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IN-FLIGHT DYNAMICS OF A FLEXIBLE MISSILE

A GOVERNMENT RESEARCH REPORT

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by

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SUMMARY

This report is an extension of the work reported in a series of STL reports entitled "Generalized Missile Dynamics Analysis." For completeness, the pertinent assumptions and derivations given in the Generalized Dynamics series are repeated in the present report.

Equations are derived for the motion of a missile in a plane, including bending, sloshing, engine swiveling, axial acceleration, and aerodynamics. The bending and sloshing are treated by means of combined bending-sloshing modes in which the first slosh mode in each tank is represented by an equivalent lateral spring-mass system. The body of the missile is represented as a main beam to which are attached sub-branches, which represent, for example, a flexible payload inside of a nose fairing. The swiveling engine or engines are treated as an integral part of the missile, clamped at zero engine swivel angle, when computing the bending-sloshing modes. Quasi-steady aerodynamics based on wind tunnel data are used to determine aerodynamic forces due to angle of attack and due to pitching, and slender body theory is used to determine the aerodynamic forces due to plunging (rate of change of angle of attack).

Aerodynamic forces resulting from missile bending are demonstrated to be negligibly small and are omitted from the equations with the exception of the angle of attack due to a "slosh" mode displacement (the external airframe often rotates as nearly rigid body in modes which consist mainly of sloshing motion) and the apparent increase in the damping ratio of the "bending" modes due to aerodynamic forces. The contribution to the velocity coefficients of the bending-sloshing modes from damping in the individual tanks is also included in the equations.

The result of the analysis is a set of differential equations which are valid for intervals of time which are short enough that changes in missile mass, bending-sloshing frequencies, etc., are negligible. Also included are equations for the moment and shear in the missile expressed in terms of a psuedo-static moment and shear plus correction terms which account for the dynamic overshoot of the bending-sloshing modes.

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ACKNOWLEDGMENT

This report is the result of putting together what is hoped to be the best features of a series of previous reports, plus the incorporation of ideas resulting from thought provoking sessions among people with varying interests and experience. The people contributing most significantly to these reports and conversations were M. V. Barton, J. G. Berry, J. A. Brooks, J. R. Fowler, Y. C. Fung, G. Gleghorn, D. C. Martin, J. W. Miles, S. Silverberg, W. T. Thomson, N. Trembath, J. D. Wood, and Dana Young. For obvious reasons, only a few of these contributors were able to review this resulting report in any detail, and it is hoped that the report is satisfactory to all the the contributors and represents the desired form of the analysis.

I. INTRODUCTION

The general design of a missile is established on the basis of over-all performance, treating the airframe as a rigid body. Using the resulting preliminary design values, such as skin thicknesses, propellant configurations, materials, etc., more detailed analyses can be made which take into account the dynamic response of the elastic missile. The purpose of this report is to provide a set of equations which can be used to obtain: (1) general dynamic performance characteristics, such as missile loads and dynamic stability; (2) detailed responses, such as structural vibration, propellant sloshing, and engine motions; and (3) exchange ratios, or the effect of changes in a parameter on the dynamic behavior of the system, such as damping of the propellant sloshing verus autopilot stability and airframe bending moments.

This analysis is intended as a direct extension of the work reported in the series of reports entitled "Generalized Missile Dynamics Analysis" (Reference 1) in which these problems were first attacked. In the present analysis, the pertinent derivations and assumptions given in Reference 1 are repeated for the sake of completeness. Most of the philosophy, methods of approach and coordinate systems will be unchanged from those used in Reference 1, however, experience with the equations has led to several modifications and extensions. The new features to be included are:

1. The use of branched beam modes.
2. The inclusion of the effect of axial acceleration in the computation of bending modes.
3. The inclusion of sloshing directly in the computation of the bending modes rather than as separate coordinates.
4. Explicit formulation of the effect of the damping in the individual tanks on the bending-sloshing modes.
5. Explicit formulation of the interaction of the "slosh" modes with the aerodynamic forces.
6. The inclusion of damping due to aerodynamics in the bending-sloshing equations.
7. The use of aerodynamics in the form of wind-tunnel data.
8. Inclusion of linearized equations of motion for small perturbation of missile attitude θ.
9. The explicit use of the "mode acceleration" method in the computation of bending moments and shears.

Again, the general procedure will be to set up the expressions for the kinetic and potential energies of the complete system and then use Lagrange's equation to determine the equation of motion for each coordinate. The definitions and assumptions to be used are:

1. The lateral bending of the missile will be described in terms of normal modes computed by treating the missile as a collection of beams attached together e.g., a payload enclosed by a nose fairing would be treated as two parallel beams with a common tangent at their attach point. The "main beam" is defined as that part of the missile to which all of sub-branches are attached, and is assumed to be the only part of the missile exposed to aerodynamics. Thus, in the above example, the nose fairing would be part of the main beam continuing on down the length of the missile. If more than one beam is exposed, special care must be taken in evaluating the integrals involving aerodynamic pressures. The bending theory used is Timoshenko beam theory modified to take axial acceleration into account. The eigenfunctions and frequencies are calculated with the swiveling engine clamped relative to the rest of the missile in its unrotated position, and the liquids in the tanks are concentrated along the corresponding sections of the missile axis. The effective rotatory inertia of the liquids (i.e., inertia due to their lateral dimension) is accounted for on a rigid tank basis by adding enough rotatory inertia to the missile, uniformly along the length of the liquid in question, so that the rigid body mass moment of inertia of the resulting equivalent fluid is the same as the "capped-tank" inertia found for an ideal liquid in the tank. For the aspect ratios of typical tanks, this rotatory inertia term is usually negative. Sloshing is represented by attached spring-masses giving the proper frequency, horizontal force, and moment of the first slosh mode of each tank.

2. The analysis is restricted to a consideration of motion in a plane.

3. Variations of mass, bending-sloshing frequencies, mode shapes, quiescent level of propellant, air density, axial acceleration, and acceleration of gravity are negelected. The equations are therefore valid only for short sections of the over-all trajectory.

4. The origin of the moving coordinate system is taken as the mass center of the entire missile including the swiveling engine and the liquids in the fuel tanks.
5. Longitudinal deformations relative to the mass center are neglected.

2. COORDINATE SYSTEM

As shown in Figure 1, the magnitude of the velocity of the mass center of the missile is denoted by v_o, and the inclination of the trajectory tangent measured from the vertical is denoted by β. The term "mass center" in this analysis is used to denote the mass center of the complete missile including the swiveling engine and the liquid in the tanks.

The missile configuration is described relative to a set of moving axes, x and y, for which the origin is at the mass center. The angle from the vertical to the x-axis is denoted by θ. Transverse deflections of the axis of the i^{th} branch of the missile relative to the x-axis are denoted by $u_i(x, t)$. Longitudinal deformations relative to the moving axes are neglected.

The orientation θ of the coordinate system is defined by imposing the condition that the angular momentum of the missile (including engine swiveling, sloshing, and elastic bending) relative to the moving axes is zero. The reason for choosing this particular condition is that it simplifies the equation of motion for the angular orientation θ. In fact it will be shown later that with this definition of coordinate system, the expression for angular acceleration takes on the simple form

$$\frac{d}{dt}(I\dot{\theta}) = \overline{M}_{c.g.} \tag{1}$$

where I is the moment of inertia of the missile about its mass center, and $\overline{M}_{c.g.}$ is the moment of the external forces about the mass center.

To express the above concepts analytically, we start by taking $u_i(x, t)$ in the following form,

$$u_i(x, t) = C_o(t) + xC_1(t) + \sum_{n=1}^{\infty} \phi_{in}(x)q_n(t) \tag{2}$$

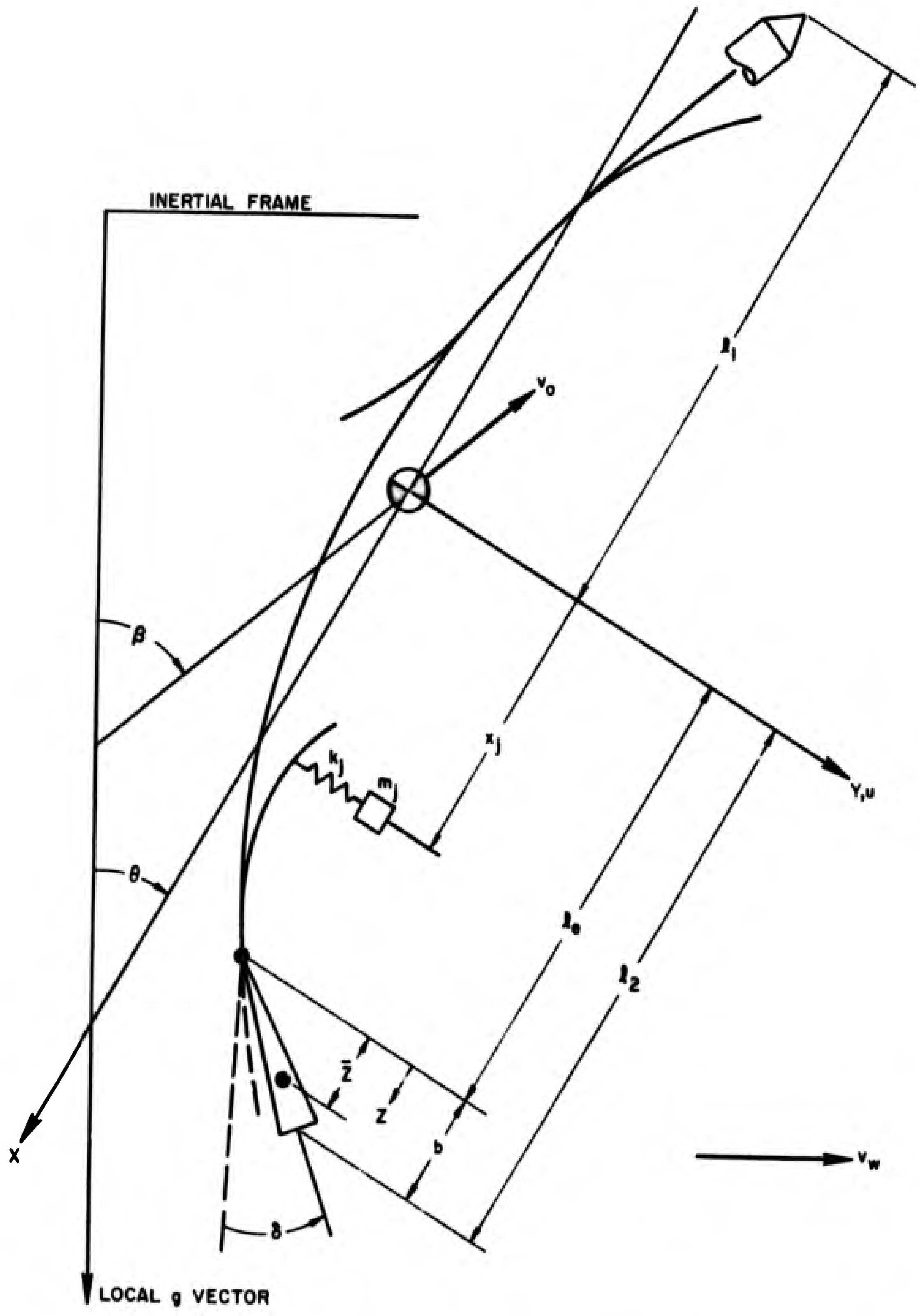

Figure 1. Coordinate System and Missile Geometry.

The parameters $C_o(t)$ and $C_1(t)$ in this expression are to be determined such that the conditions imposed on u_i are satisfied. These parameters represent a rigid body displacement are rotation, respectively, of the missile relative to the x, y - axes. The $\phi_{in}(x)$, $i = 1, 2, 3 \cdots R$, $n = 1, 2, 3 \cdots$, are the eigenfunctions of the missile considered as a collection of R beams or branches, and $q_n(t)$ are the corresponding generalized coordinates.

Before proceeding to the determination of C_o and C_1, we need to review the properties of the eigenfunctions. This is done in Section 3; in Section 4, the evaluation of C_o and C_1 is then carried out. It will be seen there that C_o is determined by the condition relating the origin to the mass center, and that C_1 is determined by the condition regarding the relative angular momentum.

3. PROPERTIES OF THE EIGENFUNCTIONS

The bending frequencies and eigenfunctions are available from a digital computer routine that uses a modified Timoshenko beam theory which includes the effect of rotatory inertia, shear, and axial acceleration.

Let

ω_n = circular frequency of the n^{th} mode.

ϕ_{in} = eigenfunction representing the total deflection (due to both bending and shear) of the i^{th} branch in the n^{th} mode.

ψ_{in} = eigenfunction representing the slope due to bending alone of the i^{th} branch in the n^{th} mode.

ζ_{jn} = deflection of the spring of the j^{th} slosh mass in the n^{th} mode, i.e., the total deflection of the j^{th} slosh mass is $\phi_{jn} + \zeta_{jn}$.

ϕ_{jn} = deflection of the beam containing the j^{th} tank at the attachment point of the j^{th} sloshing mass, in the n^{th} mode.

P = P(x) = longitudinal force at any section, positive for compression.

These quantities are calculated using the following assumptions:

1. The missile is treated as a collection of beams which extend from $-\ell_1 \leq x \leq \ell_2$.
2. The engines are clamped at $\delta = 0$ and considered as an integral part of the collection of beams.
3. The mass of the liquid in each tank, less the sloshing mass defined below, is assumed to be concentrated along the corresponding beam centerline. The rotatory inertia of the liquid is assigned as a uniform value over the length of each tank, such that the resulting moment of inertia for the equivalent fluid is equal to the "capped-tank" inertia of the liquid in a rigid container.

