ON SOME APPLICATIONS OF DYNAMIC PROGRAMMING TO MATRIX THEORY

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P-917

Revised
November 21, 1956

12 p

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SUMMARY

The purpose of this paper is to discuss some applications of the functional equation technique of dynamic programming to some questions of matrix theory.

We shall first consider the solution of a system of linear equations,

$$Ax = b, \tag{1}$$

where A is a Jacobi matrix. Then we ~~shall discuss~~ the same problem for the case where A is "almost" a block-diagonal matrix. Matrices of this type arise in the study of weakly-coupled mechanical or electrical systems. Finally, we ~~shall discuss~~ the calculation of the largest or smallest characteristic values of matrices of this type.

ON SOME APPLICATIONS OF DYNAMIC PROGRAMMING TO MATRIX THEORY

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§1. Introduction

The purpose of this paper is to discuss some applications of the functional equation technique of dynamic programming to some questions of matrix theory.

We shall first consider the solution of a system of linear equations,

$$(1) \qquad Ax = b,$$

where A is a Jacobi matrix. Then we shall discuss the same problem for the case where A is "almost" a block-diagonal matrix. Matrices of this type arise in the study of weakly-coupled mechanical or electrical systems. Finally, we shall discuss the calculation of the largest or smallest characteristic values of matrices of this type.

§2. Jacobi Matrices

There is a large body of literature connected with systems of linear equations of the form

$$(1) \qquad \begin{aligned} &a_{11}x_1 + a_{12}x_2 = b_1, \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ &\quad\vdots \\ &a_{N,N-1}\, x_{N-1} + a_{N,N}\, x_N = b_N. \end{aligned}$$

If $a_{ij} = a_{ji}$, the associated matrix A is called a Jacobi matrix. Assuming that A is positive definite, we wish to obtain the solution of this system in a form quite different from any of the solutions furnished by classical methods.

§3. Functional Equations.

If A is positive definite, the solution of the system in (1) is equivalent to that of determining the minimum of the inhomogeneous form

$$(1) \qquad Q(x) = \sum_{i,j=1}^{N} a_{ij}x_i x_j - 2\sum_{i=1}^{N} b_i x_i .$$

Let us define the auxiliary sequence of functions

$$(2) \qquad f_k(z) = \min_{\{x\}} \left[\sum_{i,j=1}^{k} a_{ij}x_i x_j - 2\sum_{i=1}^{k-1} b_i x_i - 2zx_k \right] ,$$

$k = 1, 2, \ldots, N$, $-\infty < z < \infty$. We wish to determine $f_N(b_N)$ and the point $[x_1, x_2, \ldots, x_N]$ at which the minimum is attained. It is easy to see that

$$(3) \qquad f_1(z) = \min_{x_1} \left[a_{11}x_1^2 - 2zx_1 \right] = -z^2/a_{11}.$$

We now wish to derive a recurrence relation connecting $f_k(z)$ with $f_{k-1}(z)$. If we fix x_k and then minimize over the other x_i, we obtain

$$(4) \qquad f_k(z) = \min_{x_k} \left[a_{kk}x_k^2 - 2zx_k + \min_{\{x_i\}} \left[\sum_{i,j=1}^{k-1} a_{ij}x_i x_j \right.\right.$$
$$\left.\left. -2\sum_{i=1}^{k-2} b_i x_i - 2(b_{k-1} - a_{k-1,k}x_k)x_{k-1} \right] \right]$$
$$= \min_{x_k} \left[a_{kk}x_k^2 - 2zx_k + f_{k-1}(b_{k-1} - a_{k-1,k}x_k) \right].$$

This is an application of the "principle of optimality" cf. [1].

§4. Explicit Relations.

We have thus reduced the determination of the minimizing sequence $\{x_k\}$ to the problem of computing the sequence $\{f_k(z)\}$. With the use of digital computers and systematic search techniques for determining the location of the minimum, this is easily done. However, in this case, we can determine the sequence $\{f_k(z)\}$ in a much more precise fashion.

It is easy to see inductively that each member of the sequence $\{f_k(z)\}$ is a quadratic in z. Hence we set

$$(1) \qquad f_k(z) = u_k + v_k z + w_k z^2,$$

where u_k, v_k, w_k are independent of z.

