides of the polygon. Accordingly,

*In the exceptional case where the line corresponding to max P coincides with a side of the polygon, we may still take the stress point at one of the vertices on this side.

two plastic hinges form, when the load carrying capacity is reached. This qualitative discussion is readily generalized to n independent bending moments. An n-dimensional stress space is then appropriate. The stress point is restricted to a convex polyhedron; for the maximum load intensity, it falls into a vector which, in general, lies on n faces of this polyhedron, but may exceptionally be the intersection of more than n faces. Thus, n plastic hinges form when the load carrying capacity is reached, but more plastic hinges may form in exceptional circumstances.

For $n = 2$ the relevant vertex can be found by drawing the polygon and lines of constant P; for $n > 2$, however, this geometric procedure must be replaced by an algebraic one, for instance the "simplex method" of linear programming. Interpreted in the geometric terms appropriate to Fig. 6, this purely algebraic method starts at the origin O and proceeds from there in a sequence of steps to the vertex V, each step resulting in an increase of P. The first step follows a coordinate axis and leads to a point on a side of the polygon, say T. The next step follows this side in the direction of increasing values of P to the vertex U, etc.

It will be instructive to show how this geometrical program is carried out algebraically, even though no such formal procedure is needed to solve our simple problem. The origin O corresponds to $M_A = M_C = 0$, and hence $M_D = 0$ by (2). These values of the three bending moments satisfy the inequalities (3) through (5) and furnish $P = 0$ by (1). Since we wish to maximize P, the signs in (1) indicate that we should increase M_C and decrease M_A Let us first increase M_C, keeping $M_A = 0$. Equation (2) shows that $M_D = 7\ M_0/8$ when M_C reaches its upper bound M_0. Note that this stage with $M_A = 0$, $M_C = M_0$ corresponds to the point T in Fig. 6. We next decrease M_A, keeping $M_C = M_0$. Equation (2) shows that M_D reaches its upper bound M_0 when $M_A = -M_0/2$ (point U in Fig. 6). At this stage M_C and M_D being at their upper bounds cannot be increased any further. To see what effect a decrease of these moments has on P, we eliminate M_A from (1) and (2), obtaining

$$-4\ M_C + 8\ M_D = 4\ Pa. \qquad (9)$$

The signs in (9) show that an increase in P requires a decrease in M_C, which is admissible, or an increase in M_D, which is not possible as M_D is already at its upper bound. Equation (2) shows that M_A reaches its lower bound $-M_0$ when $M_C = 6\ M_0/7$ (point V in Fig. 6). In the next step, M_A being at its lower bound could only be increased while M_D could only be decreased. To ascertain the influence of these changes on P, we eliminate M_C from (1) and (2), obtaining

$$-2\ M_A + 6\ M_D = 7\ Pa. \qquad (10)$$

The signs in (10) indicate that any admissible changes of M_A and M_D would decrease P. We have therefore reached the maximum of P, which is readily computed by substituting

$M_A = -M_0$ and $M_D = M_0$ into (10); the result agrees with (8).

The preceding analysis may be simplified by the following mechanical argument. When the point U is reached, M_C and M_D are at their upper bounds M_0, suggesting plastic hinges at C and D. If hinges are inserted at these sections and the system is given an infinitesimal displacement in which the loads do positive work (Fig. 7a), the hinge rotations at C and D correspond to negative and positive bending moments, respectively. While the hinge rotation at D agrees with the sign of $M_D = M_0$, that at C is in contradiction with the sign of $M_C = M_0$. This indicates that M_C should be decreased, while M_D is maintained at the value M_0. This means that the plastic hinge at D should be maintained, but that at C should be replaced by one at some other section, which in the present example can only be A. Figure 7b indicates the corresponding displacements, which have again been chosen so as to make the work of the loads positive. The signs of the hinge rotations in Fig. 7b suggest $M_A = -M_0$ and $M_D = M_0$. Since Equation (2) then furnishes $M_C = 6\,M_0/7$, we have found bending moments that satisfy the equation of equilibrium (2) do not exceed the bounds set by (3) through (5), and cannot be changed within these limitations so as to increase P. Figure 7b obviously shows the type of deformation that occurs when the load carrying capacity of the beam is reached; this is often referred to as the "collapse mechanism" of the beam.

