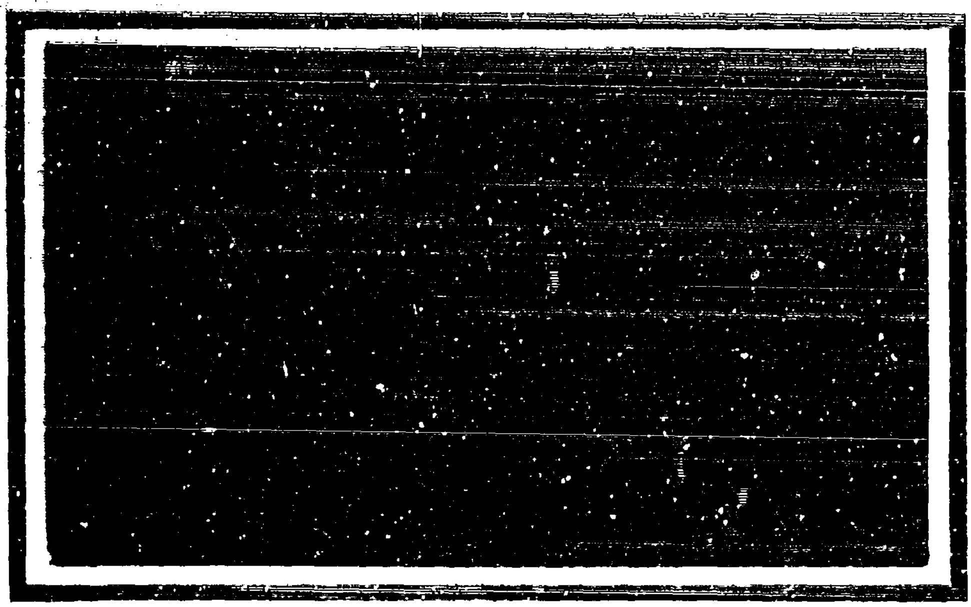
ARL 64-196

INSTITUT FÜR MECHANIK



RHEINISCH-WESTFÄLISCHE

TECHNISCHE HOCHSCHULE AACHEN

Contract AF 61 (o52) - 744
1 September 1963 - 31 August 1964

Annual Summary Report Nº 4

August 1964

Solution of the Sommerfeld-Sexl Characteristic Equation for the Stability of Couette Flow

by

F. Schultz-Grunow

Institut für Mechanik
Technical University
A a c h e n , Germany

The research reported in this document has been sponsored by the AIR RESEARCH AND DEVELOPMENT COMMAND UNITED STATES AIR FORCE, under contract AF 61 (o52)-744 through the European Office, ARDC.

C O N T E N T S

Solution of the Sommerfeld-Sexl Characteristic Equation for the Stability of Couette Flow.

1.) Introduction

One of the basic problems of hydrodynamics is the stability of Couette flow. This is shown by the importance which since ever was attributed to Couettes observation of transition to turbulence [1] as well as to G.I. Taylors investigations which discovered the occurence of cellular laminar vortices at certain Reynolds numbers [2]. The intrinsic difference of these two observations, once transition to a random eddy motion and on the other hand transition to a laminar vortex pattern is illustrated by the two limiting cases in which they occur, the transition to turbulence if the inner cylinder is at rest and the outer rotating and the generation of laminar cellular vortices in the reverse case when the inner cylinder is rotating and the outer at rest. Correspondingly the mathematical treatment is different. Whilst for the rotating outer cylinder Rayleighs perturbations [3] were introduced which still are the basis for stability investigations in relation to the generation of turbulence, for the rotating inner cylinder G.I. Taylor introduced perturbations forming rotational symmetric cellular vortices.

Refering to previous work on the rotating outer cylinder to which the following considerations are devoted, first A. Sommerfelds investigation of Couette flow between plane parallel walls [4] has to be mentioned. Although in the following Couette flow between concentric cylinders will be regarded, this work has to be mentioned as plane parallel walls represent the limiting case of a small gap between concentric cylinders. Sommerfeld proceeded to the equation which determines the eigenvalues but did not analyse or even discuss this equation. The only comment given to this equation is that it yields the time constant which itself is expected to determine critical Reynolds numbers. Sexl [5] gave the analogue derivation for centric cylinders. The equation which yields the time constant is the same as derived in the following. But also here no solution of this equation was attempted obviously because of the complicated functions occuring in this equation which are similar to Sommerfelds case Bessel functions of complex order and argument. L. Hopf's investigation on plane Couette flow [6] may be left aside as his boundary condition of constant pressure means a free surface and not a rigid surface as it is presented by the cylinders of the Couette apparatus. Finally von Mises treated the stability problem of the plane Couette flow as an oscillation problem [7]. The eigenvalues of a parameter are determined by the zeros of a polynomial. Introducing approximations only real values of this parameter were found whilst the physical meaning of it demands an imaginary quantity. This result was interpretated in the way that Couette flow should be stable for all modes and Reynolds numbers [8]. But this is in contradiction to Couettes experiments [9] as well as to more recent experiments performed by Mallock [10], Taylor [11], Wendt [12]. Sommerfeld regards this contradiction as important enough to ask in his book [13] if the well established method of small oscillations should not be applicable or if finite disturbances have to be introduced or if even the Navier Stokes equations would be insufficient. Later a more cautious interpretation of the theoretical results was given by Lin [14] which expresses that a complete stability proof still is missing.

