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HEAT TRANSFER TO A SPHERE AT THE TRANSITION FROM FREE MOLECULE FLOW

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A DIVISION OF AVCO MANUFACTURING CORPORATION

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RESEARCH NOTE 75

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HEAT TRANSFER TO A SPHERE AT THE TRANSITION FROM FREE MOLECULE FLOW

by

P. Hammerling and B. Kivel

June, 1958

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HEAT TRANSFER TO A SPHERE AT THE TRANSITION FROM FREE MOLECULE FLOW

P. Hammerling and B. Kivel

SUMMARY

Baker and Charwat[1] have recently published a semi-quantitative study of the correction to the drag of a sphere at the breakdown of the free molecule flow region. Their expression for the drag coefficient is

$$C_D = 2\left[1 + 0.44\frac{\bar{V}_e}{V_0} - \left(0.46\frac{V_0}{\bar{V}_e} + 1.3\right) a \rho_0 S_{eo}\right] \qquad (1)$$

which shows the dependence on flight velocity, V_o, mean emitted particle velocity, $\bar{V}_e$, body radius, a, ambient number density, ρ_o, and collision cross-section, S_{eo}.

Using similar assumptions to those in the above work but slightly different methods, we find the total energy given to a sphere is

$$E \doteq \pi a^2 \frac{1}{2} \rho_0 m V_0^3 \left[1 - \left(\frac{\bar{V}_e}{V_0}\right)^2\right]\left[1 - 0.2\frac{V_0}{\bar{V}_e} a \rho_0 S_{eo}\right] \qquad (2)$$

where $\rho_o m$ is the ambient mass density. Our coefficient, 0.2, is smaller than the corresponding coefficient in the drag expression 0.46, found by Baker and Charwat. The coefficient 0.2 is estimated in a manner described below. In place of the momentum recoil term, $0.44\,\bar{V}_e/V_o$, there is the factor $1 - (\bar{V}_e/V_o)^2$ to account for the energy carried away by the emitted particles. The correction to the scattering distribution for the velocity of the

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[1] R. M. L. Baker, Jr. and A. F. Charwat, Phys. of Fluids, 1, 73 (1958).

emitted particles not being negligible compared to the flight speed has not been made.

Our model starts from the observation that the angular distribution of emitted particles has simple analytic forms in the extremes of being very close to and very far from the sphere. At the surface the density of emitted particles for diffuse scattering is $2\,\rho_o V_o \cos\theta/\bar{V}_e$. The angle θ is made by the line of flight and the line connecting the center of the sphere and the point of interest. The factor $\cos\theta$ occurs because of the decrease at θ of the projected area normal to the flight direction per unit surface area of the sphere. The factor 2 arises because of the Lambert's law scattering. Far from the surface the distribution can be obtained by counting all particles emitted with the same direction. At the distance r this is

$$\frac{2\rho_o V_o a^2}{3\pi \bar{V}_e r^2}\left[(\pi-\vartheta)\cos\vartheta + \sin\vartheta\right] \qquad (3)$$

in agreement with the expression given in the paper by Baker and Charwat. Another difference lies in the approximation used to treat the probability of scattered particles returning to the sphere. Near the sphere we use $2\left[1-(1/2)\sin^4\theta\right](a/r)^2$ per collision, and at larger radial distances, $2\cos\theta\,(a/r)^2$. The former is chosen because for $a/r = 1$ at $\theta = 0$, both scattered particles hit the sphere while at $\theta = \pi/2$ just one of the scattered particles hits the sphere. However, the result does not seem sensitive to this difference.

We have used the $\cos\theta$ angular distribution with a $(a/r)^3$ dependence out to two radii from the sphere surface. At larger radial distances Eq. (3) is used. With these distributions we are able to estimate the net loss of energy by scattering. It is convenient to compare the total energy brought to the sphere by scattered particles (q_{gain}) with the energy incident particles that are hit would have given to the sphere (q_{loss}). Since some of the particles incident are deflected only slightly so that they still hit the sphere, it is necessary to point out that such events are counted twice. They are included as a total loss in q_{loss} and their contribution to the sphere is added to q_{gain}. This is used instead of giving the net loss to q_{loss} and no adjustment to q_{gain}. We find that the ratio $q_{gain}/q_{loss} \doteq 0.85$ from both the near and the distant regions. Because this ratio is the same for both regions our extrapolation to intermediate distances while crude may be called semi-quantitative. It is in this way that we estimate the factor 0.2.

The corresponding coefficients 0.2 and 0.46 should in principle be the same. In order to compute the momentum component in the flight direction and the energy that can be transferred to the sphere by a scattered particle one uses the incident values of these quantities multiplied by the same quantity, $\cos^2\theta$. Admittedly both models are semi-quantitative. That of Baker and Charwat does not emphasize the region close to and in the limb of the sphere. Our own extrapolation to intermediate distances is open to criticism.

If we calculate the number of particles hitting the sphere, we find a net gain, whereas there is a net energy loss. This results from the $\cos^2\theta$ term in the energy return expression, which greatly reduces the amount of energy return near $\theta = \pi/2$.

Unfortunately, the uncertainty in the coefficient is magnified as it results from the small net between loss and gain events. It would be worthwhile, therefore, to see this problem solved with more accuracy.

It is interesting to note the important role of surface temperature in these results. For lower surface temperature the emitted particles move away more slowly. By continuity their density is increased. Also, their mean-free path is decreased. This makes scattering events occur closer to the surface and leads to greater breakdown of free molecule flow.