While we have a unique collapse mechanism in the present example, there may be more than one collapse mechanism in special cases. If, for instance, the line for max P in Fig. 6 had coincided with the side UV of the polygon, the collapse mechanisms represented by the vertices U and V would have been possible as well as any linear positive combination of these collapse mechanisms. Note, however, the load carrying capacity has a unique value even if there is a multiplicity of collapse mechanisms.

The mechanical interpretation of the simplex method suggests an alternative procedure. Assuming a collapse mechanism, we give to the bending moments at the plastic hinges the absolute value M_0 and the signs indicated by the hinge rotations and apply the principle of virtual work. For the mechanism in Fig. 5b, for instance, we choose $M_A = -M_0$, $M_C = M_0$ and obtain

$$P = 5\,M_0/(4\,a). \tag{11}$$

From the chosen values of M_A and M_C, we now determine M_D by means of the equation of equilibrium (2); thus

$$M_D = 9\,M_0/8. \tag{12}$$

If this value of M_D had been within the bounds for this bending moment, the assumed collapse mechanism would have been correct and (11) would give the load carrying capacity of the beam. As the right-hand side of (12) exceeds the bounds for M_D, the load specified by (11) exceeds the load carrying capacity of the beam, and a plastic hinge must be introduced at D. One way of doing this is to superimpose a suitable multiple of the deflections in Fig. 5c on those in Fig. 5b. Letting the multiplier λ applied to the deflections of Fig. 5c grow gradually starting from zero, we observe that the hinge at C will close up first, at $\lambda = 3/7$, whereas the hinge at A would close up only at $\lambda = 1$. Using $\lambda = 3/7$, we obtain the collapse mechanism indicated in Fig. 7c, and this yields the load carrying capacity as before.

With the upper bound (11) there is easily associated a lower bound for the load carrying capacity. Indeed, multiplying the load (11) and the corresponding bending moments by the common factor 8/9, we obtain a reduced load and reduced bending moments that satisfy (1) through (5). The reduced load

$$P = (8/9)\,\frac{5\,M_0}{4a} = \frac{10\,M_0}{9a} \qquad (13)$$

therefore cannot exceed the load carrying capacity of the beam. This way of bounding the capacity from above and below is particularly useful because it allows the analyst to stop the procedure when the bounds are sufficiently close for practical purposes.

Note that the preceding method mixes statical and kinematical arguments. It is possible, however, to follow purely kinematical lines because any load intensity obtained by such steps as led to (11) is necessarily an upper bound for the load carrying capacity. Indeed, if all bending moments are within their bounds, the assumed mechanism is a collapse mechanism and yields the load carrying capacity; if, on the other hand, some of the moments exceed their bounds, the mechanism furnishes a load intensity in excess of the capacity of the beam. Thus, the load carrying capacity is the smallest load intensity that can be obtained by assuming a collapse mechanism (i.e., two plastic hinges) and applying the principle of virtual work. This kinematical minimum characterization of the load carrying capacity is the "dual" to the static maximum characterization developed above.

The example used throughout this lecture had to be extremely simple so that the basic ideas would not be obscured by a mass of detail. On account of its very simplicity, this example necessarily fails to bring out the power of the method discussed today. At the beginning of the next lecture, we shall consider a somewhat more complex example, as both a more realistic illustration and a recapitulation of this method. A brief survey of the development of the concepts and methods of limit analysis and design will be given at the end of the second lecture.*

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*Linear Programming and Structural Design, II. Limit Design, The RAND Corporation, Paper P-1123, July 11, 1957.

