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B. B. LIEBERMAN

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Abstract

The paper is concerned with the decay of the energy of disturbances which are propagated according to the wave equation with variable index of refraction in the exterior of a finite star-shaped reflecting body. It is shown that the energy of the disturbance decays like some power of t^{-1}. Certain conditions of growth and continuity are made on the index in order to insure some decay factor.

The energy decay is obtained by estimating the solution of an integral equation which results when one applies the Friedrichs' "A-B-C method" to the modified wave equation operator. Using the energy estimate with other familiar estimates, one obtains a rate of decay for the disturbance itself.

Introduction

Let us consider disturbances which are propagated in three-dimensional space according to the modified wave operator

$$u_{xx} + u_{yy} + u_{zz} - c^{-2}(x,y,z)\, u_{tt} = 0.$$

If, initially, the disturbance were confined in some finite region in the exterior of a finite smooth reflecting body, one would expect that after a certain fixed time most of the disturbance would have propagated to infinity. In [1] it is shown with certain restrictions on the body that this is the case for $c = 1$, and the disturbance at a point dies out exponentially.

For the non-homogeneous wave equation one also finds that if u satisfies initial data of compact support and the equation

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = g(x,y,z)e^{i\omega t}$$

in the exterior of a smooth, finite, star-shaped body on which

$$u(x,y,z,t) = 0 \qquad\qquad t \geq 0,$$

then $ue^{-i\omega t}$ tends exponentially in time to $v(x,y,z)$ at every fixed point exterior to the body. Here v satisfies

$$v_{xx} + v_{yy} + v_{zz} + w^2 v = g$$

exterior to the body,

$$v = 0$$

on the body and v satisfies the Sommerfeld radiation condition at infinity.

This result is known as the limiting amplitude principle. The proof is based on Kirchoff's theorem and on the previous energy estimate.

One expects that both the energy decay and the limiting amplitude principle are also true with a variable speed of propagation. Here an estimate will be found for the energy decay in any fixed region. A proof of the limiting amplitude principle similar in nature to [2] is not possible since Kirchoff's representation theorem is no longer valid in an inhomogeneous medium. But, it will be shown that all solutions to the initial boundary value problem for the homogeneous wave equation with variable index of refraction decay in time at any fixed point exterior to the body.

One may expect some restrictions on the index of refraction for, if the index oscillated very rapidly, the energy might be trapped between two minima of the speed of propagation. However, in the case of the earth and its atmosphere, a good approximation for the behavior of the index of refraction is:

$$c(r) = c_o - c_1 e^{-\beta r}$$

where $c_o > c_1 > 0$ and $\beta > 0$.

This case will be covered by the hypotheses on the index, and the energy does go to infinity.

The estimates are obtained by the methods used in [2]; i.e., the energy estimate will be obtained by a variation of the A- B - C method of Friedrichs. This consists in multiplying the differential operator

$$\Delta u - c^{-2} u_{tt} = 0$$

by the operator

$$Au + \vec{B}\cdot\nabla u + Cu_t$$

and integrating over the space exterior to the body. By strategically choosing the A, B, and C, and by applying the divergence theorem to the resulting integral, one obtains an integral equation. Upon estimation of certain terms in the integral equation, one finds that the energy in any fixed finite region exterior to the body decays in time like some power of t^{-1}.

The main theorem to be proved is:

Theorem 1: Let u be a smooth solution of

(1) $$u_{xx} + u_{yy} + u_{zz} - c^{-2}(x,y,z)u_{tt} = 0$$

in the exterior of a smooth, star-shaped, finite three-dimensional body B on which the boundary condition

(2) $$u = 0$$

is satisfied. Let u satisfy the initial conditions

(3) $$u(x,y,z,o) = f(x,y,z)$$

(4) $$u_t(x,y,z,o) = g(x,y,z)$$

where f and g are both twice continuously differentiable and of compact support. Suppose c(x,y,z) satisfies the following conditions:

(5) I c(x,y,z) is continuously differentiable

II c(x,y,z) is a function of the distance from the origin of the coordinate system

III $\min c(x,y,z) = \gamma > o$

IV $|\dot{c}| < \min \left(\frac{\gamma}{2r} , \frac{1}{r^4} \right)$

for all $r = \sqrt{x^2 + y^2 + z^2} \geq 0.$

Then

(6) $$t^{2-\delta} \int_R (|\nabla (ru)|^2 + c^{-2}(ru_t)^2) \sin \theta \, dr d\varphi d\theta < M$$

where M is some constant depending upon c, f, g, R and

$$(7) \qquad \alpha = \max_{0<r<\infty} \frac{4r}{c} \left| \frac{dc}{dr} \right| < \delta < 2 .$$

