THEORY OF GAMES OF STRATEGY

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SUMMARY

This paper contains a general survey of the mathematical theory of zero-sum two-person games. ( ) ~~It was prepared~~ for publication in the Applied Mechanics Reviews.

THEORY OF GAMES OF STRATEGY

In many economic, military, and operations research problems, the game factor dominates—i.e., the outcome or result can only be described in terms of the decisions made by several participants, each having a different objective. The Theory of Games, a relatively new branch of mathematics, analyzes such strategic problems by studying the following theoretical model patterned on actual parlor games such as Chess and Poker: A participant knows that one of several possible events will occur, and with respect to these events he has certain preferences. He lacks full control over the variables which determine the event. Although he has some control, other participants, who have different objectives, also influence the event. Further, the event may be influenced by random elements.

Games of chance have been studied mathematically for many years, and the mathematical theory of probability was developed from their study. Although strategic situations have long been observed and recorded, the first attempt to abstract them into a mathematical theory of strategy was made in 1921 by Emile Borel. The theory was firmly established by John von Neumann in 1928 when he proved the Minimax Theorem. However, it was the publication in 1944 of the impressive work Theory of Games and Economic Behavior, by John von Neumann and Oskar Morgenstern, that stimulated research in the mathematical theory of games.

By a game is meant a set of rules which specify unambiguously the number of players, the moves that each player may or must make under all possible circumstances, the moves that are made for the players by chance, the amount of information available to the players, and the payoff to each player. Von Neumann has given a mathematically precise definition of a game by making use of the notion of partition of sets.

Although each game is initially described in terms of its moves and the amount of information available to the players, we can normalize the game by the introduction of the concept of a strategy. In the actual play of the game, each player, instead of making his decision at each move of the game, may formulate a complete plan for playing the game from beginning to end, for every situation that may arise. Such a plan is referred to as a strategy. It takes into account any information that may become available in the course of the play of the game. No freedom of action is lost through the use of a strategy, since a strategy specifies a player's actions in terms of the information that may be available in accordance with the rules of the game.

Every game can be described in terms of the sets of strategies of the players, where each strategy is a player's method of playing a given game from beginning to end. Every combination of strategies, one from each player, will determine an outcome of the game which is measured by a payoff to each

player. Thus the game is determined by the number of players, their strategies, and their payoff functions. If a player has n different strategies, they may be identified by the numbers 1, 2, ..., n.

The fundamental problem of the theory of games of strategy is the determination of optimal strategies for each player and to evaluate the amount each player can expect to receive. No satisfactory theory exists as yet which solves the problem for an arbitrary number of players. In general, the difficulties are both computational and conceptual. However, the fundamental case of two players with opposing interests is conceptually complete and much progress has been made on the technical problems.

1. Games with Finite Number of Strategies

The mathematical model for games with two players having strictly opposing interests is deceptively simple: Player I chooses a strategy, any one of the numbers $i = 1, 2, \ldots, m$, and Player II chooses a strategy, any one of the numbers $j = 1, 2, \ldots, n$, each choice being made without any knowledge of the other. The payoff to Player I is a function of the chosen strategies, a_{ij}, while the payoff to Player II is $-a_{ij}$. The objective of Player I is to maximize a_{ij}, but he controls only the choice of i, while the objective of his opponent is to maximize $-a_{ij}$ and he controls the choice of j. What are the guiding principles which should determine the choices and what is the expected outcome of the game? We have assumed that the outcome can be measured quantitatively by a number and that the notion of expectation is applicable for this measurement.

Among Player I's strategies there exists some strategy such that he can obtain a payoff of at least $\max_i \min_j a_{ij}$. Player II has some strategy such that he pays no more than $\min_j \max_i a_{ij}$. For every matrix (a_{ij}), we have

$$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}.$$

If the game is such that

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v,$$

then Player I has a strategy which yields him at least v, and he can be kept from getting more than v by the second player. Thus in this case, there are <u>optimal</u> strategies i^* for the first player and j^* for the second player which have the following properties: (1) if Player I chooses i^*, then, no matter what Player II does, Player I will get at least v; (2) if Player II chooses j^*, then, no matter what Player I does, Player I will get at most v; and (3) if Player I were to announce in advance of the play that he plans to use strategy i^*, Player II could not take advantage of this information and thereby reduce Player I's payoff.

A necessary and sufficient condition that $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$ is that the payoff matrix (a_{ij}) have a saddle-point; i.e., that there exists an element of the matrix which is at the same time the minimum of its row and the maximum of its column.

