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ON THE PRINCIPLE OF INVARIANT IMBEDDING AND ONE-DIMENSIONAL NEUTRON MULTIPLICATION

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SUMMARY

In this paper, we wish to introduce a new method of treating problems involving neutron multiplication by fission, with special regard to questions of critical mass and distribution of neutrons. Our results derive from applications of the principle of invariant imbedding, which we have introduced and discussed in two previous papers, the first of which contains a brief discussion of the close interrelation between the method we employ here and the point of regeneration technique employed by Bellman and Harris.

ON THE PRINCIPLE OF INVARIANT IMBEDDING AND ONE-DIMENSIONAL NEUTRON MULTIPLICATION

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§1. Introduction

In this paper, we wish to introduce a new method of treating problems involving neutron multiplication by fission, with special regard to questions of critical mass and distribution of neutrons. Our results derive from applications of the principle of invariant imbedding, which we have introduced and discussed in two previous papers,[1,2] the first of which contains a brief discussion of the close interrelation between the method we employ here and the point of regeneration technique employed in Bellman and Harris[3].

In order to present the essentials of the technique, as free as possible of analytic difficulties, we shall consider first only one-dimensional flux of neutrons. This case is of interest in itself, from both the physical and mathematical point of view.

We shall consider first a one-velocity case, deriving partial differential equations for the generating functions of the probability distribution of "reflected" and "transmitted" neutrons, (those terms will be defined precisely below), and ordinary differential equations for the moments. Although the moment equations can be used to determine critical length, an inspection of the first and second moments shows that these yield only a very imperfect picture of the actual distribution of neutrons. Classical treatment is based on expected values.

We then turn to a two-velocity case, to illustrate the applicability of our methods to the more realistic situations where fission and fission products are energy dependent, as well as situations where several types of particles are involved.

It will be clear from our discussion and previous papers how our procedure can be modified to take into account absorption of neutrons and inhomogeneity of the medium.

In subsequent papers, we shall first consider the dependence on time of the distribution of neutrons and the case of continuous dependence on energy for the one-dimensional case, and then turn to a discussion of the two-dimensional plane slab, the cylindrical case, and the spherical case.

§2. <u>Description of the Process</u>.

Consider a finite one-dimensional "rod", the interval $[x,0]$, constituted of a material with the property that a neutron traversing an infinitesimal interval, $[y+dy,y]$, has, to first order terms, the probability $a\,dy$ of splitting into two neutrons, of the same type as the original, one going to the left and one to the right.

Assuming a neutron, the "trigger" neutron, enters from the left, as pictured below, we wish to determine the number of neutrons emerging from the left, "reflected" neutrons, and the number of neutrons emerging from the right, "transmitted" neutrons.

trigger neutron $\rightarrow$ |———————————|
x $\qquad\qquad\qquad\qquad$ 0

reflected neutrons $\leftarrow$ $\qquad\qquad\qquad$ $\rightarrow$ transmitted neutrons.

§3. Mathematical Formulation.

Let us define the two sequences of probabilities, $\{p_n(x)\}$, $\{q_n(x)\}$, as follows:

$p_n(x)$ = the probability that n particles will be reflected from an interval of length x, over all time, as a result of one neutron entering at x at time 0. (1)

$q_n(x)$ = the probability that n particles will be transmitted through an interval of length x, over all time, as a result of one neutron entering at x at time 0.

The essence of the technique of invariant imbedding is to regard the process described above as a member of a family of processes of the same type. The mathematical statement of this invariance will yield a set of equations governing the process. This is accomplished readily in this case by regarding x as a parameter capable of assuming all positive values.

§4. Basic Functional Equations.

Let us begin by deriving relations connecting members of the sequences $\{p_n(x+\Delta)\}$ and $\{p_n(x)\}$ for small Δ.

Referring to the diagram above, we see that, to first order terms,

$$p_0(x+\Delta) = (1-a\Delta)p_0(x). \qquad (1)$$

To obtain the equations for $p_n(x+\Delta)$, we consider reflection of n neutrons from $[x+\Delta, 0]$ as composed of the following occurrences:

a. transmission of the neutron thru $[x+\Delta, x]$, reflection of n neutrons from $[x, 0]$, transmission of n neutrons thru $[x, x+\Delta]$.

b. transmission of the neutron through $[x+\Delta,x]$, reflection of k neutrons from $[x,0]$, $k=1,2,\ldots,n$, fission of one of these k neutrons in $[x,x+\Delta]$, reflection of $(n-k)$ additional neutrons from $[x,0]$. (2)

c. fission of the neutron in $[x+\Delta,x]$, reflection of $(n-1)$ neutrons from $[x,0]$.