Recent experimental results [15] however showed that the earlier observations should be regarded with scepticism. Indeed it was found that turbulence at least not needs to occur. Objections on the method of observation [16] of this result may be put aside. It was argued that the immersed particles which made the flow visible should have been sedimentated by centrifugal forces so that no turbulence could be observed. But attention had been payed to this phenomenon. Sedimentation clearly could be observed and turbulence if it had occured would have prevented sedimentation.

The result that Couette flow with rotating outer cylinder may be stable lead the attention to side effects that could have influenced the earlier experimental results. Indeed it could be shown experimentally [15] and theoretically [17] that hidden excentricities and vibrations can be responsible for the transition to turbulence. The theoretical investigation showed that this is not an instability but a seperation effect occuring beyond certain Reynolds numbers. More recently in an already mentioned Göttinger thesis [16] attention was payed to end effects. A heavier liquid was inserted as to seperate the test liquid from the lower end of the apparatus. Obviously the feature of a ball bearing was attributed to the heavier liquid. But as slip neither occurs at walls nor at contacting surfaces of liquids by this provision in no ways the end effect is annihilated but merely reversed.

This review on recent work shows that the intrinsic stability problem of Couette flow with the view to the transition to turbulence jet is not solved satisfactorily. There remains to be deduced a conclusive theoretical stability proof excluding any presumptive side effects on the basis of a Rayleigh perturbation. This will be done in the following in the frame work of a correct linear perturbation theory. The equation which yields the eigenvalues is as was mentioned before identic with the one derived by Sexl [5]. This equation will be analysed by taking full account of the functions occuring in it.

Refering for the sake of completeness to the second problem characterized by a rotating inner cylinder and the outer cylinder at rest Taylors calculations showed excellent agreement with observations [2].

2.) Perturbation Theory

As was mentioned before two kinds of perturbations were introduced in the stability theory of Couette flow. To begin with Taylors rotational symmetric cellular vortex perturbation the following notations will be used: y the radial wall distance; z the axial coordinate; u, v, w the tangential radial and axial perturbation velocity components; t time and β, δ real constants. Then this perturbation is expressed by

$$
\begin{aligned}
u &= e^{\beta t}\, u_1\,(y)\,\cos\gamma\, z\\
v &= e^{\beta t}\, v_1\,(y)\,\cos\gamma\, z\\
w &= e^{\beta t}\, w_1\,(y)\,\sin\gamma\, z
\end{aligned}
$$

It readily may be seen that the continuity equation is satisfied. $\beta < 0$ means stability and $\beta > 0$ instability. On this perturbation bases G.I. Taylors stability criterion for a Couette flow with rotating inner cylinder and an outer cylinder at rest. Application to the reverse case of an inner cylinder at rest and an outer cylinder rotating did not yield results [2] so that for the present purpose this perturbation is not the proper one.

The other kind of perturbation was introduced by Rayleigh [18] . For its representation instead of the polar coordinates r, Ω there will be used the distance y from the outer wall with positive direction to the center and the circumferential coordinate x refering to the outer wall

$$x = r_1 \Omega \quad , \qquad (1$$

where the radius r_1 is related to the outer cylinder. Then the perturbation is represented by the streamfunction

$$\Psi = \varphi\,(y)\; e^{i(\alpha x - \beta t)} \quad . \qquad (2$$

α is a real and β a complex constant:

$$\beta = \beta_r + i\,\beta_i \tag{3}$$

α is related to the wave length λ by

$$\alpha = \frac{2\pi}{\lambda} \tag{4}$$

and to the number k of waves on the circumference

$$k = \alpha\, r_1 \tag{5}$$

β_r is the angular frequency of the perturbation whilst $\beta_i < 0$ means stability, $\beta_i > 0$ instability.

The complex velocity of wave propagation is

$$c = c_r + i\, c_i = \frac{\beta}{\alpha} = \frac{\beta_r + i\,\beta_i}{\alpha} \tag{6}$$

It is related to r_1 according to the meaning of x. The angular velocity of these perturbation waves with radial fronts is

$$\omega = \frac{c_r}{r_1} = \frac{\beta_r}{k} \tag{7}$$

This perturbation will be introduced. It may be mentioned that in this way recently the stability of a liquid rotating as rigid body could be solved in closed form including all modes [19]. The eigenfunctions were found to be the Bessel functions of the first kind. As the zeros of these functions are located on the real axis there exists stability.

