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LINEAR PROGRAMMING

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January 1962

P-2514

PREFACE

This paper was prepared from an hour talk given to the secondary mathematics teachers of Pasadena City Schools on January 9, 1962. The talk was part of the Mathematics Symposiums for 1961-1962 planned by W. G. Norris, Mathematics Supervisor of the Pasadena School System.

LINEAR PROGRAMMING

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1. Historical Notes

Linear programming is a relatively recent subject in mathematics, having developed mostly since 1947-1948 when the Air Force set up Project SCOOP (Scientific Computation of Optimum Programs) to investigate efficient organization of large-scale programming and scheduling activities. Since that time the subject has enjoyed a flourishing growth, both in theory and in applications. Most of the early work was done in the Air Force, which has also lent generous support to sponsoring various symposia on the subject.

The earliest work known of the linear programming type was a transportation problem posed by Hitchcock in 1941. In the same year a diet problem was considered by Cornfield, which was studied more intensively by Stigler in 1945. Parallel to these spotty early efforts, Leontief carried on input-output industry studies since the early '30s which later proved to be related to linear programming problems. Project SCOOP comprised most of the original contributors to the subject, including inter alia George Dantzig, Marshall Wood, Murray Geisler, Leon Goldstein, Julian Holley, Walter Jacobs, Alex Arden, and Emil Schell. A great forward leap was made when this group joined forces with two other groups: Koopmans at the Cowles Commission, who had done much independent work in activity analysis of the shipping industry, and von Neumann and Tucker at Princeton with their students (e.g., Gale, Kuhn, Goldman and Gomory).

One unique feature of linear programming as a subject in applied mathematics is its rapid growth in applications -- to problems in government, industry and business -- pari passu with its theoretical development. This

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simultaneous evolution of the subject along two fronts is generally the exception rather than the rule in the history of mathematics.

2. The Optimal Diet Problem

Linear programming deals with maximization (or minimization) of a linear function of n variables where the variables are subject to a set of linear constraints (i.e., equations or inequalities). All or some of these variables may further be required to be nonnegative. Let us illustrate the problem with a simple example.

Suppose we have four foods: corned beef, cabbage, potatoes, and milk, and wish to prepare a wholesome diet from them. To do this we ask ourselves first the question: What is it that's so important in a diet? The obvious answer is its nutritive value. So, let us simplify the problem by considering only three classes of nutrients: protein, carbohydrates and vitamins; and the nutrient contents of the four foods are given as follows:

$$
\begin{array}{llccccc}
 & & \text{Corned Beef} & \text{Cabbage} & \text{Potatoes} & \text{Milk} & \\
 & & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \\
\text{Proteins} & \xi_1 & 6 & 0 & 1 & 3 & \gamma_1 \\
\text{Carbohydrates} & \xi_2 & 2 & 2 & 4 & 1 & \gamma_2 \\
\text{Vitamins} & \xi_3 & 1 & 3 & 1 & 2 & \gamma_3 \\
 & & \beta_1 & \beta_2 & \beta_3 & \beta_4 &
\end{array}
\tag{2.1}
$$

where "6" means that 1 unit of corned beef contains 6 units of proteins, and "4" that 1 unit of potatoes contains 4 units of carbohydrates, etc. We assume, of course, that all these units have been properly defined, say by the nutritionists. Let us call (2.1) the nutrition matrix. A diet (or menu) is any combination of the four foods, say in amounts of $\eta_1, \cdots, \eta_4$. For any given diet ($\eta_1, \eta_2, \eta_3, \eta_4$),

(2.2) $$6\eta_1 + 0\cdot\eta_2 + 1\cdot\eta_3 + 3\cdot\eta_4$$

is the total amount of protein, and

(2.3) $$2\eta_1 + 2\eta_2 + 4\eta_3 + 1\cdot\eta_4$$

is the total amount of carbohydrates. Now we wish to impose an additional restriction on our diet that it provides at least γ_1 units of protein, γ_2 units of carbohydrates and γ_3 units of vitamins. Such a diet will be called a feasible diet. More abstractly, let a_{ij} denote the units of nutrient N_i ($i=1,\cdots, m$) contained in one unit of food F_j ($j=1,\cdots, n$). Then a feasible diet is a set of n numbers satisfying