Before proceeding with the proof, a few remarks will be made on the conditions on the index of refraction. Condition I is needed for the application of the divergence theorem to the differential operator. Condition II was assumed because of the simplification involved in the integration and also because of its good approximation to physical considerations. It seems possible that angular dependence could be introduced, provided certain boundedness conditions are made on the index and its derivatives. Condition III is needed to maintain the hyperbolic character of the differential equation. Condition IV puts the greatest restriction on the index. In effect, it says that for large r the index should approach a constant like $O(r^{-3})$. That this condition is reasonable can be seen from the paper by Miranker [3] in which he shows that the Rellich uniqueness theorem is valid for the reduced wave equation with variable index of refraction, provided the index approaches a constant for large r like $O(r^{-3})$. Thus this condition is a natural one if a limiting amplitude principle is to hold for the variable case. That $|\dot{c}|$ should be less than $\gamma/2r$ might be improved; however, it was the simplest necessary condition in order to get a decay in the energy. If, for example, we consider the earth's atmosphere,

$$c(r) = 1 - (1 - \gamma)e^{-\beta r},$$

we obtain the following restriction on γ. [We assume that c(r) has been normalized to be one at infinity.] By (5)

$$|\dot{c}(r)| = (1 - \gamma)\beta e^{-\beta r} < \gamma/2r$$

or

$$2(1-\gamma)\beta\gamma^{-1} < e^{\beta r} r^{-1}.$$

The minimum on the right side is obtained for $r \geq 0$ when $r = \beta^{-1}$. We obtain

$$2(1 - \gamma)\ \gamma^{-1} < e$$

or

$$2(2 + e)^{-1} < \gamma.$$

Thus the index may change by no more than $e(2 + e)^{-1}$ of its maximum value in order for our result to hold. We now proceed with the proof of the theorem.

Proof: Choose a spherical coordinate system so that the body B is star-shaped with respect to the origin, i.e., $r_n \leq 0$, where r_n is the radial component of the unit normal pointing into the body. Let D_t be the domain of influence of the initial data at time t. Writing equation (1) in spherical coordinates, one obtains

$$(8) \qquad \frac{(r^2 u_r \sin\Theta)_r}{r^2 \sin\Theta} + \frac{(u_\Theta \sin\Theta)_\Theta}{r^2 \sin\Theta} + \frac{u_{\varphi\varphi}}{r^2 \sin^2\Theta} - \frac{u_{tt}}{c^2(r)} = 0.$$

Multiplying (8) by the operator

$$(9) \qquad Au + Bu_r + Cu_t$$

and integrating over the entire domain of influence, one obtains

$$(10)\quad \int_0^{t_1}\int_{D_t} (Au + Bu_r + Cu_t)\left(\frac{(r^2u_r\sin\theta)_r}{r^2\sin\theta} + \frac{(u_\theta\sin\theta)_\theta}{r^2\sin\theta} + \frac{u_{\varphi\varphi}}{r^2\sin^2\theta} - \frac{u_{tt}}{c^2}\right) dvdt = 0.$$

Now set $ru = w$ and choose

$$(11)\qquad A = 2t$$

$$(12)\qquad B = 2rt$$

$$(13)\qquad C = t^2 + \int_{r_o}^{r} \frac{2\zeta}{c^2(\zeta)}\, d\zeta.$$

For convenience the explicit form of B and C will not be exhibited; however, we will exploit their form in the following calculations. After the change of variables, (10) becomes

(14)
$$\int_0^{t_1}\int_{D_t} (Bw_r + Cw_t)\left(\sin\theta w_{rr} + \frac{(w_\theta\sin\theta)_\theta}{r^2} + \frac{w_{\varphi\varphi}}{r^2\sin\theta} - \frac{w_{tt}\sin\theta}{c^2(r)}\right)\frac{dvdt}{r^2} = 0.$$