3.) Basic Equations

Introducing (2) in the Navier Stokes equations by taking full account of the viscous terms and limiting to linear inertia terms one obtains the differential equation [15]

$$\left[\varphi''-\left(\frac{r_1}{r_1-y}\right)^2\alpha^2\varphi-\frac{\varphi'}{r_1-y}\right]\left(U-\frac{r_1-y}{r_1}c\right)+\varphi\left(-U''+\frac{U'}{r_1-y}+\frac{U}{(r_1-y)^2}\right)$$

$$=-\frac{i\nu}{\alpha}\left[\left(\frac{r_1}{r_1-y}\right)^2\alpha^4\varphi-\frac{2r_1}{r_1-y}\alpha^2\varphi''-\frac{2r_1}{(r_1-y)^2}\alpha^2\varphi'-\frac{4r_1}{(r_1-y)^2}\alpha^2\varphi\right.$$

$$\left.+\frac{r_1-y}{r_1}\varphi^{IV}-\frac{\varphi'}{r_1(r_1-y)^2}-\frac{\varphi''}{r_1(r_1-y)}-\frac{2}{r_1}\varphi'''\right] \qquad (8$$

ν is the kinematic viscosity. U denotes the velocity of the undisturbed Couette flow. If V is the velocity of the outer cylinder, r_o the radius of the inner cylinder one has

$$U=\frac{V\,r_1}{\left(\frac{r_1}{r_o}\right)^2-1}\left(\frac{r_1-y}{r_o^2}-\frac{1}{r_1-y}\right) \qquad (9$$

Thus the velocity field is a superposition of a rigid body rotation and a potential vortex. Introducing (9) in (8) and making use of the operator

$$L(y)=\varphi''-\frac{\varphi'}{r_1-y}-\alpha^2\left(\frac{r_1}{r_1-y}\right)^2\varphi \qquad (10$$

one obtains

$$\left(U-\frac{r_1-y}{r_1}c\right)L=-\frac{i\nu}{\alpha}\left(\frac{r_1-y}{r_1}L''-\frac{1}{r_1}L'-\alpha^2\frac{r_1}{r_1-y}L\right) \qquad (11$$

With (10,11) the fourth order equation (8) is reduced to two equations of second order.

Introducing dimensionless quantities

$$\eta = \frac{y}{r_1} , \quad \xi = \frac{c}{\nu} , \quad R = \frac{V r_1}{\nu} \tag{12}$$

and (5), (10,11) transform to

$$(1-\eta)^2 \varphi'' - (1-\eta)\varphi' - k^2 \varphi = r_1^2 (1-\eta)^2 L , \tag{13}$$

$$\left[\frac{U}{V} - (1-\eta)\xi \right] L = - \frac{i}{k R} \left[(1-\eta)L'' - L' - \frac{k^2}{1-\eta} L \right] . \tag{14}$$

Now introducing (9) and the abbreviations

$$B = \frac{1}{\left(\frac{r_1}{r_0}\right)^2 - 1} \tag{15}$$

$$\rho^2 = k^2 - i k R B \tag{16}$$

$$b^2 = i k R \xi - i k R B \left(\frac{r_1}{r_0}\right)^2 , \tag{17}$$

further introducing the substitution

$$z = b (1-\eta) , \tag{18}$$

(14) transforms if multiplied by $k R/i$ to the Bessel equation [15]

$$z^2 L'' + z L' + (z^2 - \rho^2) L = 0 \tag{19}$$

Introducing the cylinder function Z_ρ of complex order ρ the solution of (19) can be written

$$\frac{b^2}{r_1^2} L = Z_\rho (z) \tag{20}$$

Inserting this solution in (13) one obtains

$$\varphi'' + \frac{1}{z}\,\varphi' - \frac{k^2}{z^2}\,\varphi = Z_\rho\,(z) \tag{21}$$

The solution of the homogenous part of (21) is

$$\varphi_h = C_3\, z^{k} + C_4\, z^{-k} \quad . \tag{22}$$

The particular solution φ_p of (21) is found by the method of variation of constants. Satisfying the boundary condition

$$\varphi_p\,(z_o) = \varphi'_p\,(z_o) = 0 \tag{23}$$

one has

$$\varphi_p = \int_{z_o}^{z} \frac{z^{-k}\xi^{k} - z^{k}\,\xi^{-k}}{-\,\xi^{k} k\,\xi^{-k-1} - k\,\xi^{k-1}\,\xi^{-k}}\, Z_\rho\,(\xi)\, d\,\xi \quad . \tag{24}$$

Here z_o corresponds to $\eta = 0$ what means according to (18)

$$z_o = b \tag{25}$$

From (24) follows

$$\varphi_p = -\,\frac{1}{2k} \int_{z_o}^{z} (z^{-k}\,\xi^{\,k+1} - z^{k}\xi^{\,-k+1})\; Z_\rho\,(\xi)\; d\,\xi \quad . \tag{26}$$

With the notations

$$I_1(z) = \int_{z_o}^{z} \xi^{k+1}\, Z_\rho\,(\xi)\, d\,\xi \;,\quad I_2(z) = \int_{z_o}^{z} \xi^{-k+1}\, Z_\rho\,(\xi)\, d\,\xi \tag{27}$$

the particular solution is

$$\varphi_p = -\,\frac{1}{2k}\Big[\, z^{-k}\, I_1 - z^{k}\, I_2 \Big] \quad .$$

The derivatives with respect to z of the integrals in (26) will be needed. One has

$$\frac{d}{dz}\int_{z_0}^{z} \xi^{k+1}\, Z_\varrho(\xi)\,d\xi = z^{k+1}\, Z_\varrho(z)\ , \quad \frac{d}{dz}\int_{z_0}^{z} \xi^{-k+1}\, Z_\varrho(\xi)\,d\xi = z^{-k+1}\, Z_\varrho(z)$$