At 100 miles altitude a 20 cm radius spherical satellite with velocity 26,000 ft./sec. and with surface temperature 300°K experiences ~ 1% reduction in the total heat received. It should be noted that this is a net effect and that for a 1% reduction in heat transfer about 6% of the particles incident to the sphere are actually scattered by collisions. The loss is considerably less, since many of the deflected particles still hit the sphere and many that would not have hit are deflected into the sphere.

We are grateful to R. F. Probstein of Brown University, A. Kantrowitz, and N. Kemp of AVCO Research Laboratory for suggestions and discussions.

On the ensuing pages we give the details of our treatment which have led us to the above conclusions.

A Model for a Rarified Gas Flow About a Sphere

P. Hammerling

INTRODUCTION AND ASSUMPTIONS

In free molecule flow the effects of intermolecular collisions can be ignored compared with collisions with a wall or an obstacle. This implies that the mean-free path in the gas is much greater than a characteristic length of the body. We shall consider the high speed flow of a rarified gas about a sphere In such a flow the reflected molecules are moving relatively slowly and form a region near the body whose density is greater than the free stream density. In the usual treatments of free molecule flow problems, one assumes that the reflected particles do not change the distribution function of the incident gas. However, the presence of the denser layer near the body introduces the possibility of collisions with the incident stream which can alter the expected momentum and energy transfer. As a first step one might look for the consequences of specifying that the incident and reflected particles make only one collision with one another. This restricted problem has been solved for the case of a flat plate by Heineman[1] and by Lunc and Lubonski[2].

The case of a sphere presents difficulties for two reasons: (1) The attenuation of the reflected particles due to collisions introduces a complicated exponential factor in the integrations, (2) The geometry involved also makes for difficult integrations. In order to gain some understanding of this problem without first having to do all the work, we have attempted to simplify it even further. The guiding spirit of the calculation is for it to be simple and yet include the physics of the problem. We

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[1] M. Heineman, Commun. Apl. Math, 1, 259 (1948)

[2] M Lunc and J Lubonski, Archiwum Mechanki Stoswanej, 4, 597 (1956)

have split things into two regions, the near and far zones. Far away from the sphere the geometrical difficulties can be overcome; close to the sphere the geometry must be considered more carefully, but the attenuation can be ignored. Since the results in these two regions should go continuously into one another, we have made an estimate of the in between region which is probably good to within 20%.

Behind the calculations are certain assumptions which we now list:

(a) Collisions are elastic and are visualized as between rigid spheres, the scattering being isotropic in the center of mass system of the two particles.

(b) Particles which hit the surface are emitted in a random direction with a probability proportional to the cosine of the angle between the normal to the surface and the direction of emission. This is known as diffuse reflection. One can regard the emitted particles as an effusive stream coming from an equilibrium gas inside the sphere.

(c) The accomodation coefficient $a = \frac{E_i - E_r}{E_i - E_b}$ is taken as unity E_i = incident energy flux, E_r = reflected energy flux, E_b = reflected energy flux corresponding to the body temperature.

(d) The thermal velocities of both the emitted particles and the gas proper are small compared with the flow velocity.

RESULTS

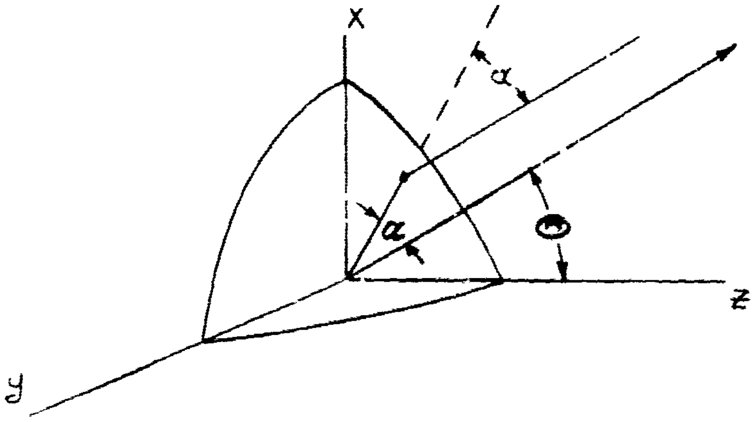
A Far Zone

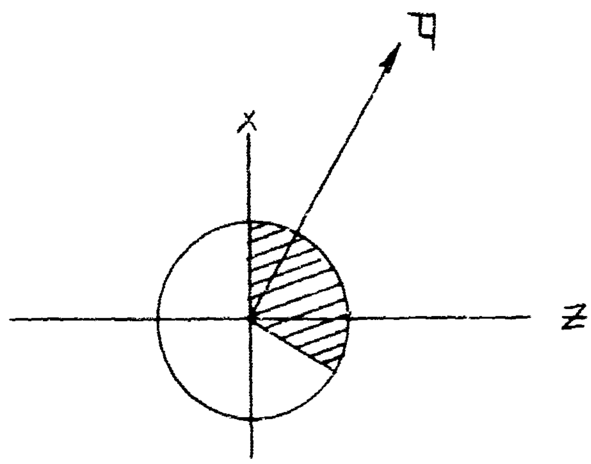
Consider the sphere in the coordinate system shown. The Z axis is in the direction of motion of the sphere With respect to a fixed direction Θ in the X Z plane a point on the sphere will have a polar angle

a. An observer at some distance from the sphere will see the reflected particles as coming in a parallel beam from that part of the sphere visible to him. In what follows we shall have to integrate over the surface of the sphere seen in this way. If one uses spherical polar coordinates about Z the surface integrals will be of the form

$$2\int_{\phi=0}^{\pi/2}\int_{\vartheta=0}^{\pi/2} f(\vartheta,\phi)\sin\vartheta\,d\vartheta\,d\phi + 2\int_{\phi=\frac{\pi}{2}}^{\pi}\int_{\vartheta=0}^{\vartheta_{max}} f(\vartheta,\phi)\sin\vartheta\,d\vartheta\,d\phi \qquad (4)$$