Using the fact that

$$2(Bw_rw_{rr}) = (Bw_r^2)_r - B_rw_r^2$$

and $2(Cw_tw_{rr}) = 2(Cw_rw_t)_r - 2C_rw_rw_t - (Cw_r^2)_t + C_tw_r^2$, etc.,

one obtains for (14)

$$\int_0^{t_1}\int_{D_t} \Bigg[\left(B\sin\theta w_r^2 + 2C\sin\theta w_rw_t - \frac{B\,w_\varphi^2}{r^2\sin\theta} + \frac{B\sin\theta}{c^2(r)}w_t^2 - \frac{B\sin\theta w_\theta^2}{r^2}\right)_r$$

$$- \left(C\sin\theta w_r^2 + \frac{Cw_\varphi^2}{r^2\sin\theta} + \frac{2\sin\theta Bw_tw_r}{c^2(r)} + \frac{C\sin\theta w_t^2}{c^2(r)} + \frac{C\sin\theta w_\theta^2}{r^2}\right)_t$$

$$+ \left(\frac{2Bw_\varphi w_r}{r^2\sin\theta} + \frac{2Cw_\varphi w_t}{r^2\sin\theta}\right)_\varphi + \left(\frac{2\sin\theta Bw_\theta w_r}{r^2} + \frac{2C\sin\theta w_\theta w_t}{r^2}\right)_\theta$$

$$+ 4w_t^2 rt\frac{\dot c(r)}{c^3(r)}\sin\theta\Bigg]\quad drd\varphi d\theta dt = 0.$$

Applying the divergence theorem to this equation one obtains

$$(15)\quad \int_S \left\{ \left[2Cw_r w_t + B\left(2w_r^2 - |\nabla w|^2 + \frac{w_t^2}{c^2(r)}\right) \right] r_v \right.$$

$$- \left[2\frac{B}{c^2(r)} w_t w_r + C\left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)}\right) \right] t_v$$

$$+ \left(\frac{2B}{r^2 \sin^2 \theta} w_\varphi w_r + \frac{2C}{r^2 \sin^2 \theta} w_\varphi w_t \right) \varphi_v$$

$$\left. + \left(\frac{2B}{r^2} w_\theta w_r + \frac{2C}{r^2} w_\theta w_t \right) \theta_v \right\} \sin\theta dS dt$$

$$+ \int_0^{t_1} \int_{D_t} \frac{4rt}{c^3(r)} c(r)\, w_t^2 \sin\theta dr d\varphi d\theta dt = 0,$$

where r_v, t_v, φ_v, and θ_v are the various components of the unit normal pointing out of the four-dimensional time-space region. The surface integral in (15) may be subdivided into three integrals ; one over the body B from time $t = 0$ to $t = t_1$; the second over the initial data up to the radius $r = a_0$; the third over the region D_{t_1}. There is no contribution for large values of r since the solution is identically zero outside the domain of influence of the initial data which are of compact support.

The integral over the body, I_1, is given by

$$(16)\quad I_1 = \int_0^{t_1} \int_B \left[B(2w_r^2 - |\nabla w|^2) r_v + \frac{2B}{r^2 \sin^2 \theta} w_\varphi w_r \varphi_v + 2\frac{B}{r^2} w_\theta w_r \theta_v \right] \sin\theta dS dt$$

$$= \int_0^{t_1} \int_B \frac{2t}{r} \left(\frac{dw}{dn}\right)^2 r_v \, d\sigma dt$$

where we have made use of (2), (12) and the fact that $t_v = 0$ on B. Since B is star-shaped $r_v \leq 0$ and therefore

(17)
$$I_1 \leq 0.$$

The integral over the initial data is

(18)
$$I_2 = \int_{D_o} \left(\int_{r_o}^{r} \frac{2\zeta}{c^2(\zeta)} d\zeta \right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta dr d\varphi d\theta \cdot$$

We make use of (13) to obtain (18). By (3), (4), and (5) I_2 is bounded.

The integral over the region D_{t_1}, I_3, is written as

(19)
$$I_3 = -\int_{D_{t_1}} \left[\frac{2B}{c^2(r)} w_t w_r + C \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \right] \sin\theta dr d\varphi d\theta.$$

Combining (16), (18), and (19) and using (17), one obtains

(20)
$$\int_{D_{t_1}} \left\{ \frac{4rt_1}{c^2(r)} w_t w_r + \left(t_1^2 + \int_{r_o}^{r} \frac{2\zeta}{c^2(\zeta)} d\zeta \right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \right\} \sin\theta dr d\varphi d\theta$$
$$- \int_o^{t_1} \int_{D_t} 4rt \frac{\dot{c}(r)}{c^3(r)} w_t^2 \sin\theta \, dr d\varphi d\theta dt \quad < I_2 \cdot$$

Since $\int_o^{t_1} \int_{D_t} \frac{4rt}{c^3(r)} |\dot{c}(r)| \, |\nabla w|^2 \sin\theta dr d\varphi d\theta dt$ is positive we can subtract it from the left side of (20) without destroying the inequality.