With these expressions one derives from (26) when introducing (27)

$$\varphi_p' = \frac{1}{2z}\,(z^{-k}\, I_1 + z^k\, I_2) \quad .$$

The complete solution and its derivative are

$$\varphi = C_1\, z^k + C_2\, z^{-k} - \frac{1}{2k}\,(z^{-k}\, I_1 - z^k\, I_2)$$

$$\varphi' = \frac{k}{z}\,(C_1\, z^k - C_2\, z^{-k}) + \frac{1}{2}\, z^{-k-1}\, I_1 + \frac{1}{2}\, z^{k-1}\, I_2$$

Introducing the boundary conditions $\varphi = \varphi' = 0$ at z_0, one has if introducing $\varphi_p = 0$ at this boundary

$$C_1\, z_0^{\ k} + C_2\, z_0^{\ -k} = 0$$

$$C_1\, z_0^{\ k} - C_2\, z_0^{\ -k} = 0\ ,$$

what means

$$C_1 = C_2 = 0 \quad . \tag{28}$$

Further the same boundary conditions at z_1 yield when regarding (28)

$$-z_1^{\ -k}\, I_1(z_1) + z_1^{\ k}\, I_2\,(z_1) = 0$$

$$z_1^{\ -k}\, I_1(z_1) + z_1^{\ k}\, I_2\,(z_1) = 0$$

which equations mean

$$z_1^{-k} I_1(z_1) = 0 , \quad z_1^{k} I_2(z_1) = 0 .$$

As z_1 is an arbitrary boundary there must be

$$I_1(z_1) = 0 , \quad I_2(z_1) = 0 \tag{29}$$

Now separating the cylinder function in its two linearily independent parts N,M with constants C_3, C_4

$$Z_\rho = C_3 N + C_4 M$$

one obtains from (27, 29)

$$C_3 \int_{z_0}^{z_1} \xi^{k+1} N(\xi)\, d\xi + C_4 \int_{z_0}^{z_1} \xi^{k+1} M(\xi)\, d\xi = 0$$

$$C_3 \int_{z_0}^{z_1} \xi^{-k+1} N(\xi)\, d\xi + C_4 \int_{z_0}^{z_1} \xi^{-k+1} M(\xi)\, d\xi = 0$$

As C_3 , C_4 are arbitrary the determinant must be zero. This condition gives

$$\int_{z_0}^{z_1} \xi^{k+1} N\, d\xi \cdot \int_{z_0}^{z_1} \xi^{-k+1} M(\xi)\, d\xi = \int_{z_0}^{z_1} \xi^{k+1} M\, d\xi \cdot \int_{z_0}^{z_1} \xi^{-k+1} N\, d\xi \tag{30}$$

which is the relation derived by Sexl for circumferential Couette flow [5] and before by Sommerfeld for rectilinear Couette flow [4] . The boundaries are according to (18, 25) if η is written for η_1

$$z_0 = b , \quad z_1 = b\,(1-\eta) \tag{31}$$

so that (30) should be satisfied at least by one pair of values b, η. It is seen that (30) is satisfied for arbitrary values η by $b = 0$. This would mean neutral stability and a wave propagation with the rigid body part of (9). But (20, 21) show that $b = 0$ would not yield a perturbation as φ would be independent from L what contradicts to the meaning of L. Thus $b = 0$ has to be excluded.

4.) Transformation of the transcendental equation of Sommerfeld-Sexl

To analyse (30) series expansions of the integrands starting at z_0 will be introduced. Writing H for M, N one has with (31)

$$\xi^{\pm k+1} H(\xi) = b^{\pm k+1} H(b) + \left(\xi^{\pm k+1} H(\xi)\right)'_{\xi=b} (\xi-b)$$

$$+ \left(\xi^{\pm k+1} H(\xi)\right)''_{\xi=b} \frac{1}{2!} (\xi-b)^2 + \ldots\ldots\ldots$$

By integration one obtains with (31)

$$\int_{z_0}^{z_1} \xi^{\pm k+1} H(\xi)\, d\xi = -\eta b^{\pm k+2} H(b) + \frac{(\eta b)^2}{2!} \left(\xi^{\pm k+1} H\right)'_{\xi=b} \quad \ldots\ldots \quad (32$$

Equating the coefficients one obtains when writing k for $\pm$ k and H for H(b) as to simplify the notations

$$\left(\xi^{k+1} H\right)'_{\xi=b} = (k+1)\, b^k H + b^{k+1} H'$$

$$\left(\xi^{k+1} H\right)''_{\xi=b} = (k+1)\, k\, b^{k-1} H + 2(k+1)\, b^k H' + b^{k+1} H''$$

$$\left(\xi^{k+1} H\right)'''_{\xi=b} = (k+1)k(k-1)b^{k-2} H + 3(k+1)k\, b^{k-1} H' + 3(k+1)b^k H'' + b^{k+1} H'''$$

- -

Rearranging now the terms of (32) with respect to the derivatives of H one obtains

$$\int_{z_0}^{z_1} \xi^{k+1} H \, d\xi = b^{k+2} H a_0 + b^{k+3} H' a_1 + b^{k+4} H'' a_2 + \ldots \tag{33}$$

The coefficients are determined by

$$a_\nu = \sum_{n=\nu+1}^{\infty} (-1)^n \frac{\eta^n}{n!} \binom{n-1}{n-\nu-1}\binom{k+1}{n-\nu-1} \tag{34a}$$