This verbal statement is equivalent to the mathematical relation

$$p_n(x+\Delta) = (1-a\Delta)\left[p_n(x)(1-a\Delta)^n + a\Delta \sum_{k=1}^{n} kp_k(x)p_{n-k}(x)\right] + a\Delta p_{n-1}(x), \quad (3)$$

to first order terms in Δ.*

Letting $\Delta \to 0$, the limiting equations are

$$p_0'(x) = -ap_0(x)$$

$$p_n'(x) = -(n+1)ap_n(x) + ap_{n-1}(x) + a\sum_{k=1}^{n} kp_k(x)p_{n-k}(x), \quad n \geq 1. \quad (4)$$

Similarly, we obtain the recurrence relations

$$q_n'(x) = -anq_n(x) + a\sum_{k=1}^{n} kq_k(x)p_{n-k}(x), \quad n=1,2,\ldots, \quad (5)$$

with $q_0(x) = 0$.

§5. Partial Differential Equations for Generating Functions.

Let us define

$$u(x,r) = \sum_{n=0}^{\infty} p_n(x)r^n, \quad v(x,r) = \sum_{n=1}^{\infty} q_n(x)r^n. \quad (1)$$

The recurrence relations of (4.4) and (4.5) yield the equations

$$u_x = au(r-1) + aru_r(u-1),$$

$$v_x = ar(u-1)v_r. \quad (2)$$

These equations can be explicitly resolved using the method of characteristics. The details are not trivial, however. We shall discuss these matters in detail subsequently.

§6. Critical Length and Moment Equations.

The concept of "critical length" can be introduced as the

* This is equivalent to the fact that in an infinitesimal interval it is sufficient to consider only first order processes.

smallest value of x for which the expected number of reflected and transmitted neutrons is infinite.

Setting

$$U(x) = u_r\big|_{r=1}, \qquad V(x) = v_r\big|_{r=1}, \tag{1}$$

we obtain from (5.2), via differentiation and setting r=1, the equations

$$U'(x) = a(1+U^2), \qquad V'(x) = aUV, \tag{2}$$

with $U(o) = 0$, $V(o) = 1$. From these equations we obtain $U(x) = \tan ax$, $V(x) = \sec ax$, showing that the critical length is $\pi/2a$.

In a similar fashion, we find the differential equation for the second moment,

$$\frac{dU_2}{dx} = 2aU + 2aU^2 + 3aUU_2, \qquad U_2(0) = 0, \tag{3}$$

which shows that $U_2 = O(1/(\cos ax)^3)$ as $x \rightarrow \pi/2a$. It follows that the expected number of neutrons is an unreliable measure for x near $\pi/2a$.

§7. Two-Velocity Case.

Following the usual approximation to the physical situation where a neutron possesses a direction and an energy varying continuously over certain limits, let us assume that there are just two types of neutrons, "fast" and "slow", and that in the fission process, either can give rise to the other.

To simplify the equations, let us assume that fast neutrons have a probability $a_F dy$ of splitting in $\{y+dy, y\}$, and that when they do split there is a probability 1/2 of a fast neutron produced going to the right and a slow neutron produced going to the left, and probability 1/2 of a slow neutron going to the right and a fast neutron to the left. Let the same situation prevail with slow neutrons with a_F replaced by a_S.

Let

$p^{(F)}_{mn}(x)$ = the probability that m fast neutrons and n slow neutrons will be reflected from an interval of length x, over all time, as a result of one fast neutron entering at x at time 0, with $p^{(S)}_{mn}(x)$ defined similarly for a slow trigger neutron. (1)

Then, as above, we have

$$p^{(F)}_{mn}(x+\Delta) = (1-a\Delta)\left[p^{(F)}_{mn}(x)(1-a\Delta)^{m+n} + \Sigma\, p^{(F)}_{k,\ell}(x)\left[\tfrac{1}{2}p^{(S)}_{m-k,n-\ell}(x) + ka_F\,\tfrac{1}{2}p^{(F)}_{m-k,n-\ell}(x)\right]\right] + a\Delta\left[\tfrac{1}{2}\left[p^{(F)}_{m-1,n}(x) + p^{(S)}_{m,n-1}(x)\right]\right], \qquad (2)$$

with a similar equation for $p^{(S)}_{mn}(x+\Delta)$.

Passing to the limit, we obtain recurrence relations which permit us to obtain partial differential equations for the generating functions, $u^{(F)}(x,r,t) = \Sigma\, p^{(F)}_{mn}(x)r^m t^n$ and $u^S(x,r,t) = \Sigma\, p^{(S)}_{mn}(x)r^m t^n$. From these we obtain ordinary differential equations for the moments, and thus can determine critical length. At the present we have not been able to solve these equations explicitly, though numerical solution would be straightforward.

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