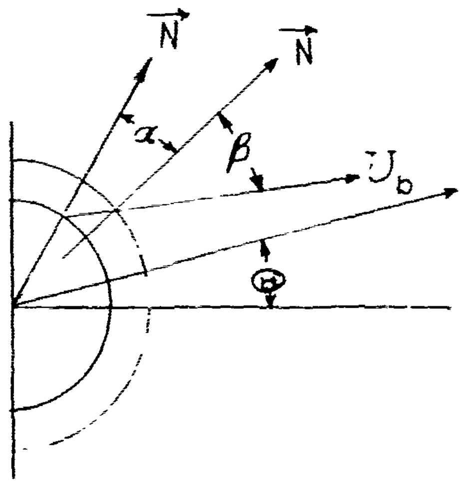
where $\operatorname{ctn}\theta_{max} = \tan\Theta\cos\phi$





The total outward flux in a given direction Θ coming from that part of the sphere that is visible can now be found. This number is equal to the number coming in at a given point multiplied by the probability of emission into the Θ direction and integrated over the visible surface.

$$dF(\Theta,\vartheta,\phi) = NU\cos\vartheta\,\frac{\cos\alpha}{\pi}\cos\beta \qquad (5)$$



The origin of $\cos\beta$ in equation (5) may be seen in Fig. (2). Consider the sphere and a concentric shell. The velocity vector of the particle emitted in the direction α makes an angle β with respect to the normal to the shell. In computing fluxes we need $\cos\beta$ as in equation (5), since we are interested in particles crossing normal to the surface. Using the law of sines we have $\cos\beta = (1 - (R/r)^2 \sin^2\alpha)^{1/2}$ For large r, $\cos\beta = 1$ which is the case under discussion. The far zone results should be good from a few sphere radii on out.

Since we require the emitted particles to go in the direction Θ the emission law is $\cos\alpha/\pi$. In our coordinate system

$$\cos\alpha = \sin\vartheta\cos\phi\sin\Theta + \cos\vartheta\cos\Theta \tag{6}$$

The outward flux is then

$$F(\Theta) = N u \pi R^2 \left(\frac{2}{\pi^2}\right)\left\{\int_0^{\pi/2}\int_0^{\pi/2} \sin\vartheta\cos\vartheta\cos\alpha\, d\vartheta\, d\phi + \int_{\phi=\pi/2}^{\pi}\int_0^{\vartheta_{max}} \sin\vartheta\cos\vartheta\cos\alpha\, d\vartheta\, d\phi\right\} \tag{7}$$

where $\operatorname{ctn}\vartheta_{max} = \tan\Theta\cos\phi$ and $\cos\alpha$ is given by (6). This turns out to be

$$F(\Theta) = N u \pi R^2 \frac{2}{3\pi^2} I(\Theta)$$

$$I(\Theta) = (\pi - \Theta)\cos\Theta + \sin\Theta \tag{8}$$

The total outward flux is found by integrating $E(\Theta)$ over all θ and ϕ since some particles are emitted in directions $\vartheta > \frac{\pi}{2}$; this results in

$$F_T = N u \pi R^2 \tag{9}$$

The emitted particle density at a distance r, $N_b(r)$ is obtained by noting that flux = $N_b(r) U_b r^2 \, d\Omega$,
hence flux/solid angle $\equiv F(\Theta) = N_b U_b r^2$ so that

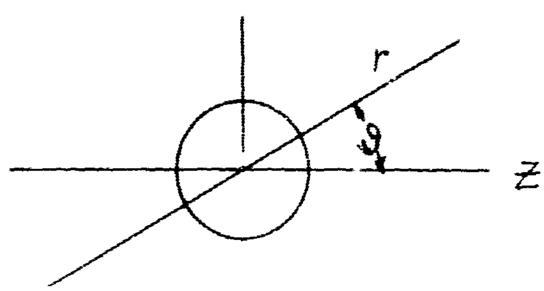
$$N_b(r) = \frac{N u \pi R^2}{U_b} \frac{2}{3\pi^2} \frac{I(\Theta)}{r^2} \tag{10}$$

Due to the possibility of collision, a particle leaving the surface will only have a certain chance of surviving unhit to a distance r; this probability is given by:

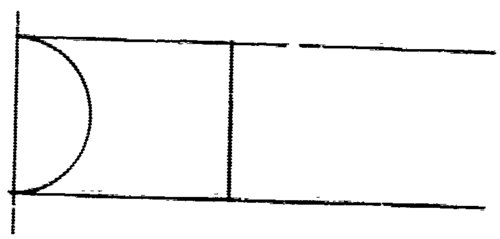
$$P = e^{-\frac{r-R}{L_b}} \tag{11}$$

, where L_b is the mean-free path. $L_b = \frac{U_b}{u} L$

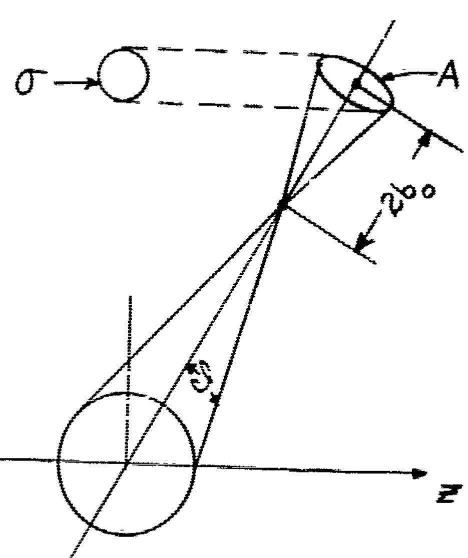
$P = 1$ when $r \approx R$



With these preliminary ideas in mind let us return to a physical consideration of the problem at hand. The incident particles may collide with the reflected particles in such a way as to send one of them into the sphere. The probability for such an event must be calculated. On the other hand, some of the incident particles that could have hit the sphere will not be able to do so because of collisions. The amount of energy per particle that can be carried to the sphere remains to be calculated also.



Consider the cross-section at the left. We shall consider all collisions within the cylinder as losses, whereas we shall integrate over the whole half space to compute gains. The losses are computed from beyond a certain radius. Closer to the sphere we must use the near zone approximation to be discussed below.



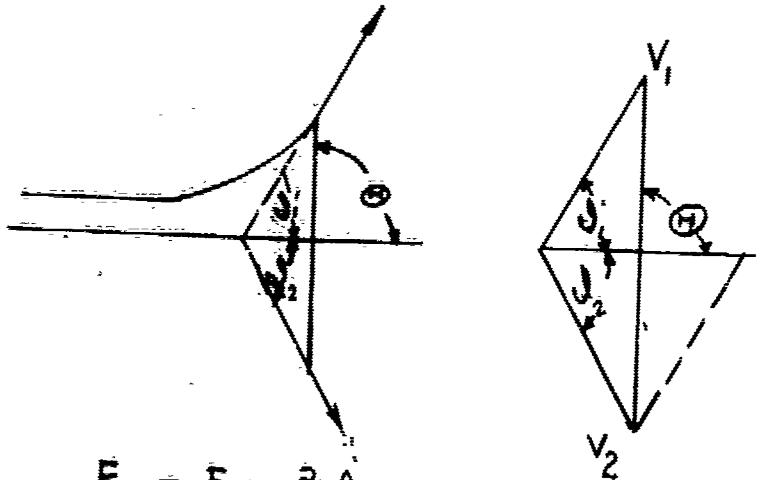
Before we compute the energy flux we first carry out the calculation mentioned above. In order that a particle colliding at a point (r, θ) hit the sphere, its final velocity vector must be in the cone shown. Of the incoming stream only those which instantaneously lie in the continued cone will be effective. The ratio of the projected area of the base of this cone to the total possible impact area is the required probability.

$$P_r = \frac{\sigma}{4\pi b_0^2} = \frac{A\cos\vartheta}{4\pi b_0^2} \qquad \therefore \qquad P_r = 2\cos\vartheta\,\frac{R^2}{r^2} \tag{12}$$

$$\frac{A}{4b_0^2} = \Omega = \frac{\pi R^2}{r^2}$$

The factor 2 arises from inverse collisions.

To obtain the energy carried per particle refer to the geometry of the encounter.



$$E_r = E\cos^2\vartheta$$

For equal masses $\vartheta_1 = \frac{\Theta}{2}$

but $\vartheta_2 = \frac{1}{2}(\pi - \Theta)$

$\vartheta_1 + \vartheta_2 = \frac{\pi}{2}$, call $\vartheta_2 = \vartheta$

Then $\frac{V_2}{V_i} = \cos\vartheta$, $\frac{V_1}{V} = \sin\vartheta$

(13)

One can now compute the energy flux in the far zone. Call q = energy flux.

$$dq = NQuN_b\,PE_r\,d^3r$$

$$NQ = \frac{1}{L} \tag{14}$$

We first calculate the expected loss from the cylindrical region. In using $N_b(r)$ we must take $I(\theta) = I(0) = \pi$. We use $E_r = E$ since we regard collisions as carrying away all the energy.

$$q_{loss} = Nu\pi R^2 E\,\frac{u}{u_b}\,\frac{2}{3}\,\alpha\,I_1$$

$$I_1 = \alpha_b\,e^{\alpha_b}\int_{r_b}^{\infty}\frac{e^{-\rho}}{\rho^2}\,d\rho \tag{15}$$

where $\alpha = \frac{R}{L}$, $\alpha_b = \frac{u}{u_b}\alpha$

The energy gain is obtained from equations (10) through (14).

$$q_{gain} = Nu\pi R^2 E\,\frac{u}{u_b}\,\frac{8}{3\pi}\,\alpha\,I_1\int_0^{\pi/2} I(\vartheta)\cos^3\vartheta\sin\vartheta\,d\vartheta \tag{16}$$

$$\frac{q_{gain}}{q_{loss}} = \frac{4}{\pi}\int_0^{\pi/2} I(\vartheta)\cos^3\vartheta\sin\vartheta\,d\vartheta \doteq .84 \tag{17}$$