(2.4) $$\eta_j \geqq 0 \qquad j = 1,\cdots, n$$

and

(2.5) $$a_{i1}\eta_1 + \cdots + a_{in}\eta_n \geqq \gamma_i \qquad i = 1,\cdots, m$$

As yet no maximization or minimization is involved, and it is obvious that (2.5) may generally be satisfied by making the η_j's large. We are therefore interested not in eating any arbitrarily large diet (i.e., large η_j's) but one which will yield the least cost

(2.6) $$\beta_1\eta_1 + \cdots + \beta_n\eta_n$$

where β_j ($j=1,\cdots, n$) is the given unit price of F_j, and consequently (2.6) is the total price to be paid for the diet $(\eta_1, \cdots, \eta_n)$. A feasible diet minimizing (2.6) is called an optimal diet. Our problem is, in summary, the following: Among all feasible diets, to find one which is optimal. This is the optimal diet problem which gives rise to the standard minimum problem of the linear programming type.

To each standard minimum problem corresponds a second linear programming problem, called the dual problem. Let us first state it formally and then try to explain it. That problem is to find numbers ξ_i $(i=1, \cdots, m)$ satisfying

(2.7) $$\xi_i \geqq 0 \qquad\qquad i = 1, \cdots, m$$

and

(2.8) $$\xi_1 a_{1j} + \cdots + \xi_m a_{mj} \leqq \beta_j \qquad\qquad j = 1, \cdots, n$$

such that

(2.9) $$\xi_1 \gamma_1 + \cdots + \xi_m \gamma_m$$

is maximized. The symmetry between (2.4) to (2.6) and (2.7) to (2.9) is immediately apparent, and this duality occupies the central place in the whole subject called linear programming.

We now give a heuristic explanation of the dual problem. Let us begin with (2.9). Since γ_i $(i=1, \cdots, m)$ are units of nutrients N_i's, (2.9) makes sense only if ξ_i denotes monetary value per unit of N_i so that the summation in (2.9) may be performed in the common denominator: money. (2.7) merely states that these unit monetary values must be non-negative and (2.8) states that the sum total of unit monetary values over all nutrients must not exceed the unit selling price β_j of F_j. Hence, ξ_i

is just the unit cost of N_i and (2.8) says that the sum total of all nutrients going into one unit of F_j must not cost more than the selling price of F_j. Subject to this restriction, we are to choose that set of ξ_i's maximized (2.9). Therefore, the dual problem is a meaningful economic problem facing the resource (i.e., nutrient) owners. They wish to set the highest possible costs on the resources subject to the condition that the food manufacturer may still continue to produce. The fundamental theorem of linear programming states that if feasible solutions exist to the primal problem and its dual, then necessarily optimal solutions exist for both such that the minimum of (2.6) equals the maximum of (2.9). This last condition leaves the food manufacturer with zero profit and is sometimes called the equilibrium condition under pure competition. Moreover, the connection between linear programming and game theory is also suggested.

3. Linear Systems and Linear Programming

Basic to the study of linear programming is the theory of linear systems (i.e., equations and/or inequalities). Generally speaking, there are two separate problems analogous to those in the study of linear equations, that is, linear algebra. The first deals with the existence (or non-existence) of solutions to a system of linear inequalities, and the second deals with the structure of the solution set. There is a beautiful algebraic theory of linear inequalities which tells us when solutions will or will not exist. In this theory we see a natural pair of linear systems such that if one has no solution, the other must have, and conversely. A general rule for finding this pair of linear systems is that the variables in one correspond to the constraints in the other, and vice versa. More specifically, any nonrestricted variable in one system corresponds to a linear equation in the other system, whereas a restricted (i.e., nonnegative or nonpositive) variable in one system corresponds to a linear inequality in the other system. This pair of linear systems is also related to orthogonal complements in the theory of vector spaces.