Introducing (33) in (30) and the notation

$$a(k) = a \ , \ a(-k) = c \tag{34b}$$

one obtains

$$\begin{aligned}
& b^5 (MN'-M'N)(a_oc_1-a_1c_o)+b^6(MN''-M''N)(a_oc_2-a_2c_o) \\
& +b^7\{(MN'''-M'''N)(a_oc_3-a_3c_o)+(M'N''-M''N')(a_1c_2-a_2c_1)\} \\
& +b^8\{(MN^{IV}-M^{IV}N)(a_oc_4-a_4c_o)+(M'N'''-M'''N')(a_1c_3-a_3c_1)\} \\
& +b^9\{(MN^V-M^VN)(a_oc_5-a_5c_o)+(M'N^{IV}-M^{IV}N')(a_1c_4-a_4c_1) \\
& \qquad +(M''N'''-M'''N'')(a_2c_3-a_3c_2)\} \\
& +b^{10}\{(MN^{VI}-M^{VI}N)(a_oc_6-a_6c_o)+(M'N^V-M^VN')(a_1c_5-a_5c_1) \\
& \qquad +(M''N^{IV}-M^{IV}N'')(a_2c_4-a_4c_2)\} \\
& + \ldots\ldots\ldots = 0
\end{aligned} \tag{35a}$$

After dividing by b^5 this equation is represented by

$$\sum_{r=0}^{\infty} h^{r} \sum_{n=0}^{n=\frac{r+1}{2}} \left(M^{(n)} N^{(r-n+1)} - M^{(r-n+1)} N^{(n)}\right)\left(a_n c_{r-n+1} - a_{r-n+1} c_n\right) = 0 \tag{35b}$$

which is the transformed Sommerfeld-Sexl equation (30). This new expression permits as will be shown the direct numerical evaluation of the stability parameters ξ_r, ξ_i for all modes k with any desired exactness.

5.) Solution of the fundamental equation (35b)

The Wronski determinant W will be introduced. One has

$$W = MN' - NM' \quad , \quad W'' = MN'' - NM'' \quad . \tag{36}$$

With the abbreviations

$$f = \frac{1}{z} \quad , \qquad g = 1 - \frac{\varrho^2}{z^2} \quad , \tag{37}$$

the equation

$$y'' + f\, y' + g = 0 \tag{38}$$

has the solutions M, N according to (19).

The differential equation for W is

$$W' = - f\, W = - \frac{W}{z} \tag{39}$$

and therefore

$$W'' = - f'W + f^2 W = \frac{2}{z^2} W$$

$$W''' = W(- f'' + 3\, ff' - f^3) = - \frac{6}{z^3} W$$

$$W^{IV} = W(- f''' + 3\, f'^2 + 4\, ff'' - 6\, f^2 f' + f^4) = \frac{24}{z^4} W \tag{40}$$

$$W^{V} = W(- f^{IV} + 10\, f'f'' + 5\, ff''' - 15\, ff'^2 - 10\, f^2 f''$$
$$+ 10\, f^3 f' - f^4 = - \frac{120}{z^5} W$$

From this one deduces

$$W^{(m)} = (-1)^m \frac{m!}{z^m} W \tag{41}$$

On the other hand it follows from (36, 39)

$$MN''' - M'''N = W'' - (M'N'' - M''N')$$

$$MN^{IV} - M^{IV}N = W''' - 2\, (M'N''' - M'''N')$$

$$MN^{V} - M^{V}N = W^{IV} - 2\, (M''N''' - M'''N'') - 3\, (M'N^{IV} - M^{IV}N') \tag{42}$$

$$MN^{VI} - M^{VI}N = W^{V} - 2\, (M''N^{IV} - M^{IV}N'') - 3\, (M''N^{IV} - M^{IV}N'')$$
$$-4\, (M'N^{V} - M^{V}N')$$

Inserting N, M in (38) and forming the m-th derivative

$$M^{(m)} + (f\, M' + g\, M)^{(m-2)} = 0$$

$$N^{(m)} + (f\, N' + g\, N)^{(m-2)} = 0 \quad ,$$

and multiplying the first equation with the n-th derivative of N and the second with the same derivative of M one obtains when subtracting

$$N^{(n)}M^{(m)}-N^{(m)}M^{(n)} = -\left[(f\ M'+g\ M)^{(m-2)}N^{(n)} - (f\ N'+g\ N)^{(m-2)}M^{(n)}\right] \tag{43}$$

With this expression the higher derivatives can be expressed by the lower ones. Finally the Wronski determinant W can be introduced so that in the basic equation (35a or 35b) all functions can be expressed by W. Thus in (35) all functions M, N and their derivatives are replaced by W, which shows to be a common factor and therefore can be canceled in (35a) und (35b) resp.