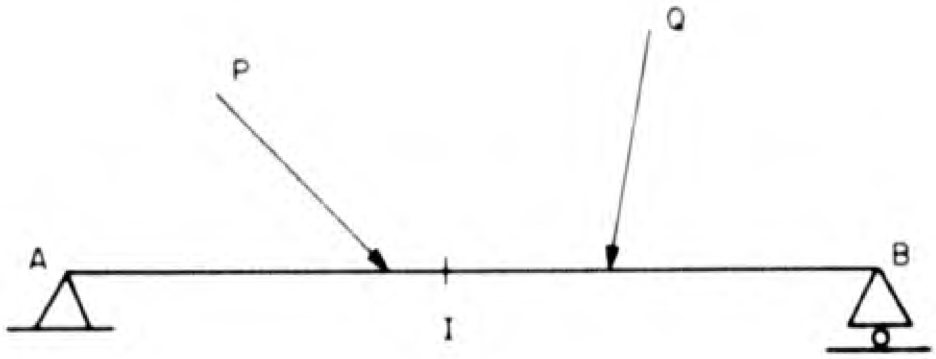

Fig. 1

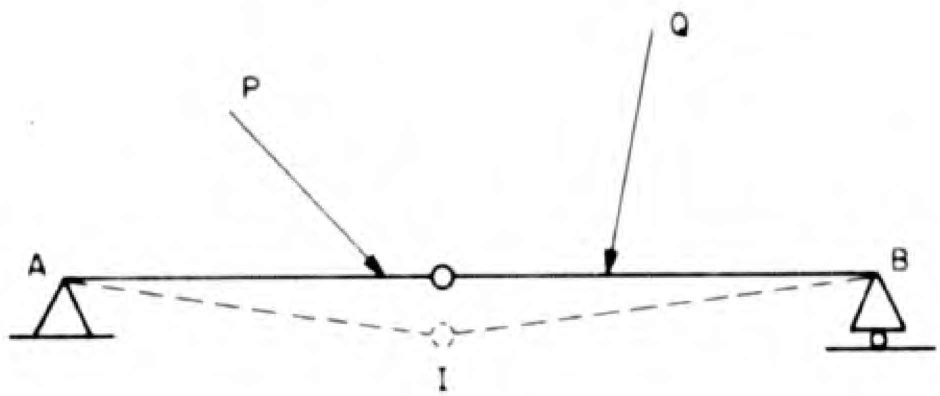

Fig. 2

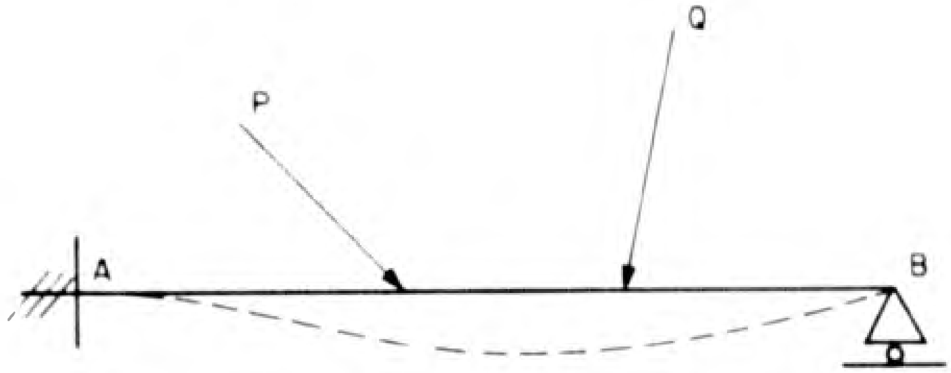

Fig. 3

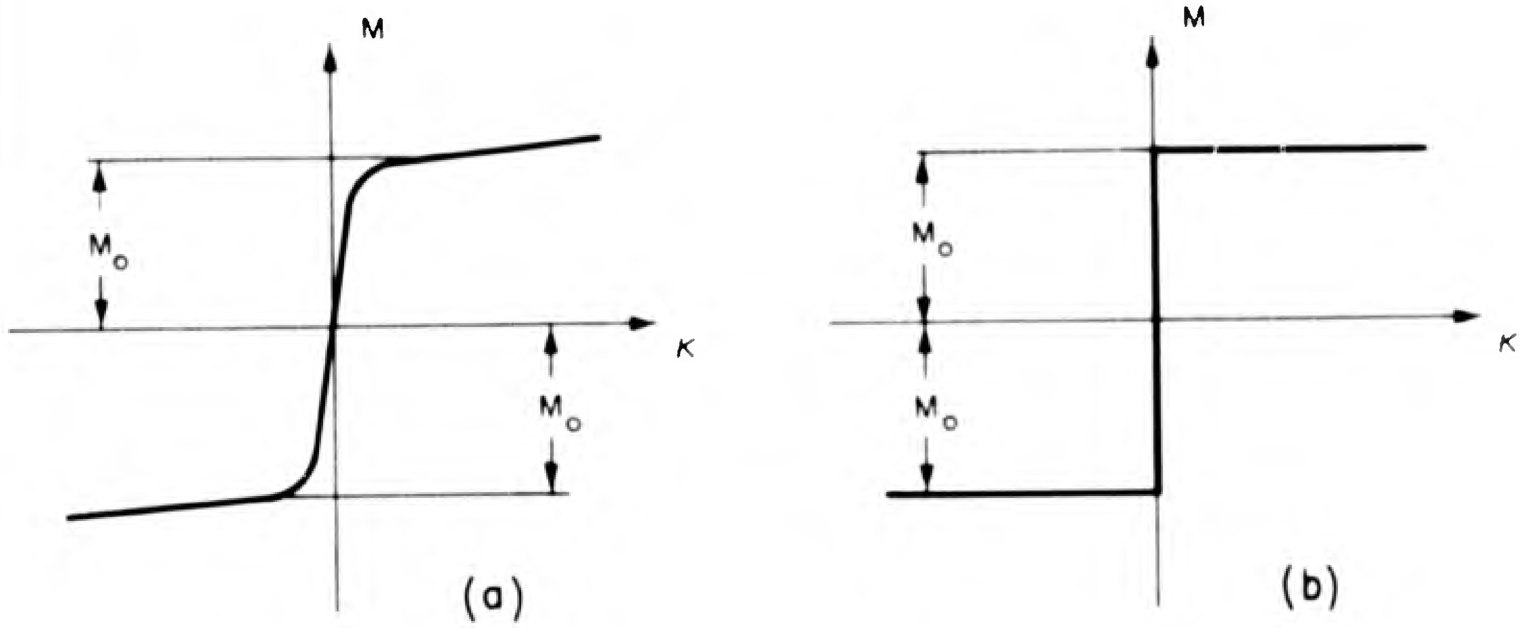