The structure of the solution set of a linear system is best studied geometrically by means of the theory of convex sets. Here we see that the solution set of a homogeneous linear system is a convex cone and that of a nonhomogeneous linear system a convex polytope. In either case, a finite set of extreme points or vectors exist such that all other solutions are convex linear combinations (i.e., centers of gravity with varying weights) of these extreme solutions. The solution set of the most general linear system is the (vector) sum of a convex cone and a convex polytope.

From the theory of linear systems follows immediately the fundamental theorem of linear programming. In terms of matrix games, this is also called the minimax theorem first proved by John von Neumann in 1928. The entire theory may be extended to nonlinear (say convex or concave) systems, which must, however, have some global property. Such extension may be found in connection with Fenchel's work.

4. Computational Methods

There is a separate body of theories related to the computational aspect of linear programming, the first of which was the elegant simplex algorithm by Dantzig. Let us illustrate it by considering the following problem: To find $\eta_j \geqq 0$ $(j=1,\cdots, n)$ minimizing

$$\beta_1\eta_1 + \cdots + \beta_n\eta_n \tag{4.1}$$

subject to

$$\begin{aligned} a_{11}\eta_1 + \cdots + a_{1n}\eta_n &= \gamma_1 \\ &\cdots \\ a_{m1}\eta_1 + \cdots + a_{mn}\eta_n &= \gamma_m \end{aligned} \tag{4.2}$$

In (4.2) we include equations only. If inequalities are involved, we merely introduce additional variables to change them into equations. These new variables, one for each inequality, will be given zero coefficients in (4.1).

There are two steps to the simplex algorithm:

1. We must find a first feasible solution, and

2. Given that a first feasible solution is found, we are to find another feasible solution which yields a possibly smaller value for (4.1). Step 2 must also tell us when to stop looking for a better solution.

We discuss Step 2 first and assume a first feasible solution exists. Let the first p columns of coefficients on the left side of (4.2) be (linearly) independent and the remaining dependent on these. Then a work sheet of the following type can be set up.

$$
(4.3)\qquad
\begin{array}{c|ccccccccccccc}
 & \text{col. } 1 & \cdots & & & \text{col. } p & \text{col. } (p+1) & \cdots & & & \text{col. } n & & \begin{array}{c}\text{right-hand}\\ \text{side}\end{array} \\
 & a^1 & \cdots & a^r & \cdots & a^p & a^{p+1} & \cdots & a^s & \cdots & a^n & & c \\
a^1 & 1 & \cdots & 0 & \cdots & 0 & \tau_{1,p+1} & \cdots & \tau_{1s} & \cdots & \tau_{1n} & \cdots & \eta_1 \\
\vdots & 0 & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
 & \vdots & & 0 & & \vdots & & & & & & & \\
a^r & \vdots & \cdots & 1 & & & \tau_{r,p+1} & \cdots & \tau_{rs} & \cdots & \tau_{rn} & \cdots & \eta_r \\
\vdots & \vdots & & 0 & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & & 0 & & & & & & & \\
a^p & 0 & \cdots & 0 & \cdots & 1 & \tau_{p,p+1} & \cdots & \tau_{ps} & \cdots & \tau_{pn} & \cdots & \eta_p
\end{array}
$$

(4.5)