Indeed from (43) one deduces

$$\begin{aligned}
N''M - M''N &= -f\ W\\
N''M' - M''N' &= g\ W\\
N''M''' - M''N''' &= (fg'-gf'-g^2)\ W\\
N''M^{IV} - N^{IV}M'' &= (ff'g+g^2f-f^2g+fg''-gf''-2gg')W\\
N'''M - M'''N &= (-f'-g+f^2)\ W\\
N'''M' - M'''N' &= (-fg+g')\ W\\
N'''M'' - M'''N'' &= (f'g+g^2-fg')\ W\\
N^{IV}M - M^{IV}N &= (-f''-2g'+3ff'+2gf-f^3)\ W\\
N^{IV}M' - M^{IV}N' &= (-2f'g-g^2+f^2g-fg'+g'')\ W
\end{aligned} \tag{44}$$

(36,42) introduced in (35a) gives

$$b^o\ W(a_oc_1-a_1c_o)$$

$$+b^1\ W'(a_oc_2-a_2c_o)$$

$$+b^2\left\{W''(a_oc_3-a_3c_o)-(M'N''-M''N')\left[(a_oc_3-a_3c_o)-(a_1c_2-a_2c_1)\right]\right\}+$$

$$
\begin{aligned}
&+b^3\Big\{W'''(a_0c_4-a_4c_0)-(M'N'''-M'''N')\Big[2(a_0c_4-a_4c_0)-(a_1c_3-a_3c_1)\Big]\Big\}\\
&+b^4\Big\{W^{IV}(a_0c_5-a_5c_0)-(M''N'''-M'''N'')\Big[2(a_0c_5-a_5c_0)-(a_2c_3-a_3c_2)\Big]\\
&\qquad\qquad -(M'N^{IV}-M^{IV}N')\Big[+3(a_0c_5-a_5c_0)-(a_1c_4-a_4c_1)\Big]\Big\}\\
&+b^5\Big\{W^{V}(a_0c_6-a_6c_0)-(M''N^{IV}-M^{IV}N'')\Big[5(a_0c_6-a_6c_0)-(a_2c_4-a_4c_2)\Big]\\
&\qquad\qquad -(M'N^{V}-M^{V}N')\Big[+4(a_0c_5-a_5c_0)-(a_1c_5-a_5c_1)\Big]\\
&+b^6\Big\{W^{VI}(a_0c_7-a_7c_0)-(M'''N^{IV}-M^{IV}N''')\Big[5(a_0c_7-a_7c_0)-(a_3c_4-a_4c_3)\Big]\\
&\qquad\qquad -(M''N^{V}-M^{V}N'')\Big[9(a_0c_7-a_7c_0)-(a_2c_5-a_5c_2)\Big]\\
&\qquad\qquad -(M'N^{VI}-M^{VI}N')\Big[4(a_0c_7-a_7c_0)-(a_1c_6-a_6c_1)\Big] = 0
\end{aligned}
\tag{45}
$$

(44) introduced in (45) gives with (4o) if the common factor W is canceled

$$
\begin{aligned}
&b^0\Big\{(a_0c_1-a_1c_0)\\
&+b^1\Big\{(a_0c_2-a_2c_0)(-f)\\
&+b^2\Big\{(a_0c_3-a_3c_0)(2f^2)-\Big[(a_0c_3-a_3c_0)-(a_1c_2-a_2c_1)\Big]g\Big\}\\
&+b^3\Big\{(a_0c_4-a_4c_0)(-6f^3)-\Big[2(a_0c_4-a_4c_0)-(a_1c_3-a_3c_1)\Big](-fg+g')\Big\}\\
&+b^4\Big\{(a_0c_5-a_5c_0)(24f^4)-\Big[2(a_0c_5-a_5c_0)-(a_2c_3-a_3c_2)\Big](f'g+g^2-fg')\\
&\qquad\qquad -\Big[3(a_0c_5-a_5c_0)-(a_1c_4-a_4c_1)\Big](-2f'g-g^2+f^2g-fg'+g'')\Big\}\\
&+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. = 0\;.
\end{aligned}
\tag{46}
$$

In f and g the boundary $z_o = b$ has to be inserted for z according to (31). This means that in the first column of (46) b cancels. Further it has to be mentioned that according to (34a) the first brackets in the two columns of (46) increase with powers of η . The first column begins with the first power and the second with the third power. Thus (46) represents a series expansion in powers of η .

6.) Discussion of the functions a_ν (η), c_ν (η)

One has according to 34(a,b)

$$\begin{matrix} a_\nu \\ = \\ c_\nu \end{matrix} \sum_{n=\nu+1}^{\infty} (-1)^n \frac{\eta^n}{n!} \binom{n-1}{n-\nu-1} \binom{\pm k+1}{n-\nu-1} \qquad (47$$

This expression shows that the functions a_ν are polynomials and that c_ν are series with terms negative for even indices ν and positive terms for the odd ones. One finds that a_o, c_o are the functions

$$\begin{matrix} a_o \\ c_o \end{matrix} = -\frac{1}{\pm k+2} + \frac{1}{\pm k+2} (1-\eta)^{\pm k+2} \qquad (48$$

This expression shows that a_o is negative. For c_o one has for k=1

$$c_o = -\eta \quad .$$

The undetermined expression (48) for k = -2 is equal to

$$c_o = \ln (1-\eta)$$

For k=3 one derives from (47)

$$c_3 = 1 - \frac{1}{1-\eta} \qquad .$$

The a's show to have first ascending and then descending terms. For odd indices the sum of the ascending terms is smaller than the largest positive term and for even indices this sum is smaller than the largest negative term if in each case the largest terms are excluded from the addition. As the descending terms are descending strongly and as there number is limited the a's with the odd indices show to be positive and with the even indices negative. This statement holds for all $0 \leq \eta \leq 1$ so that in these regions the a's have no zeros.