Substituting in (2.4), we have

$$(2) \qquad f_k(z) = \min_{x_k} \Big[a_{kk} x_k{}^2 - 2 z x_k + u_{k-1} + v_{k-1}(b_{k-1} - a_{k-1,k}$$
$$+ w_{k-1}(b_{k-1} - a_{k-1,k} x_k)^2 \Big].$$

Hence

$$(3) \qquad x_k = \frac{z + v_{k-1} a_{k-1,k} + a_{k-1,k} w_{k-1} b_{k-1}}{a_{kk} + a_{k-1,k}^2 w_{k-1}}$$

Using this value of x_k in (2), we obtain $f_k(z)$ and thus recurrence relations connecting (u_k, v_k, w_k) with $(u_{k-1}, v_{k-1}, w_{k-1})$:

(4)
$$w_k = -\frac{1}{a_{kk} + w_{k-1}a^2_{k-1,k-1}},$$

$$v_k = \frac{-v_{k-1}a_{k-1,k}}{a_{kk} + w_{k-1}a^2_{k-1,k-1}},$$

$$w_k = u_{k-1} + v_{k-1}b_{k-1} + w_{k-1}b^2_{k-1} - \frac{w_k^2\, b^2_{k-1}\, a^2_{k-1,\, k-1}}{a_{kk} + w_{k-1}a^2_{k-1,\, k-1}}.$$

§5. Slightly Intertwined Systems.

Let us now consider the problem of resolving a set of linear equations of the forms

(1)
$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_1x_4 &= c_3,\\
b_1x_3 + a_{44}x_1 + a_{45}x_5 + a_{46}x_6 &= c_4,\\
a_{54}x_4 + a_{55}x_5 + a_{56}x_6 &= c_5,\\
a_{64}x_4 + a_{65}x_5 + a_{66}x_6 + b_2x_7 &= c_6,\\
&\vdots\\
b_{N-1}x_{3N-3} + a_{1+3N,1+3N}x_{1+3N} + a_{1+3N,2+3N}x_{2+3N}&\\
+ a_{1+3N,3+3N}x_{3+3N} &= c_{1+3N},
\end{aligned}$$

$$a_{2+3N,1+3N}x_{1+3N} + a_{2+3N,2+3N}x_{2+3N} + a_{2+3N,3+3N}x_{3+3N} = c_{2+}$$

$$2_{3+3N,1+3N}x_{1+3N} + a_{3+3N,2+3N}x_{2+3N} + a_{3+3N,3+3N}x_{3+3N} = c_{3+}$$

A matrix of the type appearing above, we shall call "slightly intertwined." It arises in a variety of physical, engineering, and economic problems involving multi-component systems with weak coupling.

In addition to the question posed above, we shall also consider the eigenvalue problem. In both cases, we shall assume that the matrix is symmetric, and, in addition, that it is positive definite.

§6. Notation

Let us introduce the matrices

$$(1) \qquad A_k = (a_{i+3k,\ j+3k}), \qquad i,j = 1, 2, 3,$$

for k = 0, 1, 2, ..., and the vectors

$$(2) \qquad x^k = (x_{3k-2}, x_{3k-1}, x_{3k}), \ c^k = (c_{3k-2}, c_{3k-1}, c_{3k})$$

§7. Variational Formulation

Since the matrix of coefficients is, by assumption, positive definite, the solution of the linear system in (1.1) is equivalent to determining the minimum of the inhomogeneous quadratic form

$$(1) \qquad \begin{aligned} &(x^1, A_1x^1) + (x^2, A_2x^2) + \dots + (x^N, A_Nx^N) \\ &+ 2(c^1, x^1) + 2(c^2, x^2) + \dots + 2(c^N, x^N) \\ &+ 2b_1x_3x_4 + 2b_2x_6x_7 + \dots + 2b_{N-1}x_{3N-3}x_{3N-2} \end{aligned}$$

2+3N'

3+3N

V

§8. Dynamic Programming Formulation

For $N = 1, 2, \ldots,$ and $-\infty < z < \infty$, let us introduce the sequence of functions of the variable z defined by

$$(1) \qquad f_N(z) = \min_{x_i} \left[\sum_{i=1}^{N} (x^i, A_i x^i) + 2 \sum_{i=1}^{N} (c^i, x^i) + 2 \sum_{i=1}^{N-1} b_i x_{1+3i} x_{3i} + 2zx_{3N} \right].$$

We then have the following recurrence relation:

$$(2) \qquad f_N(z) = \min_{(x_{3N}, x_{3N-1}, x_{3N-2})} \left[(x^N, A_N x^N) + 2zx_{3N} + 2(c^N, x^N) + f_{N-1}(b_{N-1} x_{3N-2}) \right]$$

This is an application of the "principle of optimality."