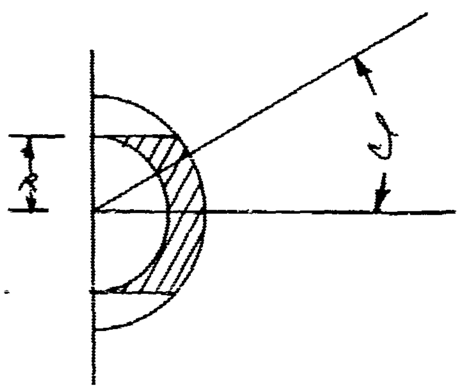
There is then a slight energy loss from the far zone. The radial integral will be needed later, it is

$$I_1 = \alpha_b e^{\alpha_b} \left[\frac{e^{-r_b}}{r_b} + Ei(-r_b) \right] \qquad (18)$$

$$r_b \equiv \frac{r}{L} \frac{u}{U_b}$$

B. Near Zone

In the zone close to the sphere, the effects of the geometry cannot be glossed over. The results for the far zone indicate a net loss; a comparable reduction in heat transfer is found to take place in the near zone. We shall present a first attempt to estimate what occurs in the near zone. The shaded portion in the figure is the region in which there can be an energy loss due to collision with particles coming from the surface.



However, particles which otherwise would not hit the sphere have some chance if they are in the unshaded region. A particle impinging at the stagnation point transfers all of its energy, whereas for $\theta = \pi/2$, the maximum energy transfer is about half the available energy. The energy given to the sphere will be taken to have the approximate distribution:

$$E_r = E \left(1 - \frac{\sin^4 \vartheta}{2} \right) \left(R/r \right)^2 \qquad (19)$$

The density of reflected particles near the surface is largely due to their low velocity and to the cosine distribution. A density function which should approximate physical conditions and is exact on the surface

$$\rho = \frac{N u \cos\vartheta}{u_b}\left(R/r\right)^3\left(\frac{1}{\pi}\int \cos^3\alpha \, d\Omega\right)^{-1} = 2\,\frac{N u \cos\vartheta}{u_b}\left(R/r\right)^3 \tag{20}$$

The value $\cos^3\alpha$ occurs because we are using the flux weighted component of the velocity; the probability of a particle leaving at the direction α is $\cos\alpha$; its contribution to the flux is reduced by $\cos\alpha$ and finally the component of the velocity is $u_b \times \cos\alpha$.

To compute the expected gain we integrate over the volume between the sphere and a concentric shell.

$$q_{gain} = \int E\cdot\rho\,(NQU)\,d^3r = 2 q_0 \frac{u}{u_b}\,\frac{2R^3}{L}\int_0^{\pi/2}\int_R^r \frac{1-\frac{1}{2}\sin^4\vartheta}{r^5}\cos\vartheta\sin\vartheta\, r^2\, d\vartheta\, dr \tag{21}$$

$$q_{gain} = q_0\,\frac{u}{u_b}\,\frac{R}{L}\left[1-\left(\frac{R}{r}\right)^2\right]\cdot 5/6$$

$$q_0 = N u \pi R^2 E \tag{22}$$

The loss is found by integrating over the volume indicated by the shading in the figure.

$$q_{loss} = 2 q_0 \frac{u}{u_b}\,\frac{R}{\pi L}\int \frac{\cos\vartheta}{\rho^3}\, d^3r$$

$$\int \frac{\cos\vartheta}{\rho^3}\,d^3r = 2\pi\int_0^{\sin^{-1}R/r} \sin\vartheta\cos\vartheta\, d\vartheta\int_R^r \frac{d\rho}{\rho} + 2\pi\int_{\sin^{-1}R/r}^{\pi/2}\sin\vartheta\cos\vartheta\int_R^{R\csc\vartheta}\frac{d\rho}{\rho}\,d\vartheta$$

$$= \frac{\pi}{2}\left[1-\left(R/r\right)^2\right] \tag{23}$$

$$q_{loss} = q_0\,\frac{u}{u_b}\,\frac{R}{L}\left[1-\left(\frac{R}{r}\right)^2\right] \tag{24}$$

The ratio of loss to gain is then

$$\frac{q_{loss}}{q_{gain}} = 6/5 \tag{25}$$

so that one also has the possibility of a decrease in heat transfer in the region close to the surface.

We now bring our results together to estimate the effect of the region in between our two extremes.

Near Zone:

$$\begin{aligned} q_{gain} &= q_0 \frac{u}{u_b} \alpha \frac{5}{6}\left[1-\left(\frac{R}{r}\right)^2\right] \\ q_{loss} &= q_0 \frac{u}{u_b} \alpha \left[1-\left(\frac{R}{r}\right)^2\right] \end{aligned} \tag{26}$$

Far Zone:

$$q_{loss} = q_0 \frac{u}{u_b} \alpha \frac{2}{3} \alpha_b e^{\alpha_b}\left[\frac{e^{-r_b}}{r_b} + Ei(-r_b)\right] \tag{27}$$