Table I gives the values of a, c for k = 1,2,3 and table II the values of the brackets occuring in (46). All these values refer to η = o,5.

7.) <u>Evaluation of the stability</u>

Inserting (37) and $z = z_o = b$ in (46) one obtains

$$\begin{aligned}
&a_oc_1-a_1c_o\\
&-(a_oc_2-a_2c_o)\\
&+2(a_oc_3-a_3c_o) - \Big[(a_oc_3-a_3c_o)-(a_1c_2-a_2c_1)\Big](b^2-\varrho^2)\\
&-6(a_oc_4-a_4c_o) - \Big[2(a_oc_4-a_4c_o)-(a_1c_3-a_3c_1)\Big](-b^2+3\varrho^2)\\
&+24(a_oc_5-a_5c_o) - \Big[2(a_oc_5-a_5c_o)-(a_2c_3-a_3c_2)\Big]\Big[-b^2-\varrho^2+(b^2-\varrho^2)^2\Big]\\
&\qquad - \Big[3(a_oc_5-a_5c_o)-(a_1c_4-a_4c_1)\Big]\Big[3b^2-11\varrho^2-(b^2-\varrho^2)^2\Big]
\end{aligned} \tag{49}$$

$$\text{---}$$

$$\text{--} = 0 \; .$$

Inserting with table II the values for $k = 1$ the real and imaginary part of (49) yield two equations. First the equation

$$18{,}68760851 + 7{,}61593802\, b^2 - 7{,}42023802\, \varrho^2 - 0{,}15278165\, (b^2 - \varrho^2)^2 = 0 \tag{50}$$

is obtained. With (15, 16, 17) and the numerical value

$$\frac{r_1}{r_o} = 2$$

[illegible] corresponds to the assumed value $\eta = 0{,}5$ the separation of the real and imaginary part of (50) gives the two equations

$$- 0{,}15278165\, R^2 \xi_1^2 - 7{,}9215013\, R\xi_1 + 11{,}11458885 + 0{,}15278165\, R^2 (\xi_r - 1)^2 = 0 \;, \tag{51}$$

$$R\xi_1 = - \frac{7{,}9215013\, (\xi_r - 1) - 0{,}0652333}{0{,}3055633\, (\xi_r - 1)} \tag{52}$$

Inserting (52) in (51) one obtains exactly a biquadratic equation in $\xi_r - 1$,

$$R^2 (\xi_r - 1)^4 + 2088{,}949384 (\xi_r - 1)^2 - 0{,}0455760 = 0$$

in which R appears as parameter. The numerical solution is

$$\begin{array}{lll} R = 1 & \qquad & \xi_r - 1 = \pm 4{,}69 \cdot 10^{-3} \\ R = 10 & & \xi_r - 1 = \pm 4{,}67 \cdot 10^{-3} \\ R = 100 & & \xi_r - 1 = \pm 4{,}67 \cdot 10^{-3} \\ R = \infty & & \xi_r - 1 = 0 \end{array}$$

Only the negative values have physical meaning. Introducing them in (52) one obtains

$R = 1$	$R\xi_1 = -71{,}44$
$R = 10$	$R\xi_1 = -71{,}63$
$R = 100$	$R\xi_1 = -71{,}63$

This result means stability and it is seen that ξ_i is inversely proportional to the Reynolds number R. The numerical values obtained for $\xi_r - 1$ show that the circumferential propagation velocity of the perturbation is extremely near to the circumferential velocity of the outer cylinder with a negligable influence of the Reynolds number. This result is analogue to the result found for the stability of a liquid rotating as rigid body for which in our notation $r_1/r_o = \infty$, where $\xi_r - 1 = 0$ was found [19].

For $k = 3$, $\eta = 0{,}5$ (49) yields by neglecting the last term of (46)

$$- 0{,}010766 = - 0{,}008207\ b^2 + 0{,}00845\ \varsigma^2$$

Separating the real and imaginary parts with (16, 17) one obtains the two equations

$$- 0{,}010766 = 0{,}008207\ kR\xi_1 + 0{,}00845\ k^2$$

$$- 0{,}008207\ kR\ (\xi_r - B(\frac{r_1}{r_o})^2 - 0{,}008716\ kRB = 0$$

The first equation gives

$$R\xi_1 = - 3{,}53$$

and the second equation, when introducing with (15) $r_1/r_o = 2$,

$$\xi_r = 0{,}981 \ .$$

Again stability and a propagation velocity near to the circumferential velocity of the outer cylinder is found. Here ξ_r appears to be independent of the Reynolds number R. Indeed the influence of R appears only in the next higher approximation that means, when the last term of (46) is considered. Therefore in any case the influence of R on ξ_r is weak as it was found numerically in the case k = 1 by considering also the last term of (46).

8.) Conclusions

The Sommerfeld-Sexl transcendental equation which yields the stability parameters of Couette flow was first transformed (see eq. 35b). Then by introducing the Wronski determinant it was possible to eliminate the unknown Bessel functions of imaginary argument and imaginary order, which until then prohibited the solution of the original Sommerfeld-Sexl equation. A series expansion in the wall distance with the stability parameters in polynomial form was obtained (see eq. 49) which can be solved numerically for any modes and widths of the gap. For a ratio of the inner and outer radius of 2 stability was found for the first and third mode.

