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KRONECKER PRODUCTS AND THE SECOND METHOD OF LYAPUNOV

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SUMMARY

The second method used by Lyapunov to study the stability of the trivial solution of $dx/dt = Ax + f(x)$ leads to the problem of solving the matrix equation $AX + XA' = C$. It is shown that this question is related to Kronecker products and Kronecker sums, and the general problem of solving $AX + XB = C$ is resolved in this fashion.

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§1. Introduction

In the second method used by Lyapunov in the investigation of the stability of the trivial solution of the nonlinear vector-matrix system

(1) $$\frac{dx}{dt} = Ax + f(x),$$

the question arises of solving the matrix equation

(2) $$AX + XA' = C,$$

for the unknown symmetric matrix X. This problem was treated by W. Hahn in a recent publication, [2], using the reduction of a matrix to triangular form.

The same problem was encountered by the author, [1], in connection with the evaluation of the integral $J = \int_0^\infty (y, By)dt$, where y is the solution of $dy/dt = Ay$, $y(0) = c$, and A is a stability matrix, i.e. all characteristic roots have negative real part. The question of existence and uniqueness of solution was resolved in a non-algebraic fashion by noting that the solution of (2) is given by

(3) $$X = -\int_0^\infty e^{At} C e^{A't} dt,$$

under the assumption that A is a stability matrix.

In this paper we shall present another approach to this problem using the concept of the Kronecker product of matrices. As we shall see, this method enables us to resolve the general problem of determining when the equation

$$(4)\qquad AX + XB = C$$

has a solution.

§2. The Kronecker Product

Let A and B be two nxn matrices, x and y characteristic vectors of A and B respectively, and λ and μ characteristic roots,

$$(1)\qquad Ax = \lambda x, \quad By = \mu y.$$

Let z be the n^2-dimensional vector formed as follows

$$(2)\qquad z = \begin{pmatrix} x_1y_1 \\ x_1y_2 \\ \vdots \\ x_1y_n \\ x_2y_1 \\ x_2y_2 \\ \vdots \\ x_2y_n \\ \vdots \\ x_ny_1 \\ x_ny_2 \\ \vdots \\ x_ny_n \end{pmatrix} = \begin{pmatrix} x_1y \\ x_2y \\ \vdots \\ x_ny \end{pmatrix}$$

Referring to (1), it is easy to see that z is a characteristic vector of the n^2-dimensional matrix

$$(3) \qquad A \times B = \begin{pmatrix} b_{11}A & b_{12}A & \cdots \\ b_{21}A & b_{22}A & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

This we recognize as the Kronecker product of the matrices A and B.* Its n^2 characteristic roots are the combinations $\lambda_i \mu_j$, with the characteristic vectors obtained as above.

§3. The Kronecker Sum

Let $A_1 = I + \epsilon A$, $B_1 = I + \epsilon B$. Then the Kronecker product is

$$(1) \qquad (I + \epsilon A) \times (I + \epsilon B) = I \times I + \epsilon \left[I \times B + A \times I \right] + O(\epsilon^2)$$

$$= \begin{pmatrix} (I + \epsilon b_{11})(I + \epsilon A) & \cdots \\ (I + \epsilon b_{21})(I + \epsilon A) & \cdots \\ \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} I + \epsilon(b_{11} I + A) & \cdots \\ I + \epsilon(b_{21} I + A) & \\ \vdots & \end{pmatrix}$$

The n^2-dimensional matrix

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* Kronecker products arise also in the study of the product of stochastic matrices. See R. Bellman, Limit Theorems for Non-Commutative Operations, Duke Math. Jour. Vol. 21 (1954), pp. 491-500.

$$(2) \qquad A \oplus B = \begin{pmatrix} A + b_{11}I & A + b_{12}I & \cdots & A + b_{1n}I \\ A + b_{21}I & \cdots & & \cdots \\ \vdots & & & \vdots \\ & \cdots & & \end{pmatrix}$$

we may call the Kronecker sum. Its characteristic roots are $\lambda_i + u_j$ and its characteristic vectors are as given in (2.2).

§4. An Alternate Derivation

This matrix may also be obtained by considering the two linear systems

$$(1) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{N} a_{ij} x_j, \qquad \frac{dy_i}{dt} = \sum_{j=1}^{N} b_{ij} x_j, \quad i = 1,2,\ldots,N,$$

and forming the n^2-dimensional linear system satisfied by the vector in (2.2). Since

$$(2) \qquad \frac{d}{dt}(x_i y_j) = \frac{dx_i}{dt} y_j + x_i \frac{dy_j}{dt},$$

we obtain the matrix in (3.2).

§5. $AX + XB = C$

Let A and B be given n-dimensional matrices, and consider the matrix equation

$$(1) \qquad AX + XB = C,$$

where X is the unknown matrix.

This system of equations for the unknown elements x_{ij} of X has the form

$$(2) \qquad (A \oplus B') \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \\ x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \\ \vdots \\ x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \\ c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \\ \vdots \\ c_{1n} \\ c_{2n} \\ \vdots \\ c_{nn} \end{pmatrix}$$

It follows that the equation in (1) possesses a unique solution for all C if and only if

$$(3) \qquad \lambda_i + \mu_j \neq 0, \quad i,j = 1,2,\ldots,n,$$

where the λ_i are the characteristic roots of A and the μ_j the characteristic roots of B.

§6. $AX + XA' = C$

It is easy to see that the same techniques apply to the equation in (1.2), so that a necessary and sufficient condition that this equation has a unique solution for all C is that $\lambda_i + \lambda_j \neq 0$, $i,j = 1,2,\ldots,n$. In particular, this condition is satisfied if A is a stability matrix.

REFERENCES

1. R. Bellman, Notes on Matrix Theory—X, A Problem in Control, Quarterly Applied Math., Vol XIV (1957), pp. 417-419.

2. W. Hahn, Eine Bemerkung Zur Zweiten Methode Von Ljapunov, Mathematische Nachrichten, 14. Band, Heft 4/6, 1956, pp. 349-354.