4. Sloshing is represented by attached spring-masses giving the proper frequency, horizontal force, and moment of the first slosh mode of each tank. (The tanks are considered as rigid containers whose walls are arbitrary surfaces of revolution when calculating the sloshing parameters.)[2]

5. The eigenfunctions or mode-shapes are normalized to the total missile mass M. (See Equation 3 below.)

The orthogonality relations for the eigenfunctions calculated under the above assumptions are given below. A derivation of these relations is given in Reference 3.

$$\sum_{i=1}^{R}\int_{\ell_i}\left[(\rho A)_i\,\phi_{im}\,\phi_{in} + (\rho I)_i\,\psi_{im}\,\psi_{in}\right]dx + \sum_{j=1}^{N} m_j(\phi_{jm}+\zeta_{jm})(\phi_{jn}+\zeta_{jn})$$

$$= \begin{cases} 0 & \text{for } m \neq n \\ M & \text{for } m = n \end{cases} \tag{3}$$

$$\sum_{i=1}^{R}\int_{\ell_i}(EI)_i\,\frac{d\psi_{im}}{dx}\,\frac{d\psi_{in}}{dx}\,dx + \sum_{i=1}^{R}\int_{\ell_i}(KAG)_i\left(\frac{d\psi_{im}}{dx}-\psi_{im}\right)\left(\frac{d\psi_{in}}{dx}-\psi_{in}\right)dx$$

$$- \sum_{i=1}^{R}\int_{\ell_i} P_i\,\frac{d\phi_{im}}{dx}\,\frac{d\phi_{in}}{dx}\,dx + \sum_{j=1}^{N} K_j\,\zeta_{jm}\,\zeta_{jn}$$

$$+ \bar{a}_x \sum_{j=1}^{N} m_j\left[\psi_{jm}(\phi_{jn}+\zeta_{jn}) + \psi_{jn}(\phi_{jm}+\zeta_{jm})\right]$$

$$= \begin{cases} 0 & m \neq n \\ M\omega_n^2 & m = n \end{cases} \tag{4}$$

It is useful to mention that the first orthogonality condition above could be simplified in form by including the contribution of the slosh masses m_j in the integrals, thus eliminating the second sum in Equation (3). In order to do this, however, the definitions of ϕ_{in} and $(\rho A)_i$ would have to be modified to contain appropriate discontinuities to account for the lumped slosh masses. Such a procedure would certainly cloud the issue for some readers so it is not used here, but nevertheless, it is useful to think of the terms containing m_j as being part of a more general integral over the missile mass.

For the missile in the free-free condition, it is useful to write down the results of applying orthogonality condition (3) between an arbitrary mode and the two rigid body modes (i.e., translation, $\phi_i = 1$, $\psi_i = 0$, $\zeta_j = 0$, and rotation, $\phi_i = x$, $\psi_i = 1$, $\zeta_j = 0$, where x is measured from the center of gravity of the entire missile).

$$\sum_{i=1}^{R} \int_{\ell_i} (\rho A)_i \, \phi_{in} \, dx + \sum_{j=1}^{N} m_j (\phi_{jn} + \zeta_{jn}) = 0 \tag{5}$$

$$\sum_{i=1}^{R} \int_{\ell_i} \left[(\rho A)_i \, x \, \phi_{in} + (\rho I)_i \, \psi_{in} \right] dx + \sum_{j=1}^{N} x_j \, m_j (\phi_{jn} + \zeta_{jn}) = 0 \tag{6}$$

Equations (5) and (6) merely state that the linear and angular momentum of the bending modes, with respect to the axes from which they are measured, are zero.

4. THE DEFLECTION FUNCTION

The location in the longitudinal direction of the mass center (and hence the origin of the x-axis) is determined from the relation

$$\sum_{i=1}^{R} \int_{\ell_i} x(\rho A)_i \, dx + \sum_{j=1}^{N} x_j m_j = 0 \tag{7}$$

As previously stated, longitudinal deformations of the missile relative to the mass center are neglected. Consequently, the longitudinal position of the mass center is fixed and independent of lateral deflections.

The lateral deflection of the axes of the various beams making up the missile is taken to have the form given by (2), namely,

$$u_i(x,t) = C_o(t) + x C_1(t) + \sum_{n=1}^{\infty} \phi_{in} q_n(t) \tag{8}$$

The corresponding angle of rotation of a missile element is given by

$$\Psi_i(x,t) = C_1(t) + \sum_{n=1}^{\infty} \psi_{in} q_n(t) \tag{9}$$

First, the constant C_o is determined. For the origin of the coordinates in the lateral direction to be at the mass center, one must impose the condition

$$\sum_{i=1}^{R} \int_{\ell_i} (\rho A)_i u_i \, dx + \sum_{j=1}^{N} (u_j + \zeta_j) m_j + m_e \bar{z} \delta = 0 \tag{10}$$

where u_j is the deflection of the beam containing the j^{th} tank at the attachment point of the j^{th} sloshing mass. The last term in (10) arises from the swiveling of the control engine. The angle δ is measured from the tangent to the missile axis at the hinge point. The total lateral displacement of the engine c.g. is

$u(\bar{z}) + \bar{z}\delta$, where $u(\bar{z})$ is the displacement of the engine treated as an integral part of the missile with $\delta = 0$. The influence of $m_e u(\bar{z})$ upon the mass center determination is included in

$$\sum_{i=1}^{R} \int_{\ell_i} (\rho A)_i u_i \, dx$$

(in fact, since the engine is allowed to be flexible in the computation of bending modes, a more exact influence on the missile mass center is accounted for in this integral). The $m_e \bar{z}\delta$ term represents the additional mass moment due to δ, treating the swiveling engines as a rigid body. No restriction is made concerning whether or not all of the engines swivel, since m_e is to be interpreted as the mass of only the swiveling engine or engines.

Substituting (8) into (10) gives

$$C_o \left[\sum_{i=1}^{R} \int_{\ell_i} (\rho A)_i \, dx + \sum_{j=1}^{N} m_j \right] + C_1 \left[\sum_{i=1}^{R} \int_{\ell_i} x(\rho A)_i \, dx + \sum_{j=1}^{N} x_j m_j \right]$$
$$+ \sum_{n=1}^{\infty} q_n \left[\sum_{i=1}^{R} \int_{\ell_i} (\rho A)_i \phi_{in} dx + \sum_{j=1}^{N} (\phi_{jn} + \zeta_{jn}) m_j \right] + m_e \bar{z} \delta = 0 \tag{11}$$

In the above expression, the coefficient of C_o is the total mass M. The coefficient of C_1 vanishes identically because of the choice of the origin of the x-axis, as seen from (7). The coefficients of q_n vanish identically because the mode shapes are orthogonal with respect to a rigid body translation, as shown by (5). Thus, the rigid body translation term C_o is merely the result of the missile moving in the opposite direction from an engine deflection δ.

$$C_o = - \frac{m_e \bar{z}}{M} \delta \overset{\Delta}{=} - K_o \delta \tag{12}$$

To determine $C_1(t)$, the angular momentum of the missile with respect to the moving set of axes is set to zero. This condition is expressed by

$$\sum_{i=1}^{R} \int_{\ell_i} \left[x(\rho A)_i \dot{u}_i + (\rho I)_i \dot{\Psi}_i \right] dx + \sum_{j=1}^{N} x_j m_j (\dot{u}_j + \dot{\zeta}_j) + \dot{\delta} \int_{0}^{b} \left[xz(\rho A)_e + (\rho I)_e \right] dz = 0 \tag{13}$$

Substituting $\dot{u}_i$ and $\dot{\Psi}_i$ from (8) and (9) into (13) gives

$$\dot{C}_o \left[\sum_{i=1}^{R} \int_{\ell_i} x(\rho A)_i \, dx + \sum_{j=1}^{N} x_j m_j \right] + \dot{C}_1 \left\{ \sum_{i=1}^{R} \int_{\ell_i} \left[x^2 (\rho A)_i + (\rho I)_i \right] dx + \sum_{j=1}^{N} x_j^2 m_j \right\} + \sum_{n=1}^{\infty} \dot{q}_n \left\{ \sum_{i=1}^{R} \int_{\ell_i} \left[x(\rho A)_i \phi_{in} + (\rho I)_i \psi_{in} \right] dx + \sum_{j=1}^{N} x_j m_j (\phi_{jn} + \zeta_{jn}) \right\} + \dot{\delta} \int_{0}^{b} \left[\ell_e z(\rho A)_e + z^2 (\rho A)_e + (\rho I)_e \right] dz = 0 \tag{14}$$

The coefficient of $\dot{C}_o$ vanishes because the origin of the x-axis is at the missile c.g. as specified by (7). The coefficient of $\dot{C}_1$ is the mass moment of inertia I_c of the missile about its c.g. The coefficients of $\dot{q}_n$ vanish identically because the sloshing-bending modes are orthogonal with respect to a rigid body rotation as expressed by (6). The coefficient of $\dot{\delta}$ was modified by substituting $x = \ell_e + z$. Integrating the resulting equation with respect to time and taking the integration constant as zero so that $C_1 = 0$ when $\delta = 0$, C_1 becomes

$$C_1 = -K_1 \delta \tag{15}$$

where

$$K_1 = \frac{1}{I_c}(I_e + m_e \ell_e \bar{z}) \tag{16}$$

Substituting C_o and C_1 from (12) and (15) back into (8) and (9), the deflection functions are

$$u_i(x, t) = -(K_o + K_1 x)\delta + \sum_{n=1}^{\infty} \phi_{in} q_n(t) \tag{17}$$

$$\Psi_i(x, t) = -K_1 \delta + \sum_{n=1}^{\infty} \psi_{in} q_n(t) \tag{18}$$

5. KINEMATICS

As shown in Figure (2), let

$\tilde{P}_i$ = Position vector from the inertial axes origin to a point on the axis of the i^{th} branch of the missile.

$\tilde{R}$ = Position vector from the inertial axes origin to the origin of the moving axes.

$\tilde{i}, \tilde{j}$ = Unit vectors in the x and y directions, respectively. At any instant, these rotatate at an angular velocity $\dot{\theta}$ with respect to the inertia axes.

From the figure, one can write

$$\tilde{P}_i = \tilde{R} + x\tilde{i} + u_i\tilde{j} \tag{19}$$

The velocity and acceleration of a point on the missile axis are found by differentiating (19). In doing this, observe that $d\tilde{i}/dt = -\dot{\theta}\tilde{j}$ and $d\tilde{j}/dt = \dot{\theta}\tilde{i}$. Also, $\dot{x} = 0$ since longitudinal displacements relative to the moving axes are neglected. Performing the differentiations gives

$$\begin{aligned} \frac{d\tilde{P}_i}{dt} &= \frac{d\tilde{R}}{dt} + u_i\dot{\theta}\tilde{i} + (\dot{u}_i - x\dot{\theta})\tilde{j} \\ \frac{d^2\tilde{P}_i}{dt^2} &= \frac{d^2\tilde{R}}{dt^2} + (u_i\ddot{\theta} + 2\dot{u}_i\dot{\theta} - x\dot{\theta}^2)\tilde{i} + (\ddot{u}_i - x\ddot{\theta} - u_i\dot{\theta}^2)\tilde{j} \end{aligned} \tag{20}$$

The velocity and acceleration of the mass center (origin of x, y-axes) can be expressed in terms of its components along the directions of the x and y axes, that is

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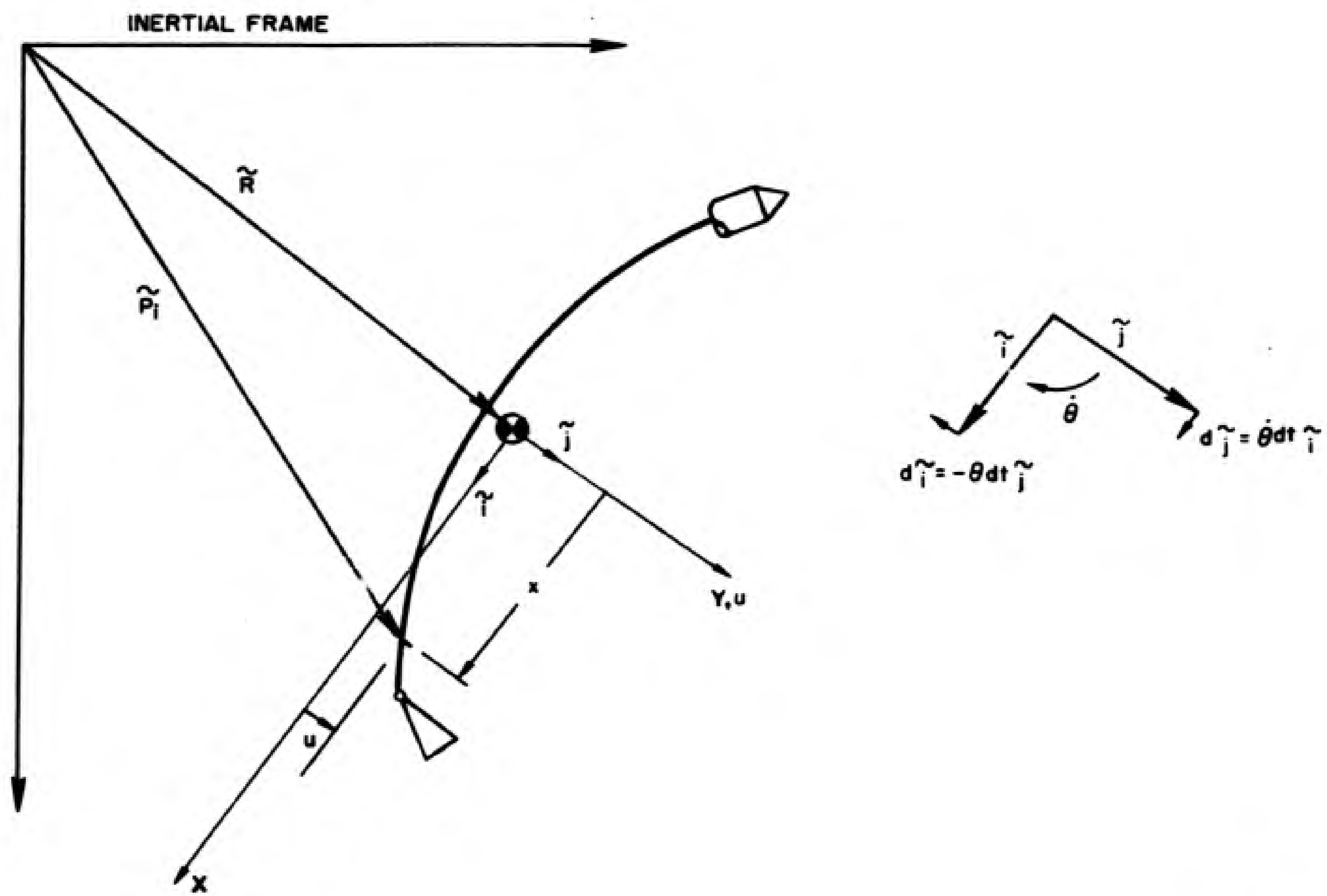