In this simplex tableau we express $a^{p+1},\cdots,\ a^n$ and c as linear combinations of $a^1,\cdots,\ a^p$. It is instructive to recall our diet problem, in which the columns represent foods. Then (4.3) says we can choose p basic foods $a^1,\cdots,\ a^p$, then form linear combinations of these to get nutritionally (4.6) equivalent substitutes of the remaining foods. For example,

$$
(4.4)\qquad a^s = \tau_{1s}\, a^1 + \cdots + \tau_{ps}\, a^p
$$

that is, one unit of food a^s is nutritionally equivalent to τ_{1s} units of a^1 plus τ_{2s} units of a^2 plus so on until τ_{ps} units of a^p. The last column in (4.3) simply states the required nutrient vector $(\gamma_1,\cdots,\ \gamma_m)$ may be exactly satisfied by η_1 units of $a^1,\cdots,\ \eta_p$ units of a^p, the first feasible solution which we assume to exist. It is interesting to note also that once $a^1,\cdots,\ a^p$ are chosen to be the basic foods, the substitutes which can be formed for them are just themselves, i.e., use a^1 as substitute (4.7) for $a^1,\cdots$, a^p as substitute for a^p. In other words, for basic foods, there is no need to find substitutes, or the substitutes are identical with the foods themselves, whereas for nonbasic foods, it is possible to find substitutes for them using combinations of the basic foods.

Our first menu therefore consists of η_1 units of $a^1, \cdots,$ η_p units of a^p and nothing else, for which the total price is

5) $$\beta_1 \eta_1 + \cdots + \beta_p \eta_p$$

There may be some nonbasic foods not yet in the menu which would be cheaper than their substitutes in terms of the basic foods. Take, for example, a^s with unit price β_s. (4.4) states that in order to find some combination of $a^1, \cdots, a^p$ to be nutritionally equivalent to a^s, we must use τ_{1s} units of $a^1, \cdots,$ τ_{ps} units of a^p so that the substitute would cost

6) $$\tau_{1s} \beta_1 + \cdots + \tau_{ps} \beta_p = \zeta_s$$

If $\beta_s < \zeta_s$, then we would save money by introducing a^s into the menu. But each unit of a^s introduced into the menu will displace τ_{1s} units of $a^1, \cdots,$ τ_{ps} units of a^p. We want to find the largest possible number of units η_s of a^s which could be introduced into the menu.

It is certainly obvious that each additional unit of a^s introduced would enable us to cut down on the amounts of $a^1, \cdots, a^p$ used in the first menu. In fact, if we use η_s units of a^s, we reduce the amounts needed for $a^1, \cdots, a^p$ to

7) $$\begin{aligned} \eta_1' &= \eta_1 - \tau_{1s}\eta_s \\ &\cdots \\ \eta_p' &= \eta_p - \tau_{ps}\eta_s \end{aligned}$$

These quantities must still remain nonnegative. This condition will certainly be fulfilled if $\tau_{is} \leqq 0$ since η_i' would then not be smaller than η_i which is nonnegative to begin with. Hence, we need to worry about those cases in which $\tau_{is} > 0$. Among these, we choose η_s so large as to reduce one η_i to zero, that is, we delete one of the old foods, say a^r, from the menu completely and replace it by a^s. Our algorithm for finding a cheaper feasible diet consists of two steps.

1. Introduce any nonbasic food a^s into the menu which is cheaper than its substitute in terms of the basic foods $a^1, \cdots, a^p$. That is, if

(4.8) $$\beta_s < \zeta_s = \tau_{1s}\beta_1 + \cdots + \tau_{ps}\beta_p$$

2. Delete a basic food a^r $(1 \leqq r \leqq p)$ from the menu by introducing η_s units of a^s where

(4.9) $$\eta_s = \frac{\eta_r}{\tau_{rs}} = \min_{\substack{i \\ \tau_{is} > 0}} \frac{\eta_i}{\tau_{is}}$$

These η_s units of a^s will reduce the amounts used of $a^1, \cdots, a^p$ from $\eta_1, \cdots, \eta_p$ to $\eta_1', \cdots, \eta_p'$ as defined by (4.7). ($\eta_r' = 0$, of course, by choice of η_s.)