The Jahnke-Emde tables list values of the exponential integral needed in Eq. (27). For small values of the argument this function can be calculated from:

$$Ei(-x) = \gamma + ln\, x - \left(x - \frac{x^2}{4}\right)$$

$$\gamma = 0.5722$$

The net of loss and gain for the near and far zones is:

$$\begin{aligned} q_{near} &= q_0 \frac{u}{u_b} \frac{\alpha}{6}\left[1-\left(\frac{R}{r}\right)^2\right] \\ q_{far} &= q_0 \frac{u}{u_b} \frac{\alpha}{10} \alpha_b e^{\alpha b}\left[\frac{e^{-r_b}}{r_b} + Ei(-r_b)\right] \end{aligned} \tag{28}$$

The net from the entire space is found by adding the above and evaluating the result at $r = 3R$. The net energy given to the sphere is then

$$q_{Net} = q_0\left[1 - \frac{u}{u_b}\,\frac{R}{L}\,(0.2)\right] \tag{29}$$

Taking into account the approximations that have been made, we feel that the factor (0.2) is known to better than $\pm$ 50%.

A correction to free molecule results may be expected in the regime where

$$\frac{u}{u_b}\,\frac{R}{L} > \frac{1}{2} \tag{30}$$

In terms of Reynolds number Eq. (30) is

$$R_e\left(\frac{T_\infty}{T_b}\right)^{1/2} > 1 \tag{31}$$

One obtains Eq. (31) from simple gas kinetic arguments

Let a = sound speed $= (\gamma RT)^{1/2}$; $\bar{c} = \left(\frac{8RT}{\pi}\right)^{1/2}$

From $\nu = \frac{1}{2} L\bar{c}$

M = Mach #

$$\frac{R}{L} = \frac{1}{2}\,\frac{R\bar{c}}{\nu} = \frac{1}{2}\left(\frac{8}{\pi\gamma}\right)^{1/2}\frac{Ra}{\nu}\,\frac{u}{u}$$

$$= \frac{1}{2}\left(\frac{8}{\pi\gamma}\right)^{1/2}\frac{R_e}{M}$$

T_∞ = Free stream temperature

$$\frac{u}{u_b} = \frac{Ma}{u_b} = M\left(\frac{\pi\gamma}{8}\right)^{1/2}\left(\frac{T_\infty}{T_b}\right)^{1/2}$$

T_b = Body temperature

$$\therefore \quad \frac{u}{u_b}\,\frac{R}{L} = \frac{1}{2}\,R_e\left(\frac{T_\infty}{T_b}\right)^{1/2}$$

ν = "kinematic" viscosity

C. Numerical Example

In order to have a feel for these quantities, consider the case of a satellite at an altitude of 100 miles.

Some pertinent numbers are

$$\rho/\rho_o = 3.906 \times 10^{-8} \exp[-h/143{,}228] \qquad h \geq 160 \text{ km}$$

$$u = 26{,}000 \text{ ft/sec} = 75 \times 10^4 \text{ cm./sec.}$$

at 100 miles $\rho/\rho_o = 12.4 \times 10^{-10}$

thus $N = 16.6 \times 10^9 \frac{\text{air molecules}}{\text{cm}^3} \doteq \rho/\rho_o L_o$

The air molecules are regarded as rigid spheres of diameter 3.72A , so that

$$Q = 4.35 \times 10^{-15} \text{ cm}^2.$$

$$\frac{1}{L} = NQ = 7.2 \times 10^{-5} \text{ cm.}^{-1}$$

$$\text{For } R = 20 \text{ cm.}; \ \alpha = 1.45 \times 10^{-3}, \ \alpha_b = \alpha \frac{u}{u_b} = .024$$

If the body has a temperature of 300°K, then

$$U_b = 2\left(\frac{2kT}{\pi m}\right)^{1/2} = 4.6 \times 10^4 \text{ cm/sec.}$$

Under these conditions we have the following results based on Eq. (26)

$$q_o = Nu\pi R^2 E = \frac{1}{2}\rho u^3 \pi R^2 = 42 \text{ watts}$$

$$\frac{q_{loss}}{q_o} = 0.0195$$

The above are for a region extending one radius from the sphere. The ratio of loss q_o from the region greater than one radius away from the sphere is:

$$q_{loss}/q_o = .007$$

At 100 miles a 20 cm sphere with surface temperature 300°K experiences a 1 to 2% reduction in total heat received because of collisions of the incident gas with rebounding molecules. About 4% of the incident particles are scattered by collisions. However, the loss is considerably less, since many of the deflected particles still hit the sphere and many that would not have hit are deflected into the sphere.

From Equation (30) we have

$$e^{-h/b} = 3.1 \times 10^{-8} \frac{1}{a} \frac{\sqrt{T_b}}{R} \tag{32}$$

where for

$$140 \leq h \leq 160 \text{ km}$$
$$b = 52,249 \text{ ft.}$$
$$a = 2.309 \times 10^{-5}$$
$$h \geq 160 \text{ km.}$$
$$b = 143,228 \text{ ft.}$$
$$a = 3.906 \times 10^{-8}$$