Figure 2. Kinematics.

$$\frac{d\tilde{R}}{dt} = \bar{v}_x \tilde{i} + \bar{v}_y \tilde{j}$$

$$\frac{d^2\tilde{R}}{dt^2} = \bar{a}_x \tilde{i} + \bar{a}_y \tilde{j} \tag{21}$$

Substituting (21) into (20), the velocity and acceleration components of a point on the axis of the j^{th} branch of the missile are

$$\left.\begin{aligned} v_{xi} &= \bar{v}_x + u_i\dot{\theta} \\ v_{yi} &= \bar{v}_y + \dot{u}_i - x\dot{\theta} \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} a_{xi} &= \bar{a}_x + u_i\ddot{\theta} + 2\dot{u}_i\dot{\theta} - x\dot{\theta}^2 \\ a_{yi} &= \bar{a}_y + \ddot{u}_i - x\ddot{\theta} - u_i\dot{\theta}^2 \end{aligned}\right\} \tag{23}$$

The velocity of a point on the axis of the swiveling engine can be found from the above by replacing u_i by $u_2 + \delta z$, where u_2 is the deflection function for the missile branch which includes the engine. There could be more than one engine and more than one branch (or two engines which can be considered as one) but by proper interpretation, the following equations are still valid.

$$v_x = \bar{v}_x + (u_2 + \delta z)\dot{\theta} \tag{24}$$

$$v_y = \bar{v}_y + \dot{u}_2 + \dot{\delta} z - x\dot{\theta} \tag{25}$$

6. KINETIC ENERGY OF THE SYSTEM

For use in developing the Lagrangian equations of motion, the expressions for the kinetic energy of the complete missile will be found. First to be considered is the kinetic energy associated with the velocity of points along the missile axis, including sloshing and the swiveling engines, assuming them to be clamped at $\delta = 0$. Denoting this portion of the kinetic energy by T_u, and using (22) and (23), this energy can be expressed by

$$T_u = \frac{1}{2} \sum_{i=1}^{R} \int_{\ell_i} (v_{xi}^2 + v_{yi}^2)(\rho A)_i \, dx + \frac{1}{2} \sum_{j=1}^{N} m_j (v_{xj}^2 + v_{yj}^2)$$

$$= \frac{1}{2} \sum_{i=1}^{R} \int_{\ell_i} \left[(\bar{v}_x + u_i \dot{\theta})^2 + (\bar{v}_y + \dot{u}_i - x\dot{\theta})^2 \right] (\rho A)_i \, dx \tag{26}$$

$$+ \frac{1}{2} \sum_{j=1}^{N} \left[(\bar{v}_x + u_j \dot{\theta})^2 + (\bar{v}_y + \dot{u}_j - x_j \dot{\theta} + \dot{\zeta}_j)^2 \right] m_j$$

In Equation (26), v_{xj} and v_{yj} are the velocity components of the slosh masses, and are found by evaluating Equations (22) and (23) at $x = x_j$, as indicated.

Next, consider the kinetic energy due to the rotatory inertia of each infinitesimal transverse slice of the missile. The angular velocity of any slice is given by $\dot{\theta} - \dot{\Psi}_i$, where Ψ_i is defined by (9) or (18). The minus sign arises only as a result of the choice of coordinate system. The corresponding kinetic energy is

$$T_\Psi = \frac{1}{2} \sum_{i=1}^{R} \int_{\ell_i} (\rho I)_i (\dot{\theta} - \dot{\Psi}_i)^2 \, dx \tag{27}$$

For the swiveling engine, the total kinetic energy is given, using (24) and (25), by

$$T_\delta = \frac{1}{2}\int_o^b \left[(\bar{v}_x + u_2\dot{\theta} + \delta z\dot{\theta})^2 + (\bar{v}_y + \dot{u}_2 + \dot{\delta}z - x\dot{\theta})^2\right](\rho A)_e\, dz$$

$$+ \frac{1}{2}\int_o^b (\dot{\theta} - \dot{\Psi}_2 - \dot{\delta})^2 (\rho I)_e\, dz \qquad (28)$$

where again, the subscript 2 refers to the branch containing the swiveling engines. This can be written in the following form,

$$T_\delta = \frac{1}{2}\int_o^b \left[(\bar{v}_x + u_2\dot{\theta})^2 + (\bar{v}_y + \dot{u}_2 - x\dot{\theta})^2\right](\rho A)_e\, dx + \frac{1}{2}\int_o^b (\dot{\theta} - \dot{\Psi}_2)^2 (\rho I)_e\, dx$$

$$+ \frac{1}{2}\int_o^b \left[2\delta z\dot{\theta}\,(\bar{v}_x + u_2\dot{\theta}) + \delta^2 z^2\dot{\theta}^2 + 2\dot{\delta}\,z(\bar{v}_y + \dot{u}_2 - x\dot{\theta}) + \dot{\delta}^2 z^2\right](\rho A)_e\, dz$$

$$+ \frac{1}{2}\int_o^b \left[\dot{\delta}^2 - 2\dot{\delta}(\dot{\theta} - \dot{\Psi}_2)\right](\rho I)_e\, dz \qquad (29)$$

The first two integrals in the above expression represent the kinetic energy of the swiveling engine when $\delta = 0$; this portion of T_δ is already included in $T_u + T_\Psi$. The second and third integrals give the additional kinetic energy due to swiveling through an angle δ. Denoting this addition kinetic energy by T_δ^*, it is given by

$$T_\delta^* = \frac{1}{2}\int_o^b \left[2\delta z\dot{\theta}(\bar{v}_x + u_2\dot{\theta}) + \delta^2 z^2\dot{\theta}^2 + 2\dot{\delta}\,z(\bar{v}_y + \dot{u}_2 - x\dot{\theta}) + \dot{\delta}^2 z^2\right](\rho A)_e\, dz$$

$$+ \frac{1}{2}\int_o^b \left[\dot{\delta}^2 - 2\dot{\delta}(\dot{\theta} - \dot{\Psi}_2)\right](\rho I)_e\, dz \qquad (30)$$

The total kinetic energy of the system is the sum of (26), (27) and (30); that is

$$T = T_u + T_\Psi + T_\delta^* \tag{31}$$

7. AERODYNAMICS

The aerodynamics to be used in this study will be based as much as possible on actual wind tunnel data. For the most part, this data consists of plots of running load versus missile station for various fixed angles of attack of a rigid missile. It has been found that these aerodynamic loads vary linearly with angle of attack up to about five degrees so that for the purposes of this analysis only a single load curve $w_a(x)$ pounds per inch along the missile per unit angle of attack is necessary.

The load distribution $a_T w_a(x)$ is the steady aerodynamic load which results when the rigid missile is held at a fixed angle of attack a_T . If the rigid missile is now permitted to have an angular velocity $\dot{\theta}$ (pitching) and a rate of change of angle of attack $\dot{a}$ (plunging) one must deal with <u>unsteady</u> aerodynamics. The aerodynamic forces due to an instantaneous angle of attack are taken to be the same as those which would result if the angle of attack were constant at its instantaneous value. The additional aerodynamic forces due to pitching and plunging can be estimated by three alternate techniques:

a. quasi-steady theory,
b. slender body theory, or
c. piston theory.

For slowly varying motions (of the order of a few cycles per second) the quasi-steady theory is recommended to give the best estimate for the aerodynamic forces due to pitching. The aerodynamic forces due to a constant pitching velocity $\dot{\theta}$ could be obtained directly from wind-tunnel tests on fixed models bent into a bannana shape to give an apparent pitching velocity, but it is anticipated that such data will seldom be obtained. Quasi-steady theory cannot be used to give the aerodynamic forces due to plunging. Slender body theory will be used to determine these forces. Piston theory for unsteady motion becomes accurate only for motions with a frequency much higher than are likely to occur in the missile dynamics considered here.

Aerodynamic forces due to an acceleration $\ddot{\theta}$ are quite small and will be neglected.

Most of the aerodynamic forces due to missile bending will be neglected as small with two exceptions:

a. angle of attack due to motion in a sloshing mode (the airframe often moves as a nearly rigid body in a sloshing mode; see below) and

b. damping of the structural modes due to aerodynamics.

Other aerodynamic forces due to missile bending have a negligible effect on the motion. For example, the ratio of aerodynamic "stiffness" (i.e., the coefficient of q_n in the bending equation, due to aerodynamics) to structural stiffness $(M\omega_n^2)$ for the first "bending" mode of a two-stage liquid propellant missile is 1:1200. This ratio for the first sloshing mode is 1:3700, based on quasi-steady aerodynamics. The ratios of aerodynamic damping (i.e., the coefficient of $\dot{q}_n$ due to aerodynamics) to structural damping (assuming $b_n = .01$) for the first three "bending" modes of this missile are 1/3, 1/12, and 1/10, respectively at maximum dynamic pressure. This effect could be considerable and will be included, as indicated.

The quasi-steady aerodynamic theory mentioned above consists simply of computing the apparent angle of attack at any station due to an effect in question, and multiplying this angle by $w_\alpha(x)$ at that station to obtain the load there. For example, the load distribution due to a pitching velocity $\dot{\theta}$ is

$$\dot{\theta} w_{\dot{\theta}}(x) = \frac{x\dot{\theta}}{v_o} w_\alpha(x) \quad . \tag{32}$$

Similarly, the load distribution due to a bending velocity $\dot{q}_n$ is

$$\dot{q}_n w_{\dot{q}_n} = \frac{\phi_{1n}(x)\dot{q}_n}{v_o} w_\alpha(x) \tag{33}$$

where again, the subscript 1 refers to the missile branch exposed to aerodynamics. Computing the generalized force on the n^{th} mode due to this last load, the contribution to the modal damping ratio b_n due to aerodynamics is

$$\Delta b_n = \frac{1}{2M\omega_n v_o} \int_{-\ell_1}^{\ell_2} w_a(x)\, \phi_{1n}^2(x) d_x \tag{34}$$

The load distribution due to a plunging $\dot{a}$, using slender body theory, is

$$\dot{a} w_{\dot{a}}(x) = \dot{a} \rho v_o S(x) \tag{35}$$

where ρ is the density of the air at the altitude of the missile and $S(x)$ is the cross-sectional area of the missile.

The total load $p(x, t)$ acting on the missile is an appropriate summation of the effects discussed above. The total rigid body angle of attack a_T is

$$a_T = \beta - \theta - K_1 \delta - \frac{K_o \dot{\delta}}{v_o} + \sum_{n=1}^{N} a_n q_n - \frac{v_w}{v_o} \cos \beta \tag{36}$$

where

a_n = "effective" angle of attack per unit "slosh" mode amplitude, i.e., if appreciable bending occurs in a slosh mode, a_n should be taken as a mean slope over the region of large $w_a(x)$.

q_n, $n = 1, 2, \cdots N$ are the bending-sloshing modes which have very large sloshing amplitudes. These have been called "slosh" modes.

v_w = wind velocity at missile altitude, assumed directed parallel to the ground, positive in the y-direction with $\theta = 0$

The total effective pitching velocity of the airframe is

$$\dot{\theta}_T = \dot{\theta} + K_1 \dot{\delta} - \sum_{n=1}^{N} a_n \dot{q}_n \tag{37}$$

This is the pitching velocity that should be used in (32) for the quasi-steady load distribution due to pitching. However, the aerodynamic load due to pitching will be small and $K_1\dot{\delta}$ and $a_n\dot{q}_n$ are small compared to $\dot{\theta}$ (a typical value of K_1 is 0.0015 and of a_n is 0.001 rad/in) so that $K_1\dot{\delta}$ and $a_n\dot{q}_n$ in the pitching aerodynamics become "higher order" terms, i.e., the product of two small effects, giving rise to forces much smaller than the uncertainty in the quasi-steady theory. For this reason only $\dot{\theta}$ itself will be considered for the determination of pitching aerodynamics.

The total $\dot{a}_T$ to be used in determining the aerodynamic load due to plunging is found by differentiating (36). Combining the aerodynamic loads due to quasi-steady angle of attack, pitching, and plunging, the total aerodynamic load distribution $p(x, t)$ is

$$p(x, t) = a_T w_a(x) + \dot{\theta} w_{\dot{\theta}}(x) + \dot{a}_T w_{\dot{a}}(x) \tag{38}$$

where

$$\begin{aligned} w_a(x) &= \text{steady load distribution per unit angle of attack,} \\ w_{\dot{\theta}}(x) &= \frac{x}{v_o} w_a(x) \\ w_{\dot{a}}(x) &= \rho v_o S(x) \end{aligned} \tag{39}$$

and a_T is given by Equation (36). One other aerodynamic load distribution is included, by implication, in addition to this $p(x, t)$. These additional forces are due to the interaction between the bending velocities $\dot{q}_n$ and the aerodynamics. These forces are small and their only significant effect is to change the apparent damping ratios b_n of the bending modes, as indicated by Equation (34).