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Fig. 4

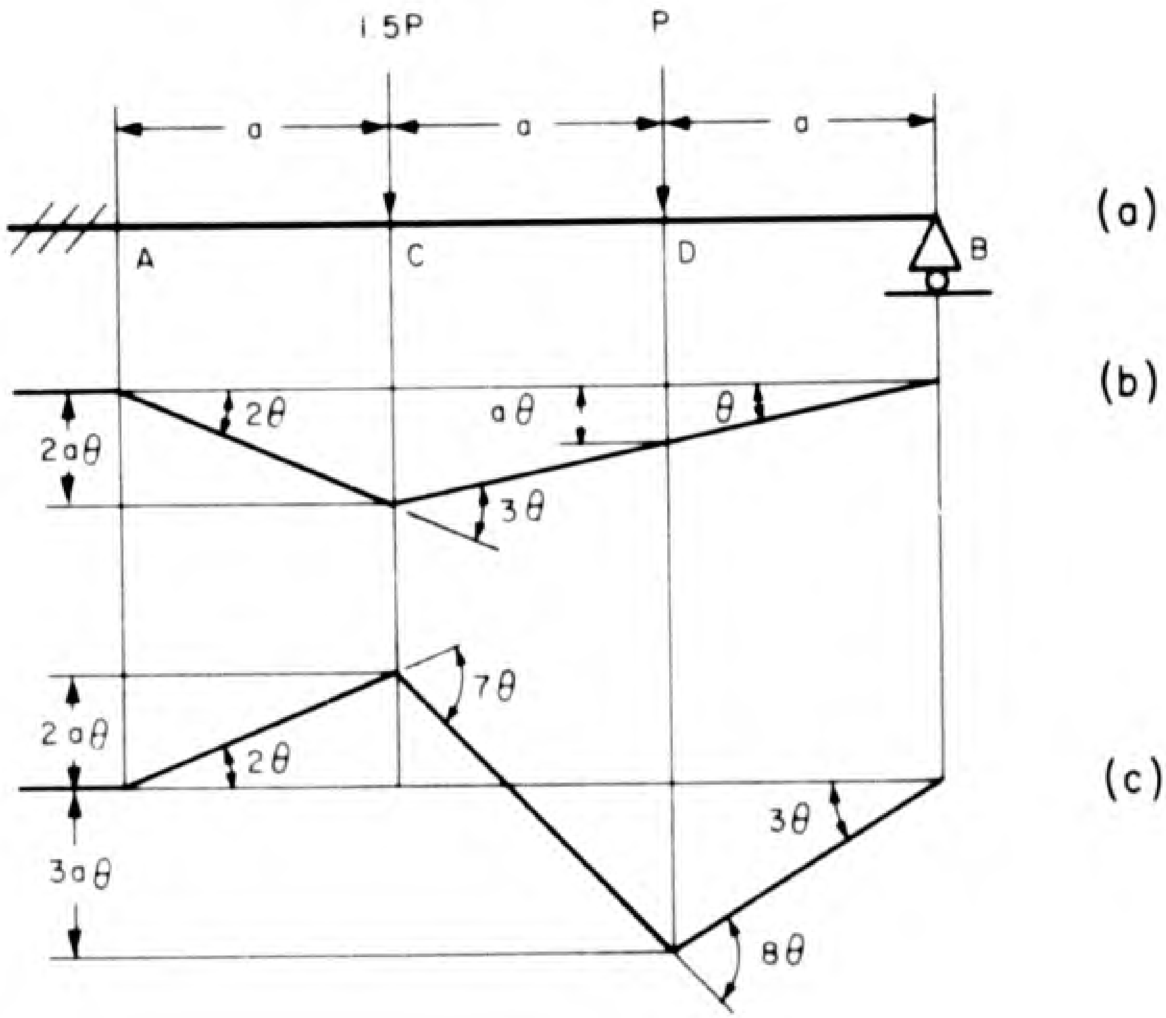
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Fig. 5