If the payoff matrix (a_{ij}) is such that it contains no saddle-point, and hence neither player has an optimal strategy, it is necessary to generalize the notion of a strategy. A player, instead of choosing a single strategy, chooses a probability distribution over his set of strategies and the particular strategy for the play of the game is chosen by some chance device satisfying this probability distribution. Each probability distribution over the set of strategies is a <u>mixed strategy</u>. This randomization by a player protects him against choosing a strategy which would benefit his opponent. Of course, any strategy can be regarded as a mixed strategy. However, to evaluate the effect of a mixed strategy, the expected value of the effect of the strategies needs to be determined.

Let S_m and S_n be the sets of mixed strategies of Player I and Player II, respectively. Let $E(X, Y)$ be the expected payoff received by Player I if he uses mixed strategy X and Player II uses mixed strategy Y; then it turns out that in any game with a finite number of strategies

$$\max_{X \in S_m} \min_{Y \in S_n} E(X, Y) = \min_{Y \in S_n} \max_{X \in S_m} E(X, Y) = v.$$

This is the Main Theorem of finite games, and was first proved by von Neumann in 1928. It shows that both players have optimal mixed strategies, i.e., there exists a probability distribution for Player I which assures him a gain of at least v on the average, and there exists a probability distribution for Player II

which insures him against a loss of more than v on the average.

2. Solution of Finite Games

The sets of optimal strategies can be characterized geometrically as closed convex polyhedra. Thus to find all optimal strategies we need only determine the extreme points of the convex set. L. S. Shapley and R. N. Snow have shown that each extreme point is associated with some nonsingular square submatrix of the payoff matrix (a_{ij}). Thus every optimal strategy which is an extreme point can be obtained as the unique solution of a suitably chosen subsystem of linear equations.

There exist iterative procedures for approximating optimal mixed strategies. The two most common procedures are due to George W. Brown and John von Neumann. The Brown procedure is a method of fictitious play which bases future decisions on relevant past history. In the von Neumann procedure the steady state solution of systems of differential equations corresponds to the optimal strategies of the game.

3. Games with Infinite Strategies

Although most parlor games require the evaluation of a finite number of strategies, many military and economic games have an infinite number of strategies. For example, in such games it is frequently required to choose an optimal percentage

from an infinite number of percentages. In an infinite game, the two players choose strategies x and y from infinite sets S_1 and S_2. The payoff is the value of a function $M(x, y)$. By analogy with the finite games, if

$$\max_{x \in S_1} \min_{y \in S_2} M(x, y) < \min_{y \in S_2} \max_{x \in S_1} M(x, y)$$

then mixed strategies are required. They are defined as probability distributions F and G over S_1 and S_2. Now Player I's expectation is given by the double integral

$$E(F, G) = \int\int M(x, y)dF(x)dG(y).$$

The existence of optimal mixed strategies, i.e.

$$\max_{F} \min_{G} E(F, G) = \min_{G} \max_{F} E(F, G)$$

now depends on the function $M(x, y)$. If $M(x, y)$ is continuous, with S_1 and S_2 unit intervals $0 \leq x \leq 1$, $0 \leq y \leq 1$, then optimal mixed strategies exist. However, if $M(x, y)$ is discontinuous, optimal strategies need not exist. Even when optimal strategies exist, as in the case of continuous $M(x, y)$, no general method exists for computing them.

The method of solution of infinite games depends upon the functional form of $M(x, y)$. In many applications the payoff

function $M(x, y)$ is strictly convex in y for each x, in which case an optimal strategy for Player II is to choose that y which minimizes $\operatorname{Max}_x M(x, y)$. However, Player I must use a mixed strategy generally consisting of two strategies. The value of the game is $\operatorname{Min}_y \operatorname{Max}_x M(x, y)$.

If the payoff function is a polynomial, i.e., if

$$M(x, y) = \sum_{i,j=0}^{m,n} a_{ij} x^i y^j,$$

then both players have optimal mixed strategies which are step functions. Each player randomizes on $1/2 \min (m + 2, n + 2)$ strategies, at most.