The resultant aerodynamic force in the y-direction found by integrating $p(x, t)$ over the length of the missile branch exposed to the atmosphere:

$$F_y^* = -\int_{-\ell_1}^{\ell_2} p(x, t)dx \tag{40}$$

Performing the integration, using $p(x, t)$ from (38), gives

$$F_y^* = - N_\alpha \alpha_T - N_{\dot{\alpha}} \dot{\alpha}_T - N_{\dot{\theta}} \dot{\theta} \tag{41}$$

where

$$N_\alpha = \int_{-\ell_1}^{\ell_2} w_\alpha(x)\, dx$$

$$N_{\dot{\alpha}} = \rho v_o \int_{-\ell_1}^{\ell_2} S(x)\, dx \tag{42}$$

$$N_{\dot{\theta}} = \frac{1}{v_o} \int_{-\ell_1}^{\ell_2} x w_\alpha(x)\, dx$$

The moment of the aerodynamic forces about the mass center is given by

$$M_{c.g.}^* = \int_{-\ell_1}^{\ell_2} x p(x, t)\, dx \tag{43}$$

Performing this integration gives

$$M_{c.g.}^* = M_\alpha \alpha_T + M_{\dot{\alpha}} \dot{\alpha}_T + M_{\dot{\theta}} \dot{\theta} \tag{44}$$

where

$$M_{\alpha} = \int_{-\ell_1}^{\ell_2} x w_{\alpha}(x) dx = v_o N_{\dot{\theta}}$$

$$M_{\dot{\alpha}} = \rho v_o \int_{-\ell_1}^{\ell_2} x S(x)\, dx \qquad (45)$$

$$M_{\dot{\theta}} = \frac{1}{v_o} \int_{-\ell_1}^{\ell_2} x^2 w_{\alpha}(x) dx$$

The generalized forces on the bending modes due directly to the aerodynamic forces are given by

$$Q_n^* = - \int_{-\ell_1}^{\ell_2} p(x, t)\, \phi_{1n}(x)\, dx \qquad (46)$$

Using (38), this integral becomes

$$\int_{\ell_1}^{\ell_2} p\, \phi_{1n}\, dx = H_{\alpha n} \alpha_T + H_{\dot{\alpha} n} \dot{\alpha}_T + H_{\dot{\theta} n} \dot{\theta} \qquad (47)$$

where

$$H_{\alpha n} = \int_{-\ell_1}^{\ell_2} w_{\alpha}(x)\, \phi_{1n}(x)\, dx$$

$$H_{\dot{\alpha} n} = \rho v_o \int_{-\ell_1}^{\ell_2} S(x)\, \phi_{1n}(x)\, dx \qquad (48)$$

$$H_{\dot{\theta} n} = \frac{1}{v_o} \int_{-\ell_1}^{\ell_2} x w_{\alpha}(x)\, \phi_{1n}(x)\, dx$$

8. MOTION OF MASS CENTER AND ROTATION

From the general principle of the motion of the mass center, it is known that the total mass times the acceleration of the mass center in a given direction equals the resultant external force in that direction. Thus, referring to Figure (3), which shows the forces acting on the missile, one can write

$$M\bar{a}_x = - T_s - T_c - Mg \cos \theta + F_x^* \tag{49}$$

$$M\bar{a}_y = - T_c \, (u'_e + \delta) - T_x u'_e + Mg \sin \theta + F_y^* \tag{50}$$

where F_x^* and F_y^* are the resultant aerodynamic forces in the x and y directions, and are approximately equal to the drag and lift respectively. An explicit expression for F_y^* is given in the previous section.

The equation of motion for rotation θ takes on a form similar to that for the motion of the c.g. because of the choice of coordinate system. Just as the equations of motion for the origin of the coordinate system (i.e., the c.g.) are formed by equating the total applied force to the mass time the acceleration of the c.g., the equation of motion for the orientation of the coordinate system (i.e., the axis about which the moment of momentum is zero) is formed by equating the total external moments to the rate of change of angular momentum of the missile. Of the quantities θ, q_n, and δ which specify the configuration of the missile, only changes in θ affect its angular momentum by definition of the coordinate system. Recall that if the engine is given a displacement δ, the rest of the missile moves through an angle $-K_i\delta$ as measured in the moving coordinate system so that there is no change in the moment of momentum about the moving axes. Similarly, the bending-sloshing modes are computed to have zero angular momentum with respect to the axes from which they are measured. Using the forces given in Figure 3, the equation of motion for θ is then given simply by

$$\frac{d}{dt} (I\dot{\theta}) = \left[T_c (u'_e + \delta) + T_s u'_e\right] \ell_e - (T_c + T_s) u_e + M^*_{c.g.} \tag{51}$$

where M^*_{cg} is the moment about the mass center of the aerodynamic forces.

In Equation 51, I is the instantaneous mass moment of inertia of the missile including the effective moment of inertia of the liquids in the tanks. This moment of inertia I varies with time since the configuration of the missile changes with respect to the x, y-axes. However, in most practical cases the magnitude of this variation will be small compared with the total and it is appropriate to take the moment of inertia as a constant given, as for Equation (14), by

$$I \approx I_c = \sum_{i=1}^{R} \int_{\ell_i} \left[(\rho A)_i x^2 + (\rho I)_i\right] dx + \sum_{j=i}^{N} x_j^2 m_j \tag{52}$$

With this approximation, the left hand side of (51) may be replaced by $I_c \ddot{\theta}$.

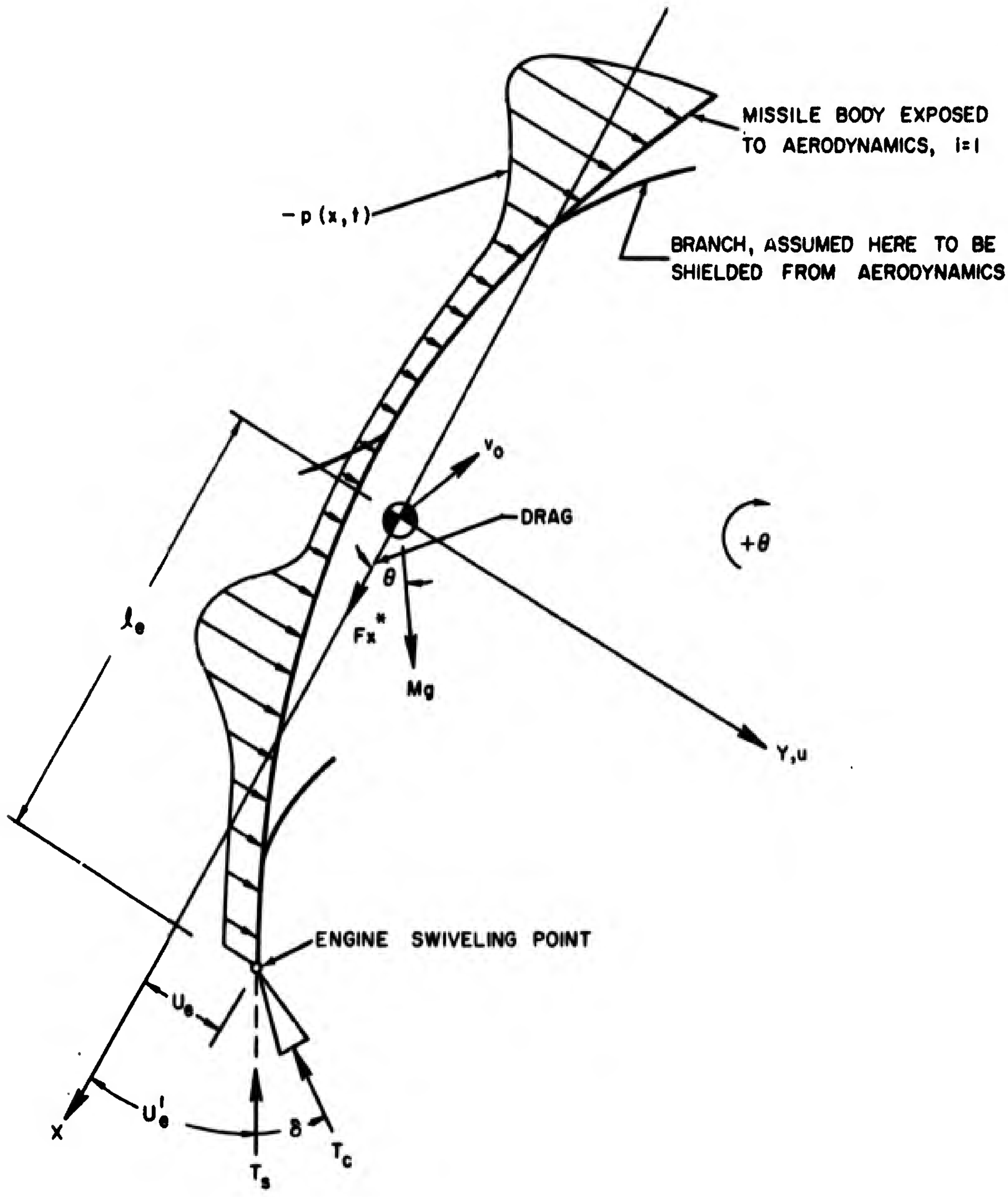


Figure 3. Forces Acting on Missile.

9. BENDING - SLOSHING EQUATIONS

The most expedient way to derive the bending-sloshing equations is to give each mode a virtual displacement and compute the generalized force on each mode due to all of the external forces acting on the missile. The inertia forces due to the swiveling motion δ of the engines must be included as an external force because such motions are "external" to the bending modes which have been computed with $\delta = 0$.

The general form of the equations of motion for the bending-sloshing modes is

$$\ddot{q}_n + 2 b_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{Q_n(t)}{M} \tag{53}$$

in which the bending modes have been normalized (i. e., scaled) to give a generalized mass equal to the total mass, as indicated in Equation (3). Equation (53) (with the exception of the $\dot{q}_n$ term) can be derived by a straight-forward application of Lagrange's equation using the kinetic energy given by Equation (31) and the potential energy of bending. The coupling terms can be eliminated by use of orthogonality conditions (3) and (4) to yield the simple form given.

The time functions $Q_n(t)$ are the generalized forces mentioned above. The constant ω_n is the circular frequency of the n^{th} mode and b_n is the damping ratio, found experimentally by shaking an assembled missile, calculated from experiments on components, or assumed conservatively small. It should be mentioned that strictly speaking, one can seldom represent the damping of a structure in this way. In order that this representation be strictly correct, all damping forces would have to be of the viscous type (i. e., proportional to velocity) and would have to be distributed along the length of the missile in the same way in which the mass and rotatory inertia are distributed, or the same way in which an appropriate combination of the elastic forces are distributed. If the damping forces are distributed in any other wav, application of the Lagrange's equations would result in the equations of motion for the bending modes having velocity coupling through the viscous damping forces. That is, the bending-sloshing equation for the n^{th} mode would have additional terms of the form $C_{mn}\dot{q}_m$, where $m = 1, 2, 3\cdots$. For reasonably small values of damping, however, (corresponding to a few percent of critical damping in any mode) these coupling terms have a negligibly small effect on the motion (see Reference 4) and can be omitted as was done in Equation (53). Similarly, if the damping is of hysteresis type (structural damping) rather than viscous, it can still be represented by an equivalent viscous damping ratio b_n for sufficiently small damping. The contribution to b_n from aerodynamics is described in Section 7.

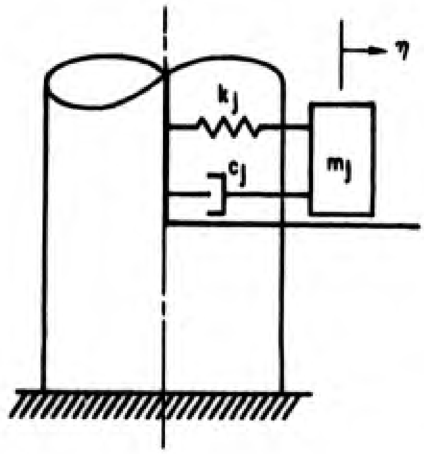
The fluid damping in the slosh tanks has a serious effect on overall missile stability (see, for example, Reference 5) and its influence on the bending-sloshing modes must be treated more explicitly. The effect of slosh damping in the individual tanks on the bending-sloshing equations is essentially different from the phenomona discussed above in that the coupling terms $C_{mn}\dot{q}_m$ between the "slosh" modes must be considered. This is because the frequencies of the "slosh" modes are quite close together, in contrast to the "bending" modes. These coupling terms are easily found by considering the generalized forces on the modes due to the damping forces in the individual tanks. Again, the detailed nature of how the damping forces act need not be considered. From the standpoint of the spring-mass models which are used in this analysis to represent sloshing, the damping in the individual tanks may be considered as coming from dashpots across each lumped spring, as shown in the sketch on the following page.

The equation of motion for the free vibration of each spring mass model, with their tanks fixed, is simply

$$m_j \ddot{\eta}_j + C_j \dot{\eta}_j + K_j \eta_j = 0 \tag{54}$$

or

$$\ddot{\eta}_j + 2\gamma_j \omega_j \dot{\eta}_j + \omega_j^2 \eta = 0$$



where γ_j is the damping ratio for the first mode of sloshing of an individual tank. This ratio is available for each tank from either direct experimental measurements or from the application of approximate theories. Comparing (54) and (55) the equivalent dashpot that would give this damping ratio can be computed from

$$C_j = 2\gamma_j m_j \omega_j \tag{56}$$

where

$$\omega_j = \sqrt{\frac{K_j}{m_j}}$$

is the frequency of the uncoupled slosh motion.

When these spring-mass-dashpot models are attached to the flexible missile, the generalized force on the n^{th} bending-sloshing mode due to the dashpots is

$$Q_{nc} = - \sum_{m=1}^{\infty} \left(\sum_{j=1}^{N} C_j \zeta_{jm} \zeta_{jn} \right) \dot{q}_m \tag{57}$$

Equation (57) is obtained by giving the n^{th} mode a virtual displacement Δq_n and computing the work done on the n^{th} mode by the dashpots C_j during the displacement. The force in the j^{th} dashpot is C_j times the relative velocity between the missile centerline and the lumped slosh mass, i.e.,

$$F_j = - C_j \sum_{m=1}^{\infty} \zeta_{jm} \dot{q}_m \tag{58}$$

where ζ_{jm} is the relative displacement of the j^{th} slosh mass in the m^{th} mode. The virtual displacement at each slosh mass is $\zeta_{jn} \Delta q_n$, so that the total work done by N such dashpots during a virtual displacement Δq_n is

$$W_n = - \sum_{j=1}^{N} \left(C_j \sum_{m=1}^{\infty} \zeta_{jm} \dot{q}_m \right) \left(\zeta_{jn} \Delta q_n \right) \tag{59}$$

Dividing this work by Δq_n gives expression (57) for the generalized force. Equation (57) is more conveniently written as

$$Q_{nc} = - \sum_{m=1}^{\infty} C_{mn} \dot{q}_m \tag{60}$$

where, using (56), the coefficients C_{mn} are given by

$$C_{mn} = \sum_{j=1}^{N} 2\gamma_j m_j \omega_j \zeta_{jm} \zeta_{jn} \tag{61}$$

The coupling coefficients C_{mn} between the "bending" modes (i.e., $m, n > N$; $m \neq n$) will more than likely have a negligible effect on the motion and may be omitted if a significant simplification results.