Consequently we have

$$(21)\quad \int_{D_t} \left\{ \left(t_1^2 + \int_{r_0}^{r} \frac{2\zeta}{c^2(\zeta)} d\zeta \right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) + \frac{4rt_1}{c^2(r)} w_t w_r \right\} \sin \theta dr d\varphi d\theta$$

$$-\int_0^{t_1} t dt \int_{D_t} \frac{4r|\dot{c}(r)|}{c(r)} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin \theta dr d\varphi d\theta \quad < I_2 \;.$$

Considering the first integral in (21) , we integrate $\int_{r_0}^{r} \frac{2\zeta}{c^2(\zeta)} d\zeta$ by parts to obtain for this term

$$(22)\quad \int_{D_{t_1}} \left\{ \left(t_1^2 + \frac{r^2}{c^2(r)} \right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) + \frac{4rt_1}{c^2(r)} w_t w_r \right\} \sin\theta dr d\varphi d\theta$$

$$+\int_{D_{t_1}} \left(\int_{r_0}^{r} 2\zeta^2 \frac{\dot{c}(\zeta)}{c^3(\zeta)} d\zeta \right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta dr d\varphi d\theta \;.$$

Applying Schwarz's inequality to the term $4rt_1 w_t w_r c^{-2}(r)$ in (22), one has

$$(23)\quad |4rt_1 w_t w_r c^{-2}(r)| \leq 2rt_1 c^{-1}(r) \, (w_r^2 + c^{-2} w_t^2) \;.$$

Subtracting the positive term

$$\int_{D_{t_1}} \frac{2rt_1}{c(r)} \left(\frac{w_\varphi^2}{r^2 \sin^2\theta} + \frac{1}{r^2} w_\theta^2 \right) \sin\theta \; dr d\varphi d\theta$$

from the left side of (21), and applying (23) to the first integrand does not alter the inequality, and one obtains

$$(24)\quad \int_{D_{t_1}} \left(t_1 - \frac{r}{c(r)}\right)^2 \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)}\right) \sin\theta \, dr d\varphi d\theta$$

$$- \int_0^{t_1} t dt \int_{D_t} \frac{4r|\dot{c}|}{c(r)} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)}\right) \sin\theta \, dr d\varphi d\theta$$

$$< I_2 - \int_{D_{t_1}} \left(\int_{r_0}^{r} \frac{2\zeta^2 \dot{c}(\zeta)}{c^3(\zeta)} d\zeta\right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)}\right) \sin\theta dr d\varphi d\theta \, .$$

Now if we suppose that $c(r)$ is a monotonic increasing function, the second term on the right side of (24) would be strictly positive and we could therefore neglect this term since the left side would then surely be less than I_2. However, if $c(r)$ is not monotonic increasing we proceed as follows. In (10) set $A = B = 0$ and $C = 1$. One thus has

$$\int_0^{t_1} \int_{D_t} w_t \left(\sin\theta \, w_{rr} + \frac{(w_\theta \sin\theta)_\theta}{r^2} + \frac{w_{\varphi\varphi}}{r^2 \sin\theta} - \frac{w_{tt} \sin\theta}{c^2(r)}\right) \frac{dv}{r^2} dt = 0 \, .$$

Integrating as before one obtains

$$\int_0^{t_1} \int_{D_t} \Bigg\{ 2(w_r w_t)_r \sin\theta + 2 \frac{(w_\varphi w_t)_\varphi}{r^2 \sin\theta} + 2 \frac{(w_\theta \sin\theta \, w_t)_\theta}{r^2}$$

$$- \left(w_r^2 \sin\theta - \frac{w_\varphi^2}{r^2 \sin\theta} - \frac{w_t^2 \sin\theta}{c^2(r)} - \frac{w_\theta^2}{r^2} \sin\theta\right)_t \Bigg\} \frac{dv dt}{r^2} = 0 \, .$$

Applying the divergence theorem, there results the following surface integral

$$\int_S \left[\left(2w_t w_r r_v + \frac{2w_\varphi w_t \varphi_v}{r^2 \sin^2\theta} + \frac{2w_\theta w_t \theta_v}{r^2} \right) \sin\theta \right.$$

$$\left. - \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) t_v \sin\theta \right] dS dt = 0.$$

As before, the surface integral is taken over the body B, the initial domain and the domain of influence at $t = t_1$. Since $w = 0$ on B, $w_t = 0$ on B. Therefore the integral reduces to the following identity

$$(25) \quad \int_{D_{t_1}} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta \, dr d\varphi d\theta$$

$$= \int_{D_o} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta \, dr d\varphi d\theta.$$

This identity expresses the notion of conservation of energy.