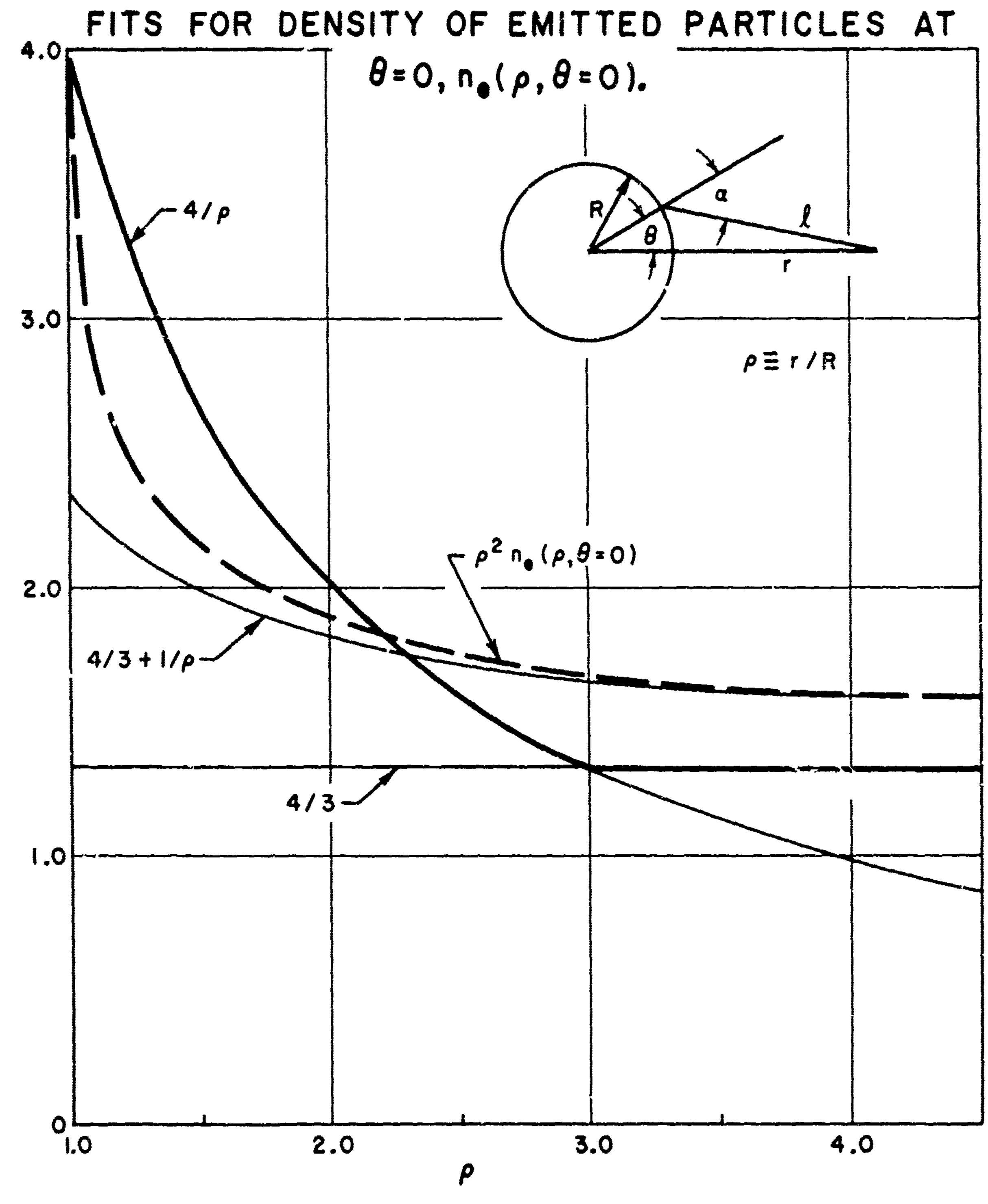
We conclude that for a 100 cm satellite traveling at 26,000 ft/sec and having a surface temperature 300°K, free molecule flow theory gives a reasonable result for the total energy imparted to the sphere as long as the altitude is greater than 115 ± 10 miles.

ADDENDUM

The density of particles as a function of distance away from the sphere r along the direction of flight ($\theta = 0$) is given by the expression

$$n(r, \vartheta = 0) = n_0 \frac{u}{u_b} 2R^2 \int_0^{\cos^{-1} R/r} \cos\vartheta \cos\alpha \sin\vartheta \, d\vartheta$$

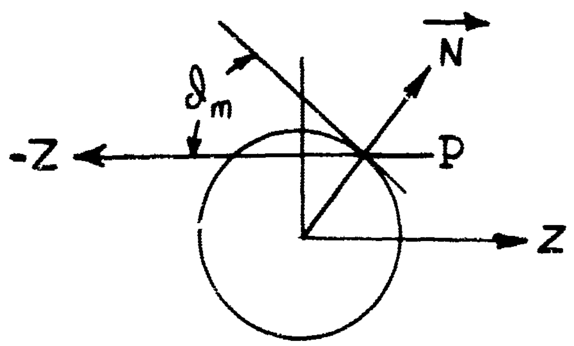
This quantity is plotted in the accompanying figure. For comparison we also plot $4R/r$ and $4/3 + R/r$. The former is our fit for the near zone. The latter is asymptotically correct for the far zone; however, we have only taken the leading term which is $4/3$. Our fit is indicated in the figure as a bold line and can be seen to be slightly larger in the near zone and smaller in the far zone than the correct density which is given by the dashed line. It is felt, however, that the explanation should give a reasonably close answer to the problem.


FITS FOR DENSITY OF EMITTED PARTICLES AT
$\theta = 0$, $n_e(\rho, \theta = 0)$.
4.0
3.0
2.0
1.0
0
1.0
2.0
3.0
4.0
ρ
$4/\rho$
$\rho^2 n_e(\rho, \theta = 0)$
$4/3 + 1/\rho$
$4/3$
R
θ
α
ℓ
r
$\rho \equiv r/R$

ADDENDUM II

In this section we find an expression for the energy return to the sphere which is exact on the surface. This modifies our work in the near zone but leaves the far zone results unchanged. The new result indicates a lower coefficient than given in Eq (29), 0.12 instead of 0.18.