Consider now the contribution to Q_n in (37) from the inertia forces due to engine swiveling. The inertia force of an elemental swiveling engine mass $(\rho A)_e dz$ is $(\rho A)_e dz \cdot z\ddot{\delta}$. Similarly, the inertia moment of such an element is $(\rho I)_e dz \cdot \ddot{\delta}$. The contribution to the generalized force Q_n from these forces and moments are determined by again giving a virtual displacement Δq_n, computing the work done by the forces, and dividing by Δq_n.

Thus

$$dQ_{n\delta} = \frac{-1}{\Delta q_n}\left[\ddot{\delta} z\,(\rho A)_e dz\ .\ \phi_{2n}\,\Delta q_n + \ddot{\delta}\,(\rho I)_e dz\ .\ \psi_{2n}\,\Delta q_n\right] \tag{62}$$

where the subscript 2n refers to the n^{th} mode, and the branch of the missile which contains the swiveling engines (merely a convention first defined in the section on kinematics). Integrating over the swiveling engines gives the total generalized force due to swiveling:

$$Q_{n\delta} = -\ddot{\delta}\int_o^b \left[z\,(\rho A)_e\ \phi_{2n} + (\rho I)_e\ \psi_{2n}\right] dz \tag{63}$$

The contribution to Q_n from the other external forces shown in Figure (3) are found in a similar way, so that the total generalized force Q_n is

$$Q_n = -\ddot{\delta}\int_c^b \left[z\,(\rho A)_e\ \phi_{2n} + (\rho I)_e\ \psi_{2n}\right] dz - \int_{-\ell_1}^{\ell_2} p\,(x,t)\ \phi_{1n}(x)\,dx$$
$$-\left[T_c\delta + (T_c + T_s)\,u'_e\right]\phi_{2n}(\ell_e) - \sum_{m=1}^{\infty} C_{mn}\dot{q}_m \tag{64}$$

where the subscript 1n refers to the deflection in the n^{th} mode, and the branch which is exposed to aerodynamic forces. Explicit expressions for the integrals $\int p\phi_{1n}\,dx$ are given in Section 7.

10. ENGINE SWIVELING EQUATION

The interaction between engine swiveling and the other variables in this analysis is quite subtle and if one were to use the Newtonian approach used above, it is probable that some of the interacting forces would be overlooked. To avoid such a possibility, the engine swiveling equations will be derived with the use of Lagrange's equation for the engine rotation angle δ, which is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\delta}} - \frac{\partial T}{\partial \delta} + \frac{\partial V}{\partial \delta} = Q_\delta \tag{65}$$

Perhaps the most straight forward way to apply this equation is to take the kinetic energy equation (21) using $u_i\ (x, t)$ and $\Psi_i\ (x, t)$ as given by (17) and (18) respectively. Thus u_i and Ψ_i are considered as functions of δ when differentiating T with respect to δ and $\dot{\delta}$. In this case, the external forces in Figure (3) contribute to Q_δ because a virtual displacement of δ is accompanied by a rigid-body translation $-K_o\delta$ and rotation $-K_1\delta$.

An alternative method of applying Equation (65) is to use u_i and Ψ_i as given in (8) and (9) respectively and consider $C_o\ (t)$ and $C_1\ (t)$ as generalized coordinates when differentiating T with respect to δ and $\dot{\delta}$. In order to be generalized coordinates, of course, $C_o\ (t)$ and $C_1\ (t)$ must be independent of the other coordinates of the system. Since both C_1 and θ represent a rigid body rotation, they are not independent of each other, and similarly, a velocity $\dot{C}_o\ (t)$ is not independent of the velocity $\overline{v}_y$ of the coordinate system. Hence, if $C_o\ (t)$ and $C_1\ (t)$ are considered as generalized coordinates, θ and the lateral location of the origin of the coordinate system cannot be considered as generalized coordinates at the same time. This simply means that if C_o and C_1 are to be considered as generalized coordinates, u_i and Ψ_i must be measured from a fixed (except for axial motions) coordinate system. Alternately, one could argue that in order to define θ and the location of the coordinate system in the manner in which it was done, C_o and C_1 cannot be independent coordinates.

The kinetic energy expression for fixed coordinates, however, is obtained by merely setting $\dot{\theta} = \bar{v}_y = 0$ in Equation (31). When using the kinetic energy Equation (31) in the Lagrange Equation (65) for δ, the same result would be obtained even if one mistakenly considered $\dot{\theta}$ and $\bar{v}_y$ as generalized velocities, but this is only because $\dot{\theta} = \bar{v}_y = 0$ is the acceptable interpretation for these velocities when $C_o(t)$ and $C_1(t)$ are considered as generalized coordinates.

Thus, the alternate method of deriving the engine swiveling equation is to substitute u_i and Ψ_i from (8) and (9) into the kinetic energy Equation (65) with $\dot{\theta} = \bar{v}_y = 0$, and use the Lagrange Equation (65) considering $C_o(t)$ and $C_1(t)$ as independent coordinates. In this case, the external forces in Figure (3) do not contribute to Q_δ. One can now substitute for these coordinates C_o and C_1 the expressions (12) and (15), which define C_o and C_1 in terms of δ, and revert back to the original coordinate system. This procedure is equivalent to assuming that u_i and Ψ_i are independent of δ until after deriving the engine swiveling equation, and not imposing the conditions (10) and (13) until after the swiveling equation is set up.

This second procedure is evidently more expedient and will be used here. Differentiating (31), the first term of (65) becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \delta} = \frac{d}{dt}\frac{\partial T_\delta^*}{\partial \delta} = \frac{d}{dt}\left[\int_o^b z(\dot{\bar{v}}_y + \dot{u}_2 - x\dot{\theta} + \dot{\delta} z)(\rho A)_e \, dz + \int_o^b (\dot{\delta} - \dot{\theta} + \dot{\Psi}_2)(\rho I)_e \, dz\right]$$

$$= \ddot{\delta}\int_o^b \left[z^2(\rho A)_e + (\rho I)_e\right] dz + \dot{\bar{v}}_y \int_o^b z(\rho A)_e \, dz - \ddot{\theta}\int_o^b \left[xz(\rho A)_e + (\rho I)_e\right] dz$$

$$+ \int_o^b \left[z\ddot{u}_2(\rho A)_e + \ddot{\Psi}_2(\rho I)_e\right] dz = I_e\ddot{\delta} + m_e\bar{z}\dot{\bar{v}}_y - (I_e + m_e \ell_e \bar{z})\ddot{\theta}$$

$$+ \int_o^b \left[z\ddot{u}_2(\rho A)_e + \ddot{\Psi}_2(\rho I)_e\right] dz \qquad (66)$$

Substituting (17) and (18) into the last term of (66) gives

$$\int_0^b \left[z\ddot{u}_2(\rho A)_e + \ddot{\Psi}_2(\rho I)_e\right]dz = -\ddot{\delta}\left[K_o m_e \bar{z} + K_1(I_e + m_e \ell_e \bar{z})\right]$$

$$+ \sum_{m=1}^{\infty} \ddot{q}_n \int_0^b \left[\phi_{2n} z(\rho A)_e + \Psi_{2n}(\rho I)_e\right]dz \tag{67}$$

Differentiating (31) with respect to δ gives

$$\frac{\partial T}{\partial \delta} = \frac{\partial T^*_\delta}{\partial \delta} = \int_0^b \left[z\dot{\theta}(\bar{v}_x + u_2\dot{\theta}) + \delta z^2 \dot{\theta}^2\right](\rho A)_e\, dz$$

$$\approx \dot{\theta}\bar{v}_x m_e \bar{z} \tag{68}$$

where terms containing $\dot{\theta}^2$ have been dropped in the approximate expression.

The potential energy to be used in (65) is derived by reference to Figure 4. Only one swiveling engine or nozzle is shown in this figure, and if more than one engine swivels, they will be assumed to swivel together so that by proper interpretation of I_e, m_e, K_A, K_D, and K_δ they can be considered as one. The potential energy V is found by computing the work done in rotating the swiveling engine from $\theta = 0$ to the deflected position shown. This potential energy is

$$V = m_e \bar{z}(g \cos\theta - a_x)\left[\cos u'_e - \cos(u'_e + \delta)\right]$$

$$- m_e g \bar{z} \sin\theta \sin\delta + K_D \delta^2/2 \tag{69}$$

which for small angles δ and u'_e becomes

$$V = m_e \bar{z}(g\cos\theta - a_x)(\delta^2/2 + \delta u'_e) - m_e \bar{z} g\, \delta \sin\theta + K_D \delta^2/2 \tag{70}$$

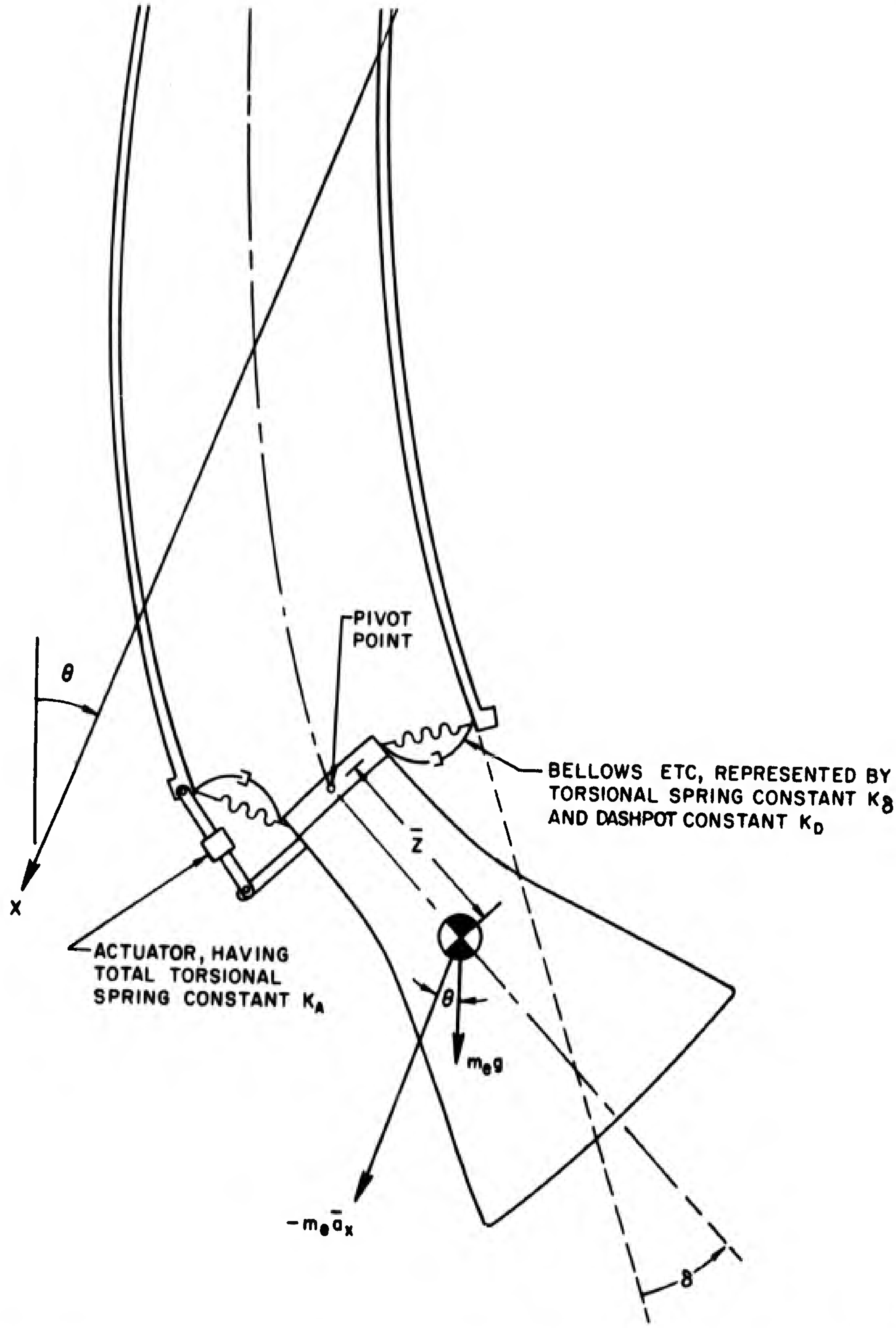


Figure 4. Schematic Showing Forces Acting on Swiveling Engines.

The potential energy due to the inertia force $m_e \bar{a}_y$ must not be included here because it is accounted for by the kinetic energy term of Lagrange's equation.

The potential energy due to the engine rotating through the angle u'_e is not included in (70) because it has been assumed for this report that the bending modes have been computed with the effects of longitudinal acceleration included. This means that the work done by longitudinal acceleration forces in rotating the engine through the angle u'_e is included in the potential energy of beam-column bending. The resulting engine swiveling equation is the same either way, but the terms $\partial V/\partial q_n$ would be different. For the bending equations as written in Section 9, however, these terms have been implicitely omitted, as was done in Reference 1. These reaction forces $\partial V/\partial q_n$, where V comes from (70), are small in comparison with other forces affecting bending in either case, but in this report are even smaller than in Reference 1.

Differentiating (70) gives

$$\frac{\partial V}{\partial \delta} = m_e \bar{z} (g \cos \theta - a_x)(\delta + u'_e) - m_e \bar{z}\, g \sin \theta + K_D \delta \tag{71}$$

The moments due to the engine actuator and the damping moment of the bellows and/or jet damping enter into equation (65) most conveniently through the generalized force Q_δ. A virtual displacement of δ leads to a generalized force of

$$Q_\delta = K_A (\delta_A - \delta) - K_D \delta \tag{72}$$

where

K_A = torsional spring constant of the actuator
K_D = equivalent viscous damping constant
δ_A = actuator position called for by the autopilot.

