TOPOLOGICAL DYNAMICS
A BOOK REVIEW

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TOPOLOGICAL DYNAMICS. By Walter Helbig Gottschalk and Gustav Arnold Hedlund. American Mathematical Society Colloquium Publications, Vol. XXXVI, Providence, R.I., 1955.

If we take a second order differential equation, such as the famed equation of Van der Pol

$$\frac{d^2x}{dt^2} + \lambda(x^2 - 1)\frac{dx}{dt} + x = 0,$$

and write $dx/dt = y$, the second order equation is converted into a system of first order equations,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\lambda(x^2 - 1)y - x.$$

As t varies, the pairs of values $x(t)$ and $y(t)$ describe a curve in the (x, y)-plane, the "phase" plane.

The study of the solutions of second order differential equations thus leads in a very natural way to the study of curves in the plane. It may be expected, then, that a systematic study of curve families in the plane will lead, under suitable specialization, to results valid for large classes of differential equations. This idea of Poincaré represents one of the great advances in the study of differential equations. Essentially, we determine the properties of an individual solution by studying the set of all solutions.

It owes its success to the fact that the uniqueness theorem for differential equations tells us that we can limit our study to non-self-intersecting curves, a great simplification. Unfortunately, in three or more dimensions, curves can be non-self-intersecting and still be of quite complicated form. The result is that there is little known about three and higher dimensional systems of differential equations, compared to the great deal that can be said about two-dimensional systems.

Considering x(0) and y(0) as parameters, the values of x(t) and y(t) represent transformations of these values, transformations induced by the differential equation. It is again a natural step, not to restrict oneself to transformations derived from differential systems, but to study general continuous transformations and, eventually, general transformation

This study was initiated by Poincaré and extensively pursued by G. D. Birkhoff, culminating in the ergodic theorems of Birkhoff, von Neumann, and Carleman. The formulation of the problems of dynamic systems in the abstract terminology of transformations was a contribution of B. O. Koopman.

The present book is a very abstract presentation of a very abstract theory, with little concession to human fallibility. For a reader with the required background of topology and analysis, it is a valuable summary of the present state of the topology of dynamical systems.

One of the interesting applications of the theory is a study of unending chess, taking into account the restriction concerning recurrent positions; see M. Morse and G. Hedlund, Symbolic Dynamics, _Amer_. _Jour_. _Math_., Vol. 60 (1938), p. 815-866.

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