A collision on the surface can send in both particles which transfers all the available energy or just one particle which carries most of the energy. The former occurs if the final velocity vectors of both particles are directed toward the sphere. This criteria is expressed in terms of the tangent plane to the sphere. Consider a sphere with a tangent plane at point **P** and a coordinate system at that point parallel to the old one. Measuring angles from the -Z axis at this point we see that for two particles to hit the sphere that the velocity vector of one of them must lie between zero and θ_m.



The limiting angle, θ_m, is in general a function of ϕ and Θ. The relationship between these quantities can be found by using the fact that the normal to the tangent plane is perpendicular to every line in it.

With respect to the coordinate system at the center of the sphere, the inward directed normal has direction cosines $(-\sin\Theta, 0, \cos\Theta)$.

In the same coordinate system a line in the plane will have direction cosines ($\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$). Since these lines are perpendicular, we have

$$
\begin{gathered}
-\sin\Theta\sin\theta\cos\phi + \cos\Theta\cos\theta = 0 \\
\tan\theta = \frac{\operatorname{ctn}\Theta}{\cos\phi}
\end{gathered}
\tag{1}
$$

One can now compute the number of particles and energy returned to the sphere. In the following we make use of the fact that for a collision such that the velocity vector of one of the particles lies within 0 and θ_m, two particles hit the sphere, whereas for the region between θ_m and $\pi/2$, only one particle goes into the surface.

The number of particles returning to the surface is

$$
\begin{aligned}
N_r &= \frac{2}{\pi}\cdot 2\int_0^{\pi/2}\left\{2\int_0^{\vartheta_m}\cos\vartheta\sin\vartheta\,d\vartheta + 1\int_{\vartheta_m}^{\pi/2}\cos\vartheta\sin\vartheta\,d\vartheta\right\}d\phi \\
&= 2 - \frac{2}{\pi}\int_0^{\pi/2}\cos^2\vartheta_m\,d\phi
\end{aligned}
\tag{2}
$$

where

$$
\int_0^{\pi/2}\cos^2\vartheta_m\,d\phi = \int_0^{\pi/2}\frac{\cos^2\phi\,d\phi}{\operatorname{ctn}^2\Theta + \cos^2\phi} = \int_0^{\pi/2}\frac{1-\sin^2\phi\,d\phi}{a^2-\sin^2\phi}
$$

$$
a^2 = \csc^2\Theta
$$

Adding and subtracting a^2 in the numerator, we have

$$
\int_0^{\pi/2}\cos^2\vartheta_m\,d\phi = \frac{\pi}{2} - (a^2-1)\int_0^{\pi/2}\frac{d\phi}{a^2-\sin^2\phi}
$$

The integral on the right hand side is listed in Groebner and Hofreiter Integraltafel, # 331.56 a. Our result is then

$$N_r = 1 + \cos\Theta \tag{3}$$

The appropriate energy return function is found from an equation similar to Eq. (2)

$$E_r = \frac{4}{\pi} E \int_0^{\pi/2} \left\{ \int_0^{\vartheta_m} \cos\vartheta \sin\vartheta \, d\vartheta + \int_{\vartheta_m} \sin^2\vartheta \cos\vartheta \sin\vartheta \, d\vartheta \right\} d\phi$$

$$= E\left[1 - \frac{1}{\pi} \int_0^{\pi/2} \cos^4\vartheta_m \, d\phi \right] \tag{4}$$

The integral is evaluated in the same manner as the one in Eq. (2).

$$\int_0^{\pi/2} \cos^4\vartheta_m \, d\phi = \int_0^{\pi/2} \frac{\cos^4\phi \, d\phi}{(a^2 - \sin^2\phi)^2}$$

$$= \frac{\pi}{2} - 2(a^2-1) \int_0^{\pi/2} \frac{d\phi}{a^2 - \sin^2\phi} + (a^2-1) \int_0^{\pi/2} \frac{d\phi}{(a^2 - \sin^2\phi)^2}$$

The first integral is the one we have already used; the second integral is proportional to the derivative of the first. It can then be verified that

$$\int_0^{\pi/2} \frac{\cos^4\phi \, d\phi}{(a^2 - \sin^2\phi)^2} = \frac{\pi}{2}\left[1 - \cos\Theta - \frac{\sin^2\Theta \cos\Theta}{2} \right] \tag{5}$$

and that the energy return is

$$E_r = \frac{1}{2}\left[1 + \cos\Theta + \frac{1}{2}\cos\Theta \sin^2\Theta \right] \tag{6}$$

If we use this function instead of Eq. (19) in the body of this report, then the net energy loss in the near zone would be

$$q_{net} = q_0 \frac{u}{u_b} \frac{R}{L} \frac{1}{10} \left[1 - (R/r)^2 \right] \tag{7}$$

The outer region result is unchanged. The total net evaluated at r/R = 3, is

$$\text{total net energy loss} = q_0 \frac{U}{U_b} \frac{R}{L} (0.12) \qquad (8)$$

It is felt that the "true" coefficient lies somewhere between this result and (0.2) from Eq. (29).