Now consider the last term in (24).

$$(26) \quad \left| \int_{D_t} \left(\int_{r_o}^{r} 2\zeta^2 \frac{\dot{c}(\zeta)}{c^3(\zeta)} d\zeta \right) \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta dr d\varphi d\theta \right|$$

$$\leq \left[\max_{r_o < r < \infty} \int_{r_o}^{r} \frac{2\zeta^2 |\dot{c}(\zeta)| d\zeta}{c^3(\zeta)} \right] \left[\int_{D_o} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta \, dr d\varphi d\theta \right].$$

The last integral is finite by conditions (3) and (4). By condition (5) IV, one has

$$\max_{r_o<r<\infty} \int_{r_o}^{r} 2\zeta^2 \frac{|\dot{c}(\zeta)|}{c^3(\zeta)} d\zeta \leq \frac{1}{\gamma^2} \int_{r_o}^{\left(\frac{2}{\gamma}\right)^{1/3}} \zeta d\zeta + \frac{2}{\gamma^3} \int_{\left(\frac{2}{\gamma}\right)^{1/3}}^{r} \frac{d\zeta}{\zeta^2}$$

$$\leq \frac{1}{2\gamma^2} \left(\frac{2}{\gamma}\right)^{2/3} + \frac{2}{\gamma^3} \left(\frac{\gamma}{2}\right)^{1/3} .$$

Therefore the left side of (26) is bounded by some finite number which will be called Γ_1.

Choose a sphere O_t which is contained in D_t. Let O_t be centered at the origin and have radius $\epsilon\, \gamma t$, where $0 < \epsilon << 1$. Using (7) and (26) in (24) one obtains

$$(27)\quad t_1^2 (1 - \epsilon)^2 \int_{O_{t_1}} \left(|\nabla w|^2 + \frac{w_t^2}{c^2}\right) \sin\theta \, dr d\varphi d\theta$$

$$- \alpha \int_0^{t_1} t dt \int_{O_t} \left(|\nabla w|^2 + \frac{w_t^2}{c^2}\right) \sin\theta \, dr d\varphi d\theta$$

$$< I_2 + \Gamma_1 + \int_0^{t_1} t dt \int_{D_t - O_t} \frac{4r|\dot{c}|}{c} \left(|\nabla w|^2 + \frac{w_t^2}{c^2}\right) \sin\theta \, dr d\varphi d\theta .$$

Using (5) IV and (25) on the last term in (27) one obtains

$$\int_0^{t_1} tdt \int_{D_t - O_t} \frac{4r|\dot{c}|}{c} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta \, dr d\varphi d\theta$$

$$< \left[\int_{D_o} \left(|\nabla w|^2 + \frac{w_t^2}{c^2} \right) \sin\theta \, dr d\varphi d\theta \right] \left[\int_0^{T_1} 2 \, tdt + \int_{T_1}^{t_1} \frac{4dt}{\epsilon^3 \gamma^4 t^3} \right]$$

$$\leq \left[\int_{D_o} \left(|\nabla w|^2 + \frac{w_t^2}{c^2} \right) \sin\theta \, dr d\varphi d\theta \right] \left[T_1^2 + 2\epsilon^{-3} \gamma^{-4} T_1^{-1} \right]$$

where $T_1 = \epsilon^{-1} \gamma^{-1} (2\gamma^{-1})^{1/3}$. The last expression is finite by (3) and (4) and is bounded by some constant Γ_2.