Table I

Coefficients a_ν , c_ν for $\eta = 0{,}5$

	k = 1	k = 2	k = 3
a_0	$-\frac{7}{24}$	$-\frac{15}{64}$	$-\frac{31}{160}$
c_0	$-\frac{1}{2}$	$-\ln 2$	-1
a_1	$\frac{11}{16\cdot 12}$	$\frac{13}{320}$	$\frac{19}{640}$
c_1	$\frac{1}{8}$	$\frac{367}{16\cdot 120}$	$\frac{563}{24\cdot 80}$
a_2	$-\frac{1}{15}$	$-\frac{7}{160}$	$-\frac{33}{1120}$
c_2	$-\frac{1}{48}$	$-\frac{113}{3360}$	$-\frac{1447}{256\cdot 105}$
a_3	$\frac{11}{96\cdot 120}$	$\frac{1}{1680}$	$\frac{163}{16\cdot 16\cdot 24\cdot 70}$
c_3	$\frac{1}{384}$	$\frac{5609}{96\cdot 16\cdot 840}$	$\frac{3071}{16\cdot 96\cdot 280}$
a_4	$-\frac{29}{96\cdot 96\cdot 35}$	$-\frac{31}{32\cdot 32\cdot 16\cdot 35}$	$-\frac{1}{54\cdot 16\cdot 35}$
c_4	$-\frac{1}{32\cdot 120}$	$-\frac{6887}{32\cdot 288\cdot 48\cdot 35}$	$-\frac{1439}{32\cdot 27\cdot 64\cdot 35}$

Table II **Coefficients of equation (49) for η = 0,5**

	k = 1	k = 2	k = 3
$a_0c_1-a_1c_0$	$-0,781250000 \cdot 10^{-2}$	$-1,664070047 \cdot 10^{-2}$	$-2,712565103 \cdot 10^{-2}$
$a_0c_2-a_2c_0$	$-2,725694445 \cdot 10^{-2}$	$-2,244293471 \cdot 10^{-2}$	$-1,903436570 \cdot 10^{-2}$
$a_0c_3-a_3c_0$	$-2,821180552 \cdot 10^{-4}$	$-6,06299866 \cdot 10^{-4}$	$-10,04478817 \cdot 10^{-4}$
$a_0c_4-a_4c_0$	$+3,100198422 \cdot 10^{-5}$	$+6,67319187 \cdot 10^{-5}$	$+11,09903320 \cdot 10^{-5}$
$a_0c_5-a_5c_0$	$-2,744967345 \cdot 10^{-6}$	$-5,733659255 \cdot 10^{-6}$	$-9,89601091 \cdot 10^{-6}$
$a_1c_2-a_2c_1$	$+7,139766944 \cdot 10^{-3}$	$+6,996372770 \cdot 10^{-3}$	$+7,041655038 \cdot 10^{-3}$
$a_1c_3-a_3c_1$	$+2,983940967 \cdot 10^{-5}$	$+6,28298804 \cdot 10^{-5}$	$+10,08509075 \cdot 10^{-5}$
$a_1c_4-a_4c_1$	$-3,68148565 \cdot 10^{-6}$	$-7,73727576 \cdot 10^{-6}$	$-12,37684242 \cdot 10^{-6}$
$a_2c_3-a_3c_2$	$-15,37181712 \cdot 10^{-5}$	$-17,01739044 \cdot 10^{-5}$	$-18,99884454 \cdot 10^{-5}$

10.) List of references

[1] M. Couette, Ann.chim.phys. 21,433 (1890)

[2] G.I. Taylor, Phil. Trans. A 223, 289 (1923)

[3] Lord Rayleigh, Sci.Papers III, No. 144, 17, 1902

[4] A. Sommerfeld, Atti del congr.internat. dei Mat. Rome 1908

[5] Th. Sexl, Ann. Phys. IV. Folge 83, 835 (1927)

[6] L. Hopf, Ann. Phys. 44,1 (1914), 59, 538 (1919)

[7] R.v.Mises, Heinrich Weber Festschrift 1912, Jahresbericht d.dt. Math.Vereinigung 1912

[8] H. Schlichting, Grenzschichttheorie, Karlsruhe, 362, 1958

[9] M. Couette, Ann. chim.phys. 21, 433 (1890)

[10] M. Mallock, Phil.Trans. A 41 (1896)

[11] G.I. Taylor, Phil. Trans. A 223, 289 (1923)

[12] F. Wendt, Ing. Arch. 4,577 (1933)

[13] A. Sommerfeld, Vorlesungen über theoretische Physik II, 267,268 (1954)

[14] C.C. Lin, The theory of hydrodynamic stability, Cambridge 1955, 12

[15] F. Schultz-Grunow, ZAMM, 39, 101 (1959)

[16] W. Tillmann, Z.angew. Phys. 13, 468-475 (1961)

[17] F. Schultz-Grunow, Annual Summary Report, Contract AF 61(052)-303

[18] Lord Rayleigh, Sci.Papers IV No. 216, 215 (1903)

[19] F. Schultz-Grunow, ZAMM, 411 (1963)