4. Examples of Infinite Games

ALLOCATION OF RESOURCES. The following attack-defense game has interpretations in economics as well as in military planning. Given n targets $T_1, T_2, \ldots, T_n$ whose values are $v_1, v_2, \ldots, v_n$, respectively. Suppose Player I has a total of A attacking units and Player II has a total of D defensive units. How should the players allocate their forces among the n targets? Let us assume that if $x = (x_1, x_2, \ldots, x_n)$, where each x_i is nonnegative and $\sum x_i = A$, is an allocation of Player I's resources among the n targets, and if $y = (y_1, y_2, \ldots, y_n)$ is an allocation of the defender's forces among the targets, then the payoff to Player I is defined by

$$M(x, y) = \sum_{i=1}^{n} v_i \max(0, x_i - y_i).$$

This is an n-1 dimensional infinite game with a continuous payoff and hence optimal strategies exist. If $A \geq D$, then the optimal strategy for the defender is to defend only the high-valued targets and leave low-valued targets undefended. However, the attacker must use a mixed strategy. He selects one of the targets at random, subject to a given probability distribution, and allocates his entire force A to that target. An interesting property of the optimal strategies is that the high-valued targets which are defended are also the targets which may, depending on the outcome of the randomization, receive the concentrated attack. The low-valued targets are undefended and never attacked. Further, the optimal strategies x^* and y^* are such that there exist no soft spots, i.e.,

$$v_i(A - y_i) = v = \text{constant}, \qquad \text{if } y_i > 0$$

$$v_i A < v, \qquad y_i = 0.$$

<u>TIMING OF DECISIONS</u>. Many timing problems can be viewed as two-player games whose rules describe the actions which the players are to take, but the timing of the actions is to be determined by the players. In these games each player wishes to delay the actions as long as possible but he is also penalized for delaying by his opponent. This conflict of interests can be resolved and a best timing of actions can be determined for each player.

Let us assume that each player has to choose only one action time. Let us also assume that each player is informed of his opponent's action as soon as it takes place. Define $P_1(x)$ as the probability that Player I will succeed if he acts at time x, and $P_2(y)$ as the probability that Player II will succeed if he acts at time y. Let the payoff be +1 to the successful player and 0 to both players if both or neither succeeds, then the expected payoff to Player I as a function of action times x and y becomes:

$$M(x, y) = \begin{cases} 2P_1(x) - 1 & \text{if } x < y \\ P_1(x) - P_2(x) & \text{if } x = y \\ 1 - 2P_2(y) & \text{if } x > y. \end{cases}$$

In this case the optimal action time for each player depends on the solution of the equation

$$P_1(t) + P_2(t) = 1.$$

Each player delays his action until t, if his opponent has not acted prior to t. If his opponent has acted prior to t, then the player acts at time x such that $P_1(x) = 1$, i.e., the player waits until he is certain of success.

If in the preceding example we remove the information aspect and assume that each player is ignorant of the action, if any, taken by his opponent, then optimal timing requires randomization by both players. Let us make the additional assumption that $P_1(x) = P_2(x)$, then the payoff function becomes

$$M(x, y) = \begin{cases} -y + (1 + y)x & \text{if } x < y \\ 0 & \text{if } x = y \\ -y + (1 - y)x & \text{if } x > y \end{cases}$$

where x and y are the probabilities of success of Player I and Player II, respectively. The two players have the same optimal strategies—delay until the probability of success is at least 1/3, then act at a time chosen at random subject to the probability distribution

$$\begin{aligned} F(x) &= 0 & 0 \leq x \leq 1/3 \\ &= \frac{1}{8}\left(9 - \frac{1}{x^2}\right) & 1/3 \leq x \leq 1. \end{aligned}$$

Bibliographic Notes

There exist several books on the theory of games of strategy as well as many articles in mathematical journals. Reference 3, below, contains a complete bibliography of books and articles.

Books on the theory of games include

1. John von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior, Princeton (1944).

2. J. C. C. McKinsey, Introduction to the Theory of Games, New York, McGraw-Hill (1952).

3. Harold W. Kuhn and Albert W. Tucker (eds.), Contributions to the Theory of Games, I (1950), II (1953), Princeton. These two volumes contain a complete bibliography of game theory publications.

4. David Blackwell and M. A. Girshick, Theory of Games and Statistical Decisions, New York, Wiley and Sons (1954).

For general reading, books on the theory of games include

1. John McDonald, Strategy in Poker, Business and War, New York, W. W. Norton and Co. (1950).

2. J. D. Williams, The Compleat Strategyst, New York, McGraw-Hill (1954).

Basic technical papers include

1. John von Neumann, "Zur Theorie der Gesellschaftsspiele," Mathematische Annalen 100: 295-320 (1928).

2. Emile Borel and Jean Ville, "Applications aux Jeux de Hasard," Traite Du Calcul Des Probabilites Et De Ses Applications, IV, pt 2 (1938).