ON POSITIVE DEFINITE MATRICES
AND STIELTJES INTEGRALS

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## SUMMARY

Let $X(t)$ be a continuous $n \times n$ symmetrix matrix function of $t$ for $0 \leq t \leq 1$, monotone in the sense that $X(t) - X(s)$ is non-negative definite for $1 \geq t \geq s \geq 0$. Denote by $[X(t) - X(s)]^{1/2}$ the unique non-negative definite square root of $X(t) - X(s)$ for $t \geq s$. Take $0 \leq t_1 \leq t_2 \leq t_N = 1$ to be a sub-division of $[0, 1]$ and consider the sum

$$S_N = \sum_{i=0}^{N-1} [X(t_{i+1}) - X(t_i)]^{1/2} F(t_i)[X(t_{i+1}) - X(t_i)]^{1/2},$$

where $F(t)$ is a given continuous matrix function of $t$ in $[0, 1]$.

It is shown that as $N \longrightarrow \infty$, with $\operatorname{Max}_i (t_{i+1} - t_i) \longrightarrow 0$, $S_n$ converges to a linear matrix function of $F$ which we write

$$L(F) = \int_0^1 (dX)^{1/2} F(t)(dX)^{1/2}.$$

This is a generalized Riemann-Stieltjes integral for matrices.

# ON POSITIVE DEFINITE MATRICES AND STIELTJES INTEGRALS

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## §1. Introduction.

In a recent paper, [1], we considered two generalizations of the Riemann-Stieltjes integral connected with the study of positive definite matrices. One extension was considered in full generality, the other only for $2 \times 2$ matrices.

In this paper, relying upon a result of Ali R. Amir-Moéz, [3], concerning the variational characterization of the eigenvalues of symmetric matrices, we shall complete the second extension.

Our final result is a Riemann-Stieltjes integral for matrices, which can be extended to many other classes of non-commutative hypercomplex number systems. This will be discussed subsequently. The motivation for the present investigation arises from an extension of classical probability theory treated in [2].

## §2. A Riemann-Stieltjes Integral for Matrices.

Let $X(t)$ be a continuous $n \times n$ symmetric matric function of $t$ for $0 \leq t \leq 1$, monotone in the sense that $X(t) - X(s)$ is non-negative definite for $1 \geq t \geq s \geq 0$. Denote by $[X(t) - X(s)]^{1/2}$ the unique non-negative definite square root of $X(t) - X(s)$ for $t \geq s$.

Take $0 \leq t_1 \leq t_2 \leq \cdots \leq t_N = 1$ to be a sub-division of the interval $[0, 1]$, and consider the sum

$$(1) \qquad S_N = \sum_{i=0}^{N-1} [X(t_{i+1}) - X(t_i)]^{1/2} F(t_i) [X(t_{i+1}) - X(t_i)]^{1/2},$$

where $F(t)$ is a given continuous matrix function of $t$ in $[0, 1]$.

We wish to demonstrate the following:

<u>Theorem 1</u>. <u>Let</u> $\max_i (t_{i+1} - t_i) \longrightarrow 0$ <u>as</u> $N \longrightarrow \infty$. <u>Then</u> $S_N$ <u>converges to a linear matric functional of</u> $F$, <u>which we write</u>

$$(2) \qquad L(F) = \int_0^1 (dX)^{1/2} F(t) (dX)^{1/2}.$$

The proof of this result for $(2 \times 2)$ matrices is contained in [1]. Below we shall present a proof of the general result.

§3. <u>Preliminaries</u>.

It is sufficient to indicate the proof for the case where the sub-divisions possess a special form, $t_k = k/N$, with $N$ assuming values of the form $2^M$, $M = 1, 2, \cdots$. In this case, every sub-division is a refinement of the preceding one. Standard techniques used in the scalar theory can be carried over to the matrix case to establish the general result.

Let us now show that $S_N$ is a uniformly bounded matrix function. We have, for any n-dimensional vector $y$,

$$(2) \quad (S_N y, y) = \sum_{i=0}^{N-1} \left( [X(t_{i+1}) - X(t_i)]^{1/2} F(t_i) [X(t_{i+1}) - X(t_i)]^{1/2} y, y \right)$$

$$= \sum_{i=0}^{N-1} \left( F(t_i) [X(t_{i+1}) - X(t_i)]^{1/2} y, \ [X(t_{i+1}) - X(t_i)]^{1/2} y \right).$$

Since $F(t)$ is continuous in $[0, 1]$, we have $(Fz, z) \leq m(z, z)$ for any $z$, for a fixed $m$. Thus

$$(3) \quad (S_N y, y) \leq m \sum_{i=0}^{N-1} \left( [X(t_{i+1}) - X(t_i)]^{1/2} y, \ [X(t_{i+1}) - X(t_i)]^{1/2} y \right)$$

$$\leq m \sum_{i=0}^{N-1} \left( y, \ [X(t_{i+1}) - X(t_i)] y \right)$$

$$\leq m \left( y, \ [X(1) - X(0)] y \right) .$$

This completes the proof of the boundedness of $S_N$.