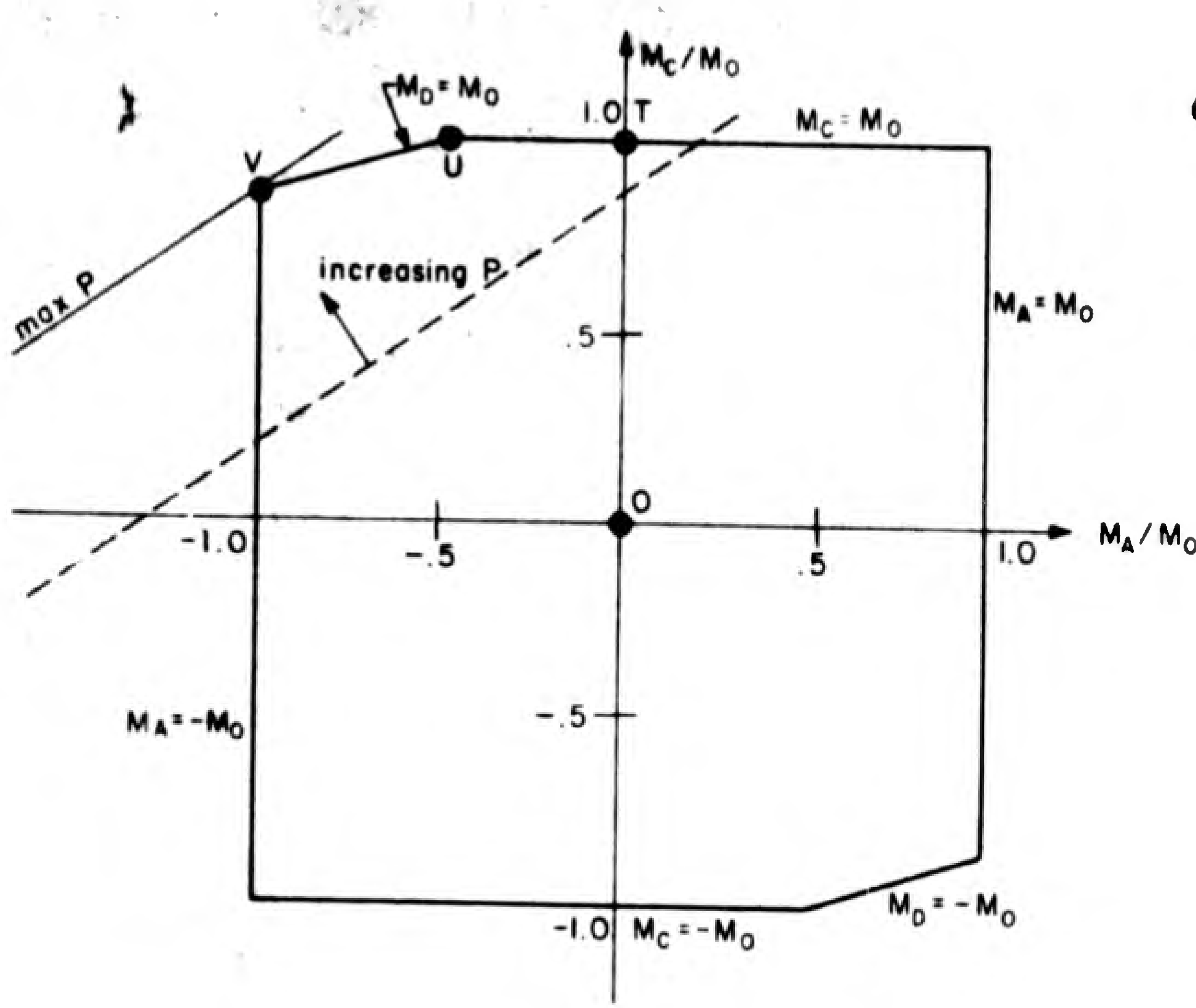
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Fig. 6