Substituting (66), (67), (68), (71), and (72) into (65), the following equation of motion is obtained for the swiveling engine:

$$(I_e - K_o m_e \bar{z} - K_1^2 I_c)\ddot{\delta} - K_1 I_c \ddot{\theta} + m_e \bar{z}(g \cos\theta - \bar{a}_x)(\delta + u'_e)$$

$$+ m_e \bar{z}(\bar{a}_y - g \sin\theta) + \sum_{n=1}^{\infty} \ddot{q}_n \int_o^b \left[z\phi_{2n}(\rho A)_e + \psi_{2n}(\rho I)_e \right] dz =$$

$$= K_A(\delta_A - \delta) - K_\delta \delta - K_D \dot{\delta} \tag{73}$$

In writing this expression, $\bar{a}_y$ has been substituted for $\dot{\bar{v}}_y - \dot{\theta}\bar{v}_x$. This follows from differentiating the expression for the velocity of the center of gravity:

$$\widetilde{\bar{v}} = \bar{v}_x \tilde{i} + \bar{v}_y \tilde{j} \tag{74}$$

$$\tilde{\bar{a}} = \frac{d\widetilde{\bar{v}}}{dt} = (\dot{\bar{v}}_x + \dot{\theta}\bar{v}_y)\tilde{i} + (\dot{\bar{v}}_y - \dot{\theta}\bar{v}_x)\tilde{j} \tag{75}$$

11. SUMMARY OF PERTURBATED EQUATIONS OF MOTION

In summarizing the equations of motion derived above, they will be written in terms of small perturbations of θ and β about their nominal values at the time of flight in question. These nominal values are found from another program which computes the trajectory taking into account slowly varying quantities such as missile mass, air density, etc., but neglecting the "high" frequency motions considered here. Denoting these nominal values by $\theta_o = \beta_o$ (only β is computed in the trajectory studies, so the nominal value of θ is defined here as $\theta_o = \beta_o$) one can write

$$\theta = \beta_o + \Theta \tag{76}$$

$$\sin\theta = \sin\beta_o \cos\Theta + \cos\beta_o \sin\Theta \tag{77}$$

$$\cos\theta = \cos\beta_o \cos\Theta - \sin\beta_o \sin\Theta \tag{78}$$

Considering only small values of the perturbation angle Θ, these last expressions can be approximated by

$$\sin\theta \approx \sin\beta_o + \Theta\cos\beta_o \tag{79}$$

$$\cos\theta \approx \cos\beta_o - \Theta\sin\beta_o \tag{80}$$

For the angle β of the velocity vector, one can write

$$\beta \approx \beta_o + \frac{\bar{v}_y}{v_o} \tag{81}$$

Using (76), (79), (80) and (81) in the expressions derived in the previous sections, one can write down the perturbated equations of motion. These are listed below along with a summary of other equations derived in the report. Equations which are merely listed here with no changes will be given the same number as in the body of the report. Those which are incorporated with the perturbation equations above will be given their number from the body of the report plus an asterisk.

Deflection Function

$$u_i(x,t) = -(K_o + K_1 x)\,\delta + \sum_{n=1}^{\infty} \phi_{in}\, q_n(t) \tag{17}$$

$$\Psi_i(x,t) = -K_1\,\delta + \sum_{n=1}^{\infty} \psi_{in}\, q_n(t) \tag{18}$$

where

$$K_o = \frac{m_e \bar{z}}{M} \tag{12}$$

$$K_1 = \frac{1}{I_c}\left(I_e + m_e \ell_e \bar{z}\right) \tag{16}$$

$$I_e = \int_o^b \left[z^2(\rho A)_e + (\rho I)_e\right] dz \qquad (14),(16)$$

$$I_c = \sum_{i=1}^{R} \int_{\ell_i} \left[x^2(\rho A)_i + (\rho I)_i\right] dx + \sum_{j=1}^{N} x_j^2\, m_j \tag{52}$$

Motion of Mass Center and Rotation

$$M\bar{a}_x = -T_s - T_c - Mg(\cos\beta_o - \Theta \sin\beta_o) + F_x^* \qquad (49)^*$$

$$M\bar{a}_y = -T_c(u_e' + \delta) - T_s u_e' + Mg(\sin\beta_o + \Theta\cos\beta_o) + F_y^* \qquad (50)^*$$

$$I_c\ddot{\theta} = \left[T_c\,\delta + (T_c + T_s)\,u_e'\right]\ell_e - (T_c + T_s)\,u_e + M_{c.g.}^* \qquad (51)^*$$

where F_x^* is the drag force on the missile, found from trajectory data for the time of flight in question. Also, F_y^* and $M_{c.g.}^*$ are the resultant aerodynamic force in the y-direction and the resultant aerodynamic moment about the mass center, respectively. Specific expressions for these are given in Section 7 on aerodynamics.

Bending-Sloshing Equations

$$M(\ddot{q}_n + 2b_n\omega_n\dot{q}_n + \omega_n^2 q_n) = -\ddot{\delta}\int_o^b \left[z(\rho A_e)\,\phi_{2n} + (\rho I_e)\,\psi_{2n}\right]dz$$
$$-\int_{-\ell_1}^{\ell_2} p(x,t)\,\phi_{1n}(x)\,dx - \left[T_c\,\delta + (T_c + T_s)u_e'\right]\phi_{2n}(\ell_e)$$
$$-\sum_{m=1}^{\infty} C_{mn}\dot{q}_m \qquad (53),(64)$$

where

$$C_{mn} = \sum_{j=1}^{N} 2\gamma_j m_j \omega_j \zeta_{jm}\,\zeta_{jn} \qquad (61)$$

and

$$b_n = b_n(\text{structural}) + \frac{1}{2M\omega_n v_o}\int_{-\ell_1}^{\ell_2} \omega_a(x)\,\phi_{1n}^2 dx \qquad (34)$$

Detailed expressions for the integrals $\int_{-\ell_1}^{\ell_2} p\,\phi_{1n}dx$ are given in Section 7.

Total Angle of Attack

$$a_T = \frac{1}{v_o}(\bar{v}_y - K_o\dot{\delta} - v_w) - \Theta - K_1\,\delta + \sum_{n=1}^{N} a_n q_n(t) \qquad (36)^*$$

where

$$\bar{v}_y = \int_o^t (\bar{a}_y + \dot{\theta}\bar{v}_x)\,dt$$

and a mean value of $\bar{v}_x$ is used in order to linearize the equation.

Engine Swiveling Equation

$$(I_e - K_o m_e \bar{z} - K_1^2 I_c)\ddot{\delta} - K_1 I_c \ddot{\Theta} + m_e \bar{z}(g \cos\beta_o - \bar{a}_x)(\delta + u_e')$$

$$+ m_e \bar{z}(\bar{a}_y - g\cos\beta_o - \Theta\cos\beta_o) + \sum_{n=1}^{\infty} \ddot{q}_n \int_o^b \left[z(\rho A)_e \phi_{2n} + (\rho I)_e \psi_{2n}\right] dz$$

$$= K_A(\delta_A - \delta) - K_\delta \delta - K_D \dot{\delta} \qquad (73)^*$$

where a nonlinear term $-m_e \bar{z} g \Theta (\delta + u_e') \sin\beta_o$ has been dropped (this is an extremely small term) from the left hand side, and a mean value is to be taken for $\bar{a}_x$ in order to make the equation linear (the terms neglected for this approximation are also negligibly small).

12. BENDING MOMENT AND SHEAR BY MODE ACCELERATION

The bending moment at any section in the i^{th} branch of the missile can be written as the sum of the bending moments at that section due to each mode:

$$M_i(x, t) = \sum_{n=1}^{\infty} M_{in}(x)\, q_n(t) \tag{82}$$

where

$$M_i(x) = (EI)_i \frac{d\psi_{in}}{dx}$$

This is called the "mode-displacement" expression for the moment. If (82) is used as it stands, however, many modes will have to be used because the convergence of moment and shear will be quite poor, particularly since from 85 percent to 95 percent of the generalized force acting on the modes is due to the concentrated engine force. The convergence is poor mainly because if (82) is used, both the "static" and dynamic loads are expanded into modes. If, for example, the missile were flying trimmed at a fixed engine angle and there were no dynamics involved, many modes would be needed to find the shear near the engine gimble point. On the other hand, if one were to forget modes and merely apply the equations of statics, computation of the moments and shears would be exact and found quite simply for this case. If the system is allowed to respond dynamically, of course, the response of the bending modes must be considered. But rather than applying (82) directly, it is quite advantageous from the point of view of convergence to use the dynamic response of the bending modes merely as a correction to the "pseudo-static" response, which is found by neglecting the inertia forces in the bending modes. To see how this can be done, recall that the bending-sloshing equations are

$$\ddot{q}_n + \omega_n^2 q_n = \frac{Q_n(t)}{M} \tag{83}$$

The reason for not including the damping terms will be discussed later. The pseudo-static moment $M_{is}(x, t)$ in the i^{th} branch of the missile will be defined as the moment that would result if the inertia forces of bending-sloshing modes are neglected. Rigid body inertia forces are not neglected because they are the only reaction forces for the rigid body modes. Neglecting the bending-sloshing inertia forces means setting $\ddot{q}_n = 0$ in (83) so that in this case, substitution of (83) into (82) gives

$$M_{is}(x,t) = \sum_{n=1}^{\infty} M_{in}(x) \frac{Q_n(t)}{M\omega_n^2} \tag{84}$$

If the inertia forces of bending are included, substitution of (83) into (82) gives

$$M_i(x,t) = \sum_{n=1}^{\infty} M_{in}(x) \left(\frac{Q_n(t)}{M\omega_n^2} - \frac{\ddot{q}_n}{\omega_n^2} \right) \tag{85}$$

Combining (84) and (85) this total moment can also be written

$$M_i(x,t) = M_{is}(x,t) - \sum_{n=1}^{\infty} M_{in}(x) \frac{\ddot{q}_n(t)}{\omega_n^2} \tag{86}$$

Equations (61) and (86) are identical, of course, and nothing has been gained if one thinks of $M_{is}(x,t)$ as coming from (84). If, however, one realizes that $M_{is}(x,t)$ is the moment that results from applying the external forces, which give rize to $Q_n(t)$, to the missile considering it as rigid (<u>and also including the rigid body inertia forces</u>) a great improvement in convergence results. When the moment $M_{is}(x,t)$ is found by such a separate calculation, the only terms in (86) which depend on the mode response have ω_n^2 in the denominator and converge faster than those in (82), especially for rather slowly varying concentrated loads, which is the case here for the force applied at the engine gimble point.

If the damping term $2b_n \omega_n \dot{q}_n$ were included in (83), a term

$$- \sum_{n=1}^{\infty} M_{in}(x) \frac{2b_n \dot{q}_n}{\omega_n}$$

would appear in (86). Since the damping ratios b_n will be small, these terms will also be small compared to the other terms in (86). To see this, one can examine the ratio between the contribution to the moment or shear from such $\dot{q}_n$ terms and from the $\ddot{q}_n$ terms in (86). If the coordinate q_n is oscillating at a frequency ω, this ratio ($\dot{q}_n$ contribution over $\ddot{q}_n$ contribution) for either moment or shear is $2b_n\omega_n/\omega$. Thus, for high driving frequencies ω, this ratio is certainly small, and for low frequencies, the psuedo-static loads predominate. At intermediate frequencies $\omega \approx \omega_n$, the fact that b_n is very small (in the range of 0.01) insures that velocity terms can be neglected and equations such as (86) are satisfactorily accurate.

Identical reasoning leads to the following "mode-acceleration" expression for shear:

$$V_i(x,t) = V_{is}(x,t) - \sum_{n=1}^{\infty} V_{in}(x) \frac{\ddot{q}_n}{\omega_n^2} \tag{87}$$

where

$$V_{in}(x) = (KAG)_i \left(\frac{d\phi_{in}}{dx} - \Psi_{in} \right)$$

It should be mentioned that the interactions between the bending displacements $q_n(t)$ and the forces acting on the missile have been treated by a "mode-displacement" procedure as seen from Equations (8) and (9) for u_i and Ψ_i. These interactions are entirely with the slopes $(d\phi_{in}/dx)q_n$ and $\Psi_{in} q_n$ and displacements $\phi_{in}q_n$, which converge much more rapidly than shear and moment so that it is not inconsistent to use "mode-acceleration" to find the moment and shear.

Expressions for $V_{is}(x,t)$ and $M_{is}(x,t)$ remain to be determined. There are four general forces to be used in determining these pseudo-static loads:

1. Aerodynamic (branch 1 only, by definition).
2. Missile rigid body inertia forces due to lateral accelerations.

3. Swiveling engine inertia forces.

4. Lateral engine thrust forces.

If one computes these loads by integrating from the tip of the missile, and is not interested in the loads in the engine bells themselves, only the first two of these forces need be considered directly, because the effect of the latter two are included through the previously derived equations of motion which insure that the missile is always in dynamic equilibrium.

At this point it is helpful to reiterate the definitions concerning the branches which make up the missile. These definitions are shown in Figure 5 and are listed below:

1. Only the main branch $(i = 1)$ is exposed to aerodynamic forces. This assumption has been made only in the interest of keeping the nomenclature within reason. If one wishes to apply these equations to a missile having more than one branch exposed, he can easily change the formalism of these derivations to fit his case.

2. The engines are defined as branch $i = 2$. If in the computation of bending modes they are actually treated as part of the main beam, there will be no $i = 2$ branch.

3. All of the other sub-branches are attached to the main branch at only one point; $x = h_i$. At this point, the bending slope ψ and total deflection ϕ of the sub-branch are the same as that of the main beam.