Since $X(t) - X(s)$ is symmetric, and non-negative definite, for $t \geq s$, we may write

$$(4) \quad X(t) - X(s) = T(t,s) \begin{pmatrix} \lambda_1(t,s) & & & 0 \\ & \lambda_2(t,s) & & \\ & & \ddots & \\ 0 & & & \lambda_n(t,s) \end{pmatrix} T'(t,s),$$

where $\lambda_i(t, s)$ are the characteristic roots of $X(t) - X(s)$, taken for the sake of definitness in the order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and $T(t, s)$ is an orthogonal transformation which may be taken to be continuous in $t$ and $s$ for $1 \geq t \geq s \geq 0$.

Then we may write

$$(5)\quad [X(t)-X(s)]^{1/2} = T(t,s)\begin{pmatrix} \lambda_1(t,s)^{1/2} & & & 0 \\ & \lambda_2(t,s)^{1/2} & & \\ & & \ddots & \\ 0 & & & \lambda_n(t,s)^{1/2} \end{pmatrix} T'(t,s$$

As in [1], we may show that the convergence of $S_N$ depends upon the convergence of sums of the form

$$(6)\qquad S_N^{(k)} = \sum_{i=0}^{N-1} g(t_i)\lambda_k(t_{i+1}, t_i),$$

$$R_N^{(j,k)} = \sum_{i=0}^{N-1} h(t_i)(\lambda_j\lambda_k)^{1/2} ,$$

for $1 \leq j, k \leq n$, where $g(t)$ and $h(t)$ are continuous functions of $t$ in $[0, 1]$. As in [1], it is sufficient to consider the case where $g$ and $h$ are constant.

The convergence of sums of the form $S_N^{(k)}$ has been considered in [1]. It remains to consider the sums $R_N^{(j,k)}$.

§4. Representation of Amir-Moéz - Hoffman.

The result we require to treat the convergence of sums involving terms of the form $(\lambda_j\lambda_k)^{1/2}$ is

Theorem 2. Let $A$ be a non-negative definite matrix with characteristic values $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let $i_1, i_2, \cdots, i_k$ be integers such that $1 \leq i_1 \leq \cdots \leq i_k \leq n$. Then

$$(1)\qquad \lambda_{i_1}\lambda_{i_2} = \sup_{\substack{M_1 \subset M_2 \\ \dim M_p = i_p}} \inf_{\substack{x_p \in M_p \\ \{x_p\}\ o.n.}} \begin{vmatrix} (Ax_1, x_1) & (Ax_1, x_2) \\ (Ax_2, x_1) & (Ax_2, x_2) \end{vmatrix}.$$

This is a particular case of a general result of Amir-Móez, [3], found independently by A. J. Hoffman.

§5. A Lemma.

Finally, we require the simple

Lemma. If A and B are 2x2 non-negative definite matrices, then

$$(1)\qquad [\det(A + B)]^{1/2} \geq (\det A)^{1/2} + (\det B)^{1/2}.$$

A proof of this is given in [1], and is readily established by direct calculation.

§6. Proof of Theorem.

It is easy to see from the inequality $(\lambda_j\lambda_k)^{1/2} \leq (\lambda_j + \lambda_k)/2$, or otherwise, that each sum of the form $R_N^{(j,k)}$ is uniformly bounded for all N. Let us now establish the inequality

$$(1)\qquad R_{2^{M+1}}^{(j,k)} \leq R_{2^M}^{(j,k)} .$$

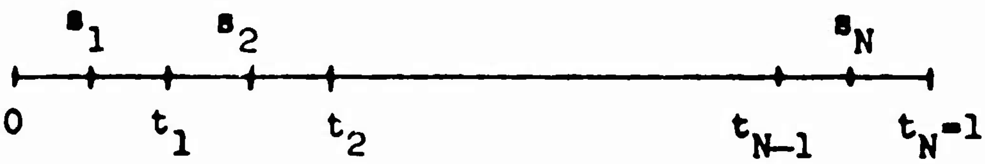
This will demonstrate the convergence of $R_N^{(j,k)}$ for $N = 2^M$.

Using Theorem 2, we see that

$$(2) \qquad [\lambda_j(t_{i+1},t_i)\lambda_k(t_{i+1},t_i)]^{1/2} = \text{Sup Inf} \begin{vmatrix} (K_i x_1, x_1) & (K_i x_1, x_2) \\ (K_i x_2, x_1) & (K_i x_2, x_2) \end{vmatrix}^{1/2},$$

where $K_i = X(t_{i+1}) - X(t_i)$.

Let $s_1, s_2, \cdots, s_N$ be the additional points added to transform the $N^{th}$ sub-division into the $(N+1)^{st}$ sub-division




Since

$$(3) \qquad K_i = X(t_{i+1}) - X(t_i) = [X(t_{i+1}) - X(s_{i+1})] + [X(s_{i+1}) - X(t_i)],$$

we see upon applying the lemma of §5 to the representation in (2) above, that

$$(4) \qquad [\lambda_j(t_{i+1},t_i)\lambda_k(t_{i+1},t_i)]^{1/2} \geq [\lambda_j(t_{i+1},s_{i+1})\lambda_k(t_{i+1},s_{i+1})]^{1/2} + [\lambda_j(s_{i+1},t_i)\lambda_k(s_{i+1},t_i)]^{1/2}.$$

This yields the desired monotonicity and completes the proof of Theorem 1.

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