§9. Computational Aspects – I

Since the function $f_1(z)$ is readily determined, we can compute the sequence $\{f_k(z)\}$ by means of (4.2), at the expense of a minimization over a 3-dimensional region. This minimization may be greatly speeded up upon using the convexity properties of the functions involved. Although no optimal methods are known for multi-dimensional problems, the one-dimensional method presented in [2] may be employed in an iterative manner.

Writing (6.2) in the form

$$(1) \qquad f_N(z) = \min_{x_{3N-2}} \left[\left\{ \min_{x_{3N}, x_{3N-1}} \left[(x^N, A_N x^N) + 2zx_{3N} + 2(c^N, x^N) \right] \right\} + f_{N-1}(b_{N-1} x_{3N-2}) \right],$$

we see that it reduces to

$$(2) \qquad f_N(z) = \min_{y} \left[g_N(z,y) + f_{N-1}(b_{N-1}y) \right],$$

where

$$(3) \qquad g_N(z,y) = \min_{x_{3N}, x_{3N-1}} \left[(x^N, Ax^N) + 2zx_{3N} + 2(c^N, x^N) \right],$$

upon identifying x_{3N-2} as y. This new relation is now well-suited to the technique described in [2].

The computation of the functions $\{g_N(z,y)\}$ is independent of the computation of the sequence $\{f_N(z)\}$. Observe that this computational approach involves no divisions.

§10. Computational Aspects – II

Another approach to the computational solution reposes upon the easily established fact that $f_N(z)$ is a quadratic in z for each N, i.e.

$$(1) \qquad f_N(z) = U_N + 2V_Nz + W_Nz^2,$$

where U_N, V_N and W_N are independent of z. This is the same device used above.

Substituting in (8.2), we obtain the equation

$$(2) \qquad U_N + V_Nz + W_Nz^2 = \min_{(x_{3N}, x_{3N-1}, x_{3N-2})} \Big[(x^N, A_Nx^N) + 2zx_{3N} + 2(c^N, x^N) + U_{N-1} + 2b_{N-1}x_{3N-2}V_{N-1} + b_{N-1}^2x_{3N-2}^2W_{N-1} \Big]$$

Upon performing the minimization and determining the minimum value of the right-hand side, we obtain recurrence relations connecting the triple (U_N, V_N, W_N) with the triple $(U_{N-1}, V_{N-1}, W_{N-1})$.

This affords an alternative computational technique.

§11. The Eigenvalue Problem

Consider the problem of determining the largest eigenvalue of the matrix appearing in (5.1). This is equivalent to determining the maximum of

$$(1) \qquad Q_N(x) = \sum_{i=1}^{N} (x^i, A_i x^i) + 2 \sum_{i=1}^{N-1} b_i x_{3i} x_{1+3i}$$

over the sphere S_N: $\sum_{i=1}^{N} (x^i, x^i) = 1$.

Define the auxiliary system of functions

$$(2) \qquad f_N(z) = \operatorname*{Max}_{S_N} \left[Q_N(x) + 2z x_{3N} \right],$$

for $-\infty < z < \infty$, and $N = 1, 2, \ldots,$.

Then it is readily seen that

$$(3) \qquad f_N(z) = \operatorname*{Max}_{(x^N, x^N) \leq 1} \left[(x^N, A_N x^N) + \left[1 - (x^N, x^N)\right]^{\frac{1}{2}} f_{N-1}(w_N(x)) \right],$$

where

$$(4) \qquad w_N(x) = 2b_{N-1} x_{3N-2} / (1 - (x^N, x^N))^{\frac{1}{2}}$$

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