4. The other ends of the sub-branches, located at $x = d_i$, are free.

5. These sub-branches are to be numbered sequentially according to their attach points, starting from the nose of the missile. Thus:

$$h_3 < h_4 < \ldots h_i \ldots < h_{R-1} < h_R$$

Consider first the psuedo-static shear and moment in the sub-branches. The only lateral forces acting on these branches, with the above assumptions, are due to the lateral acceleration a_{yi} of points along the branch. This

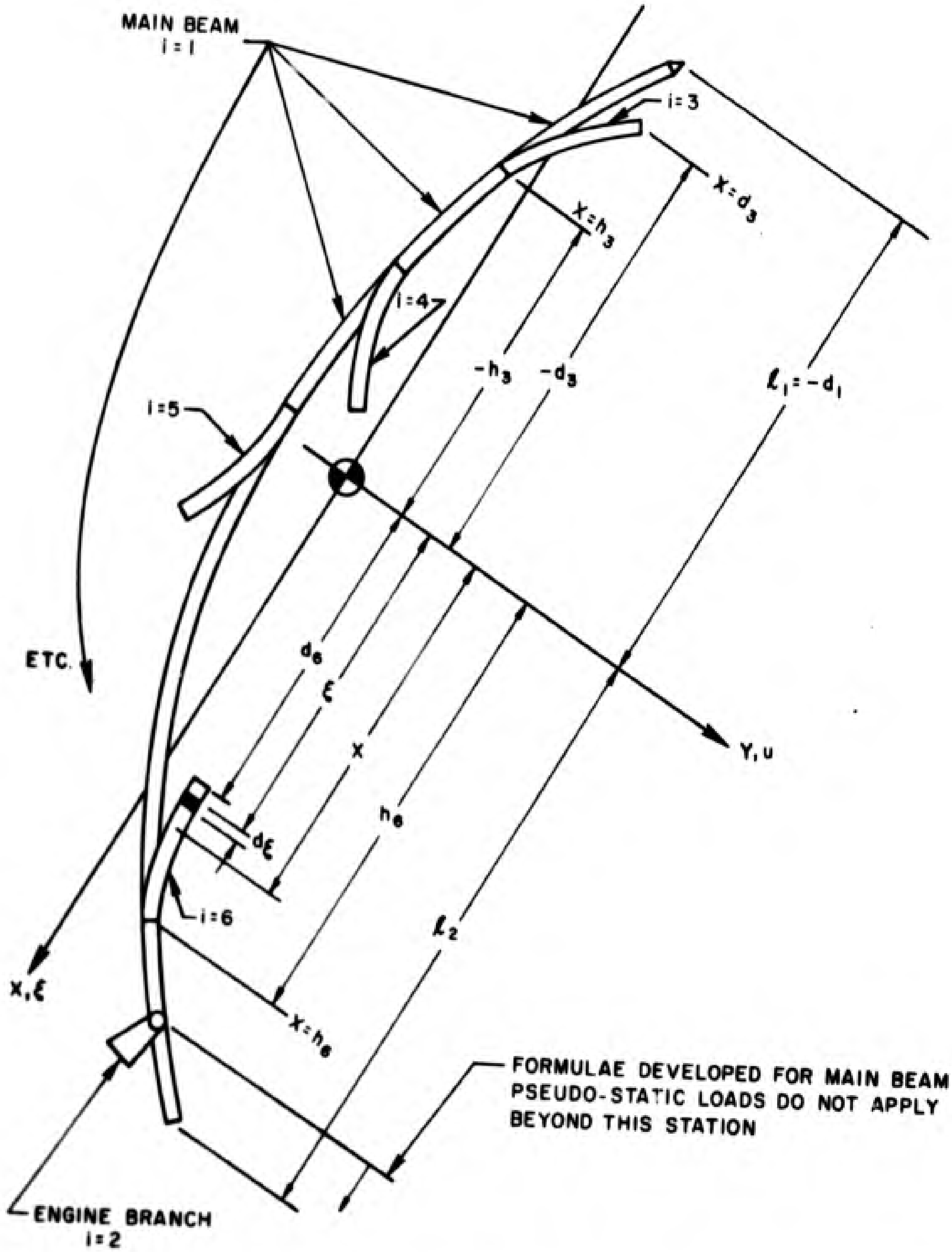


Figure 5. Branch Nomenclature Example for R = 6.

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acceleration is given by (23). Dropping the term with $\dot{\theta}^2$ and noting that $\ddot{u}_i = -(K_o + K_1 x)\ddot{\delta}$ for a rigid missile, the acceleration to be used here is given by

$$a_{yi} = \bar{a}_y - K_o\ddot{\delta} - x\ddot{\theta} - K_1 x\ddot{\delta}$$

Integrating the resulting lateral inertia loads, shown in Figure 6, the psuedo-static shear and moment in the i^{th} branch, $i = 3, 4, \ldots R$, are

$$V_{is}(x, t) = -\int_{d_i}^{x} (\rho A)_i (\bar{a}_y - K_o\ddot{\delta} - \xi\ddot{\theta} - K_1 \xi\ddot{\delta})\, d\xi \tag{89}$$

and

$$M_{is}(x, t) = -\int_{d_i}^{x} (x - \xi)(\rho A)_i (\bar{a}_y - K_o\ddot{\delta} - \xi\ddot{\theta} - K_1 \xi\ddot{\delta})\, d\xi$$
$$-\int_{d_i}^{x} (\rho I)_i (\ddot{\theta} + K_1\ddot{\delta})\, d\xi \tag{90}$$

where the dummy variable ξ is measured from the same origin as x, as shown in Figure 5. Equations (89) and (90) were written with a branch projecting toward the nose of the missile in mind, but if one writes similar equations for a rearward facing branch, the expressions are identical because the sign in front of the integrals must be changed at the same time that the limits of integration are reversed. Also, these expressions have been written as though there were no slosh masses in the sub-branches merely to avoid the clumsy nomenclature that would result for the summations over limited ranges of j. If a branch indeed contains slosh masses, all of the formulae derived here can be used if $(\rho A)_i$ is taken to have appropriate discontinuities at the slosh mass attach points.

Taking the integrals over the branches out as parameters to be determined for each missile station of interest, Equations (89) and (90) become

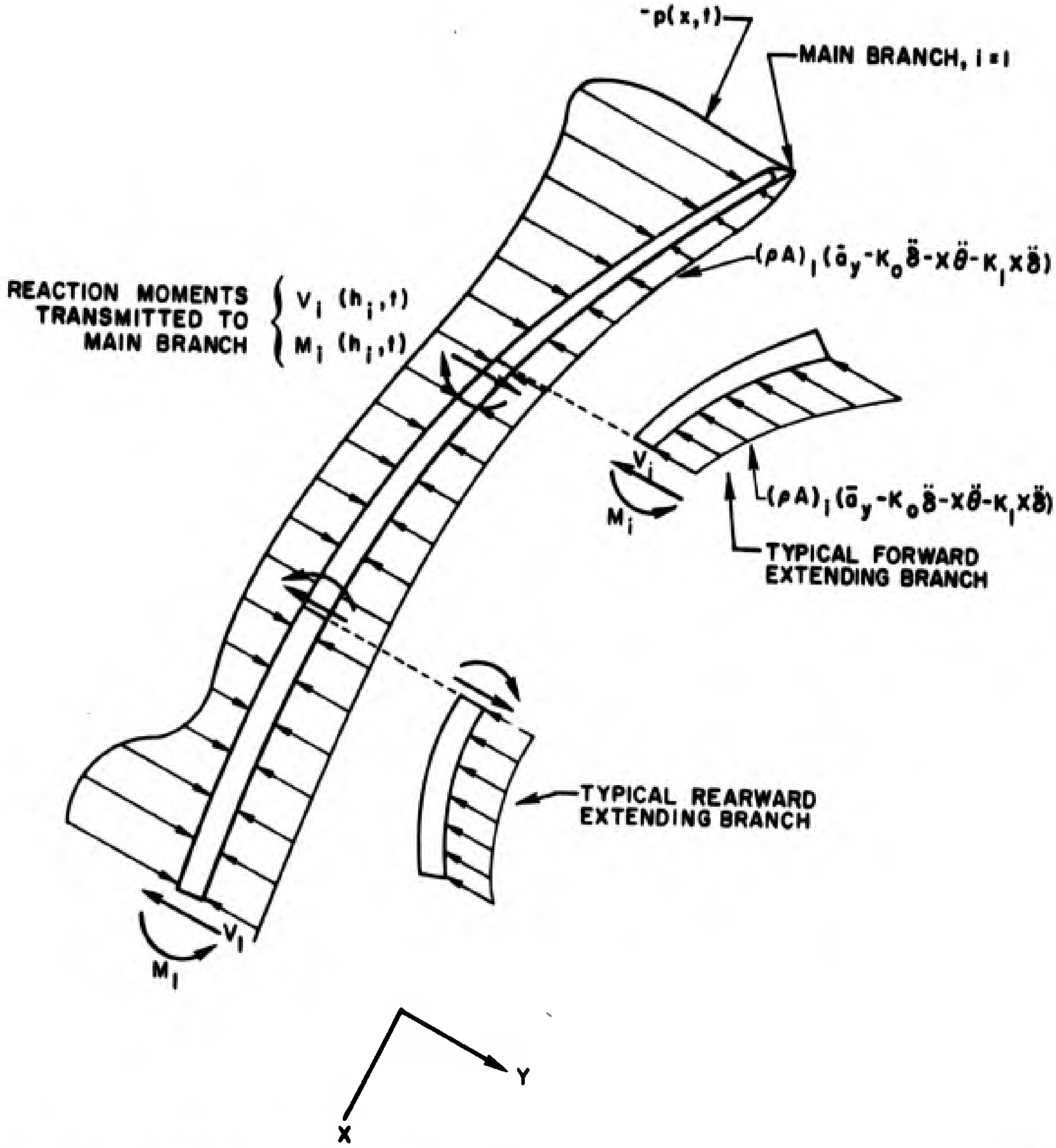

Figure 6. Free Bodies of Branches.
(All moments and shears are drawn in the positive sense for the coordinate system of Reference 6, from which the bending mode program was written.)

$$V_{is}(x,t) = -P_{1i}(\bar{a}_y - K_o\ddot{\delta}) + P_{2i}(\ddot{\theta} + K_1\ddot{\delta}) \tag{91}$$

$$M_{is}(x,t) = (P_{2i} - xP_{1i})(\bar{a}_y - K_o\ddot{\delta}) + (xP_{2i} - P_{3i})(\ddot{\theta} - K_1\ddot{\delta}) \tag{92}$$

where

$$\begin{aligned} P_{1i}(x) &= \int_{d_i}^{x} (\rho A)_i \, d\xi \\ P_{2i}(x) &= \int_{d_i}^{x} \xi(\rho A)_i \, d\xi + \sum_{j=1}^{N_1} x_j m_j \\ P_{3i}(x) &= \int_{d_i}^{x} \left[\xi^2 (\rho A)_i + (\rho I)_i\right] d\xi + \sum_{j=1}^{N_1} x_j^2 m_j \end{aligned} \tag{93}$$

In Equations (93), it has now been convenient to actually give the slosh mass contributions explicitly. The summations are to be taken over all of the slosh masses between station x and the free end of the i^{th} branch.

The shear and moment in the main beam are found in the same way, except that aerodynamic forces must be included, as well as the cumulative sum of the shears and moments transmitted to the main beam by the sub-branches. Again integrating from the nose end of the missile, the shear and moment in the main beam are given by

$$\begin{aligned} V_{1s}(x,t) = &-\int_{-\ell_1}^{x} \left[p(\xi,t) + (\rho A)_1 (\bar{a}_y - K_o\ddot{\delta} - \xi\ddot{\theta} - K_1 \xi\ddot{\delta})\right] d\xi \\ &- \sum_{j=1}^{N_1} m_j (\bar{a}_y - K_o\ddot{\delta} - x_j\ddot{\theta} - K_1 x_j \ddot{\delta}) + \sum_{i=3}^{R_1} \pm V_{is}(h_i, t) \end{aligned} \tag{94}$$

where the reaction shears $V_{is}(h_i, t)$ are to be added in the positive sence for forward facing branches, and in the negative sense for rearward facing branches, (see Figure 6). Similar care must be taken for the reaction moments in (95):

$$M_{1s}(x,t) = -\int_{-\ell_1}^{x} (x-\xi)\left[p(\xi,t) + (\rho A)_1 (\bar{a}_y - K_o\ddot{\delta} - \xi\ddot{\theta} - K_1\xi\ddot{\delta})\right] d\xi$$

$$-\int_{-\ell_1}^{x} (\rho I)_1 (\ddot{\theta} + K_1\ddot{\delta})\, d\xi - \sum_{j=1}^{N_1} x_j m_j (\bar{a}_y - K_o\ddot{\delta} - x_j\ddot{\theta} - K_1 x_j\ddot{\delta})$$

$$+ \sum_{i=3}^{R_1} \pm M_{is}(h_i, t) \tag{95}$$

where R_1 is the index of the last branch on the nose side of the station x at which the moment and shear are to be evaluated. That is, R_1 is adjusted such that $h_{R_1} < x < h_{R_1+1}$. Similarly, N_1 is the index of the last slosh mass between x and the nose of the missile. Using $p(\xi, t)$ as given by (38), the pseudo-static moment and shear in the main beam are given by

$$V_{1s}(x,t) = -L_1\alpha_T - L_2\dot{\theta} - L_4\dot{\alpha}_T - P_{11}(\bar{a}_y - K_o\ddot{\delta})$$

$$+ P_{21}(\ddot{\theta} + K_1\ddot{\delta}) + \sum_{i=1}^{R_1} \pm V_{is}(h_i, t) \tag{96}$$

$$M_{1s}(x,t) = (v_o L_2 - xL_1)\alpha_T + (L_3 - xL_2)\dot{\theta} + (L_5 - xL_4)\dot{\alpha}_T$$

$$+(P_{21} - xP_{11})(\bar{a}_y - K_o\ddot{\delta}) + (xP_{21} - P_{31})(\ddot{\theta} + K_1\ddot{\delta})$$

$$+ \sum_{i=1}^{R_1} \pm M_{is}(h_i, t) \tag{97}$$

where

$$
\begin{aligned}
L_1 &= \int_{-\ell_1}^{x} w_a(\xi)\, d\xi \\
L_2 &= \frac{1}{v_o} \int_{-\ell_1}^{x} \xi w_a(\xi)\, d\xi \\
L_3 &= \frac{1}{v_o} \int_{-\ell_1}^{x} \xi^2 w_a(\xi)\, d\xi \\
L_4 &= \rho v_o \int_{-\ell_1}^{x} S(\xi)\, d\xi \\
L_5 &= \rho v_o \int_{-\ell_1}^{x} \xi S(\xi)\, d\xi
\end{aligned}
\qquad (98)
$$

and the P's are evaluated using (93) noting that $d_1 = -l_1$. Notice that the integrals in (98) are all running integrals which were evaluated over the whole missile in the section on aerodynamics. Thus, the only complication added here is that the values of the integrals needed previously should be printed out for values of x of interest for loads, as well as over the entire missile.