Now if we set

$$\int_{O_t} \left(|\nabla w|^2 + \frac{w_t^2}{c^2(r)} \right) \sin\theta \, dr d\varphi d\theta = E(O_t),$$

one obtains the following inequality from (27)

$$\text{(28)} \quad t_1^2 (1 - \epsilon)^2 E(O_{t_1}) - \alpha \int_o^{t_1} t \, E(O_t) dt < I_2 + \Gamma_1 + \Gamma_2 = \Gamma_3.$$

This can be rewritten as

$$\text{(29)} \quad t_1^2 \, E(O_{t_1}) - \alpha_1 \int_0^{t_1} t \, E(O_t) \, dt < \Gamma_4$$

where $\alpha_1 = \alpha (1 - \epsilon)^{-2}$

$$\Gamma_4 = \Gamma_3 (1 - \epsilon)^{-2}.$$

Set $t \, E(O_t) = G(t)$ in (29) which becomes now

$$t\,G(t) - \alpha_1 \int_0^t G(\tau)d\tau < \Gamma_4 \quad \text{or}$$

$$(30) \quad t^{1+\alpha_1} \frac{d}{dt}\left(t^{-\alpha_1} \int_0^t G(\tau)d\tau\right) < \Gamma_4.$$

Integrating (30) from $t = 1$ to $t = t_1$ one obtains

$$(31) \quad t_1^{-\alpha_1} \int_0^{t_1} G(\tau)d\tau < \frac{\Gamma_4}{\alpha_1} - \frac{\Gamma_4}{\alpha_1 t^{\alpha_1}} + \int_0^1 \tau\, E(O_\tau)d\tau .$$

[Note: The choice of the initial point is quite arbitrary and $t = 1$ was chosen just for convenience.]

But by (25) the last term in (31) is finite and (31) becomes

$$(32) \quad \int_0^{t_1} G(\tau)d\tau < \left(\frac{\Gamma_4}{\alpha_1} + \Gamma_5\right) t^{\alpha_1}$$

where $\Gamma_5 = \int_0^1 \tau\, E(O_\tau)d\tau$.

Returning to (29) one obtains

$$t_1^2\, E(O_{t_1}) < \Gamma_4 + \alpha_1\left(\frac{\Gamma_4}{\alpha_1} + \Gamma_5\right) t^{\alpha_1}$$

or

$$(33) \quad E(O_{t_1}) < \frac{\Gamma_4}{t_1^2} + \frac{\alpha_1}{t_1^{2-\alpha_1}}\left(\frac{\Gamma_4}{\alpha_1} + \Gamma_5\right).$$

For large t_1, this becomes

$$E(O_{t_1}) < M\, t^{-(2-\alpha_1)}$$

or

$$E(O_{t_1}) < Mt^{-(2-\delta)}$$

$$0 < \epsilon \ll 1.$$

By letting t_1 become large enough, any finite fixed region will eventually be included in O_{t_1}. Thus for large t_1, $R \subset O_{t_1}$ and by the definition of $E\ (O_{t_1})$, one obtains the result as required in (6). Thus if

$$\alpha = \max_{r_o<r<\infty} \frac{4r}{c} |\dot{c}|$$

is less than 2, one can obtain a rate of decay for the energy in any finite region.

Appendix

It is now possible to say something about the decay of the solution u of (1) - (5). We have the following:

Theorem 2: If u is a smooth solution of (1) - (5) in the exterior of a smooth, star-shaped finite body B, then

(34) $|u| < M_2 \, t^{-(1-\delta/2)}$.

The proof makes use of two basic lemmas which will only be stated as they are quite familiar and appear in [2].

Lemma 1. Any solution u of (1) and (5) satisfies the inequality

$$(35)\quad |u(X,Y,Z,t)| < K_1\left(\iiint_R u^2 dxdydz\right)^{1/2} + K_2\left(\iiint_R \frac{u_{tt}^2}{c^2(r)}\, dxdydz\right)^{1/2}$$

where R is the region in space between B and a fixed sphere of radius R depending only on X,Y,Z. The two positive constants K_1, K_2 depend also only on X, Y, Z.

Lemma 2. Let R be a sphere of radius R, then one has

$$(36)\quad \left(\int_{r<R} u^2 dxdydz\right)^{1/2} \leq K_3\left(\int_{r<R} |\nabla u|^2\, dxdydz\right)^{1/2} .$$

Proof: Let u be a solution of (1) - (5). Then u_t is also a solution of (1) - (5) with different functions f and g. Thus u_t also satisfies (6) but with a different constant M_1. By lemmas one and two,

$$(37)\quad |u| < K_1K_3\left(\int_R |\nabla w|^2 \sin\Theta\, dr d\varphi d\Theta\right)^{1/2} + K_2\left(\int_R \frac{w_{tt}^2}{c^2(r)} \sin\Theta\, dr d\varphi d\Theta\right)^{1/2} .$$

Applying (6) to (37) for t such that $R < \epsilon\,\gamma t$, one obtains

$$|u| < M_2 t^{-(1-\delta/2)} .$$

Thus if $\alpha < 2$ one will have a decay for the solution.

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