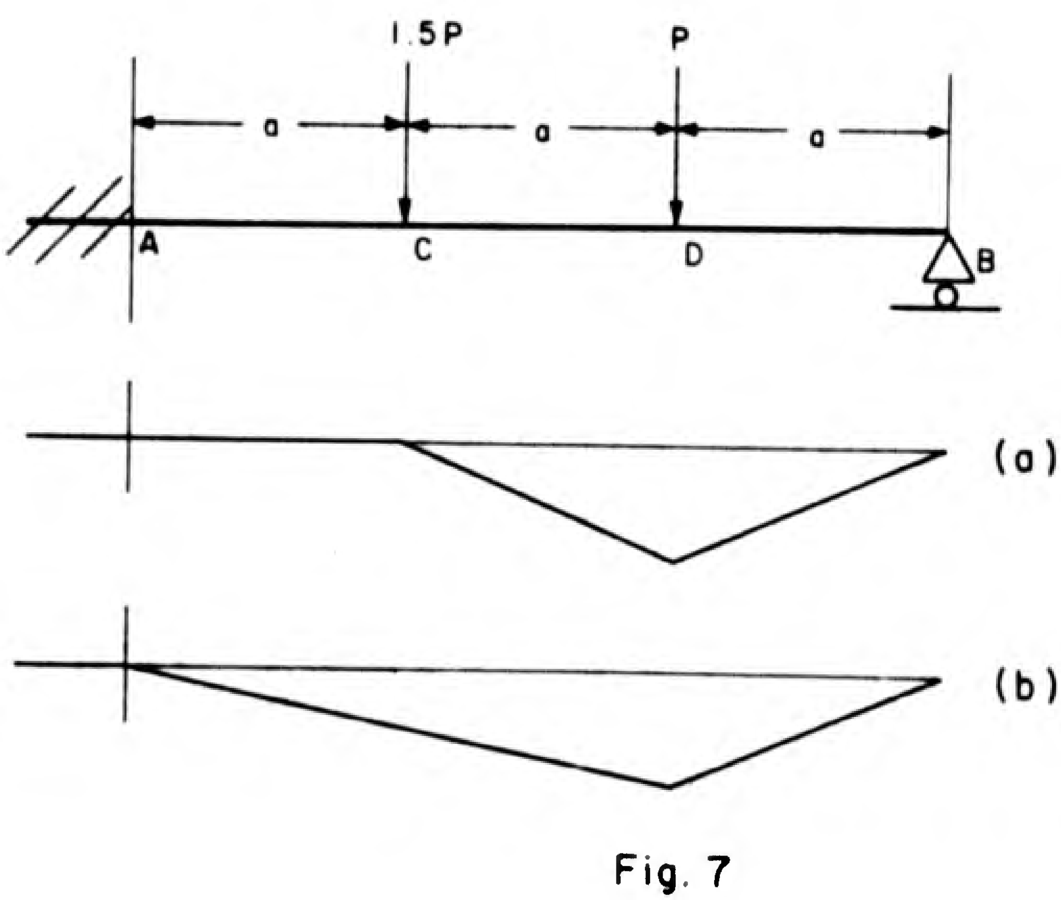
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Fig. 7

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