If one wishes to simplify these expressions for moment and shear, it is worthwhile to mention that aerodynamic forces due to pitching $\dot{\theta}$ and plunging $\dot{a}_T$ have been included in this study mainly because of their effect on rigid body stability. Their contribution to moment and shear will be quite small and if an appreciable saving in computation time or complexity can be made, it would be quite reasonable to omit the terms in (96) and (97) containing $\dot{\theta}$ $\dot{a}_T$. This would eliminate the need for tabulating the running integrals for L_3, L_4 and L_5.

13. TABULATION OF INTEGRALS

For convenience, all of the integrals which define the coefficients of the differential equations are listed here.

Integrals Involving Only the Rigid Missile

$$I_e = \int_o^b \left[z^2 (\rho A)_e + (\rho I)_e\right] dz \qquad (14), (16)$$

$$I_c = \sum_{i=1}^{R} \int_{\ell_i} \left[x^2 (\rho A)_i + (\rho I)_i\right] dx + \sum_{j=1}^{N} x_j^2 m_j \qquad (52)$$

$$P_{1i}(x) = \int_{d_i}^{x} (\rho A)_i \, d\xi$$

$$P_{2i}(x) = \int_{d_i}^{x} \xi (\rho A)_i \, d\xi + \sum x_j m_i \qquad (93)$$

$$P_{3i}(x) = \int_{d_i}^{x} \left[\xi^2 (\rho A)_i + (\rho I)_i\right] d\xi + \sum x_j^2 m_j$$

The j summation is over all slosh masses between station x and the free end of the branch in question. The "free" end of branch $i = 1$ is defined as the nose of the missile. For this branch, the location of slosh masses in a sub-branch must be interpreted as acting at $x = h_i$ when deciding whether it is between station x and the nose of the missile.

$$m_e = \int_o^b (\rho A)_e \, dz$$

$$m_e \bar{z} = \int_o^b z (\rho A)_e \, dz$$

$$L_4(x) = \rho v_o \int_{-\ell_1}^{x} S(\xi)\, d\xi \quad , \quad L_4(\ell_2) = N_{\dot{a}}$$

$$L_5(x) = \rho v_o \int_{-\ell_1}^{x} \xi S(\xi)\, d\xi \quad , \quad L_5(\ell_2) = M_{\dot{a}} \qquad (42), (98)$$

Integrals Involving the Bending-Sloshing Modes and Aerodynamic Data

Equations (3) and (4) giving the integrals for the generalized mass and for $M\omega_n^2$ are not repeated here because they are already programmed in the bending-sloshing mode package.

$$- \frac{Q_{n\delta}}{\ddot{\delta}} = \int_{o}^{b} \left[z(\rho A)_e \phi_{2n} + (\rho I)_e \psi_{2n} \right] dz \qquad (63)$$

$$2M\omega_n \Delta b_n = \int_{-\ell_1}^{\ell_2} w_a(x)\, \phi_{1n}^2(x)\, dx \qquad (c)$$

$$L_1(x) = \int_{-\ell_1}^{x} w_a(\xi)\, d\xi \quad , \quad L_1(\ell_2) = N_a \qquad (42), (98)$$

$$L_2(x) = \frac{1}{v_o} \int_{-\ell_1}^{x} \xi w_a(\xi)\, d\xi \quad , \quad L_2(\ell_2) = M_a \qquad (45), (98)$$

$$L_3(x) = \frac{1}{v_o} \int_{-\ell_1}^{x} \xi^2 w_a(\xi)\, d\xi \quad , \quad L_3(\ell_2) = M_{\dot{\theta}} \qquad (45), (98)$$

$$H_{an} = \int_{-\ell_1}^{\ell_2} w_a(x)\, \phi_{1n}(x)\, dx$$

$$H_{\dot{a}n} = \rho v_o \int_{-\ell_1}^{\ell_2} S(x)\, \phi_{1n}(x)\, dx \tag{48}$$

$$H_{\dot{\theta}n} = \frac{1}{v_o} \int_{-\ell_1}^{\ell_2} x w_a(x)\, \phi_{1n}(x)\, dx$$

Total number of integrals = 8 + 3i + 5n.

14. NOMENCLATURE

The principle notation used is defined below. Symbols used only in intermediate steps of deviations are defined where used and are not listed here.

A = cross-sectional area of missile section.

$(\rho A)_i$ = mass per unit length of i^{th} branch.

$C_o(t)$, $C_1(t)$ = variables in deflection function; defined by (12) and (15).

C_{mn} = coefficient of $\dot{q}_m$ in generalized force of n^{th} mode; defined by (61).

E = modulus of elasticity.

$(EI)_i$ = bending stiffness of i^{th} branch.

F_x^*, F_y^* = resultant aerodynamic forces in the positive x and y directions, respectively. See (41).

G = shear modulus of elasticity.

$(KAG)_i$ = shear stiffness of i^{th} branch.

$H_{\alpha n}$, $H_{\dot{\alpha} n}$, $H_{\dot{\theta} n}$ = aerodynamic coefficients in generalized force of the n^{th} mode; defined by (48).

I = total mass moment of inertia of missile.

I_c = constant part of I; defined by (52).

I_e = moment of inertia of the swiveling engine or engines about the gimbal point; defined by (14) and (16).

$(\rho I)_i$ = mass moment of inertia per unit cross section of the i^{th} branch.

K_o, K_1 = constants in the expression for the deflection function; defined by (12) and (16).

K_j = spring constant at j^{th} slosh spring-mass system.

K_A, K_D, K_δ = torsional spring constants in the generalized force of the engine swiveling equation; defined by (72).

L_1, L_2, L_3 ... = aerodynamic coefficients in expressions for pseudo-static loads; defined by (98).

$M = \sum_{i=1}^{R} \int_{\ell_i} (\rho A)_i \, dx + \sum_{j=1}^{R} m_j$ = total mass of missile.

M_i = bending moment in i^{th} branch of the missile, including dynamics; defined by (86).

M_{is} = psuedo-static bending moment in i^{th} branch; defined by (92) and for $i = 1$ by (97).

$\widetilde{M}_{c.g.}$ = moment about the mass center of all external forces; defined by (51).

$M^*_{c.g.}$ = moment about the mass center of the aerodynamic forces; defined by (44).

M_α, $M_{\dot{\alpha}}$, $M_{\dot{\theta}}$ = aerodynamics coefficients in expression for $M^*_{c.g.}$, defined by (45).

N = total number of liquid tanks or slosh masses.

$N_1 = 1, 2, \ldots N$ = number of slosh masses between the nose of the missile and section x; used in (94) and (95).

N_α, $N_{\dot{\alpha}}$, $N_{\dot{\theta}}$ = aerodynamic coefficients in expression for F^*_y; defined by (42).

$P_i = P_i(x) = -a_x \int_{d_i}^{x} (\rho A)_i \, dx - a_x \sum_{j=1}^{N_1} m_j$ = total axial compression force in i^{th} branch.

P_{1i}, P_{2i}, P_{3i} = constants in expressions for pseudo-static loads; defined by (93).

Q_n, Q_δ = generalized forces in bending and engine-swiveling equations, respectively.

R = total number of branches, including the main beam, $i=1$, and the swiveling engine, $i=2$.

$R_1 = 3, 4 \cdots R$ = number of branches with attach points between the nose of the missile and section x, used in (94) and (95)

$S = S(x) = \pi R_o^2$ = gross external crossectional area of the main branch; R_o is the outside diameter of the missile.

T_c = thrust of the swiveling control engine or engines.

T_s = thrust of engine or engines which do not swivel in the plane of motion being considered.

T = total kinetic energy of the missile.

$T_u, T_\Psi, T_\delta, T_\delta^*$ = component parts of the kinetic energy; defined in section 6.

V = potential energy.

V_i = total shear in the i^{th} branch, including dynamics; defined by (87).

V_{is} = pseudo-static shear in the i^{th} branch, defined by (91) and for $i=1$ by (96).

a_{xi}, a_{yi} = acceleration components of a point on the i^{th} branch; defined by (23).

$\bar{a}_x$, $\bar{a}_y$ = acceleration components of the mass center; defined by (75).

b_n = damping ratio of the n^{th} bending-sloshing mode, from the structure (estimated or measured) and from aerodynamics, calculated using (34).

d_j = x-coordinate of the free end of the i^{th} branch.

g = acceleration of gravity.

$i = 1, 2, \ldots R$ = index, referring to the i^{th} branch, numbered sequentially according to their attach points, starting from the nose of the missile; except for the main beam, $i = 1$; and the swiveling engine, $i = 2$.

$j = 1, 2, \ldots N$ = index, referring to the j^{th} slosh mass, numbered sequentially according to their attach points, starting from the nose of the missile.

K_j = spring constant of the j^{th} sloshing spring-mass system.

K = shape factor in shear rigidity KAG.

ℓ_1 = distance from mass center to the nose end.

ℓ_2 = distance from mass center to the tail end.

ℓ_e = distance from mass center to the engine hinge point.

m_e = total mass of the swiveling engine or engines.

$m, n = 1, 2, 3, \ldots$ = indicies, referring to the m^{th} and n^{th} bending-sloshing modes; $1 \leq m, n \leq N$ are "slosh" modes and $m, n > N$ are "bending" modes.

m_j = mass of the j^{th} sloshing spring-mass system.

$p(x, t)$ = aerodynamic force per unit length acting in the y-direction.

$q_n = q_n(t)$ = generalized coordinate of the n^{th} bending-sloshing mode.

t = time.

$u_i = u_i(x, t)$ = deflection of the i^{th} branch relative to the x, y-axes; defined by (2) or (17).

u_j = deflection of the branch containing the j^{th} slosh mass at $x = x_j$.

$u_e = u_e(\ell_e)$ = deflection at the engine hinge point.

u_e' = du/dx at $x = \ell_e$.

v_o = magnitude of the velocity of the mass center of the missile.

v_w = magnitude of the wind velocity, assumed directed parallel to the ground.

v_{xi}, v_{yi} = velocity components of a point on the axis of the i^{th} branch; defined by (22).

$\bar{v}_x, \bar{v}_y$ = velocity components of the mass center of the missile; defined by (74).

$w_\alpha, w_{\dot{\alpha}}, w_{\dot{\theta}}$ = aerodynamic forces per unit length per unit α, $\dot{\alpha}$ and $\dot{\theta}$, respectively; defined by (39).

x, y = coordinates of the moving axes.

x_j = coordinate of attach-point of j^{th} slosh mass.

z = coordinate of points along the engine, measured from the engine hinge point.
NOTE: $x = \ell_e + z$ in the region $x \geq \ell_e$.

$\bar{z}$ = distance from the engine hinge-point to the mass center of the engine.

α_T = total angle of attack which gives rise to aerodynamic forces; defined by (36).

β = angle from the vertical to the tangent of the trajectory of the mass center.

β_o = nominal value of β for region of time being studied, as determined from trajectory program.

δ = angle of swiveling of the control engine.

δ_A = equivalent swiveling angle of the engine actuator.

γ_j = damping ratio of the first mode sloshing motion in the isolated j^{th} liquid tank.

ζ_{jn} = that part of the n^{th} eigenfunction which represents the stretch in the j^{th} slosh spring.

θ = angle from the vertical to the x-axis.

Θ = $\theta - \beta_o$ = perturbed angle of the x, y-coordinate system, measured from the nominal angle $\theta = \beta_o$.

ρ = air density at nominal altitude given by trajectory program.
NOTE: when ρ refers to the mean density of the missile, it always appears as $(\rho A)_i$ or $(\rho I)_i$.

$\varphi_{in} = \varphi_{in}(x)$ = eigenfunction representing the total deflection of the i^{th} branch (due to both bending and shear) for the n^{th} bending-sloshing mode of the missile.

ϕ_{jn} = $\phi_{in}(x_j)$ = that part of the n^{th} mode eigenfunction which represents the total deflection of the branch containing the j^{th} slosh mass at the slosh mass attach point.

$\psi_{in} = \psi_{in}(x)$ = eigenfunction representing the slope due to bending alone of the i^{th} branch for the n^{th} bending-sloshing mode.

$\psi_{jn} = \psi_{in}(x_j)$ = that part of the n^{th} mode eigenfunction which represents the slope due to bending alone of the branch containing the j^{th} slosh mass, at the slosh mass attach point.

ω_n = circular frequency of the n^{th} bending-sloshing mode.

$\omega_j = \sqrt{K_j/m_j}$ = circular frequency of the first sloshing mode in the isolated j^{th} liquid tank.

$\Psi_i = \Psi_i(x, t)$ = angle of rotation relative to the x, y-axes of a transverse slice of the i^{th} branch; defined by (9) or (18).

$(\ )'$ = differentiation with respect to x.

$(\dot{\ })$ = differentiation with respect to t.

$\int_{\ell_i}$ = integration over the i^{th} branch between the limits $x = d_i$ to $x = h_i$ (more explicitly defined in Reference 3).

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IN-FLIGHT DYNAMICS OF A FLEXIBLE MISSILE

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I. INTRODUCTION

The general design of a missile is established on the basis of over-all performance, treating the airframe as a rigid body.. Using the resulting preliminary design values, such as skin thicknesses, propellant configurations, materials, etc., more detailed analyses can be made which take into account the dynamic response of the elastic missile. The purpose of this report is to provide a set of equations which can be used to obtain: (1) general dynamic performance characteristics, such as missile loads and dynamic stability; (2) detailed responses, such as structural vibration, propellant sloshing, and engine motions; and (3) exchange ratios, or the effect of changes in a parameter on the dynamic behavior of the system, such as damping of the propellant sloshing verus autopilot stability and airframe bending moments.

This analysis is intended as a direct extension of the work reported in the series of reports entitled "Generalized Missile Dynamics Analysis" (Reference 1) in which these problems were first attacked. In the present analysis, the pertinent derivations and assumptions given in Reference 1 are repeated for the sake of completeness. Most of the philosophy, methods of approach and coordinate systems will be unchanged from those used in Reference 1, however, experience with the equations has led to several modifications and extensions. The new features to be included are:

1. The use of branched beam modes.
2. The inclusion of the effect of axial acceleration in the computation of bending modes.
3. The inclusion of sloshing directly in the computation of the bending modes rather than as separate coordinates.
4. Explicit formulation of the effect of the damping in the individual tanks on the bending-sloshing modes.
5. Explicit formulation of the interaction of the "slosh" modes with the aerodynamic forces