After these two steps we will have a new feasible solution. It is easy to calculate the total price for the new menu:

(4.10) $$\beta_1\eta_1' + \cdots + \beta_p\eta_p' + \beta_s\eta_s$$

This is less than or equal to (4.5) since $\eta_r' = 0$ and so

$$\beta_1\eta_1' + \cdots + \beta_p\eta_p' + \beta_s\eta_s = \beta_1(\eta_1 - \tau_{1s}\eta_s) + \cdots + \beta_p(\eta_p - \tau_{ps}\eta_s)$$

(4.11) $$+ \beta_s\eta_s = \beta_1\eta_1 + \cdots + \beta_p\eta_p - (\beta_1\tau_{1s} + \cdots + \beta_p\tau_{ps})\eta_s + \beta_s\eta_s$$

$$= \beta_1\eta_1 + \cdots + \beta_p\eta_p - (\zeta_s - \beta_s)\eta_s$$

The last term in (4.11) shows that the price of the new menu is the difference between the price of the old menu and the amount of possible saving by introducing η_s units of a^s. We repeat the process until no nonbasic food can be found which will be cheaper than their substitutes.

The foregoing discussion gives the essence of the simplex algorithm and its extensions. We conclude this section with a brief remark on how to find a first feasible solution. To do that, we consider the following auxiliary linear programming problem: To find $\eta_j \geqq 0$ ($j=1,\cdots, n$), $\omega_i \geqq 0$ ($i=1,\cdots, m$) minimizing

(4.12) $$\omega_1 + \cdots + \omega_m$$

subject to

(4.13)
$$\begin{aligned} a_{11}\eta_1 + \cdots + a_{1n}\eta_n + \omega_1 \quad &= \gamma_1 \\ a_{21}\eta_1 + \cdots + a_{2n}\eta_n \quad + \omega_2 &= \gamma_2 \\ &\;\,\vdots \\ a_{m1}\eta_1 + \cdots + a_{mn}\eta_n \quad\quad + \omega_m &= \gamma_m \end{aligned}$$

where we may assume without loss of generality $\gamma_i \geqq 0$ ($i=1,\cdots, m$). If this minimization problem yields zero for (4.12), then all slacks in

(4.13) vanish and the corresponding $\eta_1, \cdots, \eta_n$ will form the first feasible solution to our original system, i.e. (4.2). If some slack remains, then (4.2) has no feasible solution. The problem of finding a first feasible solution to (4.2) has now been changed to one of finding a first feasible solution to (4.13). But a first feasible solution to (4.13) can be found by inspection, namely by setting

$$\eta_1 = \cdots = \eta_n = 0 \quad , \quad \omega_1 = r_1 , \ldots , \omega_m = r_m .$$

There are many variations as well as extensions of the simplex algorithm, and also other algorithms which are available for computing large-scale linear programming problems. For most general-purpose electronic computers, computer programs now exist for such use. We mention only a few as follows: Burroughs Datatron, Ferranti Limited (English) Pegasus, IBM 650, 701, 704, 705, 709, 1620, 7070, 7090. Sperry Rand Univac 1, 1103, 1103A.

5. Applications and Related Subjects

We shall mention only briefly some fields of application of linear programming. As it was noted in Section 1, the subject arose originally from practical problems in transportation, dietetics and military planning. A rather comprehensive survey of linear programming applications may be found in the Bibliography on Linear Programming and Related Techniques by Vera Riley and S. I. Gass (Johns Hopkins Press, 1958).

The following outline includes only some selected applications for illustration purposes:

1. Agricultural applications:
    a. Farm management
    b. Feed-mixing
    c. Crop rotation
2. Industrial applications:
    a. Chemical industry
    b. Coal industry
    c. Iron and steel industry
    d. Paper industry
    e. Petroleum industry
3. Commercial applications:
    a. Airline routing
    b. Communication networks
    c. Railway freight
    d. Securities selection
    e. Inventory control
4. Military applications:
    a. Weapon selection
    b. Program planning
    c. Personnel assignment

Not only has the range of application of linear programming broadened in recent years, the subject has been found related to many other branches of mathematics. Of these, we list a few:

1. Game theory
2. Network flow or circuit theory
3. Graph theory

4. Number theory

5. Mathematical statistics

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